

**DUALITY AND EQUIVALENCE OF MODULE CATEGORIES IN
NONCOMMUTATIVE GEOMETRY II:
MUKAI DUALITY FOR HOLOMORPHIC NONCOMMUTATIVE TORI**

JONATHAN BLOCK

To my parents

ABSTRACT. This is the second in a series of papers intended to set up a framework to study categories of modules in the context of non-commutative geometries. In [3] we associated to each curved dga A the basic DG category \mathcal{P}_A , which in the case where A is the Dolbeault dga on a complex manifold is equivalent to the derived category of sheaves with coherent cohomology. In this paper we enlarge this category to include objects which correspond to quasi-coherent sheaves. We then apply this framework to proving an equivalence of categories between derived categories on the noncommutative complex torus and on a holomorphic gerbe on the dual complex torus.

CONTENTS

1. Introduction	2
2. The quasi-perfect category of modules over a differential graded algebra	4
2.1. Review of the perfect DG category of a curved DGA	4
2.2. The Quasi-perfect category	6
2.3. DG-equivalences	10
3. Mukai duality for non-commutative tori	12
3.1. The Complex noncommutative torus	12
3.2. The holomorphic gerby torus	13
3.3. The deformed Poincare line bundles	14
3.4. The duality	18
References	30

1. INTRODUCTION

This is the second in a series of papers. Its purpose is twofold. First, we continue to set up a framework to study categories of modules in the context of noncommutative geometries, which are integrable in some sense. This integrability we encode in a (curved) differential graded algebra $\mathbf{A} = (\mathcal{A}^\bullet, d, c)$. The second is to apply our framework to an interesting non-trivial example. In particular, we prove an extension of Mukai duality from the classical case of complex tori, to that of noncommutative tori.

In [3] we introduced the basic DG category $\mathcal{P}_{\mathbf{A}}$, of \mathbf{A} , which corresponded to the category of coherent sheaves on a complex manifold.

One of the purposes of this paper is to develop more of the apparatus. In particular, we show how to get some of the six operations of Grothendieck in our context. They are all some version of tensoring with a bimodule. However, in order to do this, we need to enlarge our basic category of modules $\mathcal{P}_{\mathbf{A}}$ to a larger category $q\mathcal{P}_{\mathbf{A}}$, the quasi-perfect category of \mathcal{A}^\bullet . This corresponds to the category of quasi-coherent sheaves on a complex manifold. The category of quasi-coherent sheaves on a complex manifold is less well known for complex manifolds than it is for algebraic schemes.

There are at least two reasons for enlarging the category. Most pressing, the derived pushforward is most naturally defined into the quasi-perfect category. For example, in complex geometry, to define the derived pushforward, one performs the following steps:

- (1) Resolve by injective sheaves, which can only be done in the quasi-coherent category,
- (2) Push forward. This remains a quasi-coherent complex of sheaves.
- (3) Prove that the direct image is equivalent to a complex of coherent sheaves, i.e. Grauert's direct image theorem.

If we don't care to end up back in the coherent category, we can stop after step (2). This is what we do in this paper. Finiteness conditions and Grauert's direct image theorem are the subject of the third paper in the series, [4].

The second reason for introducing this larger category is that on a complex manifold, there can potentially be very few coherent sheaves. Certainly, not nearly enough to determine the manifold up to isomorphism. Quasi-coherent sheaves provides a larger category where more invariants of complex manifolds might be found.

Most of the paper is concerned with an application of our framework: to formulate and prove a deformed version of Mukai duality, which we now explain. Let X be a complex torus. Thus $X = V/\Lambda$ where V is a g -dimensional complex vector space and $\Lambda \cong \mathbb{Z}^{2g}$ is a lattice in V . Let X^\vee denote the dual complex torus. This can be described in a number of ways:

- (1) as $\text{Pic}^0(X)$, the variety of holomorphic line bundles on X with first Chern class 0 (i.e., they are topologically trivial);
- (2) as the moduli space of flat unitary line bundles on X . This is the same as the space of unitary representations of $\pi_1(X)$, but it has a complex structure that depends on that of X ;
- (3) and most explicitly as $\overline{V}^\vee/\Lambda^\vee$ where Λ^\vee is the dual lattice,

$$\Lambda^\vee = \{v \in \overline{V}^\vee \mid \text{Im } \langle v, \lambda \rangle \in \mathbb{Z} \ \forall \lambda \in \Lambda\}.$$

Here \overline{V}^\vee consists of conjugate linear homomorphisms from V to \mathbb{C} .

We now describe Mukai duality. On $X \times X^\vee$ there is a canonical line bundle, \mathcal{P} , the Poincaré bundle which is uniquely determined by the following universal properties:

- (1) $\mathcal{P}|_{X \times \{p\}} \cong p$ where $p \in X^\vee$ and is therefore a line bundle on X .
- (2) $\mathcal{P}|_{\{0\} \times X^\vee}$ is trivial.

Now Mukai duality says that there is an equivalence of derived categories of coherent sheaves

$$D^b(X) \rightarrow D^b(X^\vee)$$

induced by the functor

$$\mathcal{F} \mapsto p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{P})$$

where p_i are the two obvious projections. The relation of this statement with the Baum-Connes conjecture is discussed in [3].

Besides the usual deformations of a complex manifold X as a complex manifold, there are interesting extra deformations that are derived from the philosophy expounded by Bondal, Drinfeld and Kontsevich: The most general way to deform a space X is by deforming the derived category $D^b(X)$ of sheaves or a DG enhancement (such as $\mathcal{P}_{\mathcal{A}^0, \bullet(X)}$) as a DG category. The infinitesimal deformations of $D^b(X)$ are given by the second Hochschild cohomology of X :

$$HH^2(X) := \text{Ext}_{X \times X}^2(\mathcal{O}_\Delta, \mathcal{O}_\Delta).$$

There is a ‘Hodge type’ decomposition:

$$HH^2(X) = H^0(X; \wedge^2 T_X) \oplus H^1(X; T_X) \oplus H^2(X; \mathcal{O}_X)$$

- (1) The term $H^1(X; T_X)$ corresponds to the classical deformations of X as a complex manifold.
- (2) $H^0(X; \wedge^2 T_X)$ consists of global holomorphic Poisson structures and correspond to deformations of X as a noncommutative space.
- (3) The most mysterious term $H^2(X; \mathcal{O}_X)$ corresponds to deformations of the trivial \mathcal{O}^\times gerbe to a non-trivial \mathcal{O}^\times -gerbe.

In the case above, where $X = V/\Lambda$ is a complex torus and $X^\vee = \bar{V}^\vee/\Lambda^\vee$ its dual torus, the equivalence of categories implemented by the Poincaré bundle establishes the following identification of the terms of the Hochschild cohomology:

$$(1.1) \quad \begin{aligned} H^0(X; \wedge^2 T_X) &\cong \wedge^2 V &&\cong H^2(X^\vee; \mathcal{O}) \\ H^1(X; T_X) &\cong V \otimes \bar{V}^\vee &&\cong H^1(X^\vee; T_{X^\vee}) \\ H^2(X; \mathcal{O}) &\cong \wedge^2 \bar{V}^\vee &&\cong H^2(X^\vee; \wedge^2 T_{X^\vee}) \end{aligned}$$

What this suggests, is that if we deform X to a holomorphic noncommutative torus, then the dual will deform to a holomorphic gerby torus, and vice versa. This is where one should look for a derived equivalence of categories.

This derived equivalence was carried out in the context of deformation quantizations by Ben-Bassat, Block and Pantev, [1]. Here we will prove this derived equivalence in our context which has the advantage over [1] of applying to informal (i.e. non-formal) deformations.

The author would like to thank Oren Ben-Bassat, Andrei Caldararu, Calder Daenzer, Anton Kapustin, Tony Pantev and Bertrand Toen for conversations and suggestions during this work.

2. THE QUASI-PERFECT CATEGORY OF MODULES OVER A DIFFERENTIAL GRADED ALGEBRA

In this section we make our extension of the perfect category of a (curved) DGA.

2.1. Review of the perfect DG category of a curved DGA. Here we briefly review the DG-category that we assigned to a curved DGA. See [3] for more details. Also, see [6], [19], [10] for facts about DG-categories.

Definition 2.1. A curved DGA [24] (Schwarz calls them Q -algebras) is a triple $(\mathcal{A}^\bullet, d, c)$ where \mathcal{A}^\bullet is a (non-negatively) graded algebra over k with a derivation

$$d : \mathcal{A}^\bullet \rightarrow \mathcal{A}^{\bullet+1}$$

which satisfies the usual Leibnitz relation but

$$d^2(a) = [c, a]$$

where $c \in \mathcal{A}^2$ is a fixed element (the curvature). Furthermore we require the Bianchi identity $dc = 0$.

We will always set $\mathcal{A} = \mathcal{A}^0$.

A DGA is the special case where $c = 0$. Note that c is part of the data and even if $d^2 = 0$, that c might not be 0, and gives a non DGA example of a curved DGA. This is in fact the case for our gerby complex torus described in the introduction. The prototypical example of a curved DGA is $(\mathcal{A}^\bullet(M, \text{End}(E)), Ad\nabla, F)$ of differential forms on a manifold with values in the endomorphisms of a vector bundle E with connection ∇ and curvature F .

Let $(\mathcal{A}^\bullet, d, c)$ be a curved DGA. Let E^\bullet be a \mathbb{Z} -graded (bounded in both directions) right \mathcal{A} -module which is finitely generated and projective.

Definition 2.2. A \mathbb{Z} -connection \mathbb{A} is a k -linear map

$$\mathbb{A} : E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \rightarrow E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$$

of total degree one, which satisfies the usual Leibnitz condition

$$\mathbb{A}(e\omega) = (\mathbb{A}(e))\omega + (-1)^e ed\omega$$

Such a connection is determined by its value on E^\bullet . Let \mathbb{A}^k be the component of \mathbb{A} such that $\mathbb{A}^k : E^\bullet \rightarrow E^{\bullet-k+1} \otimes_{\mathcal{A}} \mathcal{A}^k$, thus $\mathbb{A} = \mathbb{A}^0 + \mathbb{A}^1 + \mathbb{A}^2 + \dots$. It is clear that \mathbb{A}^1 is a connection on E^\bullet in the ordinary sense and that \mathbb{A}^k is \mathcal{A} -linear for $k \neq 1$.

Definition 2.3. For a curved DGA $(\mathcal{A}^\bullet, d, c)$, an object in the *perfect DG-category* $\mathcal{P}_{\mathcal{A}^\bullet}$ (called a perfect twisted complex, or a module) is a \mathbb{Z} -graded (but bounded in both directions) right module E^\bullet over \mathcal{A} which is finitely generated and projective, a \mathbb{Z} -connection

$$\mathbb{E} : E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \rightarrow E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$$

that satisfies the integrability condition

$$\mathbb{E} \circ \mathbb{E}(e) = -e \cdot c$$

The minus appears because we are dealing with right modules.

The morphisms between two such objects $E_1 = (E_1^\bullet, \mathbb{E}_1)$ and $E_2 = (E_2^\bullet, \mathbb{E}_2)$ of degree k are

$$\mathrm{Hom}_{\mathcal{P}_{\mathcal{A}^\bullet}}^k(E_1, E_2) = \{\phi : E_1^\bullet \otimes_{\mathcal{A}^\bullet} \mathcal{A}^\bullet \rightarrow E_2^\bullet \otimes_{\mathcal{A}^\bullet} \mathcal{A}^\bullet \mid \phi(ea) = (-1)^{k|a|} \phi(e)a\}$$

with differential defined in the standard way

$$d(\phi)(e) = \mathbb{E}_2(\phi(e)) - (-1)^{|\phi|} \phi(\mathbb{E}_1(e))$$

Again, such a ϕ is determined by its restriction to E_1^\bullet and if necessary we denote the component of ϕ that maps

$$(2.1) \quad E_1^\bullet \rightarrow E_2^{\bullet+k-j} \otimes_{\mathcal{A}^\bullet} \mathcal{A}^j$$

by ϕ^j .

Proposition 2.4. *For a curved DGA $(\mathcal{A}^\bullet, d, c)$ the category $\mathcal{P}_{\mathcal{A}^\bullet}$ is a DG-category.*

This is clear from the following lemma.

Lemma 2.5. *Let E_1, E_2 be modules over the curved DGA $(\mathcal{A}^\bullet, d, c)$. Then the differential defined above*

$$d : \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}^\bullet}}^\bullet(E_1, E_2) \rightarrow \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}^\bullet}}^{\bullet+1}(E_1, E_2)$$

satisfies $d^2 = 0$.

Proof. This is a simple check, and follows because the curvature terms from E_1 and E_2 cancel. \square

Definition 2.6. A morphism $f : X \rightarrow Y$ in $\mathcal{P}_{\mathcal{A}^\bullet}$ which is closed and of degree 0 is a quasi-isomorphism if and only if for all objects A of $\mathcal{P}_{\mathcal{A}^\bullet}$ the morphism induced by post-composing with f ,

$$f^\# : \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}^\bullet}}^\bullet(A, X) \rightarrow \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}^\bullet}}^\bullet(A, Y)$$

is a quasi-isomorphism of complexes.

In the case of the Dolbeault algebra, it is classical that this recovers the usual notion of a morphism of complexes of holomorphic vector bundles being a quasi-isomorphism.

In [3] we proved the following criterion for being a quasi-isomorphism.

Proposition 2.7. *Under the additional assumption that each \mathcal{A}^p is flat as an \mathcal{A} -module, a closed morphism $\phi \in \mathrm{Hom}_{\mathcal{P}_{\mathcal{A}^\bullet}}^0(E_1, E_2)$ is a quasi-isomorphism if $\phi^0 : (E_1^\bullet, \mathbb{E}_1^0) \rightarrow (E_2^\bullet, \mathbb{E}_2^0)$ is a quasi-isomorphism of complexes of \mathcal{A} -modules.*

We also discussed how to construct functors between categories of the form $\mathcal{P}_{\mathcal{A}^\bullet}$. Let $(\mathcal{A}_1^\bullet, d, c_1)$ and $(\mathcal{A}_2^\bullet, d, c_2)$ be two curved DGAs. Consider the following data, $\mathcal{X} = (X^\bullet, \mathbb{X})$ where

- (1) X^\bullet is a graded finitely generated projective right- \mathcal{A}_2 -module,
- (2) $\mathbb{X} : X^\bullet \rightarrow X^\bullet \otimes_{\mathcal{A}_2} \mathcal{A}_2^\bullet$ is a \mathbb{Z} -connection,
- (3) \mathcal{A}_1^\bullet acts on the left of $X^\bullet \otimes_{\mathcal{A}_2} \mathcal{A}_2^\bullet$ satisfying

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b$$

and

$$\mathbb{X}(a \cdot (x \otimes b)) = da \cdot (x \otimes b) + a \cdot \mathbb{X}(x \otimes b)$$

for $a \in \mathcal{A}_1^\bullet$, $x \in X^\bullet$ and $b \in \mathcal{A}_2^\bullet$,

(4) \mathbb{X} satisfies the following condition

$$\mathbb{X} \circ \mathbb{X}(x \otimes b) = c_1 \cdot (x \otimes b) - (x \otimes b) \cdot c_2$$

on the complex $X^\bullet \otimes_{\mathcal{A}_2} \mathcal{A}_2^\bullet$.

