Lengths and volumes in Riemannian manifolds

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Abstract

We consider the question of when an inequality between lengths of "corresponding" geodesics implies a corresponding inequality between volumes. We prove this in a number of cases for compact manifolds with and without boundary. In particular, we show that for two Riemannian metrics of negative curvature on a compact surface without boundary, an inequality between the marked length spectra implies the same inequality between the areas with equality implying isometry.

1 Introduction

In this paper we consider the question of inequalities on volumes implied by inequalities on lengths of geodesics. This is in fact a collection of related problems.

The problem for a $(M, \partial M)$ a compact manifold M with boundary ∂M would be to compare the volumes, $Vol(g_0)$ and $Vol(g_1)$ (in two dimensions we use $A(g_0)$ and $A(g_1)$), of two Riemannian metrics g_0 and g_1 on M if we know that for every pair of points $x, y \in \partial M$ we have $d_{g_0}(x, y) \leq d_{g_1}(x, y)$. In the above $d_{g_i}(x, y)$ represents the distance in M with respect to the the metric g_i, i.e. the length of the g_i-shortest path in M. We sometimes call $d_{g_i}(x, y)$ the *chordal* distance between x and y. Of course, without some further assumptions on the metrics (such as some minimizing property of geodesics) there is no general comparison between the volumes.

The corresponding question for compact manifolds N without boundary involves the marked length spectrum. The marked length spectrum for a Riemannian metric g on N is a function, $MLS_g : \mathcal{C} \to \mathbb{R}^+$, from the set $\mathcal{C}$ of free homotopy classes of the fundamental group $\pi_1(N)$ to the nonnegative reals. For each $\langle\gamma\rangle \in \mathcal{C}$, $MLS_g(\langle\gamma\rangle)$ is the length of the shortest curve in $\langle\gamma\rangle$ (always a geodesic). We consider two Riemannian metrics g_0 and g_1 on N such that $MLS_{g_1}(\langle\gamma\rangle) \geq MLS_{g_0}(\langle\gamma\rangle)$ for all free homotopy classes $\langle\gamma\rangle$ (we then say $MLS_{g_1} \geq MLS_{g_0}$) and ask if $Vol(g_1)$ must be greater than or equal to $Vol(g_0)$. Again this is hopeless without further assumptions.

The natural setting in the manifold without boundary case is in negative curvature. Here there are lots of closed geodesics but exactly one for each free homotopy class (achieving the minimum length in that class). It was conjectured in [C-D-S] that for two negatively curved metrics, g_0 and g_1, on a manifold N the inequality $MLS_{g_1} \geq MLS_{g_0}$ would imply $Vol(g_1) \geq Vol(g_0)$. It was further conjectured that equality would hold if and only if g_0 and g_1 are isometric. Our most general result proves this for surfaces (two dimensional manifolds):

Theorem 1.1. *If g_0 and g_1 are metrics of negative curvature on a surface N^2 such that $MLS_{g_1} \geq MLS_{g_0}$, then $A(g_1) \geq A(g_0)$. Further, if $A(g_1) = A(g_0)$ then g_1 is isometric to g_0.*

The relation between length and volume is more natural when the metrics are pointwise conformal. The higher dimensional version of the above holds in this case.

Theorem 1.2. *Let g_0 be a metric of negative curvature on N^n and let $g_1 = f^2(x)g_0$ for some smooth positive function f on N. Then if $MLS_{g_1} \geq MLS_{g_0}$, then $Vol(g_1) \geq Vol(g_0)$. Further, if $Vol(g_1) = Vol(g_0)$ then $f \equiv 1$.*

We now turn to the case of manifolds with boundary. The problem first showed up in consideration of the boundary rigidity problem. In this problem one considers the case when $d_{g_0}(x,y) = d_{g_1}(x,y)$ for all boundary points x and y and asks if g_0 must be isometric to g_1. Again this cannot hold in general. A survey of what is known about this problem can be found in [Cr3]. There are a few natural choices for assumptions in this case. The most general natural such assumption is SGM. The SGM condition (which is given in terms of $d_g : \partial M \times \partial M \to \mathbb{R}$ only) is harder to define. A precise definition is given in [Cr1], but loosely speaking it means that all nongrazing geodesic segments are strongly minimizing. By a nongrazing geodesic segment we will mean a segment of a geodesic which lies in the interior of M except possibly for the endpoints. A segment is said to minimize if its length is the distance between the endpoints and to strongly minimize if it is the unique such path. Of course this loose definition seems to rely on more than d_g but the relationship is worked out in [Cr1]. Examples of such $(M, \partial M, g)$ are given by compact subdomains of an open ball, B, in a Riemannian manifold where all geodesics segments in B minimize. The only reason not to use the "loose" definition above is that using a definition in terms only of d_g guarantees that if $d_0(x,y) = d_1(x,y)$ and g_0 is SGM then g_1 will be as well. In fact, all theorems stated here for the SGM case also hold for manifolds satisfying the loose definition, so the reader can treat that as a definition of SGM for the purpose of this paper.

