SCATTERING DATA VERSUS LENS DATA ON SURFACES

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ABSTRACT. The scattering data of a Riemannian manifold with boundary record the incoming and outgoing directions of each geodesic passing through. We show that the scattering data of a generic Riemannian surface with no trapped geodesics and no conjugate points determine the lengths of geodesics. Counterexamples exists when trapped geodesics are allowed.

1. INTRODUCTION

1.1. **Scattering data and lens data.** Let M be a Riemannian manifold. Let $\pi : \Omega M \to M$ be the unit tangent bundle of M and $\Omega_x M$ be the set of unit tangent vectors at x for any $x \in M$. Let $\partial \Omega M$ be the boundary of the unit tangent bundle of M. In other words, $\partial \Omega M = \bigcup_{x \in \partial M} \Omega_x M$. For each $x \in \partial M$, let $\nu_M(x)$ be the unit normal vector of M pointing inwards at x. Then put $\partial_+ \Omega_x M = \{X \in \Omega_x M : (X, \nu_M(x))_{g_M} > 0\}$, $\partial_0 \Omega_x M = \{X \in \Omega_x M : (X, \nu_M(x))_{g_M} = 0\}$, and $\partial_- \Omega_x M = \{X \in \Omega_x M : (X, \nu_M(x))_{g_M} < 0\}$. Also, write $\partial_+ \Omega M = \bigcup_{x \in \partial M} \partial_+ \Omega_x M$, $\partial_0 \Omega M = \bigcup_{x \in \partial M} \partial_0 \Omega_x M$, and $\partial_- \Omega M = \bigcup_{x \in \partial M} \partial_- \Omega_x M$.

For each $X \in \partial_+ \Omega M$, there is a geodesic γ_X whose initial tangent vector is X. Extend the geodesic as long as possible until it touches the boundary ∂M again. Put $\tau(X) := \ell(\gamma_X)$, the length of γ_X.

If the geodesic γ_X is of finite length, call its tangent vector at the other end point $\alpha_M(X)$. (See Figure 1.) The map $\alpha_M : \partial_+ \Omega M \to \partial \Omega M$ defined above is called the *scattering relation* of M. Note that $\alpha_M(X)$ will be undefined if γ_X is of infinite length.
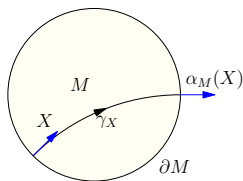

FIGURE 1. The scattering map α_M

Suppose that we have two Riemannian manifolds (M, g_M), (N, g_N) and an isometry $h : \partial M \to \partial N$ between their boundaries. Then there is a natural bundle map $\varphi : \partial \Omega M \to \partial \Omega N$ defined as

$$\varphi(aX + b\nu_M(x)) = ah_*(X) + b\nu_N(h(x)) \tag{1}$$

For any unit vector X based at x tangent to ∂M and real numbers a and b such that $a^2 + b^2 = 1$.

Definition 1.1. M and N are said to have the same *scattering data rel h* if $\varphi \circ \alpha_M = \alpha_N \circ \varphi$. If we also have $\ell(\gamma_X) = \ell(\gamma_{\varphi(X)})$, then we say M and N have the same *lens data rel h*.

We will omit "rel h" when h is clear from the context.

The difference between lens data and scattering data is quite subtle since lengths of geodesics can be recovered locally from scattering data up to a constant using the first variation of arc length [Mic]. However, the scattering data do not necessarily determine the lens data completely. For example, consider the two Riemannian manifolds in Figure 2 (which contain trapped geodesics) where the second is obtained from the first by removing a round hemisphere and identifying antipodal points on the boundary great circle. The two surfaces have the same scattering data but different lens data. If a geodesic in the first manifold does not enter the hemisphere part then the corresponding geodesic in the second looks the same and has the same length. However for geodesics that enter the hemisphere the lengths of corresponding geodesics differ by a constant equal to the (intrinsic) diameter of the hemisphere.
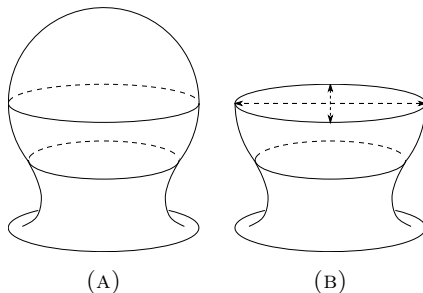

(A)   (B)

FIGURE 2. Same scattering data but different lens data. Here 2b is obtained from 2a by removing an upper hemisphere and then identifying antipodal points of the boundary component on the top.

Definition 1.2. $p \in \partial M$ is called a switch point if the geodesic curvature of ∂M is 0 at p but not always 0 in any neighborhood of p. The set of switch points of M is denoted by F_M.

In general F_M could even be a Cantor set with positive measure, which would create a lot of technical difficulties. In this paper we will only address the generic case where F_M is finite.

When we say a manifold has no conjugate points we mean that for every geodesic segment $\gamma : [0, L] \to M$ all nontrivial Jacobi fields can vanish at most once. Note that the geodesic here is allowed to be tangent to boundary at points in its interior. (Also note that this definition is not the same as the one in [ABB3].)

Theorem 1. *Suppose that we have two compact smooth Riemannian surfaces* (M, g_M), (N, g_N) *and an isometry* $h : \partial M \to \partial N$ *between their boundaries. Assume that F_M is finite, and M has no trapped geodesics (including closed geodesics)*

and no conjugate points. Then M and N have the same scattering data rel h if and only if they have the same lens data rel h.

1.2. Scattering rigidity and lens rigidity.

Definition 1.3. M is *scattering rigid* (resp. *lens rigid*) if the space M and the metric on M is determined by its scattering data (resp. lens data) up to an isometry which leaves the boundary fixed.

Showing that a manifold is scattering or lens rigid is an example of a geometric inverse problem (for which there is a vast literature).

A number of manifolds are known to be lens rigid:

(1) Simple Riemannian surfaces with boundary (L. Pestov–G. Uhlmann, [PU])
(2) Compact subdomains of $\mathbb{R}^n$ with flat metrics (M. Gromov, [Gro]) or metrics close to that (D. Burago–S. Ivanov, [BI1])
(3) Compact subdomains of open hemispheres (R. Michel , [Mic])
(4) Almost hyperbolic metrics (D. Burago–S. Ivanov, [BI2])
(5) Compact subdomains of symmetric spaces of negative curvature (G. Besson–G. Courtois–S. Gallot, [BCG])
(6) $D^n \times \mathbb{R}/\mathbb{Z}^1$ when $n > 1$ (C. Croke, [Cro2]) and when $n = 1$ (C. Croke–P. Herreros, [CH])

However, very few are known to be scattering rigid:

(1) $D^n \times \mathbb{R}/\mathbb{Z}^1$ when $n > 1$ (C. Croke, [Cro2]) (It is still not known if the flat annulus $D^1 \times \mathbb{R}/\mathbb{Z}^1$ is scattering rigid.)
(2) Simple Riemannian surfaces with boundary (H. Wen, [Wen])

Scattering rigidity and lens rigidity are equivalent when the scattering data determine the lens data. Hence we have the following corollary of Theorem 1

Corollary 1. *Suppose that M satisfies the conditions in Theorem 1, then M is scattering rigid if and only if M is lens rigid.*

Remark. Simple Riemannian manifolds [Mic] and, more generally, SGM (strongly geodesically minimizing) manifolds [Cro1] are conjectured to be lens rigid. Most SGM surfaces satisfy our conditions since they have no trapped geodesics and they have conjugate points only in some non-generic cases.

2. Space of geodesics

Geodesics on a smooth Riemannian manifold without boundary satisfy the geodesic equation and they minimize the length locally.

However, a curve on a smooth Riemannian manifold with boundary that does not satisfy the geodesic equation may still minimize the length locally if part of the curve runs along the boundary.