Let us call such a pair $\mathcal{X} = (X^\bullet, \mathbb{X})$ an $\mathcal{A}_1^\bullet - \mathcal{A}_2^\bullet$ -twisted bimodule.

Given a $\mathcal{A}_1^\bullet - \mathcal{A}_2^\bullet$ -twisted bimodule $\mathcal{X} = (X^\bullet, \mathbb{X})$, we can then define a DG-functor

$$\mathcal{X}_* : \mathcal{P}_{\mathcal{A}_1^\bullet} \rightarrow \mathcal{P}_{\mathcal{A}_2^\bullet}$$

by $\mathcal{X}_*(E^\bullet, \mathbb{E})$ is the twisted complex

$$(E^\bullet \otimes_{\mathcal{A}_1} X^\bullet, \mathbb{E}_2)$$

where $\mathbb{E}_2(e \otimes x) = \mathbb{E}(e) \cdot x + e \otimes \mathbb{X}(x)$, where the \cdot denotes the action of \mathcal{A}_1^\bullet on $X^\bullet \otimes_{\mathcal{A}_1} \mathcal{A}_2^\bullet$. One easily checks that $\mathcal{X}_*(E)$ is an object of $\mathcal{P}_{\mathcal{A}_2^\bullet}$. We will write $\mathbb{E}\#\mathbb{X}$ for \mathbb{E}_2 .

(1) Given an $\mathcal{A}_1^\bullet - \mathcal{A}_2^\bullet$ -twisted bimodule $\mathcal{X} = (X^\bullet, \mathbb{X})$ and an $\mathcal{A}_2^\bullet - \mathcal{A}_3^\bullet$ -twisted bimodule $\mathcal{Y} = (Y^\bullet, \mathbb{Y})$ we can form an $\mathcal{A}_1^\bullet - \mathcal{A}_3^\bullet$ -twisted bimodule

$$\mathcal{X} \otimes_{\mathcal{A}_2} \mathcal{Y} = (X^\bullet \otimes_{\mathcal{A}_2} Y^\bullet, \mathbb{X}\#\mathbb{Y})$$

in the same way that we defined the functor \mathcal{X}_* . Moreover, it is clear that the functors $\mathcal{Y}_* \circ \mathcal{X}_*$ and $(\mathcal{X} \otimes_{\mathcal{A}_2} \mathcal{Y})_*$ are naturally isomorphic.

(2) Given two curved DGA's, $(\mathcal{A}_1^\bullet, d_1, c_1)$ and $(\mathcal{A}_2^\bullet, d_2, c_2)$ a homomorphism from \mathcal{A}_1^\bullet to \mathcal{A}_2^\bullet is a pair (f, ω) where $f : \mathcal{A}_1^\bullet \rightarrow \mathcal{A}_2^\bullet$ is a morphism of graded algebras, $\omega \in \mathcal{A}_2^1$ and they satisfy

- (a) $f(d_1 a_1) = d_2 f(a_1) + [\omega, f(a_1)]$ and
- (b) $f(c_1) = c_2 + d_2 \omega + \omega^2$.

To such a homomorphism we associate the twisted bimodule \mathcal{X}_f where $X_f^\bullet = \mathcal{A}_2$ in degree 0. \mathcal{A}_1^\bullet acts by the morphism f and the \mathbb{Z} -connection is

$$\mathbb{X}_f(a_2) = d_2(a_2) + \omega \cdot a_2.$$

(3) As a special case of the previous example, when $\phi : X \rightarrow Y$ is a holomorphic map between complex manifolds and $f = \phi^* : \mathcal{A}^{0,\bullet}(Y) \rightarrow \mathcal{A}^{0,\bullet}(X)$ is the induced map on the Dolbeault DGAs, then the DG-functor

$$\mathcal{X}_{f*} : \mathcal{P}_{\mathcal{A}^{0,\bullet}(Y)} \rightarrow \mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}$$

is just the pullback functor of coherent sheaves.

2.2. The Quasi-perfect category. The need to define analogues of derived pushforwards of coherent sheaves drives us to introduce the larger quasi-perfect category $q\mathcal{P}$. Let X and Y be complex manifolds. Let p_1 (resp. p_2) denote the projection from $X \times Y$ to X (resp. Y). We will start by describing the twisted bimodule that should implement the pushforward

$$p_{2*} : \mathcal{P}_{\mathcal{A}^{0,\bullet}(X \times Y)} \rightarrow \mathcal{P}_{\mathcal{A}^{0,\bullet}(Y)}$$

Let

$$M^\bullet = \Gamma(X \times Y; \wedge^\bullet p_1^* T^{0,1} X)$$

be the space of smooth forms along the fiber of the projection p_2 . This has a natural right action of $\mathcal{A}(Y)$. Define a \mathbb{Z} -connection

$$(2.2) \quad \mathbb{M} : M^\bullet \rightarrow M^\bullet \otimes_{\mathcal{A}(Y)} \mathcal{A}^{0,\bullet}(Y) \cong \mathcal{A}^{0,\bullet}(X \times Y)$$

by $\mathbb{M} = \mathbb{M}^0 + \mathbb{M}^1$ where $\mathbb{M}^0 = \bar{\partial}_X$ is the $\bar{\partial}$ -operator along the fiber and $\mathbb{M}^1 = \bar{\partial}_Y$. Note that \mathbb{M}^0 is $\mathcal{A}(Y)$ -linear. To complete the construction of a twisted bimodule we need to construct an action of $\mathcal{A}^{0,\bullet}(X \times Y)$ on $M^\bullet \otimes_{\mathcal{A}(Y)} \mathcal{A}^{0,\bullet}(Y)$. Using the isomorphism in (2.2) this is just left multiplication of $\mathcal{A}^{0,\bullet}(X \times Y)$ on itself. All this is perfectly fine and this defines a twisted bimodule structure $\mathcal{M} = (M^\bullet, \mathbb{M})$, except that M^\bullet is not finitely generated over $\mathcal{A}(Y)$.

Let us see what the functor \mathcal{M}_* does on $\mathcal{P}_{\mathcal{A}^{0,\bullet}(X \times Y)}$. If E is the smooth sections of a holomorphic vector bundle, equipped with its $\mathbb{E} = \bar{\partial}$ -operator, then $\mathcal{M}_*(E)$ is the twisted $\mathcal{A}^{0,\bullet}(Y)$ -module E (considered only as a $\mathcal{A}(Y)$ -module). The \mathbb{Z} -connection $\mathbb{M}\# \mathbb{E} = (\mathbb{M}\# \mathbb{E})^0 + (\mathbb{M}\# \mathbb{E})^1$ where $(\mathbb{M}\# \mathbb{E})^0$ is the $\bar{\partial}$ -operator along the fiber and $(\mathbb{M}\# \mathbb{E})^1$ is the $\bar{\partial}$ -operator along the base. If the cohomology of E with respect to $(\mathbb{M}\# \mathbb{E})^0$ were of locally constant dimension, then it would define a complex of vector bundles on Y and $(\mathbb{M}\# \mathbb{E})^1$ would provide a \mathbb{Z} -connection over Y and we would have an object in $\mathcal{P}_{\mathcal{A}^{0,\bullet}(Y)}$. In general we would have to resolve after we take the fiberwise cohomology or perturb before we take the fiberwise cohomology so that we get a complex of vector bundles. A priori, Grauert's direct image theorem tells us that we will get something coherent. In this paper, we will merely leave the answer as the "quasi"-perfect object $\mathcal{M}_*(E)$ and deal with when the answer lies in the smaller perfect category in part III.

While the bimodule M^\bullet is not finitely generated over $\mathcal{A}(Y)$ it is still projective the sense of homological algebra over topological algebras. We are thus in a relative homological situation.

We now set up the general framework. We will work with Fréchet algebras to make things simpler. The correct generalization beyond Fréchet algebras will be bornological algebras as in [14] and has been used recently by [21] and [18]. The generalization to this context is straightforward. Fix a ground field \mathbb{k} , either \mathbb{R} or \mathbb{C} . A Fréchet space is a locally convex topological vector space over \mathbb{k} which is defined by a metric invariant under vector space addition and which is complete. Equivalently, it is a complete locally convex topological vector space which is defined by a countable collection of semi-norms $p \in \Lambda$. A Fréchet algebra is a Fréchet space \mathcal{A} which is endowed with a \mathbb{C} -bilinear map

$$\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$$

which is continuous. A Fréchet algebra is multiplicatively convex if for every continuous semi-norm $p \in \Lambda$ we have

$$p(ab) \leq p(a)p(b)$$

This notion becomes important when using perturbation techniques as in the proof of Grauert's theorem. A module over a Fréchet algebra \mathcal{A} is a module (for us, usually a right module) M which is a Fréchet space such that the module action is continuous. For two modules M and N over \mathcal{A} , a morphism $T : M \rightarrow N$ is a continuous \mathcal{A} -linear homomorphism. The space $\mathcal{L}_{\mathcal{A}}(M, N)$ is a complete locally convex topological vector space, though no longer Fréchet in general. We will usually consider $\mathcal{L}_{\mathcal{A}}(M, N)$ just as an abstract vector space. A

complex of \mathcal{A} -modules is a complex of Fréchet spaces (M^\bullet, d) such that d is continuous and \mathcal{A} -linear. If (M^\bullet, d) and (N^\bullet, d) are complexes of \mathcal{A} -modules then

$$\mathcal{L}_{\mathcal{A}}^\bullet(M^\bullet, N^\bullet)$$

is a complex with differential

$$(d\phi)(m) = d(\phi(m)) - (-1)^{|\phi|}\phi(d(m)).$$

From now on, when considering Fréchet modules we will write

$$\mathrm{Hom}_{\mathcal{A}} = \mathcal{L}_{\mathcal{A}} \text{ and } \mathrm{Hom}_{\mathcal{A}}^\bullet = \mathcal{L}_{\mathcal{A}}^\bullet.$$

This is consistent with our previous notation since for finitely generated projective modules every \mathcal{A} -linear homomorphism is continuous.

We will use \otimes for the completed projective tensor product of Fréchet spaces. (This is usually denoted $\hat{\otimes}$.) Again, this is consistent with our previous usage since for finitely generated projective modules, the algebraic tensor product and the topological tensor product agree.

For a right \mathcal{A} -module M and a left \mathcal{A} -module N , we define $M \otimes_{\mathcal{A}} N$ to be the quotient by the closure of the image of the map

$$(2.3) \quad M \otimes_{\mathcal{A}} N \rightarrow M \otimes N \quad \text{defined by} \quad m \otimes a \otimes n \mapsto ma \otimes n - m \otimes an$$

This has the universal property that $\mathrm{Hom}_{\mathbb{k}}(M \otimes_{\mathcal{A}} N, L)$ is naturally isomorphic to the space of continuous \mathbb{k} -bilinear maps $T : M \times N \rightarrow L$ such that $T(ma, n) = T(m, an)$. One often defines the tensor product over \mathcal{A} without taking closures. In our case this won't change anything as will be explained below.

Proposition 2.8. *Given a Fréchet \mathcal{A} -module of the form $V \otimes_{\mathbb{k}} A$, where V is a Fréchet space, and any other Fréchet module M there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(V \otimes_{\mathbb{k}} A, M) \cong \mathrm{Hom}_{\mathbb{k}}(V, M)$$

An \mathcal{A} -module of the form $V \otimes_{\mathbb{k}} A$ is called *relatively free*. A module P is *relatively projective* if it is a direct summand of a relatively free module. That is, there is a surjection

$$V \otimes_{\mathbb{k}} A \rightarrow P$$

that admits an \mathcal{A} -linear continuous section. From now on, we use simply free (resp. projective) instead of relatively free (resp. projective).

The following is standard.

Proposition 2.9. *Let P be a projective right \mathcal{A} -module and E any Fréchet left \mathcal{A} -module. Then $P \otimes_{\mathcal{A}} E$ is equal to what one would get by taking the quotient in (2.3) without taking the closure of the image.*

The category of bounded complexes of projective \mathcal{A} -modules and continuous maps forms a DG-category, $\mathcal{C}_{\mathcal{A}}$. For $\mathcal{A} = \mathbb{k}$ is just the category of bounded complexes of Fréchet spaces with continuous maps. As per usual terminology, a module over the category $\mathcal{C}_{\mathcal{A}}$ is a contravariant functor, that is a functor from the opposite category $M : \mathcal{C}_{\mathcal{A}}^{\circ} \rightarrow \mathcal{C}_{\mathbb{k}}$. The category of modules over $\mathcal{C}_{\mathcal{A}}$ itself forms a DG-category, $[\mathcal{C}_{\mathcal{A}}^{\circ}, \mathcal{C}_{\mathcal{A}}^{\circ} \mathrm{DG} - \mathrm{Mod}]$. The category $\mathcal{C}_{\mathcal{A}}$ embeds in $\mathcal{C}_{\mathcal{A}}^{\circ} \mathrm{DG} - \mathrm{Mod}$ via the Yoneda embedding

$$M^\bullet \mapsto \mathrm{Hom}^\bullet(\cdot, M^\bullet)$$

Even for bounded complexes of Fréchet modules M^\bullet which are not projective, we can define an object in $\mathcal{C}_\mathcal{A}^\circ DG - \text{Mod}$ using the same formula as the Yoneda embedding. We note the following obvious proposition that relates the notion of quasi-isomorphism in our DG-category $\mathcal{C}_\mathcal{A}^\circ DG - \text{Mod}$ to relative homological algebra.

Proposition 2.10. *Let*

$$0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$$

be an exact sequences of not necessarily projective \mathcal{A} -modules and consider it as a complex in $\mathcal{C}_\mathcal{A}^\circ DG - \text{Mod}$. Then it is quasi-isomorphic to 0 if and only if it is split as a sequence of topological vector spaces.

Finally we come to the definition of the quasi-perfect category.