The most general result one could hope for would be of the form: If g_0 is an SGM metric on $(M, \partial M)$ and g_1 is another metric with $d_{g_0}(x,y) \leq d_{g_1}(x,y)$ then $Vol(g_0) \leq Vol(g_1)$ with equality of volumes implying isometry of the metrics. This is still very much an open question, which as stated includes the boundary rigidity problem. In this paper we prove a two dimensional version and prove the higher dimensional version for conformal metrics.

The proof of the conformal case is essentially given in [Cr2], however as the version in [Cr2] (which was concerned only with boundary rigidity) makes stronger assumptions and since the proof is short we include a slightly different version of it here.

Theorem 1.3. *Let $(M, \partial M, g_0)$ be an SGM manifold with boundary and let $g_1 = f^2 g_0$ be such that $d_{g_1}(x, y) \geq d_{g_0}(x, y)$ for all $x, y \in \partial M$. Then $Vol(g_1) \geq Vol(g_0)$. Further, if $Vol(g_1) = Vol(g_0)$ then $f \equiv 1$*

We now consider the case where g_0 and g_1 are not assumed to be conformal. We digress for a little to discuss the higher dimensional case even though we will contribute a theorem only in two dimensions. In [Gr1] Gromov introduced the notion of Filling Volume $FillVol(N^n, d)$ for a compact manifold N with a metric d (here d is a distance function which is not necessarily Riemannian). For the actual definition one should see [Gr1], but it is shown in [Gr1] that when $n \geq 2$

$$FillVol(N^n, d) = \inf_g Vol(M^{n+1}, g)$$

where M is any fixed manifold such that $\partial M = N$ (one can even take $M = N \times [0, \infty)$), the infimum is taken over all Riemannian metrics g on M for which the boundary distance function is $\geq d$. In the case where d is the chordal distance function of some Riemannian $(M^{n+1}, \partial M = N, g_0)$ one can not only fix the topology of M^{n+1} but also restrict to metrics g which agree with g_0 when restricted to N (see [Cr3]). The filling volume is not known for many (N, d), but a positive answer to our question would say that $FillVol(\partial M, d) = Vol(M)$ for an SGM manifold $(M, \partial M, g)$, where d is the chordal distance and further that the minimal filling is unique. Gromov in [Gr1] proved this for M^n a compact subdomain of $\mathbb{R}^n$. The minimal entropy theorem of Besson, Courtois, and Gallot [B-C-G2, B-C-G1] can be used to prove the result for compact subdomains of symmetric spaces of negative curvature (see [Cr3]). For general convex simply connected manifolds $(M, \partial M, g_0)$ of negative curvature there is a C^3 neighborhood in the space of metrics such that any metric g_1 in that neighborhood with $g_1|_{\partial M} = g_0|_{\partial M}$ and $d_{g_1}(x, y) \geq d_{g_0}(x, y)$ has $Vol(g_1) \geq Vol(g_0)$ and equality of the volumes implies g_1 is isometric to g_0 (see [C-D-S]). This leads to the conjecture that for any compact subdomain of a simply connected space of negative curvature of dimension ≥ 3 the filling volume of the boundary with the chordal metric is just the volume of the domain and that the domain is the unique volume minimizing filling. Of course this conjecture is still open.

We now turn to the two dimensional case. Recently Ivanov [Iv] considered the case of compact metrics g_0 and g_1 on a disk. He assumes that g_0 is a convex metric in the sense that every pair of interior points can be joined by a unique geodesic. He proves that if $d_{g_1} \geq d_{g_0}$ then $A(g_1) \geq A(g_0)$. He also says that equality in the area would imply that $d_{g_1} = d_{g_0}$. Of course, going from here to isometry of g_0 and g_1 is the boundary rigidity problem. This boundary rigidity problem is solved [Cr2, Ot2] in the case that the metric g_0 has negative curvature. Using completely different methods, we prove a similar though somewhat different result.

