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Homotopy Theory and Generalized Duality for Spectral Sheaves

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We would like to express our sadness at the passing of Robert Thomason whose ideas have had a lasting impact on our work. The mathematical community is sorely impoverished by the loss. We dedicate this paper to his memory.

1 Introduction

In this paper we announce a Verdier-type duality theorem for sheaves of spectra on a topological space X. Along the way we are led to develop the homotopy theory and stable homotopy theory of spectral sheaves. Such theories have been worked out in the past, most notably by [Br], [BG], [T], and [J]. But for our purposes these theories are inappropriate. They also have not been developed as fully as they are here. The previous authors did not have at their disposal the especially good categories of spectra constructed in [EKM], which allow one to do all of homological algebra in the category of spectra. Because we want to work in one of these categories of spectra, we are led to consider sheaves of spaces (as opposed to simplicial sets), and this gives rise to some additional technical difficulties.

As an application we compute an example from geometric topology of stratified spaces. In future work we intend to apply our theory to equivariant K-theory.

2 Spectral sheaves

There are numerous categories of spectra in existence and our theory can be developed in a number of them.

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Thus, let $\mathcal{S}p$ be any one of the following categories:

(1) The category of spectra as constructed by [LMS].

(2) The category of S-module spectra of [EKM].

(3) The category of A-module spectra where A is an S-algebra spectrum [EKM].

(4) The category of commutative A-module spectra where A is a commutative S-algebra spectrum [EKM].

Each of these categories has a closed model category structure described in the works cited above and we will make use of these when we need to.

Let X be a topological space. Let $\mathcal{PS}p(X)$ be the category of presheaves of spectra from $\mathcal{S}p$.

Definition 2.1. A presheaf $\mathcal{F}$ is called a *sheaf* if and only if the following diagram is an equalizer:

$$\Gamma(U,\mathcal{F}) \to \prod_{\alpha} \Gamma(U_\alpha,\mathcal{F}) \rightrightarrows \prod_{\alpha_0,\alpha_1} \Gamma(U_{\alpha_0} \cap U_{\alpha_1},\mathcal{F})$$

for any open set U in X and open cover $\{U_\alpha\}$ of U. We let $\mathcal{S}p(X)$ denote the category of sheaves on X of spectra from $\mathcal{S}p$.

The next proposition describes the relationship between the category of sheaves and presheaves.

Proposition 2.1. The forgetful functor $l\colon \mathcal{S}p(X) \to \mathcal{PS}p(X)$ has a left adjoint

$$a\colon \mathcal{PS}p(X) \to \mathcal{S}p(X). \qquad \square$$

The functor a is called the *sheafification functor*. The proof that a exists is more involved than in the classical case of abelian sheaves. Instead of a two pass procedure as in Artin [Ar], it is accomplished with a transfinite number of passes, similarly to the spectrification functor in [LMS], Appendix. As usual, when doing a construction on sheaves that takes you out of the category of sheaves, one performs the sheafification functor to force one back.

For $\mathcal{F} \in \mathcal{PS}p(X)$ and $x \in X$ the stalk $\mathcal{F}_x$ is defined as usual by $\mathcal{F}_x \equiv \operatorname{co}\lim_{U \ni x} \Gamma(U,\mathcal{F})$ where the colimit is taken in $\mathcal{S}p$. Similarly the homotopy stalk is defined by $\operatorname{ho}\mathcal{F}_x = \operatorname{hoco}\lim_{U \ni x} \Gamma(U,\mathcal{F})$. Since sequential colimits and homotopy groups do not commute in general, there is indeed a difference between the stalk and the homotopy stalk. For two presheaves $\mathcal{F},\mathcal{G} \in \mathcal{PS}p(X)$, define a presheaf $\mathcal{F} \wedge_{\mathcal{P}} \mathcal{G}$ by

$$\Gamma(U,\mathcal{F} \wedge_{\mathcal{P}} \mathcal{G}) = \Gamma(U,\mathcal{F}) \wedge \Gamma(U,\mathcal{G}).$$

Here the smash product of spectra is to be taken in the appropriate category. Thus for example, if we are dealing with sheaves of A-module spectra (A a commutative S-algebra spectrum in the sense of [EKM]), then $\wedge$ means $\wedge_A$. If $\mathcal{F}, \mathcal{G} \in \mathcal{S}p(X)$ are sheaves, then we denote by $\mathcal{F} \wedge \mathcal{G}$ the sheafification of $\mathcal{F} \wedge_{\mathcal{P}} \mathcal{G}$.