Definition 2.11. Let $(\mathcal{A}^\bullet, d, c)$ (or just \mathcal{A}^\bullet for short) be a curved DGA, which is Fréchet as an algebra and such that d is continuous in the Fréchet topology. We associate to \mathcal{A}^\bullet the DG-category $q\mathcal{P}_{\mathcal{A}^\bullet}$ of quasi-perfect twisted complexes whose objects are $E = (E^\bullet, \mathbb{E})$ where E^\bullet is a bounded \mathbb{Z} -graded right Fréchet \mathcal{A} -module which is projective (but not necessarily finitely generated) and

$$\mathbb{E} : E^\bullet \rightarrow E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$$

is a \mathbb{Z} -connection, which is continuous in the respective Fréchet topologies, and it satisfies

$$\mathbb{E} \circ \mathbb{E}(e) = -e \cdot c$$

The morphisms between two such objects $E_1 = (E_1^\bullet, \mathbb{E}_1)$ and $E_2 = (E_2^\bullet, \mathbb{E}_2)$ of degree k are

$$\text{Hom}_{q\mathcal{P}_{\mathcal{A}^\bullet}}^k(E_1, E_2) = \{\phi : E_1^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \rightarrow E_2^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \mid \phi(ea) = (-1)^{k|a|}\phi(e)a\}$$

where now the morphisms are required to be continuous. The morphisms are equipped with a differential defined by

$$d(\phi)(e) = \mathbb{E}_2(\phi(e)) - (-1)^{|\phi|}\phi(\mathbb{E}_1(e))$$

When considering cohomology of the Hom-complex, we always consider the ordinary cohomology of complexes of vector spaces. That is, we do not quotient out by the closure of the boundaries and we do not consider any topology on the cohomology spaces. Of course, the topology of the modules enters when defining what Hom is.

Clearly, $\mathcal{P}_{\mathcal{A}^\bullet}$ is a full subcategory of $q\mathcal{P}_{\mathcal{A}^\bullet}$ since a homomorphisms between finitely generated \mathcal{A} -modules is automatically continuous.

Proposition 2.12. *For a curved DGA $(\mathcal{A}^\bullet, d, c)$ the category $q\mathcal{P}_{\mathcal{A}^\bullet}$ is a DG-category.*

We also have the following generalization of a perfect twisted bimodule. Let $(\mathcal{A}_1^\bullet, d, c_1)$ and $(\mathcal{A}_2^\bullet, d, c_2)$ be two curved DGAs.

Definition 2.13. A quasi-perfect twisted bimodule is the data $X = (X^\bullet, \mathbb{X})$ where

- (1) X^\bullet is a bounded \mathbb{Z} -graded projective right Fréchet \mathcal{A}_2 -module, (not necessarily finitely generated)
- (2) $\mathbb{X} : X^\bullet \rightarrow X^\bullet \otimes_{\mathcal{A}_2} \mathcal{A}_2^\bullet$ is a continuous \mathbb{Z} -connection,

(3) \mathcal{A}_1^\bullet acts continuously on the left of $X^\bullet \otimes_{\mathcal{A}_2} \mathcal{A}_2^\bullet$ satisfying

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b$$

and

$$\mathbb{X}(a \cdot (x \otimes b)) = da \cdot (x \otimes b) + a \cdot \mathbb{X}(x \otimes b)$$

for $a \in \mathcal{A}_1^\bullet$, $x \in X^\bullet$ and $b \in \mathcal{A}_2^\bullet$,

(4) \mathbb{X} satisfies the following condition

$$\mathbb{X} \circ \mathbb{X}(x \otimes b) = c_1 \cdot (x \otimes b) - (x \otimes b) \cdot c_2$$

on the complex $X^\bullet \otimes_{\mathcal{A}_2} \mathcal{A}_2^\bullet$.

Since tensoring a relatively projective right \mathcal{A}_1 -module by a \mathcal{A}_1 - \mathcal{A}_2 bimodule which is a relatively project right \mathcal{A}_2 -module yields a relatively projective right \mathcal{A}_2 -module, we have, as before, that a quasi-perfect twisted bimodule $\mathcal{X} = (X^\bullet, \mathbb{X})$ defines a DG-functor

$$\mathcal{X}_* : q\mathcal{P}_{\mathcal{A}_1^\bullet} \rightarrow q\mathcal{P}_{\mathcal{A}_2^\bullet}$$

Proposition 2.14. *For two complex manifolds X and Y and $p_2 : X \times Y \rightarrow Y$ the projection map, the pair $\mathcal{M} = (M^\bullet, \mathbb{M})$ defined at the beginning of this section defines a $\mathcal{A}^{0,\bullet}(X \times Y)$ - $\mathcal{A}^{0,\bullet}(Y)$ quasi-perfect twisted bimodule, and thus defines a DG functor*

$$\mathcal{M}_* : q\mathcal{P}_{\mathcal{A}^{0,\bullet}(X \times Y)} \rightarrow q\mathcal{P}_{\mathcal{A}^{0,\bullet}(Y)}$$

Remark 2.15. (1) For an arbitrary holomorphic map $f : X \rightarrow Y$ it is possible to define the pushforward using the techniques at hand. Namely, by factoring the map as an embedding and a projection, we can resolve the graph of f , and compose with the quasi-perfect twisted bimodule for the projection. We leave the details as an exercise.
 (2) The pushforward defined by \mathcal{M} is the derived pushforward since we always stay in the category $q\mathcal{P}$, which (for the case of the Dolbeault algebra) corresponds to fine sheaves of projectives.

By enlarging the category from $\mathcal{P}_{\mathcal{A}^\bullet}$ to $q\mathcal{P}_{\mathcal{A}^\bullet}$ we have potentially changed (made more strict) the notion of quasi-isomorphism between to objects in $q\mathcal{P}_{\mathcal{A}^\bullet}$, even if they are both in $\mathcal{P}_{\mathcal{A}^\bullet}$. However, in the case of a complex manifold, the notion of quasi-isomorphism between complexes of coherent sheaves is still preserved in the quasi-perfect category because of the following result which is an extension of the result for the perfect category.

Proposition 2.16. *Let E_1 and E_2 be two quasi-perfect twisted complexes. If for each p , \mathcal{A}^p is flat as an \mathcal{A} -module, a closed morphism $\phi \in \text{Hom}_{q\mathcal{P}_{\mathcal{A}^\bullet}}^0(E_1, E_2)$ is a quasi-isomorphism if $\phi^0 : (E_1^\bullet, \mathbb{E}_1^0) \rightarrow (E_2^\bullet, \mathbb{E}_2^0)$ is a quasi-isomorphism of complexes of \mathcal{A} -modules.*

2.3. DG-equivalences. We would like to establish a criteria for when two quasi-perfect twisted bimodules determine a DG-equivalence. Let $(\mathcal{A}^\bullet, d, c_A)$ and $(\mathcal{B}^\bullet, d, c_B)$ be two curved DGAs. Let us note that in the curved DGA case, the module (\mathcal{A}, d) is not a perfect twisted complex over \mathcal{A}^\bullet , since it does not satisfy the correct curvature conditions. It is however, a perfect twisted bimodule and it is clear that the functors it induces on $\mathcal{P}_{\mathcal{A}^\bullet}$ and $q\mathcal{P}_{\mathcal{A}^\bullet}$ are the identity functors.

Let $(\mathcal{P}^\bullet, \mathbb{P})$ be a \mathcal{B}^\bullet - \mathcal{A}^\bullet quasi-perfect twisted bimodule and $(\mathcal{Q}^\bullet, \mathbb{Q})$ an \mathcal{A}^\bullet - \mathcal{B}^\bullet quasi-perfect twisted bimodule. Let us write $\mathcal{X}^\bullet = \mathcal{Q}^\bullet \otimes_{\mathcal{B}^\bullet} \mathcal{P}^\bullet$ and $\mathcal{Y}^\bullet = \mathcal{P}^\bullet \otimes_{\mathcal{A}^\bullet} \mathcal{Q}^\bullet$. Furthermore, let $\mathbb{X} = \mathbb{Q} \# \mathbb{P}$ and $\mathbb{Y} = \mathbb{P} \# \mathbb{Q}$. Suppose we have two maps

$$\alpha : \mathcal{X}^\bullet \otimes_{\mathcal{A}^\bullet} \mathcal{A}^\bullet \rightarrow \mathcal{A}^\bullet$$

and

$$\beta : \mathcal{Y}^\bullet \otimes_{\mathcal{B}^\bullet} \mathcal{B}^\bullet \rightarrow \mathcal{B}^\bullet$$

such that α is a surjective map of \mathcal{A}^\bullet -bimodules and β is a surjective map of \mathcal{B}^\bullet -bimodules and both α and β intertwine the \mathbb{Z} -connections:

$$\alpha(\mathbb{X}(x)) = d(\alpha(x))$$

and

$$\beta(\mathbb{Y}(y)) = d(\beta(y))$$

Under these circumstances, there are natural transformations of functors

$$\alpha : \mathcal{X}_* \rightarrow \mathbb{1}_{q\mathcal{P}_{\mathcal{A}^\bullet}}$$

and

$$\beta : \mathcal{Y}_* \rightarrow \mathbb{1}_{q\mathcal{P}_{\mathcal{B}^\bullet}}$$

defined, for example, for $E = (E^\bullet, \mathbb{E}) \in q\mathcal{P}_{\mathcal{A}^\bullet}$

$$\alpha_E : E^\bullet \otimes_{\mathcal{A}^\bullet} \mathcal{X}^\bullet \otimes_{\mathcal{A}^\bullet} \mathcal{A}^\bullet \rightarrow E^\bullet \otimes_{\mathcal{A}^\bullet} \mathcal{A}^\bullet$$

by

$$\alpha_E(e \otimes x) = e \otimes \alpha(x)$$

Now according to (2.16), to show that \mathcal{P}^\bullet and \mathcal{Q}^\bullet induce DG-quasi-equivalences, we need to see that for each $E = (E^\bullet, \mathbb{E})$ that we have an isomorphism

$$\alpha_E^0 : H^*(E^\bullet \otimes_{\mathcal{A}^\bullet} \mathcal{X}^\bullet, (\mathbb{E} \# \mathbb{X})^0) \rightarrow H^*(E^\bullet, \mathbb{E}^0)$$

and similarly for \mathcal{Y}_* . We would like to have a condition that we can check about \mathcal{X}^\bullet and \mathcal{Y}^\bullet themselves. We have the following criterion.

Note first, that in the case when the curvature is not zero, that \mathbb{X} is not a differential. It is not even true that $(\mathbb{X}^0)^2$ is necessarily zero. One has $(\mathbb{X}^0)^2 = c_A$. Suppose there is an endomorphism $\Phi : \mathcal{X}^\bullet \rightarrow \mathcal{X}^\bullet$ of degree one, which is \mathcal{A} -linear on the right and the left. Suppose further, that Φ satisfies

$$(2.4) \quad [\mathbb{X}^0, \Phi] + \Phi \circ \Phi = -c_A$$

Then of course we can form $\mathbb{X}^0 + \Phi$ which now has square zero and our criterion is

Lemma 2.17. *If α^0 induces an isomorphism*

$$H^*(\mathcal{X}^\bullet, \mathbb{X}^0 + \Phi) \xrightarrow{\alpha^0} \mathcal{A}$$

then the natural transformation $\alpha : \mathcal{X}_ \rightarrow \mathbb{1}_{\mathcal{P}_{\mathcal{A}^\bullet}}$ is a DG-isomorphism of functors.*

Remark 2.18. As we will see below, one way to interpret the endomorphism Φ is as a trivialization of a “gerbe” on the product $(\mathcal{A}^\bullet, d, c_A) \otimes (\mathcal{B}^\bullet, d, c_B)$.

3. MUKAI DUALITY FOR NON-COMMUTATIVE TORI

In this section we state and prove the duality between a noncommutative complex torus and a gerby complex torus.

3.1. The Complex noncommutative torus. We start by describing the noncommutative tori. We will describe them in terms of twisted group algebras. Let V be a real vector space, and $\Lambda \subset V$ a lattice subgroup. Then we can form the group ring $\mathcal{S}^*(\Lambda)$, the Schwartz space of complex valued functions on Λ which decrease faster than any polynomial. Let $B \in \Lambda^2 V^\vee$, and form the biadditive, antisymmetric group cocycle $\sigma : \Lambda \times \Lambda \rightarrow U(1)$ by

$$\sigma(\lambda_1, \lambda_2) = e^{2\pi i B(\lambda_1, \lambda_2)}$$

In our computations, we will often implicitly make use of the fact that σ is biadditive and anti-symmetric. Now we can form the twisted group algebra $\mathcal{A}(\Lambda; \sigma)$ consisting of the same space of functions as $\mathcal{S}^*(\Lambda)$ but where the multiplication is defined by

$$[\lambda_1] \circ [\lambda_2] = \sigma(\lambda_1, \lambda_2) [\lambda_1 + \lambda_2]$$

This is a $*$ -algebra where $f^*(\lambda) = \overline{f(\lambda^{-1})}$. This is one of the standard ways to describe the (smooth version) of the noncommutative torus. Given $\xi \in V^\vee$ it is easy to check that

$$\xi(f)(\lambda) = 2\pi\sqrt{-1}\langle \xi, \lambda \rangle f(\lambda)$$

defines a derivation on $\mathcal{A}(\Lambda; \sigma)$. Note that the derivation ξ is “real” in the sense that $\xi(f^*) = -\xi(f)$. Finally define a (de Rham) DGA

$$\mathcal{A}^\bullet(\Lambda; \sigma) = \mathcal{A}(\Lambda; \sigma) \otimes \Lambda^\bullet V$$

where the differential d is defined on functions $\phi \in \mathcal{A}(\Lambda; \sigma)$ by

$$\langle d\phi, \xi \rangle = \xi(\phi)$$

for $\xi \in V^\vee$. In other words, for $\lambda \in \Lambda$ one has $d\lambda = 2\pi\sqrt{-1}\lambda \otimes D(\lambda)$ where $D(\lambda)$ denotes λ as an element of $\Lambda^1 V$. Extend d to the rest of $\mathcal{A}^\bullet(\Lambda; \sigma)$ by Leibnitz. Note that $d^2 = 0$.

We are most interested in the case where our torus has a complex structure and in defining the analogue of the Dolbeault DGA for a noncommutative complex torus. So now let V will be a vector space with a complex structure $J : V \rightarrow V$, $J^2 = -\mathbb{1}$. Let g be the complex dimension of V . Set $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Then $J \otimes 1 : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ still squares to $-\mathbb{1}$ and so $V_{\mathbb{C}}$ decomposes into $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces, $V_{1,0} \oplus V_{0,1}$. The dual $V_{\mathbb{C}}^\vee$ also decomposes as $V_{\mathbb{C}}^\vee = V^{1,0} \oplus V^{0,1}$. Let $D' : V_{\mathbb{C}} \otimes \mathbb{C} \rightarrow V^{1,0}$ and $D'' : V_{\mathbb{C}} \otimes \mathbb{C} \rightarrow V^{0,1}$ denote the corresponding projections. Explicitly

$$D' = \frac{J \otimes 1 + 1 \otimes \sqrt{-1}}{2\sqrt{-1}}$$

and

$$D'' = \frac{-J \otimes 1 + 1 \otimes \sqrt{-1}}{2\sqrt{-1}}$$

and $D = D' + D''$ where D denotes the identity. This also established a decomposition

$$\Lambda^k V_{\mathbb{C}} = \otimes_{p+q=\bullet} \Lambda^{p,q} V$$

where $\Lambda^{p,q}V = \Lambda^p V_{1,0} \otimes \Lambda^q V_{0,1}$. Complex conjugation on $V_{\mathbb{C}}$ defines an involution and identifies V with the $v \in V_{\mathbb{C}}$ such that $\bar{v} = v$.