Since we will not assume that our surfaces are simply connected we will need to worry about the homotopy class of a path between two boundary points. So

for boundary points x and y, we let $\pi_1(x, y)$ represent the homotopy class of curves in M between x and y and let

$$\mathcal{A} = \{(x, y, [\alpha]) \mid x, y \in \partial M \text{ and } [\alpha] \in \pi_1(x, y)\}.$$

Of course, in the case of a disk $\mathcal{A}$ is just $\partial M \times \partial M$. We will define $L_g(x, y, [\alpha])$ to be the length of the shortest curve from x to y in $[\alpha]$.

There is a natural measure on the space Γ_g of nongrazing g-geodesics between boundary points, and a map $F : \Gamma_g \to \mathcal{A}$ defined by $F(\gamma) = (\gamma(0), \gamma(L), [\gamma])$ where L is the g-length of γ. We get a measure μ_g on $\mathcal{A}$ by pushing the natural measure on Γ_g forward via F, and define $f_g : \mathcal{A} \to \mathbb{R}^+$ by $\mu_g = f_g(x, y, [\alpha])dxdy$ where dx and dy represent the arclength along the boundary. Note that, unless $(M, \partial M, g)$ is a convex metric on a disk, $f_g(x, y, [\alpha])$ will be 0 for many values of $(x, y, [\alpha])$; namely those for which there is no nongrazing geodesic segment in $[\alpha]$ from x to y. (In fact, when M is SGM for each $(x, y) \in \partial M \times \partial M$ there is at most one $[\alpha]$ for which $f_g(x, y, [\alpha]) \neq 0$.) On the other hand it is not hard to see that f_g will be smooth at points $(x, y, [\alpha])$ where there is such a geodesic segment and the geodesic segment is not tangent to the boundary at either endpoint. The fact that $\int_{\mathcal{A}} L_g(x, y, [\alpha])\mu_g = 2\pi A(g)$ is a consequence of Santaló's formula (see section 3). However, somewhat surprisingly, there is a formula for the difference of the areas of two SGM metrics on a surface:

Theorem 1.4. *Let g_0 and g_1 be two SGM metrics on a compact two-dimensional surface $(M, \partial M)$. Then*

$$A(g_1) - A(g_0) = \frac{1}{2\pi} \int_{\mathcal{A}} \left[L_{g_1}(x, y, [\alpha]) - L_{g_0}(x, y, [\alpha])\right] \left[\mu_{g_1} + \mu_{g_0}\right].$$

Of course the content of the proof is that the cross terms cancel. A nearly immediate consequence of this result, which relates lengths to areas, is:

Corollary 1.5. *Let g_0 and g_1 be two SGM metrics on a compact two-dimensional surface $(M, \partial M)$ such that $d_{g_1} \geq d_{g_0}$. If further the g_1-length of each nongrazing g_1-geodesic γ joining boundary points is greater than or equal to the minimal g_0-length of paths homotopic to γ with fixed endpoints, then $A(g_1) \geq A(g_0)$; moreover, if these areas coincide then $d_{g_1}(x, y) = d_{g_0}(x, y)$ for all pairs of boundary points (x, y) such that there is either a g_0 or a g_1 nongrazing geodesic between them.*

In particular, if $(M, \partial M)$ is topologically a disk and if $d_{g_1} \geq d_{g_0}$ then $A(g_1) \geq A(g_0)$.

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2 Manifolds without boundary

The main tools used in the proof of Theorem 1.1 are Bonahon's intersection of geodesic currents [Bo] and Sigmund's theorem [Si] about the denseness of measures supported on closed orbits of an Axiom A flow (in particular the geodesic

flow of a negatively curved metric) in the space of flow invariant probability measures.