Definition 2.1. A smooth curve $\gamma : [a, b] \to M$ is called a *geodesic* if it satisfies the geodesic equation $\nabla_{\gamma'}\gamma' = 0$. A curve γ in M is called a *p-geodesic* if it minimize the length locally, that is, for any $t \in [a, b]$, there is $\delta > 0$ such that $\gamma|_{[t,t']}$ is the shortest curve connecting $\gamma(t)$ and $\gamma(t')$ for any t' such that $|t - t'| < \delta$.

The basic properties of p-geodesics were studied in [ABB1] (also see [ABB2, ABB3]) (in these references p-geodesics are simply referred to as geodesics). In particular such a p-geodesic is a C^1 path. The path is the union of not only interior

segments (smooth geodesic segments in the usual sense) and boundary segments (smooth geodesic segments of the boundary) and switch points (where the path joins two of the previous types and is not twice differentiable) but also accumulation points of switch points, called intermittent points. On can even have cantor sets of positive measure of intermittent points. The boundary will have 0 curvature (as will the p-geodesic) at intermittent points. In two dimensions it is also easy to see that if $\gamma(t) \in \partial M$ then the boundary is concave at $\gamma(t)$ (though maybe not strictly) since otherwise one could locally shorten the curve.

In section 7 we will prove:

Proposition 1. *Let M be a compact 2-manifold (with or without boundary) that has no conjugate points and finite F_M, and γ a p-geodesic segment between points $x \in M$ and $y \in M$. Then for any curve τ from x to y homotopic to γ (relative to x and y) we have $L(\gamma) \leq L(\tau)$. Further $L(\gamma) = L(\tau)$ only when $\gamma = \tau$ up to parametrization.*

We should remark that that the assumption that F_M is finite in Proposition 1 is probably unnecessary. In particular it is easy to see (and is of independent interest):

Corollary 2. *Let M be a compact 2-manifold (with or without boundary) that has no conjugate points, and γ a geodesic segment between points $x \in M$ and $y \in M$. Then for any curve τ from x to y homotopic to γ (relative to x and y) we have $L(\gamma) \leq L(\tau)$. Further $L(\gamma) = L(\tau)$ only when $\gamma = \tau$ up to parametrization.*

Proof. We can extend our metric smoothly to a metric on M union a collar neighborhood of the boundary. We can then change our boundary by an arbitrarily small amount to an embedded curve that has a finite F_M and lies totally in the collar neighborhood. If the perturbation is small enough geodesics in the expanded space will still have no conjugate points so if the Proposition is true when F_M is finite then each geodesic segment γ minimizes in its homotopy class in the larger space and hence in the original space (since the perturbation can be made so as to preserve homotopy classes). $\qquad\square$

Let $\tilde{M}$ be the universal cover of M. Let $\Gamma^p_{\tilde{M}}$ be the space of p-geodesics $[0,1] \to \tilde{M}$ with the compact open topology. Define $(s_M, e_M) : \Gamma^p_{\tilde{M}} \to \tilde{M} \times \tilde{M}$ as $s_M(\gamma) = \gamma(0)$ and $e_M(\gamma) = \gamma(1)$.

Another easy consequence of Proposition 1 is:

Proposition 2. *If M has no conjugate points and finite F_M, then (s_M, e_M) is a homeomorphism.*

3. NON-CONVEX PART OF THE BOUNDARY

In the rest of the paper, M and N will be two compact smooth Riemannian surfaces with the same scattering data rel $h : \partial M \to \partial N$ where h is an isometry. M is assumed to have finitely many switch points, no trapped geodesics (including closed geodesics) and no conjugate points. $\varphi : \partial \Omega M \to \partial \Omega N$ is the induced bundle map defined in (1).

We say that ∂M is strictly convex near p if the curvature of ∂M is positive at p. We say that ∂M is strictly concave near p if the curvature of ∂M is negative at p. We say that ∂M is totally geodesic near p if the curvature of ∂M is zero

near p. Let $S_- = \{p \in \partial M : \partial M$ is strictly concave near $p\}$, $S_+ = \{p \in \partial M : \partial M$ is strictly convex near $p\}$ and $S_0 = \{p \in \partial M : \partial M$ is totally geodesic near $p\}$. Here being strictly concave means that the curvature is negative, Note that S_-, S_+ and S_0 are all open in ∂M and that $S_s := M \setminus (S_- \bigcup S_+ \bigcup S_0)$ is the set of switch points.

Proposition 3. *If $p_0 \in \overline{S_-}$, then the curvature of ∂M at p_0 is the same as the curvature of ∂N at $h(p_0)$.*

Proof. If $p_0 \in S_-$, there is a $p \in S_-$ near p_0 such that there is a geodesic in M which is tangent to ∂M at p, which intersects ∂M transversely at the two end points, and which have no other intersections with ∂M. Since M and N have the same scattering data, the same thing happens to $h(p)$. Hence the C^∞ jet of the metric near $h(p)$ is determined [UW] by the scattering data. In particular, the curvature of S_- at p is the same as the curvature of $h(S_-)$ at $h(p)$. Since M and N are assume to be smooth, the curvature of S_- at p_0 is the same as the curvature of $h(S_-)$ at $h(p_0)$. $\qquad\square$

Proposition 4. *If $p_0 \in S_0$, then ∂N is totally geodesic at $h(p_0)$.*

Proof. Let U be an open neighborhood of p_0 in S_0. Pick a unit tangent vector $X_0 \in \partial_0 \Omega_{p_0} M$. There are two choices of X_0 but either will work.

For any $\theta \in (0, \pi)$, let X_θ be the unit tangent vector in $\partial_+ \Omega_{p_0} X$ such that the angle between X_0 and X_θ is θ. We shall show that there is a $\delta > 0$ such that γ_{X_θ} is not tangent to ∂M when $\theta \in (0, \delta)$. Suppose that this is not true, then there is a monotonically decreasing sequence $\theta_i \to 0$ such that $\gamma_{X_{\theta_i}}$ is tangent to ∂M. Since M has no trapped geodesics, the lengths of geodesics in M is bounded from above universally. Hence there is a subsequence θ_{i_k} of θ_i such that $\gamma_{X_{\theta_{i_k}}}$ is tangent to ∂M at $q_k = \gamma_{X_{\theta_{i_k}}}(s_k)$ where s_k converges. Now, lift p_0 to a point $\tilde{p}_0$ in $\tilde{M}$, the universal cover of M. Then lift each X_θ to a unit tangent vector $\tilde{X}_\theta \in \Omega_{\tilde{p}_0} \tilde{M}$. Let $\tilde{q}_k = \gamma_{\tilde{X}_{\theta_{i_k}}}(s_k)$, then $\tilde{q}_k$ converges to some $\tilde{q} \in \partial \tilde{M}$. So there is $N \in \mathbb{N}$ such that $\tilde{q}_k$ and $\tilde{q}$ are on the same component of $\partial \tilde{M}$ when $k > N$. It follows that there is switch point between $\tilde{q}_k$ and $\tilde{q}_{k+1}$ when $k > N$, which contradicts our assumption that F_M is finite.

Define $\gamma_\theta : [0, 1] \to M$ as $\gamma_\theta(t) = \gamma_{X_\theta}(\ell(\gamma_{X_\theta})t)$. We will define γ_0 as the limit of γ_θ as $\theta \to 0$, if the limit exist. By Proposition 2, γ_θ converges as $\theta \to 0$ if $\gamma_\theta(1)$ converges. Let S_1 be the component of ∂M that contains $\gamma_{\frac{\delta}{2}}(1)$. The no conjugate points condition says that $\gamma_\theta(1)$ moves in a fixed direction on S_1 as θ goes to 0. Assume that $\gamma_\theta(1)$ does not converge as $\theta \to 0$, then $\gamma_\theta(1)$ must go around S_1 infinitely often. Pick any $q \in S_1$, there a sequence of positive numbers $\theta_i \to 0$ such that $\gamma_{\theta_i}(1) = q$. Substituting θ_i by a subsequence if necessary, we may assume that $\gamma'_{\theta_i}(1)/\tau(X_{\theta_i})$ converges to a unit tangent vector Y. Then γ_{-Y} is tangent to ∂M at p_0. Therefore, there is geodesic in M which goes through q and which is tangent to ∂M at p_0. However, this is impossible since q is an arbitrary point on S_1. Hence $\gamma_\theta(1)$ converges as $\theta \to 0$. Therefore, γ_θ converges to γ_0 as $\theta \to 0$.