Now let $X = V/\Lambda$, a complex torus of dimension g , and $X^{\bullet} = \bar{V}^{\vee}/\Lambda^{\vee}$ its dual torus. Let $B \in \Lambda^2 V^{\vee}$ be a real (constant) two form on X . Then B will decompose in to parts

$$B = B^{2,0} + B^{1,1} + B^{0,2}$$

where $B^{p,q} \in \Lambda^{p,q}V$, $B^{0,2} = \overline{B^{2,0}}$ and $\overline{B^{1,1}} = B^{1,1}$. Now $B^{0,2} \in \Lambda^2 V^{0,1} \cong H^{0,2}(X)$. Then it also represents a class in $\Pi \in \Lambda^2 V^{0,1} \cong H^0(X^{\vee}; \Lambda^2 T_{1,0} X)$. Let $\sigma : \Lambda \wedge \Lambda \rightarrow U(1)$ denote the group 2-cocycle given by

$$\sigma(\lambda_1, \lambda_2) = e^{2\pi\sqrt{-1}B(\lambda_1, \lambda_2)}.$$

and form as above $\mathcal{A}(\Lambda; \sigma)$ the twisted group algebra based on rapidly decreasing functions. Define the Dolbeault DGA $\mathcal{A}^{0,\bullet}(\Lambda; \sigma)$ to be

$$\mathcal{A}(\Lambda; \sigma) \otimes \Lambda^{\bullet} V_{1,0}$$

where for $\lambda \in \mathcal{A}(\Lambda; \sigma)$ we define

$$\bar{\partial}\lambda = 2\pi\sqrt{-1}\lambda \otimes D'(\lambda) \in \mathcal{A}(\Lambda; \sigma) \otimes V_{1,0}$$

We can then extend $\bar{\partial}$ to the rest of $\mathcal{A}^{0,\bullet}(\Lambda; \sigma)$ by the Leibnitz rule.

Remark 3.1. Let us make a comment about this definition. Even though we are defining the $\bar{\partial}$ operator, we are using the $(1, 0)$ component of $V_{\mathbb{C}}$. This is because of the duality. In the case of the trivial cocycle σ , this definition is meant to reconstruct the Dolbeault algebra on X^{\vee} . In this case, $\mathcal{A}^{0,\bullet}(X^{\vee}) \cong \mathcal{A}(X^{\vee}) \otimes \Lambda^{\bullet} T_0^{0,1} X^{\vee}$. But

$$T_0^{0,1} X^{\vee} = \overline{V^{\vee}} \cong V_{1,0}.$$

To check the reasonableness of this definition we have

Proposition 3.2. *If $\sigma = 1$ is the trivial cocycle, then the DGA $(\mathcal{A}^{0,\bullet}(\Lambda; \sigma), \bar{\partial})$ is isomorphic to the Dolbeault DGA $(\mathcal{A}^{0,\bullet}(X^{\vee}), \bar{\partial})$.*

3.2. The holomorphic gerby torus. The other side of the duality involves a holomorphic \mathcal{O}^{\times} gerbe over the dual of X^{\vee} , that is, over X . A gerbe is usually described in terms of a 2-cocycle with values in \mathcal{O}^{\times} . Our gerbes are topologically trivial, though not holomorphically so. Using this topological trivialization, we can express it in terms of a very simple curved DGA. See [5] for the precise relationship between a holomorphic gerbe defined in terms of a cocycle and the curved DGA we now describe.

Consider $B \in \Lambda^2 V^{\vee}$ decomposed as $B = B^{0,2} + B^{1,1} + B^{2,0}$ as above. Let $\mathcal{A}^{0,\bullet}(X; B)$ denote the curved DGA $(\mathcal{A}^{0,\bullet}(X), \bar{\partial}, 2\pi\sqrt{-1}B^{0,2})$. (We write $\mathcal{A}^{0,\bullet}(X; B)$ even though we have only used the $(0, 2)$ component. If we had constructed the bigraded Dolbeault algebra, $\mathcal{A}^{\bullet,\bullet}(X; B)$, we would have used all the components of B .) While the underlying DGA (non-curved) of $\mathcal{A}^{0,\bullet}(X, B)$ is the same as the Dolbeault algebra of X , the existence of the curvature changes the notion of a module and thus the category \mathcal{P} and $q\mathcal{P}$.

We will show there is a DG-quasi-equivalence of categories between $q\mathcal{P}_{\mathcal{A}^{0,\bullet}(\Lambda; \sigma)}$ and $q\mathcal{P}_{\mathcal{A}^{0,\bullet}(X; B)}$. This will be implemented by a pair of quasi-perfect twisted bimodules which are deformed versions of the Poincare sheaves.

3.3. The deformed Poincare line bundles. A reader familiar with C^* -algebra K -theoretic techniques will notice the similarity of the constructions in this section with the Kasparov bimodules that implement the Baum-Connes assembly map. On the other hand, our constructions are also deformations of the Poincare sheaves on $X \times X^\vee$ realized as a line bundle with a $\bar{\partial}$ -connection.

We define an $\mathcal{A}(X)$ - $\mathcal{A}(\Lambda; \sigma)$ bimodule \mathcal{P} by setting \mathcal{P} to be the vector space of Schwartz functions $\mathcal{S}(V)$ with the left action of $\mathcal{A}(X)$ to be just pulling back a function from X to V and multiplying. The right action of $\mathcal{A}(\Lambda; \sigma)$ is defined for $\lambda \in \Lambda \subset \mathcal{A}(\Lambda; \sigma)$ and for $p \in \mathcal{P}$ by

$$p \cdot \lambda(v) = \sigma(\lambda, v)p(\lambda + v)$$

Here σ has been extended to map from $\Lambda \times V \rightarrow U(1)$ using the same formula: $\sigma(\lambda, v) = e^{2\pi\sqrt{-1}B(\lambda, v)}$. This extension still satisfies the obvious ‘‘cocycle’’ relation

$$\delta\sigma(\lambda_1, \lambda_2, v) = 1$$

where

$$\delta\sigma(\lambda_1, \lambda_2, v) = \sigma(\lambda_2, v)\sigma(\lambda_1 + \lambda_2, v)^{-1}\sigma(\lambda_1, \lambda_2 + v)\sigma(\lambda_1, \lambda_2)^{-1}$$

One checks easily that this makes \mathcal{P} into an $\mathcal{A}(X)$ - $\mathcal{A}(\Lambda; \sigma)$ bimodule.

Lemma 3.3. *The module \mathcal{P} is projective as a right $\mathcal{A}(\Lambda; \sigma)$ module. It is also projective as a left $\mathcal{A}(X)$ -module.*

Proof. The proof of this is rather standard and uses a ‘‘partition of unity’’ $h : V \rightarrow \mathbb{R}$ a compactly supported nonnegative C^∞ function such that

$$\sum_{\gamma \in \Lambda} h(\gamma + v) = 1$$

for all $v \in V$. Then we use it to split the map $\mathcal{P} \otimes \mathcal{A}(\Lambda; \sigma) \rightarrow \mathcal{P}$ given by the action. Define the $\mathcal{A}(\Lambda; \sigma)$ -module map $\iota : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{A}(\Lambda; \sigma)$ (a free $\mathcal{A}(\Lambda; \sigma)$ module)

$$(3.1) \quad \iota p = \sum_{g \in \Lambda} (p \cdot (-g))(v)h(v) \otimes g$$

It is easy to check this is a splitting. We check, for $p \in \mathcal{P}$ and $\mu \in \Lambda$ that

$$(3.2) \quad \begin{aligned} \iota(p \cdot \mu)(v) &= \sum_{g \in \Lambda} ((p \cdot \mu) \cdot (-g))(v)h(v) \otimes g \\ &= \sum_{g \in \Lambda} \sigma(\mu, -g)(p \cdot (\mu - g))(v)h(v) \otimes g \end{aligned}$$

using the fact that $(p \cdot \mu) \cdot g = \sigma(\mu, g)(p \cdot (\mu + g))$. Letting $-\tau = \mu - g$ so $g = \tau + \mu$ (3.2) becomes

$$(3.3) \quad = \sum_{\tau \in \Lambda} (p \cdot (-\tau))(v)h(v) \otimes \sigma(\mu + \tau, \mu)(\tau + \mu) = (\iota(p) \cdot \mu)(v)$$

Thus ι is a module homomorphism.

The projectivity as a left $\mathcal{A}(X)$ -module is even easier. \mathcal{P} is the global C^∞ sections of an infinite dimensional Fréchet space bundle over X ,

$$\mathcal{P} \cong \Gamma(X; \mathcal{V})$$

where the total space of this bundle is

$$\mathcal{V} = V \times_\Lambda \mathcal{S}^* \Lambda$$

Then the standard fact that sections of a bundle (albeit infinite dimensional) is projective still holds. \square

To define our quasi-perfect twisted bimodule we set

$$\mathcal{P}^\bullet = \mathcal{P} \otimes \Lambda^\bullet V^{0,1}$$

The actions extend in an obvious way. The projectivity follows from Lemma 3.3.

$B^{0,2}$ defines a $\bar{\partial}$ -closed $(0, 2)$ form on X . Its pullback to V is $\bar{\partial}$ exact. In fact, $B^{0,2} = \bar{\partial}\omega$ where ω is a $(0, 1)$ form on V which can be described as follows. In coordinates, using the reality of B we have (always use the summation convention)

$$B = b_{ij} dz_i dz_j + \overline{b_{ij}} d\bar{z}_i d\bar{z}_j + c_{ij} dz_i d\bar{z}_j$$

where c_{ij} is a skew Hermitian matrix, and we may (and do) assume that b_{ij} is skew symmetric. Set

$$(3.4) \quad \omega = \overline{b_{ij} z_i} d\bar{z}_j + c_{ij} z_i d\bar{z}_j$$

Then $\bar{\partial}\omega = B^{0,2}$. We now observe the following relationships between ω and σ . Let $\lambda \in \Lambda$. Then

$$(3.5) \quad \begin{aligned} \bar{\partial}(\sigma(\lambda, \cdot)) &= \bar{\partial}(e^{2\pi\sqrt{-1}(b_{ij}\lambda_i z_j + \overline{b_{ij}\lambda_i z_j} + c_{ij}(\lambda_i \bar{z}_j - z_i \bar{\lambda}_j))}) \\ &= 2\pi\sqrt{-1}(\overline{b_{ij}\lambda_i} d\bar{z}_j + c_{ij}\lambda_i d\bar{z}_j)\sigma(\lambda, z) \end{aligned}$$

On the other hand, one easily calculates

$$(3.6) \quad \begin{aligned} 2\pi\sqrt{-1}(\omega - r_\lambda^* \omega)\sigma(\lambda, \cdot) &= 2\pi\sqrt{-1}(\overline{b_{ij}\bar{z}_i} d\bar{z}_j + c_{ij} z_i d\bar{z}_j - \overline{b_{ij}(z_i + \lambda_i)} d\bar{z}_j - c_{ij}(z_i + \lambda_i) d\bar{z}_j)\sigma(\lambda, \cdot) \\ &= -\bar{\partial}\sigma(\lambda, \cdot) \end{aligned}$$

where $r_\lambda(v) = v + \lambda$. Similarly, letting $l_\lambda(v) = v - \lambda$ we have

$$(3.7) \quad 2\pi\sqrt{-1}(\omega - l_\lambda^* \omega)\sigma(\lambda, \cdot) = \bar{\partial}\sigma(\lambda, \cdot).$$

Define a \mathbb{Z} -connection \mathbb{P} on \mathcal{P}^\bullet ,

$$\mathbb{P} : \mathcal{P}^\bullet \rightarrow \mathcal{P}^\bullet \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{A}^{0, \bullet}(\Lambda; \sigma) \cong \mathcal{P}^\bullet \otimes_{\mathbb{C}} \Lambda^\bullet V_{1,0}$$

as $\mathbb{P} = \mathbb{P}^0 + \mathbb{P}^1$ where

$$\mathbb{P}^0(p)(v) = \bar{\partial}_V(p)(v) + 2\pi\sqrt{-1}\omega(v) \wedge p(v)$$

and

$$\mathbb{P}^1 p(v) = -2\pi\sqrt{-1}p(v)D'(v)$$

Let us check that this is indeed a \mathbb{Z} -connection. There are two things to check.

First we check that \mathbb{P}^0 is $\mathcal{A}(\Lambda; \sigma)$ linear: For $\lambda \in \Lambda \subset \mathcal{A}(\Lambda; \sigma)$ and $p \in \mathcal{P}^\bullet$ we have

$$\begin{aligned}
(3.8) \quad \mathbb{P}^0(p \cdot \lambda) &= \bar{\partial}_V(p \cdot \lambda) + 2\pi\sqrt{-1}\omega \wedge (p \cdot \lambda) \\
&= \bar{\partial}_V(r_\lambda^* p \cdot \sigma(\lambda, \cdot)) + 2\pi\sqrt{-1}\omega \wedge r_\lambda^* p \cdot \sigma(\lambda, \cdot) \\
&= \bar{\partial}_V(r_\lambda^* p)\sigma(\lambda, \cdot) + r_\lambda^* p \bar{\partial}_V(\sigma(\lambda, \cdot)) + 2\pi\sqrt{-1}\omega \wedge r_\lambda^* p \cdot \sigma(\lambda, \cdot) \\
&= r_\lambda^* \bar{\partial}_V p \cdot \sigma(\lambda, \cdot) + r_\lambda^* p \wedge 2\pi\sqrt{-1}(r_\lambda^* \omega - \omega)\sigma(\lambda, \cdot) + 2\pi\sqrt{-1}\omega \wedge r_\lambda^* p \cdot \sigma(\lambda, \cdot) \\
&= r_\lambda^*(\bar{\partial}_V(p) + 2\pi\sqrt{-1}\omega \wedge p)\sigma(\lambda, \cdot) \\
&= \mathbb{P}^0(p) \cdot \lambda
\end{aligned}$$

Second we need that \mathbb{P}^1 satisfies Leibnitz with respect to $\mathcal{A}(\Lambda; \sigma)$: Again, for $\lambda \in \Lambda \subset \mathcal{A}(\Lambda; \sigma)$ and $p \in \mathcal{P}^\bullet$ we have

$$\begin{aligned}
(3.9) \quad \mathbb{P}^1(p \cdot \lambda)(v) &= -2\pi\sqrt{-1}(p \cdot \lambda)D'(v) \\
&= -2\pi\sqrt{-1}r_\lambda^* p(v)D'(v)\sigma(\lambda, v)
\end{aligned}$$

while

$$\begin{aligned}
(3.10) \quad \mathbb{P}^1(p) \cdot \lambda(v) + p\bar{\partial}(\lambda)(v) &= (-2\pi\sqrt{-1}pv)D'(v) \cdot \lambda + 2\pi\sqrt{-1}p \cdot \lambda D'(\lambda) \\
&= -2\pi\sqrt{-1}r_\lambda^* p(v)D'(v + \lambda)\sigma(\lambda, v) + 2\pi\sqrt{-1}r_\lambda^* p(v)D'(\lambda)\sigma(\lambda, v) \\
&= -2\pi\sqrt{-1}r_\lambda^* p(v)D'(v)\sigma(\lambda, v)
\end{aligned}$$

Now, $\mathcal{A}^{0,\bullet}(X; B)$ acts on the left of $\mathcal{P}^\bullet \otimes \mathcal{A}^{0,\bullet}(\Lambda; \sigma)$ through its action on \mathcal{P}^\bullet . The fact that for $\eta \in \mathcal{A}^{0,\bullet}(X; B)$ and $p \in \mathcal{P}^\bullet$ satisfies

$$\mathbb{P}(\eta \cdot p) = \bar{\partial}_X \eta \cdot p + \eta \cdot \mathbb{P}(p)$$

is easy to verify.