For two presheaves $\mathcal{F}, \mathcal{G} \in \mathcal{PS}p(X)$ (or sheaves) define $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ by the requirement that the following diagram is an equalizer:

$$\Gamma(U, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \to \prod_{V \subseteq U} F(\Gamma(V, \mathcal{F}), \Gamma(V, \mathcal{G})) \overset{f}{\underset{g}{\rightrightarrows}} \prod_{W \subseteq V \subseteq U} F(\Gamma(U, \mathcal{F}), \Gamma(W, \mathcal{G})).$$

Here f is induced by the restriction map $r\colon \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$ and g is induced by the restriction $r\colon \Gamma(V, \mathcal{G}) \to \Gamma(W, \mathcal{G})$ and F denotes the function spectra between two spectra (in the appropriate category, as per the remarks above). Notice that if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then so is $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Proposition 2.2. (1) Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathcal{PS}p(X)$. Then one has

$$\mathcal{H}om(\mathcal{F} \wedge_{\mathcal{P}} \mathcal{G}, \mathcal{H}) \cong \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})).$$

(2) If $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathcal{S}p(X)$, then one has

$$\mathcal{H}om(\mathcal{F} \wedge \mathcal{G}, \mathcal{H}) \cong \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})).$$

Thus the functor $\cdot \wedge_{\mathcal{P}} \mathcal{G}$ (and $\cdot \wedge \mathcal{G}$) is left adjoint to $\mathcal{H}om(\mathcal{G}, \cdot)$ in the category $\mathcal{PS}p(X)$ or ($\mathcal{S}p(X)$ resp.). □

Corollary 2.3. If $\mathcal{S}p$ denotes either category (2) or (4), then categories $\mathcal{PS}p(X)$ and $\mathcal{S}p(X)$ are symmetric monoidal categories.

Limits and colimits in the category of presheaves are created sectionwise. Limits in the category of sheaves are created at the presheaf level and colimits are obtained from the presheaf colimits by application of the sheafification functor. Thus we have the following proposition.

Proposition 2.4. The categories $\mathcal{PS}p(X)$ and $\mathcal{S}p(X)$ are complete and cocomplete. □

Given $f\colon X \to Y$ a continuous map between topological spaces, we construct the various functors usually associated with such a map:

(1) $f_*\colon \mathcal{S}p(X) \to \mathcal{S}p(Y)$ direct image,
(2) $f^{-1}\colon \mathcal{S}p(Y) \to \mathcal{S}p(X)$ inverse image,
(3) $f_!\colon \mathcal{S}p(X) \to \mathcal{S}p(Y)$ direct image with proper supports, (when X and Y are locally compact).

The constructions of these functors are not so different from their classical counterparts. We sketch only the construction of $f_!$. Let $f\colon X \to Y$ be a continuous map between locally compact spaces. Let $U \subseteq Y$ and $V \subseteq f^{-1}(U)$, U open and V closed in X. We say V is U-relatively compact if the map $f\colon V \to U$ is proper. Let $\mathcal{F} \in \mathcal{S}p(X)$. Define the functor, $\Gamma_V(U,\mathcal{F})$, of sections with supports in V as the fiber

$$\Gamma_V(U,\mathcal{F}) \to \Gamma(U,\mathcal{F}) \xrightarrow{r_{U,U\setminus V}} \Gamma(U\setminus V,\mathcal{F}).$$

(Note however that the restriction map need not be a (q)-fibration.) It is clear that for a closed set $V' \subseteq V$ one has a map $\Gamma_{V'}(U,\mathcal{F}) \to \Gamma_V(U,\mathcal{F})$ and this map is a (spacewise) inclusion. Now define $f_!\colon \mathcal{S}p(X) \to \mathcal{S}p(Y)$ by

$$\Gamma(U, f_!\mathcal{F}) = \operatorname{co}\lim_{V'} \Gamma_{V'}(f^{-1}(U),\mathcal{F})$$

where $U \subseteq Y$ and the colimit is taken over all U-relatively compact subsets $V' \subseteq f^{-1}(U)$. Then, in fact, $f_!\mathcal{F}$ is a sheaf. A particularly important example of pushforward with proper support is the pushforward with proper supports of a sheaf $\mathcal{F}$ on an open subset $U \subseteq X$ to X which we denote by $\mathcal{F}_U$. More concretely, $\Gamma(W,\mathcal{F}_U) \equiv \Gamma(W,\mathcal{F})$ if $W \subseteq U$ and is the trivial spectrum if no connected component of W sits inside U.

We summarize the properties of these functors in the following proposition.

Proposition 2.5. Let $f\colon X \to Y$ be a continuous map between topological spaces, $\mathcal{F},\mathcal{G},\mathcal{H} \in \mathcal{S}p(X)$. Then

(1) $f^{-1}\colon \mathcal{S}p(Y) \to \mathcal{S}p(X)$ is left adjoint to $f_*\colon \mathcal{S}p(X) \to \mathcal{S}p(Y)$.
(2) There are natural isomorphisms $f_*\mathcal{H}om(f^{-1}\mathcal{F},\mathcal{G}) \cong \mathcal{H}om(\mathcal{F}, f_*\mathcal{G})$.
(3) $(fg)^{-1} = g^{-1}f^{-1}$.
(4) $(fg)_* = f_*g_*$.
(5) $(fg)_! = f_!g_!$. □

3 Homotopy theory of sheaves and presheaves

In this section we study the homotopy theory of the category of presheaves and sheaves.

