SPECTRAL SUBSPACES OF AUTOMORPHISM GROUPS OF TYPE I FACTORS

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Introduction

A well known theorem of M. H. Stone published in 1930 [**5**] establishes a one-to-one correspondence between strongly continuous unitary representations of the real line in Hilbert space and spectral resolutions in that space. More recently, W. B. Arveson [**1**] generalized the notion of spectral subspace (corresponding to spectral projection in the Hilbert space situation) to a large class of representations in Banach space. His theory is particularly useful in the case of an ultraweakly continuous one-parameter group of *-automorphisms $\alpha = \{\alpha_t\}$ of a von Neumann algebra M. In fact, by [**1**, Lemma 2, p. 233], the spectral subspaces of M of the form $M^{\alpha}[\lambda, +\infty)$ completely determine the representation α. In view of this result it is natural to ask how to synthesize α from the $M^{\alpha}[\lambda, +\infty)$.

Although a constructive answer to this question seems hard to give in general, it can quite easily be provided in the special case where M is a type I factor. The solution (Lemma 1(b) below) is based on an idea of E. Størmer [**6**] and actually makes use of Stone's theorem in the Hilbert space of Hilbert–Schmidt operators associated with M.

Closely related to the synthesis problem there is also a characterization problem: given a family $\{M_\lambda\}_{\lambda \in \hat{\mathbb{R}}}$ of ultraweakly closed subspaces of a von Neumann algebra M, when does there exist a one-parameter group α of *-automorphisms of M such that $M_\lambda = M^{\alpha}[\lambda, \infty)$ for all $\lambda \in \hat{\mathbb{R}}$? A number of necessary conditions come immediately to mind:

(a) $\lambda < \mu \Rightarrow M_\mu \subset M_\lambda$, (b) $\bigcap_{\nu < \lambda} M_\nu = M_\lambda$,

(c) $\bigcap_{\nu \in \hat{\mathbb{R}}} M_\nu = \{0\}$ and $\left(\bigcup_{\nu \in \hat{\mathbb{R}}} M_\nu\right)^- = M$, (d) $M_\lambda M_\mu \subset M_{\lambda+\mu}$.

To avoid trivial counterexamples, we should certainly also specify that $1 \in M_0$. And finally, defining the 'spectrum' Σ of the family $\{M_\lambda\}$ as

$$\Sigma = \{\lambda \in \hat{\mathbb{R}} \mid M_{\lambda-\varepsilon} \neq M_{\lambda+\varepsilon} \text{ for all } \varepsilon > 0\},$$

the symmetry condition $\Sigma = -\Sigma$ must hold.

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These properties are sufficient if $M = \mathscr{L}(\mathbb{C}^2)$ (even without the requirement that $1 \in M_0$) but they already fail to characterize spectral subspaces in the case of the 3×3 matrix algebra $M = \mathscr{L}(\mathbb{C}^3)$. Consider the following subspaces of $\mathscr{L}(\mathbb{C}^3)$:

$$M_\lambda = \begin{cases} \mathscr{L}(\mathbb{C}^3) & \text{if } \lambda \leqslant -1\,, \\ \left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}\right\} & \text{if } -1 < \lambda \leqslant 0\,, \\ \left\{\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} & \text{if } 0 < \lambda \leqslant 1\,, \\ \{0\} & \text{if } \lambda > 1\,. \end{cases}$$

Although $\{M_\lambda\}$ satisfies all of the conditions enumerated above, it can be shown that there is no corresponding one-parameter group α.

In this paper we shall solve the characterization problem in the special case of a type I factor, by adding to the list (a)–(d) above a symmetry condition ((e) in Theorem 7) which is much stronger than $\Sigma = -\Sigma$. Along the way we are led to characterize, in terms of their spectral resolution, those unitary representations of $\mathbb{R}$ in the Hilbert space of Hilbert–Schmidt operators that are multiplicative (Theorem 4).

1.

Let H be a Hilbert space, let $\mathscr{L}(H)$ denote the von Neumann algebra of all bounded linear operators on H, and let Tr be the usual trace on $\mathscr{L}(H)$. We use $\mathfrak{H}(H)$ (or simply $\mathfrak{H}$) to refer to the space of all Hilbert–Schmidt operators on H. It is well known that $\mathfrak{H}(H)$ is an ultraweakly dense *-ideal of $\mathscr{L}(H)$, and that it also has the structure of a complete Hilbert space, the scalar product being given by

$$(x \mid y) = \operatorname{Tr}(y^* x), \qquad x, y \in \mathfrak{H}\,.$$

The algebra and Hilbert space structures combine to make $\mathfrak{H}$ into an (achieved) Hilbert algebra, the associated von Neumann algebra of which is isomorphic to $\mathscr{L}(H)$. Let us call this isomorphism π: it carries $\mathscr{L}(H)$ into $\mathscr{L}(\mathfrak{H})$ and is given by

$$\pi(x)y = xy, \qquad x \in \mathscr{L}(H),\ y \in \mathfrak{H}$$

[**2**, Chapitre I, §§5–6]. Finally we make the observation that the weak Hilbert space topology on $\mathfrak{H}$ is stronger than the topology induced by the ultraweak operator topology of $\mathscr{L}(H)$.

We consider a locally compact abelian group G and an ultraweakly continuous representation $\alpha = \{\alpha_t\}_{t \in G}$ of G by *-automorphisms of $\mathscr{L}(H)$. Let u_t denote the restriction of α_t to $\mathfrak{H}$. Then $u = \{u_t\}_{t \in G}$ is a strongly continuous representation of G by unitary operators on $\mathfrak{H}$. This can quite easily be shown directly, as is done in [**6**, Lemma]. Alternatively, u can be looked upon as the canonical implementation of α

when $\mathscr{L}(H)$ is represented in its standard form via π, so that [3, Corollary 3.6] applies.