Finally, let us note that

$$\begin{aligned}
\mathbb{P}(\mathbb{P}(p)) &= \mathbb{P}(\bar{\partial}_V p + 2\pi\sqrt{-1}\omega \wedge p - 2\pi\sqrt{-1}p \otimes D') \\
&= \bar{\partial}_V(\bar{\partial}_V p + 2\pi\sqrt{-1}\omega \wedge p - 2\pi\sqrt{-1}p \otimes D') \\
&\quad + 2\pi\sqrt{-1}\omega \wedge (\bar{\partial}_V p + 2\pi\sqrt{-1}\omega \wedge p - 2\pi\sqrt{-1}p \otimes D') \\
&\quad - 2\pi\sqrt{-1}(\bar{\partial}_V p + 2\pi\sqrt{-1}\omega \wedge p - 2\pi\sqrt{-1}p \otimes D') \wedge D' \\
&= \bar{\partial}_V^2 p + 2\pi\sqrt{-1}\bar{\partial}_V \omega \wedge p - 2\pi\sqrt{-1}\omega \wedge \bar{\partial}_V p - 2\pi\sqrt{-1}\bar{\partial}_V p \otimes D' - 2\pi\sqrt{-1}p \otimes \bar{\partial}_V D' \\
&\quad + 2\pi\sqrt{-1}\omega \wedge (\bar{\partial}_V p + 2\pi\sqrt{-1}\omega \wedge p - 2\pi\sqrt{-1}p \otimes D') \\
&\quad - 2\pi\sqrt{-1}(\bar{\partial}_V p + 2\pi\sqrt{-1}\omega \wedge p - 2\pi\sqrt{-1}p \otimes D') \wedge D'
\end{aligned}$$

After the obvious cancellations we are left with

$$2\pi\sqrt{-1} \bar{\partial}_V \omega \wedge p - 2\pi\sqrt{-1}p \otimes \bar{\partial}_V D'$$

Now $\bar{\partial}_V \omega = B^{0,2}$ and as a map from $V \rightarrow \Lambda^1 V_{1,0}$, D' is holomorphic, so $\bar{\partial}_V D' = 0$. Thus,

$$(3.11) \quad \mathbb{P}(\mathbb{P}p) = 2\pi\sqrt{-1}B^{0,2} \wedge p$$

We thus arrive at the

Proposition 3.4. *$(\mathcal{P}^\bullet, \mathbb{P})$ forms a $\mathcal{A}^{0,\bullet}(X; B) - \mathcal{A}^{0,\bullet}(\Lambda; \sigma)$ quasi-perfect twisted bimodule.*

We define a bi-twisted complex that implements a DG-functor in the opposite direction. Set $\mathcal{Q}^\bullet = \mathcal{S}(V; \Lambda^\bullet V_{1,0})$. In this case we define a left $\mathcal{A}(\Lambda; \sigma)$ -action by defining for $\lambda \in \Lambda \subset \mathcal{A}(\Lambda; \sigma)$ and $q \in \mathcal{Q}^\bullet$ the action

$$\lambda \cdot q(v) = \sigma(\lambda, -\lambda + v)q(-\lambda + v)$$

Note that in our case $\sigma(\lambda, -\lambda + v) = \sigma(\lambda, v)$ but we have written it as above because it is the correct formula, and it works in more general situations. It is straightforward to check that this is indeed a left action. The action extends to the rest of the DGA $\mathcal{A}^{0,\bullet}(\Lambda; \sigma)$ in the obvious way. $\mathcal{A}(X)$ acts again by pull pulling a function on X up to V and multiplying. Exactly as for the case of \mathcal{P}^\bullet , \mathcal{Q}^\bullet is also projective on both sides.

Define a \mathbb{Z} -connection $\mathbb{Q} = \mathbb{Q}^0 + \mathbb{Q}^1$ where

$$(\mathbb{Q}^0 q)(v) = 2\pi\sqrt{-1}q(v)D'(v)$$

and

$$(\mathbb{Q}^1 q)(v) = \bar{\partial}_V q - q \wedge 2\pi\sqrt{-1}\omega$$

Calculations similar to the ones for \mathbb{P} show that \mathbb{Q} is a \mathbb{Z} -connection and that

$$\mathbb{Q}(\mathbb{Q}q) = q \wedge (-2\pi\sqrt{-1}B^{0,2})$$

Hence

Proposition 3.5. *The pair $(\mathcal{Q}^\bullet, \mathbb{Q})$ forms a $\mathcal{A}^{0,\bullet}(\Lambda; \sigma) - \mathcal{A}^{0,\bullet}(X; B)$ quasi-perfect twisted bimodule.*

Define a homomorphism of $\mathcal{A}(\Lambda; \sigma) - \mathcal{A}(\Lambda; \sigma)$ bimodules

$$(3.12) \quad \alpha : \mathcal{Q} \otimes_{\mathcal{A}(X; B)} \mathcal{P} \rightarrow \mathcal{A}(\Lambda; \sigma)$$

by

$$\alpha(q \otimes p) = \sum_{\lambda} \left[\int_V q(v + \lambda)p(v)\sigma^{-1}(\lambda, v)dv \right] [\lambda]$$

Clearly, $\alpha(q\phi \otimes p) = \alpha(q \otimes \phi p)$ for $\phi \in \mathcal{A}(X; B)$. We check that this is a map of $\mathcal{A}(\Lambda; \sigma)$ bimodules. We check the compatibility with the right action. The left action is slightly

simpler.

(3.13)

$$\begin{aligned}
\alpha(q \otimes p \cdot \mu) &= \sum_{\lambda} \left[\int_V q(v + \lambda)(p \cdot \mu)(v) \sigma^{-1}(\lambda, v) dv \right] [\lambda] \\
&= \sum_{\lambda} \left[\int_V q(v + \lambda) p(v + \mu) \sigma(\mu, v) \sigma^{-1}(\lambda, v) dv \right] [\lambda] \\
\text{Setting } w = v + \mu \text{ we get } & \sum_{\lambda} \left[\int_V q(w + \lambda - \mu) p(w) \sigma(\mu, w - \mu) \sigma^{-1}(\lambda, w - \mu) dw \right] [\lambda] \\
\text{Substituting } \tau = \lambda - \mu \text{ we get } & \sum_{\tau} \left[\int_V q(w + \tau) p(w) \sigma(\mu - \lambda, w) \sigma^{-1}(\lambda, -\mu) dw \right] [\tau + \mu] \\
&= \sum_{\tau} \left[\int_V q(w + \tau) p(w) \sigma^{-1}(\tau, w) dw \right] [\tau + \mu] \sigma(\tau, \mu) \\
&= \alpha(q \otimes p) \cdot \mu
\end{aligned}$$

3.4. The duality. Our main theorem is

Theorem 3.6. *The quasi-perfect twisted bimodules \mathcal{P} and \mathcal{Q} define DG functors*

$$\mathcal{P}_* : q\mathcal{P}_{\mathcal{A}^0, \bullet(X; B)} \rightleftarrows q\mathcal{P}_{\mathcal{A}^0, \bullet(\Lambda, \sigma)} : \mathcal{Q}_*$$

which implement DG-quasi-equivalences of DG-categories.

The rest of this section will be devoted to proving this theorem. We prove it by calculating the composition of the twisted bimodules and showing that in each direction they induce functors equivalent to the identity functor (shifted by g).

Write \mathcal{X} for the composed \mathbb{Z} -connection $\mathbb{Q} \# \mathbb{P}$ on the quasi-perfect twisted bimodule $\mathcal{X}^\bullet = \mathcal{Q}^\bullet \otimes_{\mathcal{A}(X; B)} \mathcal{P}^\bullet$. We calculate the zero component $\mathcal{X}^0 = (\mathbb{Q} \# \mathbb{P})^0 = \mathbb{Q} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{P}^0$.

We have $(\mathbb{Q} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{P}^0)(q \otimes p) =$

$$(3.14) \quad 2\pi\sqrt{-1} q D' \otimes p + \bar{\partial}_V(q) \otimes p - 2\pi\sqrt{-1} q \wedge \omega \otimes p + q \otimes \bar{\partial}_V(p) + 2\pi\sqrt{-1} q \otimes \omega \wedge p$$

Now $\mathcal{X}^\bullet \cong \mathcal{S}(V \times \Lambda; \Lambda^\bullet V_{1,0} \otimes \Lambda^\bullet V^{0,1})$ via the isomorphism from $\mathcal{S}(V) \otimes_{\mathcal{A}(X; B)} \mathcal{S}(V) \cong \mathcal{S}(V \times_X V) \cong \mathcal{S}(V \times \Lambda)$ sending $(v_1, v_2) \mapsto (v_1, v_2 - v_1)$. Under this isomorphism, the left and right $\mathcal{A}(\Lambda; \sigma)$ -actions can be written as

$$(3.15) \quad \begin{aligned} (\mu \cdot \phi)(z, \lambda) &= \sigma(\mu, z + \lambda) \phi(z - \mu, \lambda + \mu) \\ (\phi \cdot \mu)(z, \lambda) &= \sigma(\mu, z) \phi(z, \lambda + \mu) \end{aligned}$$

Furthermore, using this isomorphism and writing (3.39) in coordinates we can rewrite for $\phi \in \mathcal{S}(V \times \Lambda; \Lambda^\bullet V_{1,0} \otimes \Lambda^\bullet V^{0,1})$

(3.16)

$$\mathcal{X}^0(\phi)(z, \lambda) = \sum_{j=1}^g \left[2\pi\sqrt{-1} z_j d\bar{c}_j \wedge \phi + d\bar{z}_j \wedge \frac{\partial}{\partial \bar{z}_j} \phi + 2\pi\sqrt{-1} \sum_i (\bar{\lambda}_i b_{ij} + c_{ij} \lambda_i) d\bar{z}_j \wedge \phi \right]$$

Here we are writing $D' : V \rightarrow \Lambda V_{1,0}$ in coordinates as

$$D'(z) = \sum_j z_j d\bar{\zeta}_j$$

where the notation $d\bar{\zeta}_j$ is a basis for $V_{1,0}$. We write it this way since they are really anti-holomorphic basis of \overline{V}^{\vee} . If we write

$$(3.17) \quad B_j(\lambda) = \sum_i \lambda_i b_{ij} + \overline{\lambda_i c_{ij}}$$

then (3.39) can be written

$$(3.18) \quad \mathbb{X}^0(\phi)(z, \lambda) = \sum_j 2\pi\sqrt{-1} z_j d\bar{\zeta}_j \wedge \phi + d\bar{z}_j \wedge \frac{\partial}{\partial z_j} \phi + 2\pi\sqrt{-1} \overline{B_j(\lambda)} d\bar{z}_j \wedge \phi$$

Now we equip \mathcal{X}^\bullet with an inner product. First, define on $\Lambda^\bullet V_{1,0} \otimes \Lambda^\bullet V^{0,1}$ a Hermitian product by declaring $d\bar{z}_j, d\bar{\zeta}_k$ to be orthonormal. Then for $\phi_1, \phi_2 \in \mathcal{X}^\bullet$ set

$$(3.19) \quad \langle \phi_1, \phi_2 \rangle = \int_{V \times \Lambda} \langle \phi_1, \phi_2 \rangle dv$$

where dv is Lebesgue measure on V times counting measure on Λ . We calculate the corresponding Laplacian with respect to this inner product, $(\mathbb{X}^0)^* \mathbb{X}^0 + \mathbb{X}^0 (\mathbb{X}^0)^*$. The adjoint

$$(\mathbb{X}^0)^*(\phi)(z, \lambda) = \sum_j -2\pi\sqrt{-1} \bar{z}_j \iota_{\frac{\partial}{\partial \bar{\zeta}_j}} \phi - \frac{\partial}{\partial z_j} \iota_{\frac{\partial}{\partial \bar{z}_j}} \phi - 2\pi\sqrt{-1} B_j(\lambda) \iota_{\frac{\partial}{\partial \bar{z}_j}} \phi$$

(ι_ξ denotes contraction with respect to the vector field ξ .) We first calculate $(\mathbb{X}^0)^* \mathbb{X}^0$ on functions. It follows from (3.18) that for ϕ a function that

$$(3.20) \quad (\mathbb{X}^0)^* \mathbb{X}^0 \phi(z, \lambda) = \sum_j \left(-\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2\pi\sqrt{-1} \overline{B_j(\lambda)} \frac{\partial}{\partial z_j} - 2\pi\sqrt{-1} B_j(\lambda) \frac{\partial}{\partial \bar{z}_j} + 4\pi^2(|z_j|^2 + |B_j(\lambda)|^2) \right) \phi$$

Write $Y_j(\lambda)$ for the first order differential operator

$$Y_j(\lambda)(\phi) = \overline{B_j(\lambda)} \frac{\partial}{\partial z_j} \phi + B_j(\lambda) \frac{\partial}{\partial \bar{z}_j} \phi$$

We can then rewrite (3.20) as

$$(3.21) \quad (\mathbb{X}^0)^* \mathbb{X}^0 \phi(z, \lambda) = \sum_j \left(-\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2\pi\sqrt{-1} Y_j(\lambda) + 4\pi^2(|z_j|^2 + |B_j(\lambda)|^2) \right) \phi$$

Note that $Y_j(\lambda)(z) = \overline{B_j(\lambda)}$ and $Y_j(\lambda)(\bar{z}) = B_j(\lambda)$. Define the deformed Gaussian, for $\mu \in \Lambda$

$$b_\mu(z, \lambda) = \begin{cases} \exp\left(-2\pi(|z|^2 + \sum_j \sqrt{-1} B_j(\lambda) z_j + \sum_j \sqrt{-1} \overline{B_j(\lambda)} \bar{z}_j)\right) & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$$