3.1 Presheaves

Let $\mathcal{F} \in \mathcal{PS}p(X)$. The presheaf $\widetilde{\pi}.(\mathcal{F})$ of graded abelian groups on X is defined by $\Gamma(U,\widetilde{\pi}.(\mathcal{F})) = \widetilde{\pi}.(\Gamma(U,\mathcal{F}))$ for U open in X and we will call it the *homotopy presheaf* of $\mathcal{F}$. Similarly, if $F \in Sp(X)$ define the *homotopy sheaf* of $\mathcal{F}$ to be the sheafification of $\widetilde{\pi}.(\mathcal{F})$, and denote it by $\pi.(\mathcal{F})$.

A map $f\colon \mathcal{F} \to \mathcal{G}$ in $\mathcal{PS}p(X)$ is called a *weak equivalence* if the induced map $f\colon \tilde{\pi}.(\mathcal{F}) \to \tilde{\pi}.(\mathcal{G})$ is an isomorphism of presheaves. And a map $f\colon \mathcal{F} \to \mathcal{G}$ in $\mathcal{S}p(X)$ is a weak equivalence if the induced map on $\pi.(\mathcal{F}) \to \pi.(\mathcal{G})$ is an isomorphism of sheaves, or equivalently, it induces a weak equivalence on homotopy stalks. Consider the constant sheaf with stalks isomorphic to $\Sigma^\infty(I_+)$ where I_+ denotes the unit interval plus a base point and Σ^∞ denotes the associated suspension spectrum. We will denote this sheaf by $\mathbf{I}_+$.

Let $\mathcal{F}, \mathcal{G} \in \mathcal{S}p(X)$. A homotopy in $\mathcal{S}p(X)$ is a map $\varphi\colon \mathcal{F} \wedge \mathbf{I}_+ \to \mathcal{G}$. The maps f and g are called homotopic if there is a homotopy connecting them in the usual manner. It is easy to see that this determines an equivalence relation on maps in $\mathcal{S}p(X)$. At this point, we will need to use the closed model category structure on Sp. Thus if we are working in the category of S-module spectra of [EKM], then a *q-fibration* is a map of S-module spectra $f\colon E \to F$ such that the diagonal arrow exists in any diagram of the form

$$\begin{array}{ccc} CS^n \wedge \{0\}_+ & \to & E \\ \downarrow & \nearrow & \downarrow \\ CS^n \wedge \mathbf{I}_+ & \to & F. \end{array}$$

We will preserve the q-terminology of [EKM], reserving the words fibration and cofibration to mean the classical concepts, namely, maps satisfying the homotopy lifting and extension properties respectively. A map $f\colon \mathcal{E} \to \mathcal{F}$ in $\mathcal{PS}p(X)$ is called a *q-fibration* if and only if for any two open sets $V \subseteq U \subseteq X$ the map $\Gamma(U, \mathcal{E}) \to \Gamma(U, \mathcal{F}) \times_{\Gamma(V,\mathcal{F})} \Gamma(V, \mathcal{E})$ is a q-fibration in $\mathcal{S}p$. A map $\mathcal{G} \to \mathcal{H}$ is called a *q-cofibration* if it has the left lifting property with respect to all acyclic fibrations. That is, any diagram of the form, where $\mathcal{E} \to \mathcal{F}$ is an acyclic fibration,

$$\begin{array}{ccc} \mathcal{G} & \to & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{H} & \to & \mathcal{F} \end{array}$$

admits a lifting $\mathcal{H} \to \mathcal{E}$.

Theorem 3.1. The category $\mathcal{PS}p(X)$ with these notions of weak equivalences, q-cofibrations, and q-fibrations forms a closed model category.

It is important to have on hand a large supply of q-cofibrations. To that end, we say that a map $f\colon \mathcal{F} \to \mathcal{G}$ in $\mathcal{PS}p(X)$ represents $\mathcal{G}$ as an $\mathcal{F}$-*relative cell presheaf* if $\mathcal{G} = \operatorname{colim} \mathcal{F}_n$ where $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_n$ is obtained from $\mathcal{F}_{n-1}$ as the pushout of a sum of attaching maps $(S^q_U \vee_{S^q_V} CS^q_V) \to \mathcal{F}_n$ along the coproduct of the natural maps $(S^q_U \vee_{S^q_V} CS^q_V) \to CS^q_U$ and where $V \subseteq U \subseteq X$ are open sets and S^q_U denotes the pushforward with proper supports of the constant sheaf on U with stalk S^q and $C\mathcal{F}$ denotes the cone of $\mathcal{F}$. Such a map is a q-cofibration. This also gives meaning to a cell presheaf. Thus a cell $(e_p, \partial e_p)$ is a pair of presheaves $(CS^{p-1}_U, S^{p-1}_U \vee_{S^{p-1}_V} CS^{p-1}_V)$. Unlike the case of topological spaces or spectra, the cell $e_p = CS^{p-1}_U$ does not determine its boundary $S^{p-1}_U \vee_{S^{p-1}_V} CS^{p-1}_V$ so we have to incorporate it into the definition of a cell. In the proof of the next proposition, the following formula for the smash product of two cells is useful: For two cells $(e_p, \partial e_p)$ and $(e_q, \partial e_q)$, $(e_p \wedge e_q, e_p \wedge \partial e_q \cup \partial e_p \wedge e_q) \cong (e_{p+q}, \partial e_{p+q})$ where

$$\partial e_{p+q} = S^{p+q-1}_{U\cap U'} \vee_{S^{p+q-1}_{(V\cap U')\cup(V'\cap U)}} CS^{p+q-1}_{(V\cap U')\cup(V'\cap U)}.$$

So the smash product of two cells is again a cell.