It was first pointed out by E. Størmer [6] that there is a close link between the spectral subspaces $M^\alpha(\cdot)$ of $\mathscr{L}(H)$ associated to α (in the sense of Arveson [1]), and the projection-valued measure (or resolution of the identity) $P^u(\cdot)$ defined by u in virtue of Stone's theorem. We recall Størmer's result in the first statement of the following lemma. We use the assumptions and notation introduced above.

1 LEMMA. *Let E be a closed subset of $\hat{G}$. Then*

(a) $M^\alpha(E) = \bigcap_N (P^u(E+N)\mathfrak{H})^-$, *where the closure is taken in the ultraweak topology on $\mathscr{L}(H)$ and N ranges over all compact neighbourhoods of the identity in $\hat{G}$,*

(b) $P^u(E)\mathfrak{H} = M^\alpha(E) \cap \mathfrak{H}$,

(c) *if E is a neighbourhood of the identity $0 \in \hat{G}$, and 1 denotes the identity operator on H, then $1 \in (M^\alpha(E) \cap \mathfrak{H})^-$.*

Proof. First we make the following remark: if $x \in \mathfrak{H}$ and $f \in L^1(G)$, the integral $\alpha_f(x) = \int_G f(t)\alpha_t(x)dt$ exists, not only in the sense of the ultraweak topology on $\mathscr{L}(H)$ but also in the (stronger) sense of the weak topology on $\mathfrak{H}$. In particular, $\int_G f(t)\alpha_t(x)dt = \int_G f(t)u_t(x)dt$ is a Hilbert–Schmidt operator.

Now $x \in P^u(E)\mathfrak{H}$ if and only if x is Hilbert–Schmidt and $\int_G f(t)u_t(x)dt = 0$ for all $f \in L^1(G)$ such that its Fourier transform $\hat{f}$ vanishes on a neighbourhood of E [1, p. 225]. By the above remark, this means exactly that $x \in M^\alpha(E) \cap \mathfrak{H}$, and (b) follows.

Since $1 \in M^\alpha(\{0\})$, (c) is a consequence of (a) and (b).

The second part of this lemma shows how one can reconstruct α from its spectral subspaces: knowledge of the spectral subspaces $M^\alpha(E)$ implies knowledge of the projections $P^u(E)$, hence (by Stone's theorem) of the canonical implementation u of α. On the other hand, it is clear that not all representations of G by unitary operators on $\mathfrak{H}$ give rise to a representation of G by *-automorphisms of $\mathscr{L}(H)$. A necessary and sufficient condition to that effect is readily established.

2 LEMMA. *Let $u = \{u_t\}_{t\in G}$ be a strongly continuous unitary representation of G in $\mathfrak{H}$. Then u is the restriction to $\mathfrak{H}$ of some ultraweakly continuous representation $\alpha = \{\alpha_t\}_{t\in G}$ of G by *-automorphisms of $\mathscr{L}(H)$ if and only if*

$$u_t(xy) = u_t(x)u_t(y) \qquad \textit{for all } x, y \in \mathfrak{H} \textit{ and } t \in G.$$

Proof. The necessity is obvious. Suppose conversely that the condition $u_t(xy) = u_t(x)u_t(y)$ holds for all $x, y \in \mathfrak{H}$, $t \in G$. Define $\beta = \{\beta_t\}_{t\in G}$ to be the group of

*-automorphisms of $\mathscr{L}(\mathfrak{H})$ implemented by $u: \beta_t(A) = u_t A u_t^{-1}$ for all $A \in \mathscr{L}(\mathfrak{H})$ and $t \in G$. Let $x, y \in \mathfrak{H}$. Then, with $\pi: \mathscr{L}(H) \to \mathscr{L}(\mathfrak{H})$ as above,

$$\beta_t(\pi(x))(y) = u_t \pi(x) u_t^{-1}(y) = u_t(x u_t^{-1}(y)) = u_t(x) y,$$

hence $\beta_t(\pi(x)) = \pi(u_t(x))$.

Since $\pi(\mathfrak{H})$ is ultraweakly dense in $\pi(\mathscr{L}(H))$, we conclude that $\beta_t(\pi(\mathscr{L}(H))) = \pi(\mathscr{L}(H))$ for all $t \in G$. Define an ultraweakly continuous representation $\alpha = \{\alpha_t\}_{t \in G}$ of G as a group of *-automorphisms of $\mathscr{L}(H)$ by

$$\pi(\alpha_t(x)) = \beta_t(\pi(x)), \qquad x \in \mathscr{L}(H),\ t \in G.$$

For x in $\mathfrak{H}$ we have $\pi(\alpha_t(x)) = \beta_t(\pi(x)) = \pi(u_t(x))$, hence $\alpha_t(x) = u_t(x)$.

2.

Motivated by the previous lemmas we now turn to a study of the spectral subspaces of strongly continuous unitary representations of the real line $\mathbb{R}$ in $\mathfrak{H}(H)$ such that the condition of Lemma 2 holds (we shall call these representations 'multiplicative').