We first review the notion of geodesic current in our setting. Let (N^2, g) be a compact surface of negative curvature and $(\tilde{N}, \tilde{g})$ its universal covering with the induced Riemannian metric, and denote by $G(g)$ the space of unoriented geodesics of $\tilde{g}$ furnished with the compact-open topology; $G(g)$ is homeomorphic to the open Möbius band. (It is just $(S^\infty \times S^\infty - Diag)/\{(\theta_1, \theta_2) \sim (\theta_2, \theta_1)\}$ where S^∞ is the circle at infinity.) A *geodesic current* of (N, g) is a locally finite Borel measure on $G(g)$ invariant under the (isometric) action of the fundamental group $\pi_1(N)$. A geodesic current can also be interpreted as an invariant transverse measure on the geodesic foliation $\mathcal{F}$ of the projective tangent bundle $PT(N)$(see [B]). We will let $C(N)$ represent the space of geodesic currents endowed with the weak* topology. The set $\mathcal{C}$ of conjugacy classes of the fundamental group $\pi_1(N)$ embeds in $C(N)$ as follows. Represent each prime conjugacy class (i.e., not an integer multiple of another class) $\langle \gamma \rangle$ by the unique closed geodesic, γ, on N representing the class. The lifts of γ to $\tilde{N}$ form a set of geodesics on $\tilde{N}$ which is invariant under the action of $\pi_1(N)$ and is discrete in the space $G(g)$. We associate $\langle \gamma \rangle$ with the geodesic current $j(\langle \gamma \rangle)$ that is the Dirac measure concentrated on the so-obtained set of geodesics of $\tilde{N}$. For an arbitrary conjugacy class of the fundamental group, the corresponding geodesic current is defined by homogeneity: $j(\langle \gamma^p \rangle) = |p| j(\langle \gamma \rangle)$. Geodesic currents can be added and multiplied by nonnegative reals to yield other geodesic currents.

The following, observed in [Bo], is a straightforward corollary to Sigmund's theorem [Si]:

Proposition 2.1. [Bo, Proposition 2] *The space $C(S)$ is complete and the real multiples of the geodesic currents $j(\langle \gamma \rangle)$, $\gamma \in \pi_1(S)$, coming from closed geodesics are dense in $C(S)$.*

The canonical Liouville form λ_g on the projective tangent bundle $PT(N)$ induces an invariant transverse measure on the geodesic foliation $\mathcal{F}$ of the projective tangent bundle $PT(N)$ which is generated by the 2-form equal to half of the interior product of the Liouville form with the geodesic flow vector field. The resulting geodesic current is called the *Liouville current* of the surface (N, g) and denoted by the same symbol λ_g (see [Bo, Ot1]).

In [Bo] Bonahon constructed a continuous symmetric bilinear form $i : C(S) \times C(S) \to \mathbb{R}^+$, called the *intersection form*. (On the dense set of the currents coming from closed geodesics γ_1 and γ_2, $i(\langle \gamma_1 \rangle, \langle \gamma_2 \rangle)$ is just the number of intersection points of γ_1 with γ_2.) Bonahon showed that i enjoys the following properties relative to the Liouville current λ_g:

Proposition 2.2. (a) *For every conjugacy class $\langle \gamma \rangle$ of the fundamental group $\pi_1(N)$ we have $i(j(\langle \gamma \rangle), \lambda_g) = L_g(\langle \gamma \rangle)$.*
(b) $i(\lambda_g, \lambda_g) = (\pi/2) A(g)$.

In the above $L_g(\langle \gamma \rangle)$ represents the length of the closed geodesic in the class $\langle \gamma \rangle$. Property (a) was proved in [Bo, Proposition 14], [Ot1, Proposition 3]. Property (b) was proved in [Bo, Proof of Proposition 15].

The equality case of Theorem 1.1 relies on a recent positive Livschits type theorem of Lopes and Thieullen [L-T]. The consequence (special case) that we will use is:

Proposition 2.3. *Let A be a Hölder function on the unit tangent bundle of an n-dimensional negatively curved manifold whose integral along every closed geodesic is nonnegative. If the integral of A over the unit tangent bundle is 0 then the integral of A along every closed geodesic is identically 0.*

Proof. To apply the result in [L-T] we note that the geodesic flow is Anosov, and that our assumption along with Sigmund's theorem [Si] implies that the integral of A with respect to any flow invariant probability measure is nonnegative. Thus [L-T] gives us a Hölder function V on the unit tangent bundle which is differentiable (with Hölder derivative V') in the direction of the flow such that $A(v) \geq V'(v)$. Thus, since $0 \leq \int_{UN}(A(v) - V'(v))\,d\lambda_{g_0} = \int_{UN} A(v)\,d\lambda_{g_0} = 0$, we see that $A(v) = V'(v)$ and the result follows. $\square$

Proof of Theorem 1.1. We consider two metrics g_0 and g_1 of negative curvature on the compact surface N. The geodesic flows of g_0 and g_1 are orbit equivalent [Gr2, §8]. In particular, there exists a unique $\pi_1(N)$-equivariant homeomorphism between $G(g_0)$ and $G(g_1)$. Denote by $\lambda_{g_1}^*$ the pull-back of the Liouville current λ_{g_1} of the metric g_1 under this homeomorphism. Then $\lambda_{g_1}^*$ is a geodesic current of the surface (N, g_0). From the hypothesis of the theorem and Proposition 1.2(a) it follows that for all $\gamma \in \pi_1(N)$

$$i(j(\langle\gamma\rangle), \lambda_{g_0}) = L_{g_0}(\langle\gamma\rangle) \leq L_{g_1}(\langle\gamma\rangle) = i(j(\langle\gamma\rangle), \lambda_{g_1}^*).$$