Propostition 3.2. Let $\mathcal{F}, \mathcal{G} \in \mathcal{PS}p(X)$ be cell presheaves. Then $\mathcal{F} \wedge \mathcal{G}$ is also a cell presheaf with the sequential filtration $(\mathcal{F} \wedge \mathcal{G})_n = \bigcup_{p+q=n}(\mathcal{F}_p \wedge \mathcal{G}_q)$, $n \geq 0$. □

We can refine Theorem 3.1 using the notion of topological closed model category.

Theorem 3.3. The category $\mathcal{PS}p(X)$ has the structure of a topological closed model category. □

At this point we introduce the homotopy category $ho\mathcal{PS}p(X)$. It is a triangulated category. Recall that a *localizing subcategory* D in a triangulated category C is a full subcategory of C closed under the formation of arbitrary coproducts and such that if any two members of a distinguished triangle belong to D, then so does the third. It is clear that the only localizing subcategory of $ho\mathcal{PS}p(X)$ containing the collection $\{S_U\}_{U\subseteq X}$ in $ho\mathcal{PS}p(X)$ is $ho\mathcal{PS}p(X)$ itself. Indeed, using the operations of taking the homotopy cofiber and coproducts, one can build up any cell and then one can approximate any presheaf by a cell presheaf. It is also clear that the objects S_U are small, that is for any collection of presheaves $\{\mathcal{F}i\}$, the following equality holds in $ho\mathcal{PS}p(X)$:

$$\left[S_U, \coprod \mathcal{F}_i\right] = \coprod_i [S_U, \mathcal{F}_i]\,.$$

As usual, $[F, G]$ means homotopy classes of maps, that is, Hom in the homotopy category. The category $ho\mathcal{PS}p(X)$ inherits the structure of a symmetric monoidal category from $\mathcal{PS}p(X)$. The functors smash product and internal Hom are exact because they are sectionwise exact. Summarizing, we get the following theorem.

Theorem 3.4. The category $ho\mathcal{PS}p(X)$ is a triangulated category generated by the sheaves S_U with a symmetric monoidal structure compatible with the triangulation (in the sense of [HPS]) in which coproducts of arbitrary families exist. Furthermore, every cohomology functor on $ho\mathcal{PS}p(X)$ is representable. □

In other words, the category $ho\mathcal{PS}p(X)$ is almost a stable homotopy category in the sense of [HPS] except that it does not have a set of strongly dualizable generators. The main thing to check is the representability of cohomology functors. And this follows from the standard proof, [HPS], which only uses the smallness of the generators S_U and not strong dualizabililty.

3.2 Sheaves

We now turn to the category $\mathcal{S}p(X)$ which is of primary interest to us. We already have the notion of weak equivalence in $\mathcal{S}p(X)$. Say that a map $f\colon \mathcal{F} \to \mathcal{G}$ in $\mathcal{S}p(X)$ is a q-fibration if it is one in the category $\mathcal{PS}p(X)$. Say that a map in $\mathcal{S}p(X)$ is a q-cofibration if it has the left lifting property (LLP) with respect to all acyclic q-fibrations in $\mathcal{S}p(X)$. The notion of relative cell sheaf and cell sheaf is inherited from the category of presheaves by sheafification. A q-fibrant sheaf will also be called *flabby*. We cannot hope that $\mathcal{S}p(X)$ will be a closed model category without any assumptions on the space X. To that end we propose to consider two essentially different types of spaces for which the category of sheaves will be a closed model category.

(1) X is locally compact and any point of X has a neighborhood embeddable into $\mathbb{R}^n$ (for some n, which can depend on the point of X). We call such spaces *locally embeddable*. If X embeds in $\mathbb{R}^n$, we call X *embeddable*.

(2) X is a locally Noetherian topological space of locally finite Krull dimension.

Theorem 3.5. For a space X of one of the two types above, the category $\mathcal{S}p(X)$ is a topological closed model category.

To avoid notational quagmires, we will deal from now on with the situation of a space X locally embeddable in $\mathbb{R}^n$. Recall that we have an adjoint pair of functors

$$\mathcal{PS}p(X) \underset{\iota}{\overset{a}{\rightleftarrows}} \mathcal{S}p(X)$$

between the categories of sheaves and presheaves. By definition, ι preserves q-fibrations. It also preserves acyclic q-fibrations. This formally implies that ι is left exact and a is right exact in the sense of Quillen, [Q], that is ι preserves weak equivalences between fibrant objects and a preserves weak equivalences between cofibrant objects. Furthermore it implies that the total derived functors (see the next section for a discussion of derived functors) $La\colon ho\mathcal{PS}p(X) \to ho\mathcal{S}p(X)$ and $R\iota\colon ho\mathcal{S}p(X) \to ho\mathcal{PS}p(X)$ exist and form an adjoint pair of functors. Thus to obtain a homotopically invariant sheafification of a presheaf, one must first replace it by a cofibrant presheaf (e.g., a cell presheaf). Recall the notion of a (Bousfield) localization functor, [HPS], which is a pair (L, ι) of an endofunctor