Since M and N have the same scattering data, $\gamma_{\varphi(X_\theta)}$ also converges to a geodesic ray γ_1 whose initial tangent vector is $\varphi(X_0)$. Since p_0 is in the interior of S_0, $\ell(\gamma_0) > 0$. Since M has no closed geodesics, $X_0 \neq \gamma'_0(1)/\ell(\gamma_0)$. Hence $\varphi(X_0) \neq \lim_{\theta \to 0} \varphi(\alpha(X_\theta))$, which implies that $\ell(\gamma_1) > 0$. We have $\gamma'_1(1)/\ell(\gamma_1) =$

$\lim_{\theta \to 0} \varphi(\alpha(X_\theta)) = \varphi(\lim_{\theta \to 0} \alpha(X_\theta)) = \varphi(\gamma_0'(1)/\ell(\gamma_0))$. Since ∂M is totally geodesic near p_0, there is $\delta > 0$ such that $\gamma_0|_{[0,\delta]} \subset \partial M$. For any $p \in \gamma_0((0,\delta))$, since ∂M is totally geodesic near p_0, there is $\delta > 0$ such that $\gamma_0|_{[0,\delta]} \subset \partial M$. Using the same argument as above, we can show that there is a geodesic γ_3 in N starting at $h(p)$ such that $\gamma_3'(1)/\ell(\gamma_3) = \varphi(\gamma_0'(1)/\ell(\gamma_0))$, which implies that $h(p)$ is also on γ_2. Hence $h(\gamma_0([0,\delta]))$ is totally geodesic in N.

There is geodesic ray γ_4 in M whose initial tangent vector is $-X_0$. Using the same argument as above, there is a small $\delta' > 0$ such that $\gamma_4([0,\delta]) \subset \partial N$ and that $h(\gamma_4([0,\delta]))$ is totally geodesic. Therefore, ∂N is totally geodesic near $h(p_0)$.

$\square$

4. Space of geodesics, continued

Let Γ_M' be the space of maximal geodesics which are tangent to ∂M at a switch point. We may assume that geodesics in $\Gamma' M$ are not tangent to S_-. If any geodesic in $\Gamma' M$ is tangent to S_-, we may extend M near that tangent point in a collar neighborhood to reduce the number of such tangent points without introducing new switch points. Since there are only finite many switch points, all such tangent points can be eliminated by extending M to a new manifold M'. If the extension is small enough, the M' will also have no trapped geodesics and no conjugate points. Now, do the same extension for N. Namely, glue N and $M' \setminus M$ to obtain an extension N' of N. By proposition 3, N' is C^2 (actually C^∞). Since M and N have the same scattering data, M' and N' also have the same scattering data. If M' and N' have the same lens data, then M and N also have the same lens data. Hence it suffices to prove that M' and N' have the same lens data. Thus, without loss of generality, we assume that geodesics in Γ_M' are not tangent to S_-.

4.1. **A map between space of geodesics.** Recall that Γ_M^p is the space of p-geodesics $[0,1] \to M$ whose end points are on ∂M, and we define Γ_N^p similarly. For our convenience, any reparametrization of a p-geodesic $\gamma : [0,1] \to M$ whose end points are on ∂M will also be viewed as $\gamma \in \Gamma_M^p$. Let $\Gamma_M^0 := \{\gamma \in \Gamma_M^p : \gamma$ is non-constant and not tangent to ∂M at points in $\partial S_+.\}$.

Proposition 5. *There is a map* $\Phi : \Gamma_M^0 \to \Gamma_N^p$ *which satisfies the following conditions.*

(1) Φ *is continuous with respect to the compact open topology.*
(2) *For any* $\gamma(t) \in \partial M$, *reparametrizing* $\Phi(\gamma)$ *if necessary, we have* $\Phi(\gamma)(t) = h(\gamma(t))$ *and* $\frac{\Phi(\gamma)'(t)}{|\Phi(\gamma)'(t)|} = \varphi(\frac{\gamma'(t)}{|\gamma'(t)|})$.

Remark. We will extend Φ to Γ_M^p in the next section.

Proof. Let Γ_M^1 be the space of non-constant geodesics $[0,1] \to M$ whose end points are on ∂M and which are not tangent to ∂M anywhere.

Let Γ_M^c be the space of constant geodesics on ∂M, and define $\Gamma_M^2 := \overline{\Gamma_M^1} \setminus (\Gamma_M^c \bigcup \Gamma_M')$.

Let Γ_M^b be the space of non-constant p-geodesics in M which run along $\overline{S_- \bigcup S_0} \setminus \partial S_+$, the non-convex part of ∂M.

Defined $\Phi : \Gamma_M^1 \to \Gamma_N^p$ as $\Phi(\gamma_X) = \gamma_{\varphi(X)}$. Φ is obviously continuous. Next, extend Φ to a continuous map from Γ_M^2 to Γ_N^p by taking limits. If a geodesic $\gamma = \lim_{i \to \infty} \gamma_i$ is in Γ_M^2 where $\gamma_i \in \Gamma_M^1$, then define $\Phi(\gamma) := \lim_{i \to \infty} \Phi(\gamma_i)$. This is well-defined because M and N have the same scattering data.

Note that we exclude constant curves and geodesics tangent to switch points when defining Γ_M^2 because $\lim_{i \to \infty} \Phi(\gamma_i)$ would not necessarily converge if γ were a constant curve or a geodesic in Γ_M'. For example, let γ be a constant curve and γ_i^1 a sequence of geodesics in Γ_M^1 such that $\gamma_i^1(0) = \gamma(0)$ and that $\gamma_i^1(1)$ converges to $\gamma(0)$ from one side. Then $\lim_{i \to \infty} \Phi(\gamma_i^1)$ might be a closed geodesic in N tangent ∂M at $h(\gamma(0))$. Let γ_i^2 a sequence of geodesics in Γ_M^1 such that $\gamma_i^2(0) = \gamma(0)$ and that $\gamma_i^2(1)$ converges to $\gamma(0)$ from the other side. Then $\lim_{i \to \infty} \Phi(\gamma_i^2)$ will be a closed geodesic which goes in the other direction. We will resolve this issue by showing that there are no closed geodesics in N tangent to ∂N in the next section.

For each $\gamma \in \Gamma_M^b$, $h \circ \gamma$ is also a p-geodesic by Proposition 3 and Proposition 4. Hence we may define $\Phi(\gamma) := h \circ \gamma$.

Now, pick any $\gamma \in \Gamma_M^0$. Since F_M is finite, we have a decomposition $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_n$ where each γ_k is either in Γ_M^2 or in ∂M. Define $\Phi(\gamma) = \Phi(\gamma_1) * \Phi(\gamma_2) * \cdots * \Phi(\gamma_n)$. $\qquad \square$

Define $e : \Gamma_M^0 \to \mathbb{R}$ as $e(\gamma) = \ell(\Phi(\gamma)) - \ell(\gamma)$. The two manifolds will have the same length data if $e = 0$.