In view of Proposition 2.1 and the continuity of i

$$i(\mu, \lambda_{g_0}) \leq i(\mu, \lambda_{g_1}^*)$$

for all $\mu \in C(N)$. Using this fact twice (with the symmetry of i) and using Proposition 2.2(b) we get the inequality:

$$(\pi/2)A(g_0) = i(\lambda_{g_0}, \lambda_{g_0}) \leq i(\lambda_{g_0}, \lambda_{g_1}^*) \leq i(\lambda_{g_1}^*, \lambda_{g_1}^*) = (\pi/2)A(g_1).$$

Now assume that equality holds. Let Φ^t be the geodesic flow of g_0 on the g_0-unit tangent bundle UN and let Ψ^t be the the geodesic flow of g_1 on the g_1-unit tangent bundle $U_1 N$. Let f be the orbit equivalence, i.e $f : UN \to U_1 N$ is a homeomorphism, homotopic to the mapping induced by the identity mapping $N \to N$, such that the flow $f \circ \Phi^t \circ f^{-1}$ is a time change of Ψ^t. It means that for some $\alpha : UN \times \mathbb{R} \to \mathbb{R}$ we have

$$f(\Phi^t(v)) = \Psi^{\alpha(v,t)}(f(v))$$

for all $v \in UN$ and $t \in \mathbb{R}$. Moreover, by [KH, Chap. 19] we may assume that f is Hölder continuous, and so is α.

Now, observe that by the calculations in [Bo]

$$i(\lambda_{g_0}, \lambda_{g_1}^*) = \frac{1}{4} \int_{UN} \alpha(v, 1)\,d\lambda_{g_0}.$$

Another way to see this is to notice that for every closed g_0-geodesic γ

$$i(\langle\gamma\rangle, \lambda_{g_1}^*) = \int_0^{L_{g_0}(\gamma)} \alpha(\dot\gamma(s), 1)\, ds,$$

because the left-hand side equals $L_{g_1}(\gamma)$ by Proposition 2.2(a) while the right-hand side equals (putting $v = \dot\gamma(0)$ and $L = L_{g_0}(\gamma)$)

$$\int_0^L \alpha(\Phi^s(v), 1)\, ds = \int_0^L (\alpha(v, s+1) - \alpha(v, s))\, ds$$

$$= \int_0^1 (\alpha(v, L+s) - \alpha(v, s))\, ds = \alpha(\Phi^L v, L) = \alpha(v, L) = L_{g_1}(\gamma).$$

Next, one should use Sigmund's theorem to approximate λ_{g_0}.

This also shows that under the assumptions of the theorem the function $A(v) = \alpha(v, 1) - 1$ on UN has nonnegative integrals with respect to all invariant measures on UN. Now, if $A(g_0) = A(g_1)$, it follows that $i(\lambda_{g_0}, \lambda_{g_1}^*) = i(\lambda_{g_0}, \lambda_{g_0}) = \frac{\pi}{2}A(g_0)$. Therefore,

$$\int_{UN} \alpha(v, 1)\, d\lambda_{g_0} - \int_{UN} 1\, d\lambda_{g_0} = 4i(\lambda_{g_0}, \lambda_{g_1}^*) - 2\pi A(g_0) = 0.$$

Thus $A(v)$ integrates to 0 and Proposition 2.3 says that the marked length spectra are the same. The result now follows from the marked length spectrum rigidity theorem for a negatively curved surface of [Cr2, Ot1]. (For marked length spectrum rigidity theorems for nonpositively curved surfaces see [C-F-F, Ba].) $\qquad\square$

Proof of Theorem 1.2. Let N be compact without boundary and g_0 be a Riemannian metric of negative curvature on N and $g_1 = f^2(x)g_0$ be a pointwise conformal metric whose marked length spectrum is greater than or equal to that of g_0. We show:

$Vol(g_1) \geq Vol(g_0)$ with equality if and only if $f(x) = 1$.