$L\colon ho\mathcal{PS}p(X) \to ho\mathcal{PS}p(X)$ and a natural transformation $\iota\colon Id \to L$ such that

(1) the natural transformation $L\iota\colon L \to L^2$ is an equivalence,

(2) for all objects $\mathcal{F}, \mathcal{G}$ the map $[L\mathcal{F}, L\mathcal{G}] \to [\mathcal{F}, L\mathcal{G}]$ is an isomorphism, and

(3) if $L\mathcal{F} = 0$, then $L(\mathcal{F} \wedge \mathcal{G}) = 0$ for all $\mathcal{G}$.

Proposition 3.6. The functor $Rl \circ La$ considered as an endofunctor of $ho\mathcal{PS}p(X)$ is a Bousfield localization. The subcategory of local objects with respect to $Rl \circ La$ is $ho\mathcal{S}p(X)$ and the category of $Rl \circ La$-acyclic objects will be the subcategory of presheaves with trivial homotopy stalks. □

The following proposition is an easy consequence of the definition of a localization functor and the fact that Brown representability holds in $ho\mathcal{PS}p(X)$.

Proposition. In $ho\mathcal{S}p(X)$, any cohomology functor is representable. □

4 Derived functors

In this paragraph we will show how to derive the natural functors on sheaves that were considered in Section 2. Throughout the rest of this paper we will assume X is locally embeddable in $\mathbb{R}^n$, though the results still hold for finite Krull dimension Noetherian spaces. We will call a functor $F\colon C \to D$ from a closed model category C to another category D *left (resp. right) exact* if it takes weak equivalences between fibrant (resp. cofibrant) objects into isomorphisms in D. For the functor F to have a right (resp. left) derived functor $RF\colon hoC \to D$ (and resp. $LF\colon hoC \to D$), it is sufficient it be left (resp. right) exact. In that case the derived functor is formed in the following manner. For $\mathcal{F} \in C$, find a weak equivalence $\mathcal{F} \to \mathcal{I}$ (resp. $\mathcal{P} \to \mathcal{F}$) where $\mathcal{I}$ is a fibrant object (resp. $\mathcal{P}$ is cofibrant) and define $RF(\mathcal{F}) = F(\mathcal{I})$ (resp. $LF(\mathcal{F}) = F(\mathcal{P})$).

Proposition 4.1. Let X_1 and X_2 be topological spaces and $f\colon X_1 \to X_2$ a continuous map. Then the functor $f^{-1}\colon \mathcal{S}p(X_2) \to \mathcal{S}p(X_1)$ admits a left derived functor $Lf^{-1}\colon ho\mathcal{S}p(X_2) \to ho\mathcal{S}p(X_1)$. The functor $f_*\colon \mathcal{S}p(X_1) \to \mathcal{S}p(X_2)$ admits a right derived functor $Rf_*\colon ho\mathcal{S}p(X_1) \to ho\mathcal{S}p(X_2)$. Moreover, the functors Lf^{-1} and Rf_* are adjoint. □

Corollary 4.2. The functors of taking the stalk at a point $x \in X$, $\mathcal{F} \to \mathcal{F}_x$ and the global sections functor $\mathcal{F} \to \Gamma(X, \mathcal{F})$ are right and left exact respectively. We can thus derive these functors to get $L\mathcal{F}_x\colon ho\mathcal{S}p(X) \to ho\mathcal{S}p$ and $R\Gamma(X, \mathcal{F})\colon ho\mathcal{S}p(X) \to ho\mathcal{S}p$. □

We denote $R\Gamma(X, \mathcal{F})$ by $\mathbb{H}(X, \mathcal{F})$ and $H^i(X, \mathcal{F}) = \pi_{-i}(\mathbb{H}(X, \mathcal{F}))$. To get a derived stalk functor, one first approximates the given sheaf by a cofibrant one (e.g., a cell sheaf) and takes the stalk at the given point. At first, it may appear odd to derive the stalk functor, since classically, exactness is defined in terms of stalks. In the context of ordinary sheaves or simplicial sheaves as in [Br] or [J], this derived functor never appears since the stalk functor is exact. Now recall that the homotopy stalk $\mathcal{F}$ at x is defined as $\operatorname{hoco\,lim}_{x\in U} \Gamma(U, \mathcal{F})$. It is not clear that for a cell sheaf the homotopy stalk and the actual stalk are weakly equivalent. Nevertheless, this is in fact true.