Proposition 6. *e is constant on each component of Γ_M^0*

Proof. This follows from the assumption that M and N have the same scattering data. Since the first variation of arc length of geodesics in Γ_M^p and Γ_N^p only depends on the scattering data, e is constant on each component of Γ_M^0. $\qquad \square$

4.2. **Pairs of p-geodesics.** Recall that Γ_M^p is the space of p-geodesics $[0,1] \to M$ whose end points are on ∂M. For any $p, q \in \partial M$, define $P_M(p,q) := \{(\gamma_1, \gamma_2) \in \Gamma_M^p \times \Gamma_M^p : \gamma_1(0) = p, \gamma_2(0) = q, \gamma_1(1) = \gamma_2(1)\}$. Each element in $P_M(p,q)$ will be called a *pair* (of geodesics based at p and q), and the end point $\gamma_1(1)$ will be called the *root* of the pair, denoted by $r(\gamma_1, \gamma_2)$.

Put $P_M^0(p,q) = P_M(p,q) \bigcap (\Gamma_M^0 \times \Gamma_M^0)$. Then define $l : P_M^0(p,q) \to \mathbb{R}$ as $l(\gamma_1, \gamma_2) = e(\gamma_1) - e(\gamma_2)$. Since e is constant on each component of Γ_M^0, l is constant on each component of $P_M^0(p,q)$.

Let $Q_M(p,q) \subset P_M(p,q)$ be the space of pairs which do not overlap at the end. (γ_1 and γ_2 are said to overlap at the end if there are $a_1 \in [0,1)$ and $a_2 \in [0,1)$ such that $\gamma_1|_{[a_1,1]}$ and $\gamma_2|_{[a_2,1]}$ coincide.) Suppose that $(\gamma_1, \gamma_2) \in P_M(p,q)$ and that γ_1 and γ_2 has an overlapping part γ_3 at the end. If we remove γ_3 from γ_1 and γ_2, then we obtain another pair $(\gamma_4, \gamma_5) \in P_M$ where $\gamma_1 = \gamma_4 * \gamma_3$ and $\gamma_2 = \gamma_5 * \gamma_3$. Define $b : P_M \to Q_M$ as $b(\gamma_1, \gamma_2) = (\gamma_4, \gamma_5)$. View b as a quotient map and use the quotient topology on Q_M induced by the compact open topology on P_M.

Since $e(\gamma_1) = \ell(\gamma_1) - \ell(\Phi(\gamma_1)) = \ell(\gamma_4) + \ell(\gamma_3) - \ell(\Phi(\gamma_4)) - \ell(\Phi(\gamma_3)) = e(\gamma_4) + e(\gamma_3)$, we have

$$
\begin{aligned}
l(\gamma_1, \gamma_2) &= e(\gamma_1) - e(\gamma_2) \\
&= e(\gamma_4) + e(\gamma_3) - e(\gamma_5) - e(\gamma_3) \\
&= e(\gamma_4) - e(\gamma_5) \\
&= l(\gamma_4, \gamma_5) \\
&= l(b(\gamma_1, \gamma_2)).
\end{aligned}
$$

Hence l is also well-defined on $Q_M^0(p,q) := b(P_M^0(p,q))$, and l is constant on each component of $Q_M^0(p,q)$.

Proposition 7. *Each component of $Q_M(p,q)$ is a 1-manifold without boundary.*

Proof. When the two geodesics are not tangent to each other at the root, nearby pairs in $Q_M(p,q)$ are determined by their roots. Hence a neighborhood of this pair is homeomorphic to a neighborhood of its root in the boundary, which is a 1-manifold. (See Figure 3.)
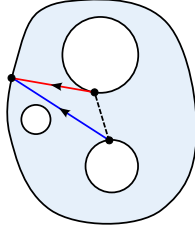

FIGURE 3. The two geodesic are not tangent to each other at the root.

If we move the root along the boundary, we may get a pair which has an overlapping part like the one in Figure 4. Note that this pair is not in $Q_M(p,q)$ since it has an overlapping part.
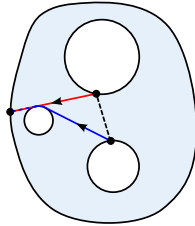

FIGURE 4. The two geodesic in a pair have an overlapping part.

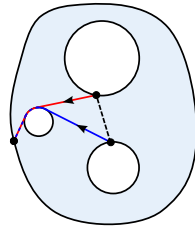The overlapping part will persist if we move the root further (Figure 6). Such

FIGURE 5. Not in $Q_M(p,q)$.

pairs are not in $Q_M(p,q)$ since they have overlapping part at the end.

b of the pair in Figure 4 gives us a pair in $Q_M(p,q)$ (Figure 6). This pair will be called q_1.

Now, we can move the root in two directions, but only one of them gives elements in $Q_M(p,q)$. For example, in Figure 6, if we move the root to the left, the pair will have an overlapping part while moving the root to the right gives pairs in $Q_M(p,q)$ (Figure 7).
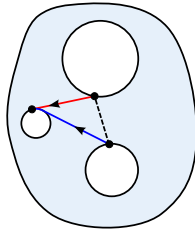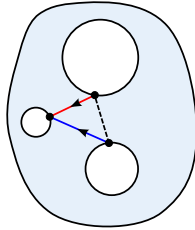
FIGURE 6. q_1

FIGURE 7

So far, we have deformed the pair in Figure 3 to the pair Figure 6 by moving the root continuously except when going from Figure 4 to Figure 6. Denote the pairs in between (except the one in Figure 4) by U. U is actually a neighborhood of q_1 in $Q_M(p, q)$. q_1 separates U into two parts, and each part is a 1-manifold since roots in each part are taken from a 1-manifold continuously.

Now, consider any pair q_2 consisting of geodesics which are tangent to each other at the root (Figure 8). Extending the geodesics in q_2 as long as possible,
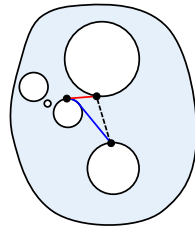
FIGURE 8. q_2

we obtain a new pair $q_3 \in P_M(p, q)$ (Figure 9). q_2 will have a neighborhood like above if the overlapping part of q_3 is not tangent to the boundary in the middle. Assume the overlapping part of q_3 is tangent to the boundary in the middle. Call the overlapping part l. Here $l : [0, 1] \to M$ is a geodesic starting at the root of q_2. Let $X : [0, 1] \to \Omega M$ be a unit normal vector field along l which is pointing inwards (as a unit normal vector of ∂M) at the root of q_2. Pick the smallest $a \in (0, 1]$, if it exists, such that l is tangent to ∂M at $l(a)$ and that $X(a)$ is pointing outwards. If there is no such a, put $a = 1$. We obtain a pair q_4 by adding $l|_{[0,a]}$ to pairs in q_2 (Figure 10). We may move the root of q_4 along the boundary in two directions, obtaining pairs in $P_M(p, q)$. If we move the root along the boundary in
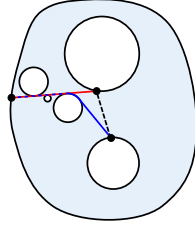
FIGURE 9. q_3
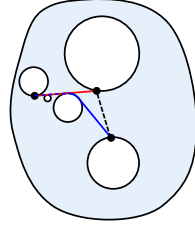
FIGURE 10. q_4

the direction of $-l'(a)$, we will get pairs in $Q_M(p, q)$. If we move the root along the boundary in the direction of $l'(a)$, the overlapping part will persist. Notice that small neighborhoods of any other extensions of q_2 do not contain elements in $Q_M(p, q)$ because all their elements are pairs of geodesics with overlapping part at the ends. Hence q_2 has a neighborhood homeomorphic to $(0, 1)$, where one half of the neighborhood comes from half of a small neighborhood of q_2 in $P_M(p, q)$, and another half of the neighborhood comes from half of a small neighborhood of q_4 in $P_M(p, q)$. Note that $b(q_4) = q_2$.

Therefore, $Q_M(p, q)$ is a 1-manifold without boundary.

$\square$

Suppose that there is a geodesic γ from p to q. Let γ_q be the constant curve at q, γ_p be the constant curve at p, and $-\gamma$ be γ with the opposite direction.