We do this in two steps by considering the quantity

$$I = \int_{UN} |u|_1\, d_0u$$

where UN is the g_0-unit tangent bundle of N, $|u|_1$ is the length of the g_0-unit vector u with respect to the metric g_1, and d_0u represents the Liouville measure with respect to g_0. That is $|u|_1 = f(x)$ where $x \in M$ is the base point of u.

Step 1 is just the Hölder inequality (where $\omega(n-1)$ represents the volume of the unit $n-1$ sphere):

$$I = \omega(n-1)\int_N f(x)\, dx \leq \omega(n-1)\left\{\int_N 1\, dx\right\}^{\frac{n-1}{n}}\left\{\int_N f^n(x)\, dx\right\}^{\frac{1}{n}}$$

$$= \omega(n-1)Vol(g_0)^{\frac{n-1}{n}}Vol(g_1)^{\frac{1}{n}}.$$

which says that $I \leq \omega(n-1)Vol(g_0)^{\frac{n-1}{n}}Vol(g_1)^{\frac{1}{n}}$ with equality holding if and only if f is constant.

Step 2 shows that $I \geq \omega(n-1)Vol(g_0)$ which when combined with step 1 yields the result. We let $f(u) : UN \to R$ be the function induced by f (i.e.

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$f(u) = f(x)$ where x is the base point of u). So $I = \int_{UN} f(u) \, d_0 u$. Using Sigmund's theorem [Si] there is a sequence, $\{\gamma_i\}$, of g_0-closed geodesics of lengths L_i such that for any function $h : UN \to R$ we have:

$$\frac{1}{Vol(UN)} \int_{UN} h(u) \, d_0 u = \lim_{i \to \infty} \frac{1}{L_i} \int_0^{L_i} h(\gamma_i'(t)) \, dt.$$

For every g_0-closed geodesic γ, $\int_0^{L_i} f(\gamma'(t)) \, dt$ represents the length of γ in the g_1 metric, which is longer than the shortest closed g_1-geodesic freely homotopic to γ, which in turn, by the condition on the marked length spectrum, is greater than the g_0-length of γ. Thus

$$\frac{I}{Vol(UN)} = \lim_{i \to \infty} \frac{1}{L_i} \int_0^{L_i} f(\gamma_i'(t)) \, dt \geq 1.$$

$\square$

3   Manifolds with Boundary

Let $(M, \partial M, g)$ be a compact SGM Riemannian manifold with boundary. There is a canonical measure $d\gamma$ on the space Γ_g of nongrazing unit speed geodesics (which in the SGM case run from a boundary point to a boundary point) such that for every integrable function f on the unit tangent bundle we have:

$$\int_{\Gamma_g} \int_0^{L(\gamma)} f(\gamma'(t)) \, dt \, d\gamma = \int_{UM} f(u) \, du, \qquad (3.1)$$

where du is the Liouville measure and $L(\gamma)$ is the length of γ. In particular, by plugging in $f = 1$ and letting $\omega(n-1)$ represent the volume of the unit $(n-1)$-sphere,

$$\omega(n-1)Vol(M) = Vol(UM) = \int_{\Gamma_g} L(\gamma) \, d\gamma. \qquad (3.2)$$

Given a codimension 1 submanifold N of M with normal vector ν, the geodesics passing through N can be parameterized by the unit vectors V tangent to M with base point on N. Santaló's formula [Sa1], [Sa2, Chap. 19] allows us to express $d\gamma$ on this subset by $d\gamma = |\langle \nu, v \rangle| \, dv$, where dv is the measure which locally is the product of the Riemannian measure on N (the base point) with the standard measure on a unit sphere. We will use this in two ways.

In the SGM case we can parameterize the geodesic segments by their initial tangents $v \in U^+ \partial M$ where $U^+ \partial M$ represents the unit tangent vectors v to M whose base points are on ∂M and which point "inward" (i.e. $\langle v, \nu \rangle \geq 0$, where ν is the inward normal). Santaló's formula allows us to express $d\gamma$ as $\langle \nu, v \rangle \, dv$. Another immediate application is Crofton's formula for geodesics which says that $C(n-1)Vol(N)$, for a codimension one submanifold N, is just the total $d\gamma$-measure of the set of geodesics that intersect N with each geodesic counted with multiplicity equal to the number of intersections with N. Here $C(n-1)$ is the integral over the unit sphere of $|\langle v, \nu \rangle|$. We will use this in 2-dimensions where N is a curve to conclude that 4 times the length is measure of such geodesics counted with multiplicity.