Proposition 4.3. The functor $f_!\colon \mathcal{S}p(X_1) \to \mathcal{S}p(X_2)$ is left exact and therefore there exists a right derived functor $Rf_!\colon \operatorname{ho}\mathcal{S}p(X_1) \to \operatorname{ho}\mathcal{S}p(X_2)$. □

Proposition 4.4. Suppose that $\mathcal{S}p$ is either (2) or (4). Then the functor $\wedge\colon \mathcal{S}p(X) \times \mathcal{S}p(X) \to \mathcal{S}p(X)$ is right exact in either of its two variables. Therefore, there exists a left derived functor $\overset{L}{\wedge}\colon \operatorname{ho}\mathcal{S}p(X) \times \operatorname{ho}\mathcal{S}p(X) \to \operatorname{ho}\mathcal{S}p(X)$. □

Proposition 4.5. Let $\mathcal{I} \in \mathcal{S}p(X)$ be a flabby sheaf or $\mathcal{P} \in \mathcal{S}p(X)$ a cell or more generally a q-cofibrant sheaf. Then the functors $\mathcal{H}om(\mathcal{P}, \cdot)$ and $[\mathcal{P}, \cdot]$ are left exact and $\mathcal{H}om(\cdot, \mathcal{I})$ and $[\cdot, \mathcal{I}]$ are right exact. Thus, $\mathcal{H}om$ and $[\cdot, \cdot]$ can be derived by resolving the first variable by cell sheaves and the second variable by flabby ones. □

In nature we often see presheaves of spectra (or spaces), not sheaves. (This is probably one of the reasons that topological sheaves have not been studied systematically until now.) It is convenient to extend the class of fibrant sheaves allowing certain types of presheaves which are suitable for forming fibrant resolutions.

Definition 4.1. A presheaf $\mathcal{P} \in X$ is called *quasiflabby* if the following diagram is a homotopy equalizer:

$$\Gamma(U, \mathcal{P}) \to \prod_{\alpha} \Gamma(U_\alpha, \mathcal{P}) \rightrightarrows \prod_{\alpha_0, \alpha_1} \Gamma(U_{\alpha_0} \cap U_{\alpha_1}, \mathcal{P})$$

for any open set U in X and open cover $\{U_\alpha\}$ of U.

Remark. In the context of schemes this notion of a quasiflabby sheaf was introduced by Brown and Gersten, [BG] who called it pseudoflasque.

Proposition 4.6. The homotopy invariant sheafification of a quasiflabby presheaf $\mathcal{P}$ is sectionwise weakly equivalent (i.e., in the category $\operatorname{ho}\mathcal{P}\mathcal{S}p(X)$) to the presheaf $\mathcal{P}$. □

The proof is essentially the same as in [BG].

Theorem 4.7. Let X be a locally compact embeddable space. For $\mathcal{E} \in \mathcal{S}p(X)$, there is a strongly convergent local to global spectral sequence

$$E_2^{pq} = H^p(X, \pi_q(\mathcal{E})) \Rightarrow \pi_{q-p}\mathbb{H}(X, \mathcal{E}).$$

The differentials are of bidegree $(r, r-1)$. The spectral sequence respects all the structure of $\mathcal{S}p$. For example, if E is a sheaf of ring spectra, then the spectral sequence is a spectral sequence of algebras. □

The proof of the theorem follows the pattern of Brown and Gersten, [BG], by filtering the sheaf E by Postnikov stages. Since we consider general (possibly non-connective) spectra, their Postnikov towers may be infinite in both directions and special care must be taken to ensure the proper behavior of the direct limit.

5 Duality

We now take up the discussion of duality in the category $\mathcal{S}p(X)$ or, more properly, in its homotopy category $ho\mathcal{S}p(X)$. Here we will assume we are working in the category (3) of A-module spectra where A is an S-algebra spectrum [EKM], and we will denote this category $\mathcal{S}p_A(X)$ to emphasize the presence of A. Let X and Y be two locally compact, locally embeddable spaces.

Theorem 5.1. Let $f\colon X \to Y$. In the category $ho\mathcal{S}p_A(X)$, the functor $Rf_!$ has a right adjoint $f^!\colon ho\mathcal{S}p_A(Y) \to ho\mathcal{S}p_A(X)$. In addition, if A is a commutative S-algebra spectrum, then $\mathcal{RH}om(Rf_!\mathcal{F}, \mathcal{G}) \cong Rf_*\mathcal{RH}om(\mathcal{F}, f^!\mathcal{G})$ in $ho\mathcal{S}p_A(Y)$. □

The existence of $f^!$ is equivalent to the representability of the functor $\mathcal{F} \longmapsto \mathrm{Hom}(Rf_!\mathcal{F}, \mathcal{G})$ for any $\mathcal{G} \in ho\mathcal{S}p_A(Y)$. Since in $ho\mathcal{S}p_A(Y)$ any cohomology functor is representable, it follows from the fact that $Rf_!$ respects coproducts. The idea of using Brown representability to derive duality results can be found in [N].

For $f\colon X \to pt.$ denote also by A the constant sheaf $f^{-1}A$ and by $\mathbb{D}_A = f^!A$ and is called the dualizing sheaf of X over A. For any sheaf $\mathcal{F} \in \mathcal{S}p_A(X)$ we define its dual $\mathbb{D}_A\mathcal{F}$ to be the sheaf $\mathcal{H}om(\mathcal{F}, \mathbb{D}_A)$. (Notice our abuse of notation in using $\mathbb{D}_A$ for both the dualizing sheaf and the dualizing operator.)