We calculate

$$\begin{aligned}
(3.22) \quad -\frac{\partial^2}{\partial z_j \partial \bar{z}_j} b_\mu(z, \lambda) &= -\frac{\partial}{\partial z_j} (-2\pi b_\mu(z, \mu)(z_j + \sqrt{-1} B_j(\mu))) \\
&= -4\pi^2 b_\mu(z, \mu)(\bar{z}_j + \sqrt{-1} B_j(\mu))(z_j + \sqrt{-1} B_j(\mu)) + 2\pi b_\mu \\
&= -4\pi^2 b_\mu(z, \mu)(|z_j|^2 + \sqrt{-1} B_j(\mu) z_j + \sqrt{-1} \overline{B_j(\mu)} z_j - |B_j(\mu)|^2) + 2\pi b_\mu
\end{aligned}$$

if $\lambda = \mu$ and is 0 for $\lambda \neq \mu$. And we see that

$$\begin{aligned}
(3.23) \quad -2\pi\sqrt{-1} Y_j(\lambda) b_\mu(z, \lambda) &= -2\pi\sqrt{-1} (-2\pi b_\mu) Y_j(\lambda) \left(|z_j|^2 + \sqrt{-1} B_j(\mu) z_j + \sqrt{-1} \overline{B_j(\mu)} z_j \right) \\
&= 4\pi^2 \sqrt{-1} b_\mu(z, \mu) \left(\overline{B_j(\mu)} z_j + B_j(\mu) z_j + \sqrt{-1} \overline{B_j(\mu)} B_j(\mu) + \sqrt{-1} B_j(\mu) \overline{B_j(\mu)} \right) \\
&= 4\pi^2 b_\mu(z, \mu) \left(\sqrt{-1} \overline{B_j(\mu)} z_j + \sqrt{-1} B_j(\mu) z_j - 2|B_j(\mu)|^2 \right)
\end{aligned}$$

if $\lambda = \mu$ and is 0 for $\lambda \neq \mu$. Adding (3.22) and (3.23) together we get

$$-4\pi^2 b_\mu(z, \mu) (|z_j|^2 + |B_j(\mu)|^2) + 2\pi b_\mu(z, \mu)$$

if $\lambda = \mu$ and is 0 for $\lambda \neq \mu$. And finally we get that

$$\begin{aligned}
(3.24) \quad (\mathbb{X}^0)^* \mathbb{X}^0 b_\mu(z, \lambda) &= \sum_{j=1}^g \left(-\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2\pi\sqrt{-1} Y_j(\lambda) + 4\pi^2 (|z_j|^2 + |B_j(\lambda)|^2) \right) b_\mu \\
&= \sum_j -4\pi^2 (|z_j|^2 + |B_j(\lambda)|^2) b_\mu + 2\pi b_\mu + 4\pi^2 (|z_j|^2 + |B_j(\lambda)|^2) b_\mu \\
&= 2\pi g b_\mu
\end{aligned}$$

The full Laplacian $\Delta^0 = (\mathbb{X}^0)^* \mathbb{X}^0 + \mathbb{X}^0 (\mathbb{X}^0)^*$ can be calculated in a straightforward manner as

$$(3.25) \quad \Delta^0 \phi(z, \lambda) = \sum_{j=1}^g \left(-\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2\pi\sqrt{-1} Y_j(\lambda) + 4\pi^2 (|z_j|^2 + |B_j(\lambda)|^2) - 2\pi\sqrt{-1} (d\bar{z}_j \circ \iota_{\frac{\partial}{\partial \bar{\zeta}_j}} + \iota_{\frac{\partial}{\partial \bar{z}_j}} \circ d\bar{\zeta}_j) \right) \phi.$$

Let us call $L_j = d\bar{z}_j \circ \iota_{\frac{\partial}{\partial \bar{\zeta}_j}} + \iota_{\frac{\partial}{\partial \bar{z}_j}} \circ d\bar{\zeta}_j$. Then we can find a basis of eigenvectors for $L = \sum_j L_j$ acting on $\Lambda^\bullet V_{1,0} \otimes \Lambda^\bullet V^{0,1}$. Set

$$e_j^\pm = d\bar{z}_j \pm \sqrt{-1} d\bar{\zeta}_j$$

Then

$$L(e_j^\pm) = \pm \sqrt{-1} e_j^\pm$$

And more generally, for $I = (i_1 < i_2 < \dots < i_k)$ and $J = (j_1 < j_2 < \dots < j_l)$ we have

$$L(e_I^+ \wedge e_J^-) = (k-l)\sqrt{-1}(e_I^+ \wedge e_J^-)$$

Hence we have the eigenvector decomposition

$$\Lambda^\bullet V_{1,0} \otimes \Lambda^\bullet V^{0,1} = \oplus_{I,J} \text{span } e_I^+ \wedge e_J^-$$

and on the $e_I^+ \wedge e_J^-$ component, we have $\Delta^0 \phi(z, \lambda)$

$$(3.26) \quad = \left\{ \sum_j \left(-\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2\pi\sqrt{-1} Y_j(\lambda) + 4\pi^2(|z_j|^2 + |B_j(\lambda)|^2) \right) + 2\pi(k-l) \right\} \phi.$$

We now solve this deformed Harmonic oscillator. Indeed the solution is a deformation of the classical solution of the Harmonic oscillator by Hermite functions, constructed using creation and annihilation operators. We will carry out the details in the case of one complex dimension, the higher dimensional case following easily because the variables in our equation are all separable.

Thus, consider the operator

$$(3.27) \quad \mathbb{H}\phi = \left(-\frac{\partial^2}{\partial z \partial \bar{z}} - 2\pi\sqrt{-1} (B(\lambda) \frac{\partial}{\partial \bar{z}} + \overline{B(\lambda)} \frac{\partial}{\partial z}) + 4\pi^2(|z|^2 + |B(\lambda)|^2) \right) \phi$$

defined on the Schwarz space $\mathcal{S}(\mathbb{C} \times \Lambda)$. We calculate the eigenvalues and eigenvectors of \mathbb{H} . Define operators

$$(3.28) \quad \begin{aligned} \mathbb{A}\phi(z, \lambda) &= \left(\frac{\partial}{\partial \bar{z}} + 2\pi(z + \sqrt{-1} \overline{B(\lambda)}) \right) \phi \\ \mathbb{A}^*\phi(z, \lambda) &= \left(-\frac{\partial}{\partial z} + 2\pi(\bar{z} - \sqrt{-1} B(\lambda)) \right) \phi \\ \mathbb{B}\phi(z, \lambda) &= \left(\frac{\partial}{\partial z} + 2\pi(\bar{z} + \sqrt{-1} B(\lambda)) \right) \phi \\ \mathbb{B}^*\phi(z, \lambda) &= \left(-\frac{\partial}{\partial \bar{z}} + 2\pi(z - \sqrt{-1} \overline{B(\lambda)}) \right) \phi \end{aligned}$$

These operators satisfy the following commutation relations.

$$(3.29) \quad \begin{aligned} [\mathbb{H}, \mathbb{A}] &= -2\pi\mathbb{A}, & [\mathbb{H}, \mathbb{A}^*] &= 2\pi\mathbb{A}^* \\ [\mathbb{H}, \mathbb{B}] &= -2\pi\mathbb{B}, & [\mathbb{H}, \mathbb{B}^*] &= 2\pi\mathbb{B}^* \\ [\mathbb{A}, \mathbb{A}^*] &= 4\pi, & [\mathbb{B}, \mathbb{B}^*] &= 4\pi \\ [\mathbb{A}, \mathbb{B}] &= 0, & [\mathbb{A}, \mathbb{B}^*] &= 0 \\ [\mathbb{A}^*, \mathbb{B}] &= 0, & [\mathbb{A}^*, \mathbb{B}^*] &= 0 \end{aligned}$$

Set $b_\mu^{0,0}(z, \lambda) = b_\mu(z, \lambda)$ from above. Define recursively,

$$(3.30) \quad b_\mu^{i+1,j}(z, \lambda) = \mathbb{A}^* b_\mu^{i,j}(z, \lambda), \quad b_\mu^{i,j+1}(z, \lambda) = \mathbb{B}^* b_\mu^{i,j}(z, \lambda)$$

This is well defined since \mathbb{A}^* and \mathbb{B}^* commute. Moreover, $\mathbb{A} b_\mu^{0,0} = \mathbb{B} b_\mu^{0,0} = 0$. The following follows from well-known techniques as in [27].

Theorem 3.7. *The functions $b_\mu^{i,j} \in \mathcal{S}(\mathbb{C} \times \Lambda)$ form an orthogonal complete basis of the closure $L^2(\mathbb{C} \times \Lambda)$ of $\mathcal{S}(\mathbb{C} \times \Lambda)$. Furthermore, we have*

$$\mathbb{H}b_\mu^{i,j} = 2\pi(i+j+1)b_\mu^{i,j}$$

It follows that in g -dimensions, that the ground states b_μ satisfy

$$\mathbb{H}b_\mu = 2\pi gb_\mu$$

Thus we see that there is a kernel for Δ^0 only for $k-l = -g$.

Theorem 3.8. (1) *The kernel of Δ^0 on \mathcal{X}^\bullet is zero except in dimension g where it has an orthogonal basis consisting of η_μ^0 for $\mu \in \Lambda$ where*

$$(3.31) \quad \begin{aligned} \eta_\mu^0 &= b_\mu(z, \lambda) e_1^- \wedge \cdots \wedge e_g^- \\ &= b_\mu(z, \lambda) (d\bar{z}_1 - \sqrt{-1}d\bar{\zeta}_1) \wedge \cdots \wedge (d\bar{z}_g - \sqrt{-1}d\bar{\zeta}_g) \end{aligned}$$

(2) *The cohomology of $(\mathcal{X}^\bullet, \mathbb{X}^0)$ is zero except in dimension g where it has an orthogonal basis consisting of η_μ^0 .*

Now one should note that b_μ is nothing other than $b_0 \cdot (-\mu)$. Moreover, \mathbb{X}^0 is linear with respect to the right action of $\mathcal{A}(\Lambda; \sigma)$. Hence

Corollary 3.9. *The cohomology of $(\mathcal{X}^\bullet, \mathbb{X}^0)$ is zero except in dimension g where it is a free $\mathcal{A}(\Lambda; \sigma)$ module of rank one, with generator η_0^0 .*

It is now time to get the rest of \mathbb{X} (not just \mathbb{X}^0) into the picture. Recall $\mathbb{X} = \mathbb{Q}\#\mathbb{P}$ and we see that $\mathbb{X} = \mathbb{X}^0 + \mathbb{X}^1$ where for $\phi \in (\mathcal{Q}^\bullet \otimes_{\mathcal{A}(X;B)} \mathcal{P}^\bullet) \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{A}^{0,\bullet}(\Lambda; \sigma)$ and using the isomorphisms described above of this with $\mathcal{S}(V \times \Lambda; \Lambda^\bullet V_{1,0} \otimes \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0})$ we have

$$\mathbb{X}^1 \phi(z, \lambda) = (1 \otimes \mathbb{P}^1) \phi(z, \lambda) = \sum_{j=1}^g -2\pi\sqrt{-1} (z_j + \lambda_j) d\bar{\tau}_j \wedge \phi(z, \lambda)$$

Here $d\bar{\tau}_j$ is the same basis of $V_{1,0}$ as $d\bar{\zeta}_j$ but considered in the second copy of $V_{1,0}$. Now

$$(3.32) \quad \begin{aligned} \mathbb{X}(\eta_\mu^0) &= \mathbb{X}(\eta_0^0 \cdot (-\mu)) \\ &= \mathbb{X}(\eta_0^0) \cdot (-\mu) + \eta_0^0 \bar{\partial}(-\mu) \\ &= \sum_j -2\pi\sqrt{-1} z_j d\bar{\tau}_j \wedge \eta_0^0 \cdot (-\mu) - 2\pi\sqrt{-1} \eta_0^0 \mu_j d\bar{\tau}_j \end{aligned}$$

So none of the η_μ^0 are closed in the complex $(\mathcal{X}^\bullet \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{A}^{0,\bullet}(\Lambda; \sigma), \mathbb{X})$.

$$(3.33) \quad \begin{aligned} \mathbb{X}(\eta_0^0)(z, \lambda) &= \mathbb{X}^0 \eta_0^0(z, \lambda) + \mathbb{X}^1 \eta_0^0(z, \lambda) \\ &= \mathbb{X}^1 \eta_0^0(z, \lambda) \\ &= \sum_j -2\pi\sqrt{-1} z_j d\bar{\tau}_j \wedge \eta_0^0(z, 0) \end{aligned}$$

for $\lambda = 0$ and is zero for $\lambda \neq 0$. Therefore, letting

$$(3.34) \quad \begin{aligned} \eta &= b_0(z, \lambda) (e_1^- + \sqrt{-1}d\bar{\tau}_1) \wedge \cdots \wedge (e_g^- + \sqrt{-1}d\bar{\tau}_g) \\ &= b_0(z, \lambda) (d\bar{z}_1 - \sqrt{-1}d\bar{\zeta}_1 + \sqrt{-1}d\bar{\tau}_1) \wedge \cdots \wedge (d\bar{z}_g - \sqrt{-1}d\bar{\zeta}_g + \sqrt{-1}d\bar{\tau}_g) \end{aligned}$$

one sees after a somewhat lengthy but simple computation that

$$\mathbb{X}(\eta) = 0$$

Corollary 3.10. *The cohomology of the complex $(\mathcal{X}[g]^\bullet \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{A}^{0, \bullet}(\Lambda; \sigma), \mathbb{X})$ is zero, except in dimension 0 where it is one (complex) dimensional.*

Proof. Define a map of complexes

$$(\mathcal{A}(\Lambda; \sigma) \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{A}^{0, \bullet}(\Lambda; \sigma), \bar{\partial}) \rightarrow (\mathcal{X}[g]^\bullet \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{A}^{0, \bullet}(\Lambda; \sigma), \mathbb{X})$$

by

$$1 \mapsto \eta$$

Now we use the natural spectral sequence for the complex. According to the previous corollary, 3.9, map above induces an isomorphism on the E_1 term of this spectral sequence. and so the cohomology identifies with $H^*(\mathcal{A}^{0, \bullet}(\Lambda; \sigma))$. \square

As a result of the two corollaries, 3.9 and 3.10 we have

Proposition 3.11. *$(\mathcal{X}[g]^\bullet, \mathbb{X})$ viewed as an object in $q\mathcal{P}_{\mathcal{A}^{0, \bullet}(\Lambda; \sigma)}$ (i.e. just as a right module) is quasi-isomorphic to $(\mathcal{A}, \bar{\partial})$.*

Unfortunately, this is not quite enough to see that it induces quasi-equivalence of categories. We need a map of quasi-perfect twisted bimodules.