3 PROPOSITION. *Let u be a multiplicative strongly continuous unitary representation of $\mathbb{R}$ in $\mathfrak{H}$, let $P^u(\,\cdot\,)$ be the corresponding resolution of the identity in $\mathfrak{H}$, and put $\mathfrak{H}_\lambda = P^u[\lambda, +\infty)\mathfrak{H}$, $\lambda \in \hat{\mathbb{R}}$. Then the following properties hold for all $\lambda, \mu \in \hat{\mathbb{R}}$:*

(i) *if $\lambda < \mu$, then $\mathfrak{H}_\mu \subset \mathfrak{H}_\lambda$,*

(ii) $\bigcap_{\nu < \lambda} \mathfrak{H}_\nu = \mathfrak{H}_\lambda$,

(iii) $\bigcap_{\nu \in \hat{\mathbb{R}}} \mathfrak{H}_\nu = \{0\}$,

(iv) $\left(\bigcup_{\nu > \lambda} \mathfrak{H}_\nu^*\right)^\perp = \mathfrak{H}_{-\lambda}$,

(v) $\mathfrak{H}_\lambda \mathfrak{H}_\mu \subset \mathfrak{H}_{\lambda+\mu}$.

Moreover, if (i) *through* (v) *hold for a family $\{\mathfrak{H}_\lambda\}_{\lambda \in \hat{\mathbb{R}}}$ of closed subspaces of $\mathfrak{H}$, then one also has*

(vi) $\mathfrak{H}_\lambda^\perp \mathfrak{H}_\mu^\perp \subset \mathfrak{H}_{\lambda+\mu}^\perp$.

(Of course, $\perp$ denotes the orthogonal complement with respect to the scalar product $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{H}$.)

Proof. Statements (i), (ii) and (iii) correspond to the following properties of $P^u(\,\cdot\,)$:

(a) if $\lambda < \mu$, then $P^u[\mu, +\infty) \leqslant P^u[\lambda, +\infty)$,

(b) $\lim_{\nu \lesssim \lambda} P^u[\nu, +\infty) = P^u[\lambda, +\infty)$,

(c) $\lim_{\nu \to +\infty} P^u[\nu, +\infty) = 0$

(the limits are considered in the strong operator topology).

To prove (iv) and (v), let α be the extension of u to an ultraweakly continuous one-parameter group of *-automorphisms of $\mathscr{L}(H)$ (Lemma 2). Then $\mathfrak{H}_v = M^\alpha[v, +\infty) \cap \mathfrak{H}$ for all $v \in \hat{\mathbb{R}}$ (Lemma 1(b)). Hence

$$\mathfrak{H}_v^* = (M^\alpha[v, +\infty) \cap \mathfrak{H})^* = M^\alpha[v, +\infty)^* \cap \mathfrak{H}$$

$$= M^\alpha(-\infty, -v] \cap \mathfrak{H} \qquad [4, \text{Lemma } 8.3.3]$$

$$= P^u(-\infty, -v]\mathfrak{H}.$$

Consequently

$$\left(\bigcup_{v>\lambda} \mathfrak{H}_v^*\right)^\perp = \left(\bigcup_{v>\lambda} P^u(-\infty, -v]\mathfrak{H}\right)^\perp = (P^u(-\infty, -\lambda)\mathfrak{H})^\perp = P^u[-\lambda, +\infty)\mathfrak{H} = \mathfrak{H}_{-\lambda}.$$

Likewise, property (v) is a consequence of

$$M^\alpha[\lambda, +\infty)M^\alpha[\mu, +\infty) \subset M^\alpha[\lambda+\mu, +\infty)$$

[**1**, Lemma 1, p. 232].

Finally, suppose that $\{\mathfrak{H}_\lambda\}_{\lambda \in \hat{\mathbb{R}}}$ is any family of closed subspaces of $\mathfrak{H}$ for which (i) through (v) hold. Let μ, v be real numbers. By (v), we have

$$\mathfrak{H}_v \mathfrak{H}_{\mu-v} \subset \mathfrak{H}_\mu$$

and hence

$$\mathfrak{H}_v^* \mathfrak{H}_\mu^\perp \subset \mathfrak{H}_{\mu-v}^\perp.$$

As a consequence,

$$\left(\bigcup_{v>-\lambda} \mathfrak{H}_v^*\right)\mathfrak{H}_\mu^\perp \subset \bigcup_{v>-\lambda} \mathfrak{H}_{\mu-v}^\perp \subset \left(\bigcap_{v>-\lambda} \mathfrak{H}_{\mu-v}\right)^\perp = \mathfrak{H}_{\lambda+\mu}^\perp,$$

where the last equality follows from (ii). Finally $\bigcup_{v>-\lambda} \mathfrak{H}_v^*$ is dense in $\mathfrak{H}_\lambda^\perp$ by (i) and (iv) and multiplication is continuous in $\mathfrak{H}$, hence we obtain $\mathfrak{H}_\lambda^\perp \mathfrak{H}_\mu^\perp \subset \mathfrak{H}_{\lambda+\mu}^\perp$, as desired.

The aim of the sequel is to show that the necessary conditions (i)–(vi) on the $\mathfrak{H}_\lambda$ described in the statement of Proposition 3 are also sufficient to define a multiplicative one-parameter group of unitaries on $\mathfrak{H}$ (they are actually redundant, as is apparent from the proof of the proposition). However, due to the symmetry around the origin expressed in 3(iv), it is both more natural and more convenient to start with a 'one-sided' family of subspaces $\{\mathfrak{H}_\lambda\}_{\lambda \geqslant 0}$. Sufficient conditions obtained in that case will immediately lead to conditions in terms of a 'two-sided' family of subspaces, indexed by the whole of $\hat{\mathbb{R}}$ (Corollary 6).