Proposition 8. (γ, γ_q) and $(\gamma_p, -\gamma)$ *are in the same component of* $Q_M(p, q)$. *Moreover, that component is homeomorphic to a circle.*

Remark. This proposition also holds if γ is a p-geodesic.

Proof. Without loss of generality, we assume that M is simply connected but not necessarily compact. The curve γ separate M into two components. Namely, $M = D_1 \bigcup D_2$ where $D_1 \bigcap D_2 = \gamma$. Without loss of generality, assume that D_1 is on the left of γ. Let $Q_1 = \{(\gamma_1, \gamma_2) \in Q_M : \gamma_1(1) \in D_1\}$, and $Q_2 = \{(\gamma_1, \gamma_2) \in Q_M : \gamma_1(1) \in D_2\}$. Let Q_1^0 be the component of Q_1 which contains (γ, γ_q). If $(\gamma_p, -\gamma) \notin Q_1^0$, then Q_1^0 has only one end point (γ, γ_q). It follows that Q_1^0 is homeomorphic to $[0, \infty)$. Let $(\alpha, \beta) : [0, \infty) \to Q_1^0$ be a homeomorphism. Here we reparametrize $\alpha(t)$ and $\beta(t)$ using the length parameter. We shall show that $\alpha(t)$ converges to a infinitely long p-geodesic ray α_0 as $t \to \infty$.

Pick any $s \in [0, \infty)$. Notice that the space of p-geodesics starting at $\gamma_1(0)$ whose length is less or equal to s is homeomorphic to the closed ball $\overline{B_s(\tilde{p})}$ where $\tilde{p}$ is a lift

of $\gamma_1(0)$ in $\tilde{M}$, which is compact. Hence there is $T_s \in [0,\infty)$ such that $\ell(\alpha(t)) > s$ when $t > T_s$. When $t > T_s$, $\alpha(t)|_{[0,s]}$ can either wrap around a concave boundary part in a fixed direction or stay still as t increases. Notice that $\alpha(t)|_{[0,s]}$ can not wrap around a concave boundary part forever since it only has finite length s. Hence $\alpha(t)|_{[0,s]}$ converges pointwisely to a p-geodesic as t converges to ∞. Therefore, $\alpha(t)$ converges to a infinitely long p-geodesic α_0 pointwisely as $t \to \infty$. $\beta(t)$ converges to some β_0 similarly.

Since $\alpha(t)$ and $\beta(t)$ only intersect at the root, α_0 and β_0 do not intersect transversely. Use the length parameter on α_0 such that $\alpha_0(0) = p$ and that $\alpha_0(s)$ is defined for $s \geq 0$.

Consider the p-geodesic "triangle" bounded by α_0, β_0 and γ. For any $s > 0$, there is a maximal geodesic g_s in the triangle which is tangent to α_0 at $\alpha_0(s)$. Since M has no trapped geodesics, g_s has two end points. If one of the end points is on α_0, then there will be two different p-geodesics between that end point and $\alpha_0(s)$, which contradicts Proposition 2. If both of these two end points are on β_0, then there will be two different p-geodesics between these two end points, which contradicts Proposition 2 again. Therefore, at least one of the end points are on γ. Let X_s be the unit tangent vector based at that end point which is tangent to g_s and pointing towards the inside of the triangle. There is a sequence $s_k \to \infty$ such that X_{s_k} converges. By triangle inequality, $\ell(g_s) \geq s - \ell(\gamma)$. Hence g_{s_k} converges to a geodesic with infinite length, which contradicts our assumption that M has no trapped geodesics.

Therefore, Q_1 is homeomorphic to $(0,1)$. Similarly, Q_2 is also homeomorphic to $(0,1)$. Thus the component of $Q_M(p,q)$ which contains (γ, γ_q) and $(\gamma_p, -\gamma)$ is a circle. □

5. CONVEX PART OF THE BOUNDARY

The goal of this section is to prove the following proposition.

Proposition 9. *For any $p_0 \in S$, the curvature of ∂M at p_0 is the same as the curvature of ∂N at $h(p_0)$.*

Pick any $p_0 \in S_+$. We aim to show that ∂N is also convex at $h(p_0)$. Pick a unit tangent vector $X_0 \in \partial_0 \Omega_{p_0} M$. There are two choices of X_0 but either works.

For any $\theta \in (0,\pi)$, let X_θ be the unit tangent vector in $\partial_+ \Omega_{p_0} X$ such that the angle between X_0 and X_θ is θ. Since S is convex at p_0, there is $\delta > 0$ such that γ_{X_θ} is not tangent to ∂M when $\theta \in (0,\delta)$. Since M and N have the same scattering data, $e(\gamma_{X_\theta}) = \ell(\gamma_{\Phi(X_\theta)}) - \ell(\gamma_{X_\theta})$ is equal to a fixed constant $L > 0$ when $\theta \in (0,\delta)$. If $L = 0$, then N is also convex at $h(p_0)$ [Mic].

Assume that $L \neq 0$, then there is a closed geodesic in N of length L which is tangent to ∂N at $h(p_0)$. We shall show that this is impossible in this section.

Let S_1 be the component of $\partial M \setminus S_-$ which contains p_0. S_1 is either a closed circle, or a curve with two ends.

5.1. S_1 **is a curve with two ends.** For any $Y \in \Omega \partial M$, let $\bar{\gamma}_Y$ be the maximum geodesic ray whose initial tangent vector is Y. If $\bar{\gamma}_Y$ is just a point, or if it only runs along a totally geodesic part of ∂M and never leaves the boundary, then Y is called a convex direction. Otherwise, Y is called a concave direction.

∂M is orientable since it is one-dimensional. Fix an orientation on ∂M. Then at each $p \in \partial M$, there is a positively oriented unit tangent vector $Y_+(p) \in \Omega_p \partial M$, and a negatively oriented unit tangent vector $Y_-(p) \in \Omega_p \partial M$.

Let Y_n be a sequence of a unit tangent vectors based at p, pointing inwards and converging to $Y_+(p)$. Then define

$$L_+(p) := \begin{cases} \lim_{n\to\infty} e(\gamma_{Y_n}) & \text{if } Y_+(p) \text{ is a convex direction,} \\ 0 & \text{if } Y_+(p) \text{ is a concave direction.} \end{cases}$$

$L_-(p)$ is defined similarly.

Proposition 10. $L_+ = L_-$ *and they are constant on each component of* ∂M.

Proof. $L_+(p) = L_-(p) = 0$ when both $Y_+(p)$ and $Y_-(p)$ are concave directions. It is also obvious that $L_+(p) = L_-(p)$ when both $Y_+(p)$ and $Y_-(p)$ are convex directions. (Both are equal to the length of a closed geodesic tangent to the boundary.) Also, by the first variation formula, $L_\pm$ are constant near p if p is not a switch point.

Suppose that p_1 is a switch point such that L_+ is non-constant near p_1. Without loss of generality, assume that $Y_+(p_1)$ is a convex direction Pick a small open neighborhood of p_1 in ∂M. The neighborhood is separated into two parts by p_1 Pick a point p_2 from the part which $Y_+(p)$ points to, and a point p_3 from the other part. Since there are only finite many switch points, when p_2 and p_3 are close enough, any geodesic going through p_2 or p_3, except possibly a geodesic going though p_3 and p_1, are not tangent ∂M at switch points.