We now show that the quasi-perfect twisted bimodule $(\mathcal{X}[g]^\bullet, \mathbb{X})$ induces a functor which is naturally quasi-equivalent to the identity functor. It is clear that the identity functor is implemented by the perfect twisted bimodule $(\mathcal{A}(\Lambda; \sigma), \bar{\partial})$. Using α from (3.12), we define a morphism (also called α

$$(3.35) \quad \alpha \in \text{Hom}_{q\mathcal{P}_{\mathcal{A}^{0, \bullet}(\Lambda; \sigma)}}^0(\mathcal{X}[g]^\bullet, \mathcal{A}(\Lambda; \sigma))$$

where for $q \otimes p \in \mathcal{X}[g]^\bullet \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{A}^{0, \bullet}(\Lambda; \sigma) = (\mathcal{Q}^\bullet \otimes_{\mathcal{A}(X; B)} \mathcal{P}^\bullet) \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{A}^{0, \bullet}(\Lambda; \sigma)$

$$(3.36) \quad \alpha(q \otimes p) = \sum_{\lambda} \frac{1}{(2\sqrt{-1})^g} \left[\int_V q(z + \lambda)p(z)\sigma^{-1}(\lambda, z) \wedge dz_1 \wedge \cdots \wedge dz_g \right] [\lambda]$$

Here the $d\bar{z}$'s in the integrand combine with the dz 's and are integrated. Only the top degree in $d\bar{z}$'s contribute to the integral and thus α indeed maps $\mathcal{X}[g]^\bullet \rightarrow \mathcal{A}(\Lambda; \sigma)$. We also note that $d\bar{\zeta}_j$ and $d\bar{\tau}_j$ are both mapped to $d\bar{\tau}_j$ in $\mathcal{A}^{0, \bullet}(\Lambda; \sigma)$. In terms of the isomorphism of $(\mathcal{Q}^\bullet \otimes_{\mathcal{A}(X; B)} \mathcal{P}^\bullet) \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{A}^{0, \bullet}(\Lambda; \sigma)$ with $\mathcal{S}(V \times \Lambda; \Lambda^\bullet V_{1,0} \otimes \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0})$ we have for $\phi(z, \lambda) \in \mathcal{S}(V \times \Lambda; \Lambda^\bullet V_{1,0} \otimes \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0})$

$$(3.37) \quad \alpha(\phi) = \sum_{\lambda} \frac{1}{(2\sqrt{-1})^g} \left[\int_V \phi(z + \lambda, -\lambda)\sigma^{-1}(\lambda, z) \wedge dz_1 \wedge \cdots \wedge dz_g \right] [\lambda]$$

We now show that

$$(3.38) \quad \alpha(\mathbb{X}(\phi)) = \bar{\partial}(\alpha(\phi))$$

and thus A is a map of twisted bimodules. We compute

$$\begin{aligned}
\alpha(\mathbb{X}(\phi)) &= \sum_{\lambda} \frac{1}{(2\sqrt{-1})^g} \left[\int_V \mathbb{X}(\phi)(z + \lambda, -\lambda) \sigma^{-1}(\lambda, z) \wedge dz_1 \wedge \cdots \wedge dz_g \right] [\lambda] \\
&= \sum_{\lambda, i} \left[\frac{1}{(2\sqrt{-1})^g} \int_V \{ 2\pi\sqrt{-1} d\bar{\zeta}_i \wedge (z + \lambda)_i \phi(z + \lambda, -\lambda) \right. \\
(3.39) \quad &+ d\bar{z}_i \wedge \frac{\partial \phi}{\partial \bar{z}_i}(z + \lambda, -\lambda) \\
&+ 2\pi\sqrt{-1} d\bar{z}_i \wedge (\overline{B_i(-\lambda)} \phi(z + \lambda, -\lambda)) \\
&\left. - 2\pi\sqrt{-1} (z + \lambda - \lambda)_i d\bar{\tau}_i \phi(z + \lambda, -\lambda) \right] \sigma^{-1}(\lambda, z) \wedge dz_1 \wedge \cdots \wedge dz_g [\lambda]
\end{aligned}$$

Now we send both $d\bar{\zeta}_i$ and $d\bar{\tau}_i$ to $d\bar{\tau}_i$

$$\begin{aligned}
&= \sum_{\lambda, i} \left[\frac{1}{(2\sqrt{-1})^g} \int_V \{ 2\pi\sqrt{-1} d\bar{\tau}_i \wedge (z + \lambda)_i \phi(z + \lambda, -\lambda) \right. \\
(3.40) \quad &- 2\pi\sqrt{-1} (z + \lambda - \lambda)_i d\bar{\tau}_i \phi(z + \lambda, -\lambda) \\
&+ d\bar{z}_i \wedge \frac{\partial \phi}{\partial \bar{z}_i}(z + \lambda, -\lambda) \\
&\left. + 2\pi\sqrt{-1} d\bar{z}_i \wedge (\overline{B_i(-\lambda)} \phi(z + \lambda, -\lambda)) \right] \sigma^{-1}(\lambda, z) \wedge dz_1 \wedge \cdots \wedge dz_g [\lambda]
\end{aligned}$$

which is equal to

$$\begin{aligned}
&\sum_{\lambda, i} \left[\frac{1}{(2\sqrt{-1})^g} \int_V \{ 2\pi\sqrt{-1} \lambda_i d\bar{\tau}_i \wedge \phi(z + \lambda, -\lambda) \sigma^{-1}(\lambda, z) \right. \\
(3.41) \quad &+ d\bar{z}_i \wedge \frac{\partial \phi}{\partial \bar{z}_i}(z + \lambda, -\lambda) \sigma^{-1}(\lambda, z) \\
&\left. + 2\pi\sqrt{-1} d\bar{z}_i \wedge (\overline{B_i(-\lambda)} \phi(z + \lambda, -\lambda)) \sigma^{-1}(\lambda, z) \right] \wedge dz_1 \wedge \cdots \wedge dz_g [\lambda]
\end{aligned}$$

And finally we have

$$\begin{aligned}
&\sum_{\lambda, i} \left[\frac{1}{(2\sqrt{-1})^g} \int_V \{ 2\pi\sqrt{-1} \lambda_i d\bar{\tau}_i \wedge \phi(z + \lambda, -\lambda) \sigma^{-1}(\lambda, z) \right. \\
&+ d\bar{z}_i \wedge \frac{\partial(\sigma^{-1}(\lambda, z) \phi(z + \lambda, -\lambda))}{\partial \bar{z}_i} \left. \right] \wedge dz_1 \wedge \cdots \wedge dz_g [\lambda] \\
(3.42) \quad &= \bar{\partial}(\alpha(\phi)) + \sum_{\lambda, i} \frac{1}{(2\sqrt{-1})^g} \int_V d\bar{z}_i \wedge \frac{\partial(\sigma^{-1}(\lambda, z) \phi(z + \lambda, -\lambda))}{\partial \bar{z}_i} \wedge dz_1 \wedge \cdots \wedge dz_g \\
&= \bar{\partial}(\alpha(\phi)) + \sum_{\lambda} \frac{1}{(2\sqrt{-1})^g} \int_V \bar{\partial}(\sigma^{-1}(\lambda, z) \phi(z + \lambda, -\lambda)) \wedge dz_1 \wedge \cdots \wedge dz_g \\
&= \bar{\partial}(A(\phi))
\end{aligned}$$

This last equality holds since for any differential form $f \in \mathcal{S}(V; \Lambda^\bullet V^{0,1})$ it follows that

$$\begin{aligned}
 \int_V \bar{\partial}(f) dz_1 \wedge \cdots \wedge dz_g &= \int_V \bar{\partial}(f dz_1 \wedge \cdots \wedge dz_g) \\
 &= \int_V (d - \partial)(f dz_1 \wedge \cdots \wedge dz_g) \\
 &= \int_V d(f dz_1 \wedge \cdots \wedge dz_g) \\
 &= 0
 \end{aligned}
 \tag{3.43}$$

by Stokes theorem and since f is Schwartz. Thus we may apply our criterion (2.17) to conclude that $(\mathcal{X}^\bullet, \mathbb{X})$ implements a functor equivalent to the identity functor. (Since there is no curvature on this side of the equivalence, $\Phi = 0$.)

In the classical case of Mukai duality on tori, the proof that the composition one direction gives the identity is exactly the same calculation as the other, since both are tori. In our case, one is a noncommutative torus and the other is a gerby torus and things are not quite as symmetric, though as we carry the details out below, one will see a significant overlap.

Write \mathbb{Y} for the composed \mathbb{Z} -connection $\mathbb{P} \# \mathbb{Q}$ on the quasi-perfect twisted bimodule

$$\mathcal{Y}^\bullet = \mathcal{P}^\bullet \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{Q}^\bullet.$$

We calculate the zero component $\mathbb{Y}^0 = (\mathbb{P} \# \mathbb{Q})^0 = \mathbb{P} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{Q}^0$.

We have $(\mathbb{P} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{Q}^0)(p \otimes q) =$

$$\bar{\partial}_V(p) \otimes q + 2\pi\sqrt{-1}\omega \wedge p \otimes q - 2\pi\sqrt{-1}D'p \otimes q + 2\pi\sqrt{-1}p \otimes D'q$$

Now we would like to write down more explicitly quasi-perfect twisted bimodule $(\mathcal{Y}^\bullet, \mathbb{Y})$.

$$\begin{aligned}
 \mathcal{Y}^\bullet &\equiv \mathcal{P}^\bullet \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{Q}^\bullet \\
 &\cong \mathcal{S}(V; \Lambda^\bullet V^{0,1}) \otimes_{\mathcal{A}(\Lambda; \sigma)} \mathcal{S}(V; \Lambda^\bullet V_{1,0})
 \end{aligned}
 \tag{3.45}$$

This last expression is the quotient of $\mathcal{S}(V; \Lambda^\bullet V^{0,1}) \otimes_{\mathbb{C}} \mathcal{S}(V; \Lambda^\bullet V_{1,0})$ by the closure of the relation $p\lambda \otimes q = p \otimes \lambda q$ or what is the same thing, $p\lambda \otimes \lambda^{-1}q = p \otimes q$. This is the coinvariants by the right action of Λ on

$$\mathcal{S}(V \times V; \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0})$$

given by

$$(\phi \cdot \lambda)(z, w) = \phi(z + \lambda, w + \lambda)\sigma(\lambda, z - w)$$

where $\phi \in \mathcal{S}(V \times V; \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0})$, $\lambda \in \Lambda$. Let us emphasize that here by the coinvariants we mean

$$\mathcal{S}(V \times V; \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0}) / (\text{closure of span } (\phi - \phi \cdot \lambda))$$

Proposition 3.12. (1) $\mathcal{Y}^\bullet \cong (\mathcal{S}(V \times V; \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0}))_\Lambda$

$\cong \{\phi \in C^\infty(V \times V; \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0})^\Lambda \mid \phi \text{ satisfies the Schwartz estimates in } z - w\}$.

That is, ϕ is invariant and satisfies

$$(z - w)^\alpha \frac{\partial^{\beta, \gamma} \phi}{\partial z^\beta \partial w^\gamma} \in L^\infty(V \times V; \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0})$$

for all multi-indices α, β and γ .

(2) Under this isomorphism, \mathbb{Y}^0 is

$$\mathbb{Y}^0\phi(z, w) = \bar{\partial}_z\phi(z, w) + 2\pi\sqrt{-1}\omega(z) \wedge \phi - 2\pi\sqrt{-1}D'(z)\phi(z, w) + 2\pi\sqrt{-1}D'(w)\phi(z, w)$$

and using the same conventions as before (in particular, (3.17)) this can be expressed as

$$\mathbb{Y}^0\phi(z, w) = \sum_j d\bar{z}_j \wedge \frac{\partial\phi}{\partial\bar{z}_j}(z, w) + 2\pi\sqrt{-1}d\bar{z}_j \wedge \overline{B_j(z)}\phi(z, w) + 2\pi\sqrt{-1}d\bar{\zeta}_j \wedge (w_j - z_j)\phi(z, w)$$

Proof. Let us call the space of invariants described in the proposition W . The map implementing the isomorphism is $\tau : (\mathcal{S}(V \times V; \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0}))_\Lambda \rightarrow W$ is

$$\tau(\phi)(z, w) = \sum_\lambda \phi(z + \lambda, w + \lambda)\sigma(\lambda, z - w)$$

Clearly this map is well defined on the coinvariants and injective and one checks that the image is in W . One defines a section $\rho : W \rightarrow (\mathcal{S}(V \times V; \Lambda^\bullet V^{0,1} \otimes \Lambda^\bullet V_{1,0}))$ by

$$\rho(\psi)(z, w) = h(z)\psi(z, w)$$

where h is a function as in the proof of 3.3. The image of this in the coinvariants is a section.

The computation of \mathbb{Y} under this isomorphism is clear. \square

Remark 3.13. It seems to be a general phenomenon that the space of coinvariants of a space of functions under a proper group action can be expressed as a space of invariants. One can also write W as the sections of an infinite dimensional vector bundle over X . From now on, when we talk about \mathcal{Y}^\bullet , we will implicitly use the isomorphism with W .

We will be applying the criterion(2.17) to conclude that \mathcal{Y}_* is naturally DG-quasi-equivalent to the identity functor. Unlike the situation for \mathbb{X} , our connection \mathbb{Y} has curvature. In particular $(\mathbb{Y}^0)^2 = 2\pi\sqrt{-1}B^{0,2}$ We define an endomorphism

$$\Phi : \mathcal{Y}^\bullet \rightarrow \mathcal{Y}^\bullet$$

of degree one by

$$\Phi(\phi)(z, w) = 2\pi\sqrt{-1} \sum_j \overline{B_j(w - z)}d\bar{z}_j \wedge \phi(z, w)$$

Then $\Phi \circ \Phi = 0$ and we have $[\mathbb{Y}^0, \Phi] = -2\pi\sqrt{-1}B^{0,2}$. Now we need to calculate

$$H^*(\mathcal{Y}^\bullet, \mathbb{Y}^0 + \Phi).$$

We have $(\mathbb{Y}^0 + \Phi)\phi(z, w)$

$$= \sum_j d\bar{z}_j \wedge \frac{\partial\phi}{\partial\bar{z}_j}(z, w) + 2\pi\sqrt{-1}d\bar{z}_j \wedge \overline{B_j(w)}\phi(z, w) + 2\pi\sqrt{-1}d\bar{\zeta}_j \wedge (w_j - z_j)\phi(z, w)$$

We calculate the Laplacian of $\mathbb{Y}^0 + \Phi$. The adjoint

$$(\mathbb{Y}^0 + \Phi)^*(\phi)(z, w) = \sum_j -\frac{\partial}{\partial z_j} \iota_{\frac{\partial}{\partial \bar{z}_j}} \phi(z, w) - 2\pi\sqrt{-1}B_j(w) \iota_{\frac{\partial}{\partial \bar{z}_j}} \phi(z, w) - 2\pi\sqrt{-1} \overline{(w_j - z_j)} \iota_{\frac{\partial}{\partial \bar{\zeta}_j}} \phi(z, w)$$

Write $Y_j(w)$ for the first order differential operator

$$Y_j(w)(\phi)(z, w) = \overline{B_j(w)} \frac{\partial}{\partial z_j} \phi + B_j(w) \frac{\partial}{\partial \bar{z}_j} \phi(z, w)$$