4 THEOREM. *Let H be a Hilbert space, and let $\{\mathfrak{H}_\lambda\}_{\lambda \geqslant 0}$ be a family of closed subspaces of $\mathfrak{H}(H)$ satisfying*

(a) *if* $0 \leqslant \lambda < \mu$, *then* $\mathfrak{H}_\mu \subset \mathfrak{H}_\lambda$, (b) *for all* $\lambda > 0$, $\bigcap_{0<\nu<\lambda} \mathfrak{H}_\nu = \mathfrak{H}_\lambda$,

(c) $\bigcap_{\nu>0} \mathfrak{H}_\nu = \{0\}$, (d) $\left(\bigcup_{\nu>0} \mathfrak{H}_\nu^*\right)^\perp = \mathfrak{H}_0$,

(e) $\mathfrak{H}_\lambda \mathfrak{H}_\mu \subset \mathfrak{H}_{\lambda+\mu}$ *for all* $\lambda, \mu \geqslant 0$, (f) $\mathfrak{H}_\lambda^\perp \mathfrak{H}_\mu^\perp \subset \mathfrak{H}_{\lambda+\mu}^\perp$ *for all* $\lambda, \mu > 0$.

Then there exists a unique multiplicative strongly continuous unitary representation u of $\mathbb{R}$ in $\mathfrak{H}$ such that

$$\mathfrak{H}_\lambda = P^u[\lambda, +\infty)\mathfrak{H} \qquad \textit{for all } \lambda \geqslant 0\,.$$

To prove the theorem, we first make the definition forced upon us by (iv) in Proposition 3: for $\lambda < 0$, we put

$$\mathfrak{H}_\lambda = \left(\bigcup_{\nu>-\lambda} \mathfrak{H}_\nu^*\right)^\perp.$$

5 LEMMA. *Under hypotheses* (a)–(d) *above, and with the previous definition, the family $\{\mathfrak{H}_\lambda\}_{\lambda\in\mathbb{R}}$ has all the properties* (i)–(iv) *of Proposition* 3. *If moreover* (e) *and* (f) *hold, then so do* (v) *and* (vi).

Proof. (i) Suppose that $\lambda < \mu$. If $0 \leqslant \lambda$ then $\mathfrak{H}_\mu \subset \mathfrak{H}_\lambda$ by hypothesis (a). If $\mu \leqslant 0$,

$$\begin{aligned}\mathfrak{H}_\mu &= \left(\bigcup_{\nu>-\mu} \mathfrak{H}_\nu^*\right)^\perp && \text{by hypothesis if } \mu = 0, \text{ by definition otherwise}\\ &\subset \left(\bigcup_{\nu>-\lambda} \mathfrak{H}_\nu^*\right)^\perp && \text{because } -\lambda > -\mu\\ &= \mathfrak{H}_\lambda && \text{by definition.}\end{aligned}$$

The case when $\lambda < 0 < \mu$ follows by combining the two cases above.

(ii) If $\lambda > 0$, we have

$$\begin{aligned}\bigcap_{\nu<\lambda} \mathfrak{H}_\nu &= \bigcap_{0<\nu<\lambda} \mathfrak{H}_\nu && \text{by (i)}\\ &= \mathfrak{H}_\lambda && \text{by hypothesis (b).}\end{aligned}$$

If $\lambda \leqslant 0$, then

$$\begin{aligned}\bigcap_{\nu<\lambda} \mathfrak{H}_\nu &= \bigcap_{\nu<\lambda}\left(\bigcap_{\mu>-\nu} \mathfrak{H}_\mu^{*\perp}\right) && \text{by definition}\\ &= \bigcap_{\mu>-\lambda} \mathfrak{H}_\mu^{*\perp} = \mathfrak{H}_\lambda && \text{by definition if } \lambda < 0, \text{ by hypothesis (d) if } \lambda = 0\,.\end{aligned}$$

(iii) $\bigcap_{\nu\in\mathbb{R}} \mathfrak{H}_\nu = \bigcap_{0<\nu} \mathfrak{H}_\nu = \{0\}$ by (i) and (c).

(iv) Suppose that $\lambda < 0$. Then we have

$$\left(\bigcup_{\nu > \lambda} \mathfrak{H}_\nu^*\right)^\perp = \bigcap_{\nu > \lambda} \mathfrak{H}_\nu^{\perp *}$$

$$= \bigcap_{\lambda < \nu < 0} \mathfrak{H}_\nu^{\perp *} \qquad \text{by (i)}$$

$$= \bigcap_{\lambda < \nu < 0} \left(\bigcup_{\mu > -\nu} \mathfrak{H}_\nu\right)^- \qquad \text{by definition and by (a)}$$

(we also used the simple fact that $\mathfrak{K}^{*\perp} = \mathfrak{K}^{\perp *}$ for every subspace $\mathfrak{K}$ of $\mathfrak{H}$). Now if $\lambda < \nu$, obviously $\mathfrak{H}_{-\lambda} \subset \bigcup_{\mu > -\nu} \mathfrak{H}_\mu \subset \left(\bigcup_{\mu > -\nu} \mathfrak{H}_\mu\right)^-$, and hence $\mathfrak{H}_{-\lambda} \subset \bigcap_{\lambda < \nu < 0} \left(\bigcup_{\mu > -\nu} \mathfrak{H}_\mu\right)^-$. On the other hand if $\nu < 0$ we have $\left(\bigcup_{\mu > -\nu} \mathfrak{H}_\mu\right)^- \subset \mathfrak{H}_{-\nu}$ by hypothesis (a) and because $\mathfrak{H}_{-\nu}$ is closed. Thus

$$\bigcap_{\lambda < \nu < 0} \left(\bigcup_{\mu > -\nu} \mathfrak{H}_\mu\right)^- \subset \bigcap_{\lambda < \nu < 0} \mathfrak{H}_{-\nu} = \mathfrak{H}_{-\lambda}$$

by (b). Altogether we have $\bigcap_{\lambda < \nu < 0} \left(\bigcup_{\mu > -\nu} \mathfrak{H}_\mu\right)^- = \mathfrak{H}_{-\lambda}$, and that finishes the proof of (iv) in the case when $\lambda < 0$. The cases when $\lambda = 0$ and $\lambda > 0$ follow by hypothesis (d) and by definition, respectively.