Proof of Theorem 1.3. We let M be an n-dimensional manifold with boundary which is endowed with two SGM metrics: g_0 with volume $V_0 =$

$\int_M dx_0$ and $g_1 = f(x)g_0$ of volume $V_1 = \int_M f^n(x)\,dx_0$. For $\gamma \in \Gamma_{g_0}$ and $i = 1, 2$, we let $L_i(\gamma)$ represent the length of γ in the g_i metric while $\mathcal{L}_i(\gamma)$ will represent the distance in g_i between the endpoints of γ. Of course $L_0(\gamma) = \mathcal{L}_0(\gamma)$ while $L_1(\gamma) \geq \mathcal{L}_1(\gamma)$. Note that $L_1(\gamma) = \int_0^{L_0(\gamma)} f(\gamma(t))\,dt$. In what follows $c_1 = \omega(n-1)^{-1}$, d_0x is the g_0 Riemannian volume form on M, and $d_0\gamma$ is the g_0 measure on Γ_{g_0}.

By Hölder's inequality:

$$\int_M f^2(x)\,d_0x \leq \left\{ \int_M f^n(x)\,d_0x \right\}^{\frac{2}{n}} \left\{ \int_M d_0x \right\}^{\frac{n-2}{n}} = Vol(g_1)^{\frac{2}{n}} Vol(g_0)^{\frac{n-2}{n}}.$$
(3.3)

Using (3.1) (twice) we see:

$$\int_M f(x)\,d_0x = c_1 \int_{\gamma \in \Gamma_{g_0}} \int_0^{L_0(\gamma)} f(\gamma(t))\,dt\,d_0\gamma = c_1 \int_{\gamma \in \Gamma_0} L_1(\gamma)\,d_0\gamma$$

$$= c_1 \int_{\gamma \in \Gamma_{g_0}} L_0(\gamma)\,d_0\gamma + c_1 \int_{\gamma \in \Gamma_{g_0}} (L_1(\gamma) - L_0(\gamma))\,d_0\gamma$$

$$= \int_M d_0x + c_1 \int_{\gamma \in \Gamma_{g_0}} (L_1(\gamma) - L_0(\gamma))\,d_0\gamma$$

and hence

$$2 \int_M (1 - f(x))\,d_0x = 2c_1 \int_{\gamma \in \Gamma_{g_0}} (L_0(\gamma) - L_1(\gamma))\,d_0\gamma$$

$$\leq 2c_1 \int_{\gamma \in \Gamma_{g_0}} (\mathcal{L}_0(\gamma) - \mathcal{L}_1(\gamma))\,d\gamma_0.$$
(3.4)

Thus since

$$\int_M (1 - f(x))^2\,d_0x = \int_M f(x)^2\,d_0x + 2 \int_M (1 - f(x))\,d_0x - Vol(g_0)$$

using (3.3) and (3.4) we arrive at the stability estimate:

$$\int_M (1 - f(x))^2 d_0x \leq (Vol(g_1)^{\frac{2}{n}} - Vol(g_0)^{\frac{2}{n}}) Vol(g_0)^{\frac{n-2}{n}}$$

$$+ 2c_1 \int_{\gamma \in \Gamma_{g_0}} \mathcal{L}_0(\gamma) - \mathcal{L}_1(\gamma)\,d_0\gamma.$$

In particular, if $\mathcal{L}_1 \geq \mathcal{L}_0$ (i.e $d_{g_1} \geq d_{g_0}$) then $V_{g_1} \geq V(g_0)$, yielding the inequality of the theorem. Further if the volumes are equal then the right hand side is nonpositive and hence $f \equiv 1$. Note that if we assume $\mathcal{L}_0 = \mathcal{L}_1$ then by (3.2) we also have $V(g_0) = V(g_1)$ and we get the boundary rigidity result that $f \equiv 1$. □

We now consider the case of compact SGM surfaces $(M, \partial M, g)$ with boundary. We will make an intersection type argument. We will call $(x, y, [\alpha]) \in \mathcal{A}$ (see the introduction for a definition) g-regular if there is a nongrazing geodesic segment from x to y and this segment is not tangent to the boundary at either endpoint. The g-regular set is of full measure with respect to μ_g. The main tool in our study is:

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Proposition 3.1. *If g_0 and g_1 are SGM Riemannian metrics on a compact surface $(M, \partial M)$ with boundary ∂M then*

$$\int_{\mathcal{A}} L_{g_0}(x, y, [\alpha]) f_{g_1}(x, y, [\alpha]) dx dy = \int_{\mathcal{A}} L_{g_1}(x, y, [\alpha]) f_{g_0}(x, y, [\alpha]) dx dy.$$