Proposition 5.2. Let X be locally contractible in the sense that any point $x \in X$ has a fundamental system of contractible neighborhoods. Then the following isomorphism holds in the category of spectra

$$R\Gamma(X, A) = F(\Sigma^\infty U_+, A).$$

In particular, $H^\cdot(X, A) = A^\cdot(X)$, the generalized cohomology of X. □

We now give a more concrete description of the sheaf $\mathbb{D}_A$.

Proposition 5.3. In the category $ho\mathcal{S}p_A(X)$, the dualizing sheaf $\mathbb{D}_A$ is isomorphic to the sheafification of the presheaf $U \longmapsto F_A(R\Gamma_c(U, A), A)$.

Theorem 5.4. Let X be a compact manifold of dimension n, A an S-algebra spectrum. Suppose that X has a fundamental class with respect to the generalized cohomology theory determined by A. Then the dualizing sheaf is isomorphic to $\Sigma^n A$ the nth suspension of the constant sheaf A. □

5.1 Duality on stratified spaces

Let $X = X_n \supset X_{n-1} \supset \cdots \supset X_{-1} = \emptyset$ be a stratified space as in [B]. It is required that any point $x \in X_k \backslash X_{k-1}$ admit a neighborhood $U_x = B_k \times \overset{\circ}{c}L$, where B_k is homeomorphic to a k-dimensional ball and $\overset{\circ}{c}L$ is an open cone on a stratified space of dimension $n - k - 1$. We call such a neighborhood a distinguished neighborhood.

A sheaf $\mathcal{F} \in \mathcal{S}p_A(X)$ is called *constructible* if for any $x \in X_k \backslash X_{k-1}$, there is a neighborhood $U_x \subset X_k \backslash X_{k-1}$ so that $F|_{U_x}$ is weakly equivalent to a constant sheaf whose stalk is a finite cell A-module spectrum.

In some ways the subcategory of constructible sheaves is analogous to the subcategory of strongly dualizable objects in a stable homotopy category: it inherits the triangulated structure from the category $ho\mathcal{S}p_A(X)$ and is closed under formation of internal hom and smash products. However, constructible sheaves only have a weak dualizability property which we will now explain.

Let us call an object $\mathcal{F}$ *weakly dualizable* (with respect to the dualizing sheaf) if for any object the natural map $\mathcal{F} \to \mathbb{D}_A \mathbb{D}_A \mathcal{F}$ is an isomorphism. Recall that there is a notion of strong dualizability which implies weak dualizabilility (albeit wih respect to the constant sphere sheaf).

Proposition 5.5. Let X be a stratifiable space and $\mathcal{F} \in ho\mathcal{S}p_A(X)$ and $\mathcal{F}$ is constructible. Then $\mathcal{F}$ is weakly dualizable. □

Proposition 5.6. Let X be a stratified space, $\mathbb{D}_A$ the dualizing sheaf of X over A. Then $\mathbb{D}_A$ is constructible. □

Corollary 5.7. We have an isomorphism $\mathbb{D}_A \mathbb{D}_A = A$, that is, the constant sheaf is weakly dualizable. □

We see, thus, that Verdier duality is a perfect duality on the subcategory of constructible sheaves.

6 Čech cohomology and homology

In the category $\mathcal{PS}p(X)$, Čech cohomology works similarly as in the classical situation and is based on the treatment in [T]. Let $\mathcal{F} \in \mathcal{PS}p(X)$ and let $\mathcal{U}$ be a covering of X. We form the cosimplicial spectrum $\mathcal{F}^{\cdot}_{\mathcal{U}}$ by

$$\prod_{\alpha_0} \Gamma(U_{\alpha_0}, \mathcal{F}) \rightleftarrows \prod_{\alpha_0,\alpha_1} \Gamma(U_{\alpha_0} \cap U_{\alpha_1}, \mathcal{F}) \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} \prod_{\alpha_0,\alpha_1,\alpha_2} \Gamma(U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}, \mathcal{F}) \cdots .$$

For a presheaf $\mathcal{F}$ and a cover $\mathcal{U}$ of X, we form the Čech cohomology of $\mathcal{F}$ on $\mathcal{U}$ as

$$\check{\mathbb{H}}_{\bullet}(\mathcal{U}, \mathcal{F}) = \mathrm{holim}_{\Delta} \mathcal{F}^{\cdot}_{\mathcal{U}}$$

where the homotopy limit is formed over the simplicial category Δ.

If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a map of spectra $\check{\mathbb{H}}^{\bullet}(\mathcal{V}, \mathcal{F}) \to \check{\mathbb{H}}^{\bullet}(\mathcal{U}, \mathcal{F})$ and this map is independent up to homotopy of the choice of the refinement map. For $\mathcal{U}_\lambda$ a cofinal system of covers of X, let $\mathcal{F}^{\cdot} = \mathrm{hocolim}_{\lambda} \mathcal{F}^{\cdot}_{\mathcal{U}_\lambda}$ the homotopy colimit of cosimplicial spectra. Finally define the Čech cohomology of X with coefficients $\mathcal{F}$ as

$$\check{\mathbb{H}}^{\bullet}(X, F) = \mathrm{holim}_{\Delta} \mathcal{F}^{\cdot}.$$

Theorem 6.1. Let X be a locally compact, embeddable paracompact space. For $\mathcal{F} \in \mathcal{PS}p(X)$ there is a weak equivalence between the Čech cohomology of $\mathcal{F}$ and the sheaf cohomology of its sheafification, i.e. $\check{\mathbb{H}}^{\bullet}(X, F) \cong \mathbb{H}^{\bullet}(X, aF)$ in $h\mathcal{S}p$.