Thus far we only used hypotheses (a) through (d). Let us assume from now on that (e) and (f) are valid as well.

(v) Again we have to distinguish between several cases:

(1) $\lambda \geqslant 0$ and $\mu \geqslant 0$: this is hypothesis (e),

(2) $0 \leqslant \lambda \leqslant -\mu$ or $0 \leqslant \mu \leqslant -\lambda$.

First suppose that $\lambda \geqslant 0$ and that $\nu > -\lambda-\mu \geqslant 0$. By hypothesis (e), $\mathfrak{H}_\nu \mathfrak{H}_\lambda \subset \mathfrak{H}_{\nu+\lambda}$, hence $\mathfrak{H}_\lambda^* \mathfrak{H}_\nu^* \subset \mathfrak{H}_{\nu+\lambda}^*$ and $\mathfrak{H}_\lambda(\mathfrak{H}_{\nu+\lambda}^*)^\perp \subset (\mathfrak{H}_\nu^*)^\perp$. Taking intersections over all $\nu > -\lambda-\mu$, we obtain

$$\mathfrak{H}_\lambda\left(\bigcap_{\nu > -\lambda-\mu} \mathfrak{H}_{\nu+\lambda}^{*\perp}\right) \subset \bigcap_{\nu > -\lambda-\mu} \mathfrak{H}_\nu^{*\perp}.$$

By definition (or by hypothesis (d) if $\lambda+\mu = 0$), this means exactly

$$\mathfrak{H}_\lambda \mathfrak{H}_\mu \subset \mathfrak{H}_{\lambda+\mu}.$$

The proof of the case when $0 \leqslant \mu \leqslant -\lambda$ is analogous.

(3) $0 < -\lambda < \mu$ or $0 < -\mu < \lambda$.

Suppose that $0 < -\lambda < \nu < \mu$. By hypothesis (f), we have $\mathfrak{H}_\nu^\perp \mathfrak{H}_{\mu-\nu}^\perp \subset \mathfrak{H}_\mu^\perp$, or $\mathfrak{H}_\nu^{*\perp}\mathfrak{H}_\mu = \mathfrak{H}_\nu^{\perp *}\mathfrak{H}_\mu \subset \mathfrak{H}_{\mu-\nu}$. Hence $\left(\bigcap_{-\lambda<\nu<\mu} \mathfrak{H}_\nu^{*\perp}\right)\mathfrak{H}_\mu \subset \bigcap_{-\lambda<\nu<\mu} \mathfrak{H}_{\mu-\nu}$. Using (i) and the definition of $\mathfrak{H}_\lambda$ in the left hand side and hypothesis (b) in the right hand side, the previous inclusion yields

$$\mathfrak{H}_\lambda \mathfrak{H}_\mu \subset \mathfrak{H}_{\lambda+\mu}\,.$$

The case when $0 < -\mu < \lambda$ is treated similarly.

(4) $\lambda < 0$ and $\mu < 0$.

Put $\nu > -\lambda-\mu$. Since $0 < -\mu < \nu$, we have by (3) above that $\mathfrak{H}_\mu\mathfrak{H}_\nu \subset \mathfrak{H}_{\mu+\nu}$, $\mathfrak{H}_\nu^*\mathfrak{H}_\mu^* \subset \mathfrak{H}_{\mu+\nu}^*$, and $\mathfrak{H}_{\mu+\nu}^{*\perp}\mathfrak{H}_\mu \subset \mathfrak{H}_\nu^{*\perp}$. As before we conclude, by considering intersections over all $\nu > -\lambda-\mu$, that $\mathfrak{H}_\lambda\mathfrak{H}_\mu \subset \mathfrak{H}_{\lambda+\mu}$. This ends the proof of property (v) and indeed of the whole lemma, since it was shown in Proposition 3 that (i) through (v) entail (vi).

Proof of Theorem 4. First we claim that $\left(\bigcup_{\nu\in\hat{\mathbb{R}}} \mathfrak{H}_\nu\right)^- = \mathfrak{H}$. Indeed, $\bigcup_{\nu\in\hat{\mathbb{R}}} \mathfrak{H}_\nu$ is a subspace of $\mathfrak{H}$ by property (i) and

$$\begin{aligned}
\left(\bigcup_{\nu\in\hat{\mathbb{R}}} \mathfrak{H}_\nu\right)^\perp &= \bigcap_{\nu\in\hat{\mathbb{R}}} \mathfrak{H}_\nu^\perp \\
&= \bigcap_{\nu<0} \mathfrak{H}_\nu^\perp, \qquad \text{again by (i)} \\
&= \bigcap_{\nu<0}\left(\bigcup_{\lambda>-\nu} \mathfrak{H}_\lambda^*\right)^- \quad \text{by definition} \\
&\subset \bigcap_{\nu<0} \mathfrak{H}_{-\nu}^* = \{0\}\,,
\end{aligned}$$

by hypotheses (a) and (c). Together with properties (i), (ii) and (iii) (shown to hold in Lemma 5), this implies, by Stone's theorem, the existence of a strongly continuous one-parameter group $u = \{u_t\}_{t\in\mathbb{R}}$ of unitary operators on $\mathfrak{H}$ with associated projection-valued measure $P^u(\cdot)$ determined by $P^u[\lambda, +\infty)\mathfrak{H} = \mathfrak{H}_\lambda$ for all $\lambda \in \hat{\mathbb{R}}$.