Proof. We consider $\mathcal{A} \times \mathcal{A}$ as the measure space with the product measure $\mu_{g_0} \times \mu_{g_1}$ and consider the geometric intersection number $i((x_0, y_0, [\alpha_0]), (x_1, y_1, [\alpha_1]))$ which is the minimum number of intersection points of two curves, one representing $[\alpha_0]$ and one representing $[\alpha_1]$. (Here if for two curves σ and τ there are parameter values s_1, s_2 and t such that $\sigma(s_1) = \tau(t)$ and $\sigma(s_2) = \tau(t)$ then this counts as two intersection points.) We claim that both sides of the equation in the statement of the proposition represent

$$\frac{1}{4} \int_{\mathcal{A} \times \mathcal{A}} i((x_0, y_0, [\alpha_0]), (x_1, y_1, [\alpha_1])) \, d\mu_{g_0}(x_0, y_0, [\alpha_0]) \times d\mu_{g_1}(x_1, y_1, [\alpha_1]).$$

We will show this for the right hand side. The other side is similar. By Fubini's theorem we need only show that for each fixed g_0-regular $(x_0, y_0, [\alpha_0])$:

$$\frac{1}{4} \int_{\mathcal{A}} i((x_0, y_0, [\alpha_0]), (x_1, y_1, [\alpha_1])) \, d\mu_{g_1}(x_1, y_1, [\alpha_1]) = L_{g_1}(x_0, y_0, [\alpha_0]).$$

In this integral we can ignore $\{(x_1, y_1, [\alpha_1]) \mid \{x_0, y_0\} \cap \{x_1, y_1\} \neq \emptyset\}$ since it has measure 0. Let τ be a shortest g_1 path from x_0 to y_0 in $[\alpha_0]$. We will show that if $(x_1, y_1, [\alpha_1])$ is g_1-regular and γ is the g_1-geodesic segment from x_1 to y_1 in $[\alpha_1]$ then γ will intersect τ in precisely $i((x_0, y_0, [\alpha_0]), (x_1, y_1, [\alpha_1]))$ points. The result will then follow from the version of Crofton's formula we discussed above. We note that, since $(x_1, y_1, [\alpha_1])$ is g_1-regular and since γ and τ do not share endpoints, γ will intersect τ at finitely many interior points of M. To count the number of intersection points of γ and τ we can instead lift τ to a curve $\tilde{\tau}$ in the universal cover $\tilde{M}$ and count the number, n, of lifts, $\tilde{\gamma}_i$, of γ whose intersection with $\tilde{\tau}$ is nonempty. The homotopy minimizing properties of τ and γ imply that $\tilde{\tau}$ and $\tilde{\gamma}_i$ in fact minimize distance between endpoints and hence they intersect at at most one point. So τ and γ intersect in exactly n points. Now if τ' and γ' are curves homotopic to τ and γ relative to the endpoints then the corresponding lifts (i.e. lifts having the same endpoints) $\tilde{\tau}'$ and $\tilde{\gamma}'$ will have oriented intersection number 1 and hence must intersect. So τ' and γ' intersect at least n times. Thus n must be equal to $i((x_0, y_0, [\alpha_0]), (x_1, y_1, [\alpha_1]))$. $\square$

The proof of Theorem 1.4 follows immediately from this proposition and the area formula $\int_{\mathcal{A}} L_g(x, y, [\alpha]) \mu_g = 2\pi A(g)$.

Proof of Corollary 1.5. From Theorem 1.4 we need only see the inequality $L_{g_1}(x, y, [\alpha]) \geq L_{g_0}(x, y, [\alpha])$ whenever either $f_{g_1}(x, y, [\alpha])$ or $f_{g_0}(x, y, [\alpha])$ is not zero. If $f_{g_0}(x, y, [\alpha])$ is not zero then the general assumption $d_{g_1} \geq d_{g_0}$ gives the result. If $f_{g_1}(x, y, [\alpha])$ is not zero then the other assumption is exactly the condition $L_{g_1}(x, y, [\alpha]) \geq L_{g_0}(x, y, [\alpha])$.

Equality holds above if and only if for all $(x, y, [\alpha])$ that are either g_0-regular or g_1-regular we have $L_{g_1}(x, y, [\alpha]) = L_{g_0}(x, y, [\alpha])$. Other pairs x, y of boundary points in the equality case of the corollary that are not covered by the above are limits of ones above. $\square$

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