Pick $q \in \partial M \setminus F_M$ such that there is a geodesic γ_1 which intersects ∂M transversely at p_1 and q. When p_2 and p_3 are close enough, there is also a geodesic γ_2 which intersects ∂M transversely at p_2 and q, a geodesic γ_3 which intersects ∂M transversely at p_3 and q, and a continuous family of geodesics between them which intersect ∂M transversely. Then we have $e(\gamma_2) = e(\gamma_3)$ by Proposition 6.
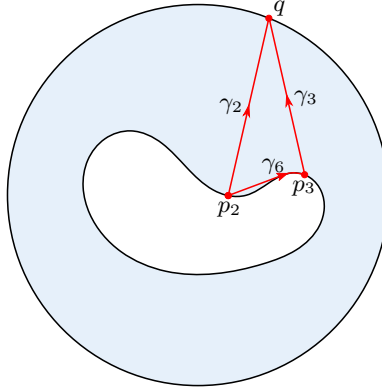

FIGURE 11. $H_1(\theta_0, \cdot) = H_2(\theta_0, \cdot)$

For any p-geodesic $\gamma \in \Gamma_M^p$ starting at p_2, by Proposition 2, there is a unique p-geodesic $H(\gamma)$ close to γ such that $H(\gamma)(0) = p_3$ and that $H(\gamma)(1) = \gamma(1)$.

Recall that l is constant on each component of $Q_M^0(p_2, q)$. Pick any $(\gamma_4, \gamma_5) \in Q_M(p_2, q) \setminus Q_M^0(p_2, q)$ where both γ_4 and γ_5 are non-constant.

We claim that $(H(\gamma_4), \gamma_5) \in Q_M(p_3, q) \setminus Q_M^0(p_3, q)$. The root of (γ_4, γ_5) is a switch point and either γ_4 or γ_5 is tangent to ∂M at that point. If γ_5 is tangent

to ∂M at the that switch point, then we have $(H(\gamma_4), \gamma_5) \in Q_M(p_3, q) \setminus Q_M^0(p_3, q)$. If γ_4 is tangent to ∂M at that switch point, then γ_4 is not a geodesic, since we assume that geodesics going through p_2 are not tangent to ∂M at switch points. Hence the ending part of γ_4 is a geodesic tangent to ∂M at both ends. Therefore, when p_2 and p_3 are close enough, the ending part of γ_4 and $H(\gamma_4)$ will coincide, which implies that $H(\gamma_4)$ is also tangent to ∂M at $\gamma_4(1)$. Thus, $(H(\gamma_4), \gamma_5) \in Q_M(p_3, q) \setminus Q_M^0(p_3, q)$.

Now, consider the two components I_2 and J_2 of $Q_M(p_2, q) \setminus Q_M^0(p_2, q)$ which are adjacent to (γ_4, γ_5), and corresponding components I_3 and J_3 of $Q_M(p_3, q) \setminus Q_M^0(p_3, q)$ which are adjacent to $(H(\gamma_4), \gamma_5)$. If γ_4 is tangent to ∂M at the root, then the ending part of γ_4 and $H(\gamma_4)$ coincide, which implies that

$$(2) \qquad\qquad l(J_2) - l(I_2) = l(J_3) - l(I_3).$$

If γ_4 is transverse to ∂M at the root, then $H(\gamma_4)$ is also transverse to ∂M if p_2 and p_3 are close enough, so we also have (2).

Let γ_6 be the shortest p-geodesic from p_2 to p_3. By proposition 8, (γ_2, γ_q) and $(\gamma_{p_2}, -\gamma_2)$ are in the same component of $Q_M(p_2, q)$ which is homeomorphic to a circle. Call that component Q_2. Since $(\gamma_6, -\gamma_3)$ and $(\gamma_{p_2}, -\gamma_2)$ are close to each other, $(\gamma_6, -\gamma_3) \in Q_2$. Since Q_2 is a circle, $(\gamma_6, -\gamma_3)$ and (γ_2, γ_q) separate Q_2 into two parts. Denote the part which does not contain $(\gamma_{p_2}, -\gamma_2)$ by U_2. Here U_2 does not contain end points.

By proposition 8, (γ_3, γ_q) and $(\gamma_{p_3}, -\gamma_3)$ separate the component of $Q_M(p_3, q)$ which contains them into two parts. Denote the part which does not contain $(-\gamma_6, -\gamma_2)$ by U_3. Here U_3 does not contain end points.

Let V_2 be the component of $U_2 \setminus Q_M^0(p_2, q)$ which is adjacent to (γ_2, γ_q), and V_3 be the component of $U_3 \setminus Q_M^0(p_3, q)$ which is adjacent to (γ_3, γ_q). Then $l(V_2) = e(\gamma_2) - L_\pm(q) = e(\gamma_3) - L_\pm(q) = l(V_3)$. Note that choice between $L_+(q)$ and $L_-(q)$ depends on the orientation of the boundary, but it does not affect the computation here. Let V_2' be the component of $U_2 \setminus Q_M^0(p_2, q)$ which contains $(\gamma_6, -\gamma_3)$, and V_3' be the component of $U_3 \setminus Q_M^0(p_3, q)$ which is adjacent to $(\gamma_{p_3}, -\gamma_3)$. By (2), $l(V_2') = l(V_2) = l(V_3) = l(V_3')$. Notice that $l(V_3) = L_+(p_3) - e(\gamma_3)$ but $l(V_3') = e(\gamma_6) - e(\gamma_3) = L_+(p_2) - e(\gamma_3)$. Hence $L_+(p_2) = L_+(p_3)$. Hence L_+ is constant on each component of ∂M. Similarly, L_- is also constant on each componnet of ∂M. Since M has no trapped geodesics, there is always a point p on each component of ∂M such that ∂M is either strictly convex near p or strictly concave near p, which implies that $L_+(p) = L_-(p)$. Hence $L_+ = L_-$ and they are constant on each component of ∂M. $\qquad\square$

From now, we shall replace both L_+ and L_- by L. When S_1 is a curve with two ends, then the component of ∂M which contains S_1 must contain a concave part, and thus $L = 0$ on S_1. Hence Proposition 9 holds.

5.2. S_1 **is closed.** Now we assume that S_1 is closed.

If every geodesic ray starting at p_0 intersects ∂M only on S_1, then the old argument for simple manifolds works automatically. Assume that there is a geodesic ray starting at p_0 intersects ∂M at a point on $\partial M \setminus S_1$. Assume that $L > 0$.

Fix an orientation on ∂N and let $X_0(x)$ be the unit vector tangent to ∂N at $x \in \partial N$ such that $X_0(x)$ and ∂N have the same orientation. Let $h_1 : \mathbb{R}/\mathbb{Z} \to h(S_1)$ be an orientation preserving diffeomorphism.

For each $x \in \partial N$ and $\theta \in (0, \pi)$, let $X_\theta(x) \in \partial_+ \Omega N$ be the unit tangent vector such that the angle between $X_0(x)$ and $X_\theta(x)$ is θ. For each $\theta \in (-\pi, 0)$, put $X_\theta(x) = -X_{\pi-\theta}(x)$. Pick a small $\delta_1 > 0$ such that $\gamma_{X_\theta(x)}$ intersects ∂N transversely for any $x \in h(S_1)$ and $\theta \in (0, \delta_1)$. Pick a small $\delta \in (0, \delta_1)$ Write $Y_1(x) = X_\delta(x)$. Define $T : h(S_1) \to h(S_1)$ as $T(x) := \pi(\alpha_N(Y_1(x)))$. Here we choose δ small enough such that the two end points of $\gamma_{X_\delta(x)}$ move in the same direction as x moves on $h(S_1)$. In other words, T is a homeomorphism.