Next we prove that

$$(*) \qquad (P^u(-\infty,\lambda]\mathfrak{H})(P^u(-\infty,\mu]\mathfrak{H}) \subset P^u(-\infty,\lambda+\mu]\mathfrak{H}$$

for all $\lambda, \mu \in \hat{\mathbb{R}}$. Let $\nu > \lambda$ and $\xi > \mu$. Since

$$\mathfrak{H}_\nu^\perp = (P^u[\nu, +\infty)\mathfrak{H})^\perp = P^u(-\infty,\nu)\mathfrak{H}\,,$$

by Lemma 5 we have

$$(P^u(-\infty,\nu)\mathfrak{H})(P^u(-\infty,\xi)\mathfrak{H}) \subset P^u(-\infty,\nu+\xi)\mathfrak{H}\,.$$

Hence

$$\left(\bigcap_{\nu > \lambda} P^u(-\infty, \nu)\mathfrak{H}\right)(P^u(-\infty, \xi)\mathfrak{H}) \subset \bigcap_{\nu > \lambda} P^u(-\infty, \nu+\xi)\mathfrak{H},$$

or

$$(P^u(-\infty, \lambda]\mathfrak{H})(P^u(-\infty, \xi)\mathfrak{H}) \subset P^u(-\infty, \lambda+\xi]\mathfrak{H}$$

by well known properties of projection-valued measures. Taking intersections over all $\xi > \mu$ and repeating the same reasoning yields (*).

We are now in a position to show that u is multiplicative. To that end, let $\mathscr{L}_w(\mathfrak{H}, \mathscr{L}(\mathfrak{H}))$ denote the space of bounded linear maps from $\mathfrak{H}$ into $\mathscr{L}(\mathfrak{H})$ that are continuous with respect to the weak (Hilbert space) topology on $\mathfrak{H}$ and the ultraweak (operator) topology on $\mathscr{L}(\mathfrak{H})$. Let β be the group of *-automorphisms of $\mathscr{L}(\mathfrak{H})$ implemented by u (as in the proof of Lemma 2). Define a one-parameter group of isometries $\phi = \{\phi_t\}_{t \in \mathbb{R}}$ of $\mathscr{L}_w(\mathfrak{H}, \mathscr{L}(\mathfrak{H}))$ by

$$\phi_t(\psi) = \beta_t \circ \psi \circ u_t^{-1}$$

for all $\psi \in \mathscr{L}_w(\mathfrak{H}, \mathscr{L}(\mathfrak{H}))$ and $t \in \mathbb{R}$. Let $M^\beta(\cdot)$ and $M^\phi(\cdot)$ be the spectral subspaces of $\mathscr{L}(\mathfrak{H})$, respectively $\mathscr{L}_w(\mathfrak{H}, \mathscr{L}(\mathfrak{H}))$, corresponding to β and ϕ. Finally, let $\pi_0 : \mathfrak{H} \to \mathscr{L}(\mathfrak{H})$ denote the restriction of π to $\mathfrak{H}$ (that is $\pi_0(x)y = xy$ for all $x, y \in \mathfrak{H}$). Clearly $\pi_0 \in \mathscr{L}_w(\mathfrak{H}, \mathscr{L}(\mathfrak{H}))$, and we are in the situation described in the preliminaries to [**1**, Theorem 2.3].

Suppose that $\lambda \in \hat{\mathbb{R}}$ and $x \in \mathfrak{H}_\lambda = P^u[\lambda, +\infty)\mathfrak{H}$. Then

$$\pi_0(x)(P^u[\mu, +\infty)\mathfrak{H}) = xP^u[\mu, +\infty)\mathfrak{H} \subset P^u[\lambda+\mu, +\infty)\mathfrak{H}$$

for all $\mu \in \hat{\mathbb{R}}$, by property (v). Hence by Corollary 2 of [**1**, Theorem 2.3], $\pi_0(x) \in M^\beta[\lambda, +\infty)$. Thus we have shown that

$$\pi_0(P^u[\lambda, +\infty)\mathfrak{H}) \subset M^\beta[\lambda, +\infty)$$

for all $\lambda \in \hat{\mathbb{R}}$. But by [**1**, Theorem 2.3] this implies that $\pi_0 \in M^\phi[0, +\infty)$. On the other hand, using (*) above, we obtain in a similar way that

$$\pi_0(P^u(-\infty, \lambda]\mathfrak{H}) \subset M^\beta(-\infty, \lambda]$$

for all $\lambda \in \hat{\mathbb{R}}$, and then $\pi_0 \in M^\phi(-\infty, 0]$. The conclusion is that $\pi_0 \in M^\phi(\{0\})$, or equivalently that $\beta_t \circ \pi_0 = \pi_0 \circ u_t$ for all $t \in \mathbb{R}$. If now $x, y \in \mathfrak{H}$, we have

$$\begin{aligned} u_t(xy) &= \beta_t(\pi_0(x))(u_t(y)) \qquad \text{by definition of } \beta_t, \\ &= \pi_0(u_t(x))(u_t(y)) = u_t(x)u_t(y), \end{aligned}$$

and u is indeed a multiplicative unitary representation of $\mathbb{R}$ in $\mathfrak{H}$.

To prove uniqueness, suppose that v is a multiplicative strongly continuous unitary representation of $\mathbb{R}$ in $\mathfrak{H}$, with corresponding spectral resolution $P^v(\cdot)$, such that

$$\mathfrak{H}_\lambda = P^v[\lambda, +\infty)\mathfrak{H} \qquad \text{for all } \lambda \geq 0.$$

Then obviously $P^v[\lambda, +\infty) = P^u[\lambda, +\infty)$ for all $\lambda \geqslant 0$. This equality actually holds for all real λ, by Proposition 3(iv). Hence $v = u$.