Up until now, we have not considered (pre)cosheaves of spectra; this is partly because we do not think it likely that such a theory can be developed as completely as the corresponding sheaf theory. So we merely content ourselves with Čech homology and therefore concentrate on precosheaves of spectra. The definition of these Čech homology groups is dual to the construction of the Čech cohomology as given above and can be found in [Q1] or [W]. We show how they can be expressed in our language as sheaf cohomology via duality. First, if $\mathcal{C}$ is a precosheaf of A-module spectra on a topological space X, where A is an $\mathcal{S}$-algebra then we can form a naive dual presheaf $\mathcal{C}'$ by $\Gamma(U, \mathcal{C}') = F_A(\Gamma(U, \mathcal{C}), A)$. We can also define the naive dual of a presheaf (which is then a precosheaf) similarly in which case $\mathcal{C}''$ is back to being a precosheaf and there is a map $\mathcal{C} \to \mathcal{C}''$.

Theorem 6.2. For X a paracompact locally compact locally embeddable space and $\mathcal{C}$ a precosheaf of spectra, then the Čech homology of $\mathcal{C}$,

$$\check{\mathbb{H}}_{\bullet}(X, \mathcal{C})'' \cong \mathbb{H}^{\bullet}(X, \mathbb{D}_A(a\mathcal{C}')).$$

(The $''$ indicates the double dual of the spectrum $\check{\mathbb{H}}_{\bullet}(X, \mathcal{C})$.) If $\mathcal{C}$ is constructible over A, then one has

$$\check{\mathbb{H}}_{\bullet}(X, \mathcal{C}) \cong \mathbb{H}^{\bullet}(X, \mathbb{D}_A \mathcal{C}'). \qquad \square$$

7 Examples

In this section we describe some of the examples that motivated our study. Thus let X be a stratified space. According to Weinberger [W], there is a precosheaf of spectra $\underline{\mathbb{L}}^{BQ}$on X which fits into a surgery exact sequence as follows: p. 130: Let X be a PL stratified space. Then under certain technical assumptions stated in the reference above there is a fibration for computing $S^{PL}(X)$ =(simple homotopy transverse simple homotopy equivalences $Y \to X$) modulo PL homeomorphism

$$S^{PL}(X) \to \mathbb{H}_\bullet(X, \underline{\mathbb{L}}^{BQ}) \to \mathbb{L}^{BQ}(X) \times \oplus[H_{i-4}(X_i; \mathbb{Z}/_2) \times \mathbb{Z}].$$

Notice that the characteristic class data (namely the normal invariants) are expressed in homology rather than cohomology, as would be expected for a singular space.

According to the previous section and the following proposition, these homology spectra can be described in terms of sheaf cohomology of the sheaf $\mathbb{D}_{\mathbb{L}^*(e)}\underline{\mathbb{L}}^{BQ'}$.

Proposition 7.1. The ring spectrum $\mathbb{L}^*(e)$ can be provided with the structure of an S-algebra spectrum (or equivalently, an E_∞ ring spectrum structure). Moreover, $\underline{\mathbb{L}}^{BQ}$ has the structure of a precosheaf of $\mathbb{L}^*(e)$-modules. □

The proof is based upon the calculations in [TW] together with the knowledge that BU and Eilenberg-MacLane spectra are E_∞ ring spectra.

Finally, we mention the following pretty example of Verdier duality in a geometric setting, which is alluded to in [W], Theorem 14.4.1. Here we see an example of where the cosheaf might be quite complicated, yet its Verdier dual is much simpler.

Theorem 7.2. Suppose X is a stratified space such that the links of all strata are aspherical manifolds for which the Borel conjecture is true(or more generally crigid stratified spaces, in the terminology of [W]). Then $\mathbb{D}_{\mathbb{L}^*(e)}\underline{\mathbb{L}}^{BQ'}$ is weakly equivalent to the constant sheaf $\underline{\mathbb{L}}_*(e)$. □

To see this, assume as in the theorem that the links of all strata are aspherical manifolds for which the Borel conjecture is true. Then one knows for every strata S with link L_S that the assembly map $L_S^+ \wedge \mathbb{L}_*(e) \to \mathbb{L}_*(\pi_1(L_S))$ is a weak equivalence. On the other hand, the Verdier dual of the constant sheaf $\underline{\mathbb{L}}_*(e)$ has stalk at $x \in S$ equal to $L_S^+ \wedge \mathbb{L}_*(e)$ which is then weakly equivalent to $\mathbb{L}_*(\pi_1(L_S))$ which, by the construction of the cosheaf $\underline{\mathbb{L}}^{BQ}$, [W], is what its costalk is.

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