Let $Y_2(x) = \alpha_N(Y_1(T^{-1}(x)))$. Then $Y_1(x)$ and $Y_2(x)$ separate the circle $\Omega_x N$ into two segments. Let $A(x)$ be the segment containing $X_0(x)$ (which is the shorter segment). Then $A = \bigcup_{x \in \partial N} A(x)$ is an annulus with boundaries $Y_1(\partial N)$ and $Y_2(\partial N)$. Thus we can can break A down to a family of *disjoint* curves $\eta_x : [0, 1] \to A$ from $Y_1(x)$ to $Y_2(T(x)) = \alpha_N(Y_1(x))$. See Figure 12.
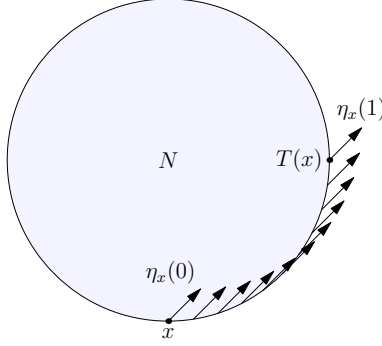

FIGURE 12. Values of η_x from 0 to 1

Definition 5.1. The *unit tangent vector field* of a smoothly immersed curve γ on any Riemannian surface N^2 (possibly with boundary) is a smoothly immersed curve $\tilde{\gamma}$ in ΩN defined as

$$(3) \qquad \tilde{\gamma}(t) = \left(\gamma(t), \frac{\gamma'(t)}{|\gamma'(t)|} \right).$$

Definition 5.2. Let $P : \Omega N \to P\Omega N$ be the quotient map on the unit tangent bundle which identifies the opposite vectors based at the same point. For any smoothly immersed curve γ in N^2, $P \circ \tilde{\gamma}$ is called the *projectivized unit tangent vector field* (or the *tangent line field*) of γ.

Define

$$f : \partial N \times \mathbb{R}/\mathbb{Z} \to P\Omega N$$

as

$$f(x, t) = \begin{cases} P\left(\tilde{\gamma}_{Y_1(x)}(2t) \right) & \text{if } 0 \le t \le \frac{1}{2}, \\ P\left(\eta_x(2 - 2t) \right) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

See Figure 13.

Proposition 11. $f : \partial N \times \mathbb{R}/\mathbb{Z} \to P\Omega N$ *is an embedding.*

Proof. The proof is the same as the proof of [Wen, Proposition 4.2]. $\qquad \square$
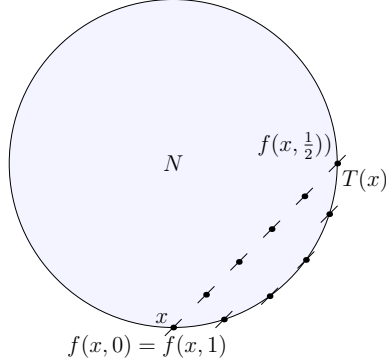
FIGURE 13. Values of $f(x, \cdot)$ from 0 to 1. This might be deceiving since $\gamma_{Y_1(x)}(2t)$ should have self-intersections.

Proposition 12. $f(x, \cdot)$ *is contractible in $P\Omega N$.*

Proof. Put $x_0 = h(p_0)$. Let θ_0 be the smallest positive number such that $\gamma_{\Phi^{-1}(X_{\theta_0})}$ is tangent to ∂M at a point $p_1 \in \partial M \setminus S_1$. For any $\theta \in (0, \theta_0)$, $\gamma_{\Phi^{-1}(X_\theta)}$ intersects ∂M transversely. Hence $\gamma_{\Phi^{-1}(X_\theta)}$ is a continuously varying family of disjoint curves, which implies that their union is a simply connected convex region. Thus each geodesic ray starting at p_1 intersects S_1 transversely because there are no conjugate points. For each $\theta \in (0, \theta_0)$, let $\gamma_\theta : [0,1] \to N$ be the geodesic from $\pi(\alpha_N(X_\theta(p_0)))$ to $h(p_1)$. Also define γ_0 and γ_{θ_0} by taking limits. Let $a(\theta)$ be the angle between γ_{θ_0} and γ_θ.

We will construct two homotopies $H_1 : [\delta, \theta_0] \times \mathbb{R}/\mathbb{Z} \to \Omega N$ and $H_2 : [0, \theta_0] \times \mathbb{R}/\mathbb{Z} \to \Omega N$. $P \circ H_1(\delta, \cdot)$ will be homotopic to $f(x_0, \cdot)$. $H_1(\theta_0, \cdot)$ and $H_2(\theta_0, \cdot)$ will be the same. Then we will show that $H_2(0, \cdot)$ is contractible, which will imply that $f(x_0, \cdot)$ is contractible.
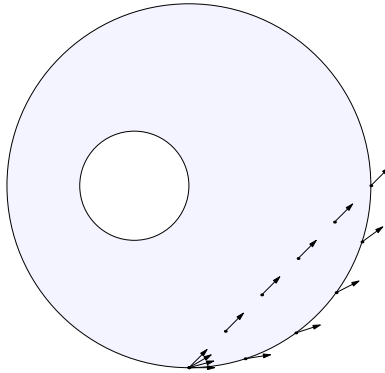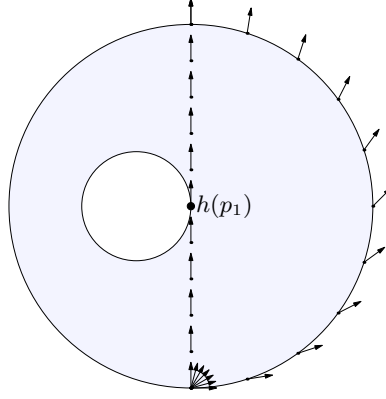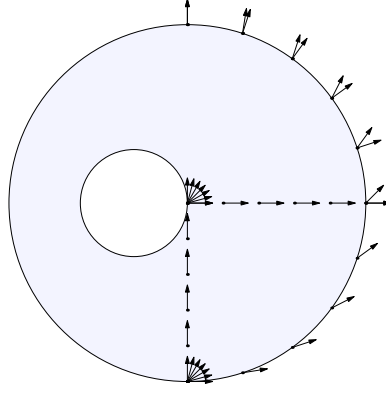

FIGURE 14. This is $H_1(\delta, \cdot)$. $P \circ H_1(\delta, \cdot)$ is homotopic to $f(x_0, \cdot)$

FIGURE 15. $H_1(\theta_0, \cdot) = H_2(\theta_0, \cdot)$

FIGURE 16. $H_2(\frac{\theta_0}{2}, \cdot)$

Consider the homotopy $H_1 : [\delta, \theta_0] \times \mathbb{R}/\mathbb{Z} \to \Omega N$ defined as

$$H_1(s, t) = \begin{cases} \tilde{\gamma}_{X_s}(3t) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \alpha_N(X_{(2-3t)s}(x_0)) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ X_{(3t-2)s}(x_0) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

Here $P \circ H_1(\delta, \cdot)$ is homotopic to $f(x_0, \cdot)$.

Define $H_2 : [0, \theta_0] \times \mathbb{R}/\mathbb{Z} \to \Omega N$ as

$$H_2(s, t) = \begin{cases} \tilde{\gamma}_0(5t) & \text{if } 0 \leq t \leq \frac{1}{5}, \\ -\tilde{\gamma}_{\theta_0-(6t-1)(\theta_0-s)}(1) & \text{if } \frac{1}{6} \leq t \leq \frac{2}{6}, \\ -\tilde{\gamma}_s(3-6t) & \text{if } \frac{2}{6} \leq t \leq \frac{3}{6}, \\ -\tilde{\gamma}_{\theta_0-(4-6t)(\theta_0-s)}(0) & \text{if } \frac{3}{6} \leq t \leq \frac{4}{6}, \\ \alpha_N(X_{(5-6t)\theta_0}(x_0)) & \text{if } \frac{4}{6} \leq t \leq \frac{5}{6}, \\ X_{(6t-5)\theta_0}(x_0) & \text{if } \frac{5}{6} \leq t \leq 1. \end{cases}$$

Here $H_2(\theta_0, \cdot)$ is a reparametrization of $H_1(\theta_0, \cdot)$.
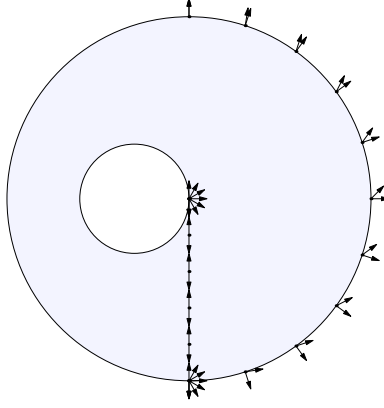
FIGURE 17. $H_2(0, \cdot)$ is contractible in this graph since no vector points to the left.