From Theorem 4 we can easily deduce criteria for a family of subspaces $\{\mathfrak{H}_\lambda\}_{\lambda \in \hat{\mathbb{R}}}$ of $\mathfrak{H}$ to define a multiplicative one-parameter group of unitaries of $\mathfrak{H}$.

6 COROLLARY. *Let $\{\mathfrak{H}_\lambda\}_{\lambda \in \hat{\mathbb{R}}}$ be a family of closed subspaces of $\mathfrak{H}$ satisfying*

(a) *if* $0 < \lambda < \mu$ *then* $\mathfrak{H}_\mu \subset \mathfrak{H}_\lambda$,

(b) *for all* $\lambda > 0$, $\bigcap_{0 < \nu < \lambda} \mathfrak{H}_\nu = \mathfrak{H}_\lambda$,

(c) $\bigcap_{\nu > 0} \mathfrak{H}_\nu = \{0\}$,

(d) $\left(\bigcup_{\nu > \lambda} \mathfrak{H}_\nu^*\right)^\perp = \mathfrak{H}_{-\lambda}$ *for all* $\lambda \geqslant 0$,

(e) $\mathfrak{H}_\lambda \mathfrak{H}_\mu \subset \mathfrak{H}_{\lambda+\mu}$ *for all* $\lambda, \mu \in \hat{\mathbb{R}}$.

Then there exists a unique multiplicative strongly continuous unitary representation of $\mathbb{R}$ in $\mathfrak{H}$ such that $\mathfrak{H}_\lambda = P^u[\lambda, +\infty)\mathfrak{H}$ for all $\lambda \in \hat{\mathbb{R}}$.

Proof. By Lemma 5, hypotheses (a) through (d) imply that properties (i) through (iv) of Proposition 3 hold (notice that, for $\lambda > 0$, hypothesis (d) coincides with the definition of $\mathfrak{H}_{-\lambda}$ made just before Lemma 5). Consequently, using (e) above (which is nothing but 3(v)) we obtain 3(vi) as well, by Proposition 3. *A fortiori* the conditions in the statement of Theorem 4 are met and the corollary follows from the proof of that theorem.

3.

Let us now consider the original problem of characterizing spectral subspaces of one-parameter groups of *-automorphisms of $\mathscr{L}(H)$.

7 THEOREM. *Let H be a Hilbert space, and let $\{M_\lambda\}_{\lambda \in \hat{\mathbb{R}}}$ be a family of ultraweakly closed subspaces of $\mathscr{L}(H)$ satisfying*

(a) *if* $\lambda < \mu$, *then* $M_\mu \subset M_\lambda$,

(b) $\bigcap_{\nu < \lambda} M_\nu = M_\lambda$ *for all* $\lambda \in \hat{\mathbb{R}}$,

(c) $\bigcap_{\nu \in \hat{\mathbb{R}}} M_\nu = \{0\}$,

(d) $M_\lambda M_\mu \subset M_{\lambda+\mu}$ *for all* $\lambda, \mu \in \hat{\mathbb{R}}$,

(e) $\left(\bigcup_{\nu > \lambda} (M_\nu^* \cap \mathfrak{H})\right)^\perp = M_{-\lambda} \cap \mathfrak{H}$ *for all* $\lambda \geqslant 0$.

*Then there exists a unique ultraweakly continuous one-parameter group α of *-automorphisms of $\mathscr{L}(H)$ such that $M_\lambda = M^\alpha[\lambda, +\infty)$ for all $\lambda \in \hat{\mathbb{R}}$.*

Proof. Uniqueness follows from [**1**, Lemma 2, p. 233]. To prove existence, define $\mathfrak{H}_\lambda = M_\lambda \cap \mathfrak{H}$ for all $\lambda \in \hat{\mathbb{R}}$. Notice that $\mathfrak{H}_\lambda$ is closed in $\mathfrak{H}$. Moreover the family $\{\mathfrak{H}_\lambda\}_{\lambda \in \hat{\mathbb{R}}}$ clearly satisfies the conditions of Corollary 6, hence there is a multiplicative strongly continuous unitary representation u of $\mathbb{R}$ in $\mathfrak{H}$ such that $\mathfrak{H}_\lambda = P^u[\lambda, +\infty)\mathfrak{H}$. Let α be the extension of u to an ultraweakly continuous one-parameter group of $*$-automorphisms of $\mathscr{L}(H)$ (Lemma 2). By Lemma 1(b),

$$M_\lambda \cap \mathfrak{H} = \mathfrak{H}_\lambda = P^u[\lambda, +\infty)\mathfrak{H} = M^\alpha[\lambda, +\infty) \cap \mathfrak{H}.$$

We show that in fact $M_\lambda = M^\alpha[\lambda, +\infty)$.

For all $\lambda \in \hat{\mathbb{R}}$ we clearly have

$$(M^\alpha[\lambda, +\infty) \cap \mathfrak{H})^- = (M_\lambda \cap \mathfrak{H})^- \subset M_\lambda$$

(where the closure is taken in the ultraweak topology). Consequently

$$\bigcap_{\varepsilon > 0} (M^\alpha[\lambda - \varepsilon, +\infty) \cap \mathfrak{H})^- \subset \bigcap_{\varepsilon > 0} M_{\lambda - \varepsilon}.$$

But the left hand side is equal to $M^\alpha[\lambda, +\infty)$ by Lemma 1(a), whereas the right hand side equals M_λ by hypothesis (b) in the statement. Hence we have shown that $M^\alpha[\lambda, +\infty) \subset M_\lambda$. To establish the converse inclusion, let $x \in M_\lambda$ and $\varepsilon > 0$. By Lemma 1(c) there exists a net $\{y_i\} \subset M^\alpha[-\varepsilon, +\infty) \cap \mathfrak{H} = M_{-\varepsilon} \cap \mathfrak{H}$ converging ultraweakly to the identity operator on H. Hence $\{y_i x\}$ converges ultraweakly to x. On the other hand $y_i x \in (M_{-\varepsilon} \cap \mathfrak{H})M_\lambda$. Since

$$(M_{-\varepsilon} \cap \mathfrak{H})M_\lambda \subset M_{\lambda-\varepsilon} \cap \mathfrak{H} = M^\alpha[\lambda - \varepsilon, +\infty) \cap \mathfrak{H},$$

we conclude that

$$x \in \bigcap_{\varepsilon > 0} (M^\alpha[\lambda - \varepsilon, +\infty) \cap \mathfrak{H})^- = M^\alpha[\lambda, +\infty),$$

and this ends the proof of the theorem.