Then the Laplacian $\square^0(\phi)(z, w) = ((\mathbb{Y}^0 + \Phi)^*(\mathbb{Y}^0 + \Phi) + (\mathbb{Y}^0 + \Phi)(\mathbb{Y}^0 + \Phi)^*)(\phi)(z, w)$

$$(3.46) \quad \begin{aligned} &= \sum_{j=1}^g \left(-\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2\pi\sqrt{-1} Y_j(w) + 4\pi^2(|z_j - w_j|^2 + |B_j(w)|^2) + 2\pi\sqrt{-1} (d\bar{z}_j \circ \iota_{\frac{\partial}{\partial \bar{\zeta}_j}} + \iota_{\frac{\partial}{\partial \bar{z}_j}} \circ d\bar{\zeta}_j) \right) \phi(z, w) \\ &= \sum_{j=1}^g \left(-\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2\pi\sqrt{-1} Y_j(w) + 4\pi^2(|z_j - w_j|^2 + |B_j(w)|^2) + 2\pi\sqrt{-1} L_j \right) \phi(z, w) \end{aligned}$$

where we recall that $L_j = d\bar{z}_j \circ \iota_{\frac{\partial}{\partial \bar{\zeta}_j}} + \iota_{\frac{\partial}{\partial \bar{z}_j}} \circ d\bar{\zeta}_j$ and that we have for $I = (i_1 < i_2 < \dots < i_k)$ and $J = (j_1 < j_2 < \dots < j_l)$

$$L(e_I^+ \wedge e_J^-) = (k - l)\sqrt{-1}(e_I^+ \wedge e_J^-)$$

where $L = \sum_j L_j$ and

$$e_j^\pm = d\bar{z}_j \pm \sqrt{-1}d\bar{\zeta}_j$$

Hence we have the eigenvector decomposition for L

$$\Lambda^\bullet V_{1,0} \otimes \Lambda^\bullet V^{0,1} = \oplus_{I,J} \text{span } e_I^+ \wedge e_J^-$$

Thus we have that on the $e_I^+ \wedge e_J^-$ component, $\square^0\phi(z, w)$

$$(3.47) \quad = \left\{ \sum_j \left(-\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2\pi\sqrt{-1} Y_j(w) + 4\pi^2(|z_j - w_j|^2 + |B_j(w)|^2) \right) - 2\pi(k - l) \right\} \phi(z, w).$$

Now we solve this equation. First, note that in the definition of the 2-cocycle σ , we may insert elements from V into both arguments. Thus $\sigma(z, w)$ is a well defined function on $V \times V$. One has the following formulae:

$$(3.48) \quad \begin{aligned} \frac{\partial}{\partial \bar{z}_j} \sigma(z, w) &= -2\pi\sqrt{-1} \overline{B_j(w)} \sigma(z, w) \\ \frac{\partial}{\partial z_j} \sigma(z, w) &= -2\pi\sqrt{-1} B_j(w) \sigma(z, w) \end{aligned}$$

Define

$$(3.49) \quad a(z, w) = \sigma(z, w) \exp(-2\pi(|z - w|^2))$$

Then

$$(3.50) \quad \begin{aligned} a(z + \lambda, w + \lambda) \sigma(\lambda, z - w) &= \sigma(z + \lambda, w + \lambda) \exp(-2\pi|(z + \lambda) - (w + \lambda)|^2) \sigma(\lambda, z - w) \\ &= \sigma(\lambda, w) \sigma(z, \lambda) \sigma(z, w) \exp(-2\pi|z - w|^2) \sigma(\lambda, z - w) \\ &= a(z, w) \end{aligned}$$

so $a \in \mathcal{Y}^0$.

Let's write \mathbb{L} for the operator

$$\mathbb{L}\phi(z, w) = \sum_j \left(-\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2\pi\sqrt{-1}Y_j(w) + 4\pi^2(|z_j - w_j|^2 + |B_j(w)|^2) \right) \phi(z, w)$$

defined on \mathcal{Y}^0 .

We now show that a is an eigenvector for \mathbb{L} . We have from (3.48) the following formulae

- (1) $\frac{\partial a}{\partial \bar{z}_j}(z, w) = -2\pi((z_j - w_j) + \sqrt{-1}\overline{B_j(w)})a(z, w)$
- (2) $\frac{\partial a}{\partial z_j}(z, w) = -2\pi(\overline{(z_j - w_j)} + \sqrt{-1}B_j(w))a(z, w)$
- (3) $\frac{\partial^2 a}{\partial z_j \partial \bar{z}_j}(z, w) = -2\pi a(z, w)$
 $+ 4\pi^2 \left(|z_j - w_j|^2 + \sqrt{-1}(z_j - w_j)B_j(w) + \sqrt{-1}\overline{(z_j - w_j)}\overline{B_j(w)} - |B_j(w)|^2 \right) a(z, w)$
- (4) $Y_j(z)a(z, w) = -2\pi B_j(w)((z_j - w_j) + \sqrt{-1}\overline{B_j(w)})a(z, w)$
 $+ -2\pi\overline{B_j(w)}(\overline{(z_j - w_j)} + \sqrt{-1}B_j(w))a(z, w)$

So

$$\begin{aligned} \mathbb{L}a(z, w) &= \sum_j 2\pi a(z, w) - \sum_j 4\pi^2 |z_j - w_j|^2 \\ &\quad - \sum_j 4\pi^2 \sqrt{-1} \left(B_j(w)(z_j - w_j) + \overline{B_j(w)}(\overline{(z_j - w_j)}) \right) a(z, w) \\ &\quad + \sum_j 4\pi^2 (|B_j(w)|^2) a(z, w) \\ (3.51) \quad &+ \sum_j 4\pi^2 \sqrt{-1} (B_j(w)((z_j - w_j) + \sqrt{-1}\overline{B_j(w)}) \\ &\quad + 4\pi^2 \sqrt{-1}\overline{B_j(w)}(\overline{(z_j - w_j)} + \sqrt{-1}B_j(w))a(z, w) \\ &\quad + \sum_j 4\pi^2 (|z_j - w_j|^2 + |B_j(x)|^2)a \\ &= 2\pi g a(z, w) \end{aligned}$$

We calculate the eigenvalues and eigenvectors of \mathbb{L} . As before, we do this in the one dimensional case. Define operators

$$\begin{aligned} \mathbb{A}\phi(z, w) &= \left(\frac{\partial}{\partial \bar{z}} + 2\pi((z - w) + \sqrt{-1}B(w)) \right) \phi \\ \mathbb{A}^*\phi(z, w) &= \left(-\frac{\partial}{\partial z} + 2\pi(\overline{(z - w)} - \sqrt{-1}\overline{B(w)}) \right) \phi \\ (3.52) \quad \mathbb{B}\phi(z, w) &= \left(\frac{\partial}{\partial z} + 2\pi(\overline{(z - w)} + \sqrt{-1}\overline{B(w)}) \right) \phi \\ \mathbb{B}^*\phi(z, w) &= \left(-\frac{\partial}{\partial \bar{z}} + 2\pi((z - w) - \sqrt{-1}B(w)) \right) \phi \end{aligned}$$

These operators satisfy the following commutation relations.

$$\begin{aligned}
 (3.53) \quad & [\mathbb{L}, \mathbb{A}] = -2\pi\mathbb{A}, \quad [\mathbb{L}, \mathbb{A}^*] = 2\pi\mathbb{A}^* \\
 & [\mathbb{L}, \mathbb{B}] = -2\pi\mathbb{B}, \quad [\mathbb{L}, \mathbb{B}^*] = 2\pi\mathbb{B}^* \\
 & [\mathbb{A}, \mathbb{A}^*] = 4\pi, \quad [\mathbb{B}, \mathbb{B}^*] = 4\pi \\
 & [\mathbb{A}, \mathbb{B}] = 0, \quad [\mathbb{A}, \mathbb{B}^*] = 0 \\
 & [\mathbb{A}^*, \mathbb{B}] = 0, \quad [\mathbb{A}^*, \mathbb{B}^*] = 0
 \end{aligned}$$

Set $a^{0,0}(z, w) = a(z, w)$ from above. Define recursively,

$$(3.54) \quad a^{i+1,j}(z, w) = \mathbb{A}^* a^{i,j}(z, w), \quad a^{i,j+1}(z, w) = \mathbb{B}^* a^{i,j}(z, w)$$

This is well defined since \mathbb{A}^* and \mathbb{B}^* commute. Moreover, $\mathbb{A}a^{0,0} = \mathbb{B}a^{0,0} = 0$. Then as before we have

Theorem 3.14. *For each $w \in V$, the functions $a^{i,j}(\cdot, w) \in \mathcal{S}(\mathbb{C})$ form an orthogonal complete basis of the closure $L^2(\mathbb{C})$ of $\mathcal{S}(\mathbb{C})$. Furthermore, we have*

$$\mathbb{L}a^{i,j} = 2\pi(i + j + 1)a^{i,j}$$

It follows that in g -dimensions, that the ground states a satisfy

$$\mathbb{L}a = 2\pi ga$$

Thus we see that there is a kernel for \square^0 only for $k - l = g$.

Theorem 3.15. *The cohomology of $(\mathcal{Y}^\bullet, \mathbb{Y}^0 + \Phi)$ is zero except in dimension g where it is a free $\mathcal{A}(\Lambda; \sigma)$ module of rank one, with generator a .*

Finally, define a map of $\mathcal{A}^{0,\bullet}(X; B)$ -bimodules

$$\beta : \mathcal{Y}^\bullet \otimes_{\mathcal{A}(X;B)} \mathcal{A}^{0,\bullet}(X; B) \rightarrow \mathcal{A}^{0,\bullet}(X; B)$$

by

$$\beta(\phi)(z) = \iota_{\Xi}(\phi)(z, z)$$

As before, this needs a little explaining. We have that

$$\mathcal{Y}^\bullet \otimes_{\mathcal{A}(X;B)} \mathcal{A}^{0,\bullet}(X; B) \cong \mathcal{Y}^\bullet \otimes \Lambda^\bullet V^{0,1}.$$

In this last factor of $\Lambda^\bullet V^{0,1}$ we denote the basis by $d\bar{w}_j$. Also, Ξ is the alternating multivector $\frac{\partial}{\partial \bar{\zeta}_1} \wedge \cdots \wedge \frac{\partial}{\partial \bar{\zeta}_g}$. Then the map β does the following. It picks off any factor containing $d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_g$, it sends both $d\bar{z}_j$ and $d\bar{w}_j$ to $d\bar{z}_j$ in $\mathcal{A}^{0,\bullet}(X; B)$ and it restricts this to the diagonal.

Proposition 3.16. *The map β is a map for $\mathcal{A}^{0,\bullet}(X; B)$ -bimodules and it commutes with the \mathbb{Z} -connections.*

This completes the proof of theorem 3.6.

REFERENCES

- [1] Ben-Bassat, O., Block, J., Pantev, T., Noncommutative tori and Fourier-Mukai duality, preprint, math.AG/0509161
- [2] Berthelot, P., Grothendieck, A., Illusie, L., Theorie des intersections et theoreme de Riemann-Roch, Springer Lect. Notes in Math., 225, (1971).
- [3] Block, J., Duality and equivalence of module categories in noncommutative geometry I, preprint math.QA/0509284.
- [4] Block, J., Duality and equivalence of module categories in noncommutative geometry III, in preparation.
- [5] Block, J., Daenzer, C., in preparation.
- [6] Bondal, A. I., Kapranov, M. M., Enhanced triangulated categories, *Math. USSR Sbornik*, Vol. 70 (1991), (No. 1), 93-107.
- [7] Connes, A., Noncommutative Geometry, Academic Press, 1993.
- [8] Connes, A., Gravity coupled with matter and the foundations of noncommutative geometry, Preprint Hep-Th 9603053.
- [9] Demailly, J-P., L^2 -Hodge theory and vanishing theorems, *SMF/AMS Text and Monographs vol 8*, pp 1-97, 1996.
- [10] Drinfeld, V., DG quotients of DG-categories, preprint.
- [11] Evens, S., Lu, J., Weinstein, A., Transverse measures, the modular class and a cohomology pairing for Lie algebroids, *Quart. J. Math. Oxford Ser. (2)* 50 (1999), no. 200, 417–436.
- [12] Griffiths, P. and Harris, J., *Principles of algebraic geometry*, Wiley-Interscience, 1987.
- [13] Hitchin, N., Generalized Complex structures. Preprint.
- [14] Houzel, C., Espaces analytiques relatifs et thorme de finitude. *Math. Ann.* 205 (1973), 13–54.
- [15] Gualtieri, R., Generalized Complex Structures. Preprint.
- [16] Kapustin, A., Topological strings on noncommutative manifolds, preprint hep-th/0310057.
- [17] Kasparov, G. G. The operator K -functor and extensions of C^* -algebras. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 44 (1980), no. 3, 571–636, 719.
- [18] M. Kashiwara, P. Schapira, Constructibility and duality for simple holonomic modules on complex symplectic manifolds, preprint QA/0512047v1.
- [19] Keller, B., Deriving DG categories, *Ann. scient. Ec. Norm. Sup*, 4^e-serie, t. 27, 1994, p. 63-102.
- [20] Kontsevich, M., Course on deformation theory, University of California, Berkeley, 1994.
- [21] Meyer, R., Embeddings of derived categories of bornological modules, Preprint.
- [22] O'Brian, N., Toledo, D., Tong, Y.L., Grothendieck-Riemann-Roch for complex manifolds. *Bull. Amer. Math. Soc. (N.S.)* 5 (1981), no. 2, 182-184.
- [23] Pali, N, Faisceaux $\bar{\partial}$ -coherents sur les varietes complexes, arXiv.org.AG/0305422 v.3, May 2003.
- [24] Polishchuk, Positselski, Quadratic Algebras, Preprint.
- [25] Polishchuk, Schwarz, Holomorphic vector bundles on noncommutative tori.
- [26] Quillen, D., K_0 for nonunital rings and Morita invariance, *J. reine angew. Math.* **472** (1996), 197-217.
- [27] Reed, Simon, B, *Methods in Modern Mathematical Physics*, vol 1 and 2, Academic Press Inc.
- [28] Schneiders, J. P., A coherence criterion for Frechet modules, in *Index Theorems for Elliptic Pairs*, Asterisque 224, Societe Math. de France, 1994.
- [29] Schwarz, A., Noncommutative supergeometry, duality and deformations, *Nuclear Physics B* 650 [PM], (2003) 475-496.
- [30] Simpson, C., Higgs bundles and local systems, *Pub. Math. IHES*, tome 75, (1992), p. 5-95.
- [31] Toledo, D., Tong, Y. L., A parametrix for $\bar{\partial}$ and Riemann-Roch in Čech theory. *Topology* 15 (1976), no. 4, 273–301.
- [32] Toledo, D., Tong, Y.L., Duality and intersection theory in complex manifolds. I. *Math. Ann.* 237 (1978), no. 1, 41–77.
- [33] Verbitsky, M., Coherent sheaves on generic compact tori, preprint AG/0310329.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104
E-mail address: `blockj@math.upenn.edu`