The conditions in the theorem are clearly necessary. Notice however that the above proof uses the formally weaker versions

(a′) if $\lambda < \mu$ then $M_\mu \cap \mathfrak{H} \subset M_\lambda \cap \mathfrak{H}$

and

(c′) $\bigcap_{\nu \in \hat{\mathbb{R}}} (M_\nu \cap \mathfrak{H}) = \{0\}$

of conditions (a) and (c).

8. *Concluding remarks*

(i) If H is finite-dimensional, the conditions of Theorem 4 are not fully independent: (a), (c) and (e) imply that $\mathfrak{H}_0 \subset \left(\bigcup_{\nu > 0} \mathfrak{H}_\nu^*\right)^\perp$. To show this, first notice that, by (a) and (c), there exists $\lambda_0 > 0$ such that $\mathfrak{H}_\lambda = \{0\}$ whenever $\lambda > \lambda_0$.

Suppose $\nu > 0$, $x \in \mathfrak{H}_0$, $y \in \mathfrak{H}_\nu$. By (e) we have $yx \in \mathfrak{H}_\nu$ and $(yx)^n \in \mathfrak{H}_{n\nu}$ for all positive integers n. Since $\mathfrak{H}_{n\nu} = \{0\}$ as soon as $n\nu > \lambda_0$, yx is nilpotent. This implies that $(x \mid y^*) = \mathrm{Tr}(yx) = 0$, or equivalently that $x \perp y^*$.

Incidentally, a similar argument using the same hypotheses shows that $1 \in \left(\bigcup_{\nu > 0} \mathfrak{H}_\nu^* \right)^\perp$. Hence (a), (c), (d) and (e) entail $1 \in \mathfrak{H}_0$.

(ii) Next we want to establish that each of the conditions (d), (e) and (f) of Theorem 4 are independent of the other two, given (a), (b) and (c). Let us consider the following examples.

(1) Let $H = \mathbb{C}^3$; identify $\mathfrak{H}$ with the algebra of 3×3 matrices $(a_{ij})_{1 \leqslant i,j \leqslant 3}$, and define subspaces $\mathfrak{K}_0, \mathfrak{K}_1$ and $\mathfrak{K}_2$ of $\mathfrak{H}$ by

$$\mathfrak{K}_0 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \right\} \qquad \mathfrak{K}_1 = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$\mathfrak{K}_2 = \left\{ \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Fix two real numbers $0 < \lambda_1 < \lambda_2$ and consider the following family $\{\mathfrak{H}_\lambda\}_{\lambda \geqslant 0}$ of subspaces of $\mathfrak{H}$:

$$\mathfrak{H}_0 = \mathfrak{K}_0\,,$$

$$\mathfrak{H}_\lambda = \begin{cases} \mathfrak{K}_1 & \text{if } 0 < \lambda \leqslant \lambda_1\,, \\ \mathfrak{K}_2 & \text{if } \lambda_1 < \lambda \leqslant \lambda_2\,, \\ \{0\} & \text{if } \lambda > \lambda_2\,. \end{cases}$$

Clearly conditions (a), (b), (c) and (d) of Theorem 4 are satisfied. However, (e) holds if and only if $2\lambda_1 \leqslant \lambda_2$, and (f) holds if and only if $2\lambda_1 \geqslant \lambda_2$.

(2) Let $H = \mathbb{C}^4$, and define the following subspaces of $\mathfrak{H}$:

$$\mathfrak{K}_0 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \right\} \qquad \mathfrak{K}_1 = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \right\}$$

$$\mathfrak{K}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Define $\{\mathfrak{H}_\lambda\}_{\lambda \geqslant 0}$ as in the previous example, with $0 < \lambda_1 < \lambda_2$. Now (a), (b) and (c) still hold, but (d) does not (however $\mathfrak{H}_0 \subset \left(\bigcup_{\nu > 0} \mathfrak{H}_\nu^*\right)^\perp$ and $1 \in \mathfrak{H}_0$). Yet (e) and (f) are satisfied, provided that $2\lambda_1 = \lambda_2$ (in fact (e) requires that $2\lambda_1 \leqslant \lambda_2$ and (f) requires that $2\lambda_1 \geqslant \lambda_2$).

(iii) These examples, together with the one given in the introduction, indicate that there is little hope of characterizing the family of spectral subspaces $\{M^\alpha[\lambda, +\infty)\}_{\lambda \in \mathbb{R}}$ of a one-parameter group α of *-automorphisms of $\mathscr{L}(H)$ without some condition involving the trace. Moreover in the above proofs advantage has been taken of the exceptional fact that the space of Hilbert–Schmidt operators is complete as a Hilbert space (cf. [**2**, Chapitre I, §8, no. 5]). Consequently there is no obvious way to obtain a similar characterization when more general von Neumann algebras are acted upon by $\mathbb{R}$ (not even type II_1 factors).

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