Notice that $H_2(0, \cdot)|_{(0, \frac{1}{2})}$ lies entirely on γ_0. Hence we can homotope $H_2(0, \cdot)|_{[0, \frac{1}{2}]}$ to a curve in $\Omega_{x_0} N$. More precisely, $H_2(0, \cdot)|_{[0, \frac{1}{2}]}$ is homotopic to $\beta : [0, \frac{1}{2}] \to \Omega_{x_0} N$ defined as $\beta(t) = X_{(1-4t)\theta_0}(x_0)$. Similarly, $H_2(0, \cdot)|_{(\frac{1}{2}, 1)}$ is homotopic to $-\beta$. Hence $H_2(0, \cdot)$ is contractible. Since $P \circ H_2(0, \cdot)$ is homotopic to $f(x_0, \cdot)$, $f(x_0, \cdot)$ is contractible in $P\Omega N$.   □

We can prove the following proposition using the same proof of [Wen, Proposition 4.4].

Proposition 13. $f(x_0, \cdot)$ *is isotopically trivial in* $P\Omega N$.

However, by [Wen, Theorem 3.14], $f(x_0, \cdot)$ is isotopically non-trivial, which contradicts Proposition 13. Hence $L = 0$.

6. PROOF OF THE MAIN THEOREM

Proposition 9 enable us extending Φ to Γ_M^p.

Proposition 14. *There is a map* $\Phi : \Gamma_M^p \to \Gamma_N^p$ *which satisfies the following conditions.*

(1) *Φ is continuous with respect to the compact open topology.*
(2) *For any $\gamma(t) \in \partial M$, reparametrizing $\Phi(\gamma)$ if necessary, we have $\Phi(\gamma)(t) = h(\gamma(t))$ and $\frac{\Phi(\gamma)'(t)}{|\Phi(\gamma)'(t)|} = \varphi(\frac{\gamma'(t)}{|\gamma'(t)|})$.*

Proof. Recall that Γ_M^1 is the space of non-constant geodesics $[0, 1] \to M$ whose end points are on ∂M and which are not tangent to ∂M.

Let Γ_M^4 be the space of p-geodesics in M which run along the non-convex part of ∂M. Note that all constant geodesics are in Γ_M^4.

Recall that $\Phi : \Gamma_M^1 \to \Gamma_N^p$ is defined as $\Phi(\gamma_X) = \gamma_{\varphi(X)}$. We extend Φ to a continuous map from $\overline{\Gamma_M^1}$ to Γ_N^p by taking limits. This is well-defined because M and N have the same scattering data. For each $\gamma \in \Gamma_M^4$, we define $\Phi(\gamma) := h \circ \gamma$.

Now, pick any $\gamma \in \partial \Gamma_M^p$. Since F_M is finite, we have a decomposition $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_n$ where each γ_k is either in Γ_M^2 or in ∂M. Define $\Phi(\gamma) = \Phi(\gamma_1) * \Phi(\gamma_2) * \cdots * \Phi(\gamma_n)$.   □

Now, we may extend e to Γ^p and extend l to $Q_M(p, q)$. As before, l is constant on each component of $Q_M(p, q)$. Finally, the main theorem is a easy consequence of Proposition 8.

Proof of Theorem 1. Since (γ, γ_q) and $(\gamma_p, -\gamma)$ are in the same component of $Q_M(p, q)$, $l(\gamma, \gamma_q) = l(\gamma_p, -\gamma)$. Hence $e(\gamma) - e(\gamma_q) = e(\gamma_p) - e(-\gamma)$. $e(\gamma_q) = \ell(\Phi(\gamma_q)) - \ell(\gamma(q)) = 0 - 0 = 0$. Similarly, $e(\gamma_p) = 0$. It is also obvious that $e(\gamma) = e(-\gamma)$. Hence $e(\gamma) = 0$, which implies that M and N have the same lens data.

$\square$

7. Manifolds with boundary and no conjugate points

In this section we consider compact Riemannian manifolds M with smooth boundary ∂M such that no geodesic segment of M has conjugate points. Our main goal is to prove Proposition 1.

Standard applications of the second variation formula (see for example [dC]) about a geodesic segment γ having no conjugate points yields that for every nontrivial differentiable one parameter family $\gamma_s(t)$ of curves in the space $\mathcal{C}_{(x,y)}$ of curves from $x = \gamma(0)$ to $y = \gamma(L)$ with $\gamma_0 = \gamma$ we have for the energy function $E(\gamma_s) = \int_0^L |\gamma_s'(t)|^2 dt$ that $\frac{d^2}{ds^2}|_{s=0} E(\gamma_s) > 0$. (Here nontrivial means the variation field is not the 0 field.) The fact that $\frac{d}{ds}|_{s=0} E(\gamma_s) = 0$ just follows from the fact that γ is a geodesic. This tells us (since we can reduce to a finite dimensional space of piecewise geodesics) that γ is a strict local minimum of the energy (and the length) in the space of piecewise smooth paths between the endpoints.

The finiteness condition on F_M eliminates the problem of intermittent points. Thus we will be able to assume that a p-geodesic consists of a finite number of segments each of which is either a geodesic (possibly with interior points where it grazes the boundary) or a geodesic on the boundary (i.e. a segment of the boundary in the two dimensional case).

We will show (Lemma 1) that the local minimizing property of geodesics is inherited by p-geodesics for two dimensional manifolds with boundary and no conjugate points.. The main problem is that the distance function (and hence the energy function) is not C^2 even for the distance between interior points (since the minimizing path can run along the boundary for part of the time). Thus we need to be careful making second variation arguments. On the other hand, we will be able to reduce to a finite dimensional case (using piecewise p-geodesics) since it was shown in [ABB1] that for any compact $K \subset M$ there is a b such that if $p \in K$ and $q \in K$ have $d(p, q) \le b$ then there is a unique minimizing p-geodesic between p and q and it varies continuously with p and q. Call b the uniqueness radius of K. Thus if $\gamma : [0, 1] \to M$ is any path in K and $0 = t_0 < t_1 < t_2 < ... < t_{k-1} < t_k = 1$ is a partition such that $d(\gamma(t_i), \gamma(t_{i+1})) \le b$ then replacing $\gamma|[t_i, t_{i+1}]$ with the p-geodesic from $\gamma(t_i)$ to $\gamma(t_{i+1})$ (parameterized on $[t_i, t_{i+1}]$) yields a piecewise p-geodesic curve with at most the same energy.

We will consider the space of piecewise C^1 curves $\gamma : [0, 1] \to M$ between two fixed points x and y in M such that $E(\gamma) \le E$ for some fixed E. Since the length of any such curve is less than or equal to $E^{1/2}$, all such curves lie in the closed (hence compact) ball $B(x, E^{1/2})$ of radius $E^{1/2}$. We let b be the uniqueness radius of that ball. Now for any partition $0 = t_0 < t_1 < t_2 < ... < t_{k-1} <$

$t_k = 1$ such that $t_{i+1} - t_i < \frac{b^2}{E}$ and any such γ, $L(\gamma|_{[t_i, t_{i+1}]}) = \int_{t_i}^{t_{i+1}} |\gamma'(t)| dt \leq$
$\{\int_{t_i}^{t_{i+1}} |\gamma'(t)|^2 dt\}^{1/2} (t_{i+1} - t_i)^{1/2} < E^{1/2} \frac{b}{E^{1/2}} = b$.

On The space M^{k+1} (the product of $k + 1$ copies of M) for each partition $0 = t_0 < t_1 < t_2 < ... < t_{k-1} < t_k = 1$ we define the energy function $E_{(t_0, t_1, t_2, ..., t_k)} :$ $M^{k+1} \to R$ by

$$E_{(t_0, t_1, t_2, ..., t_k)}(x_0, x_1, ..., x_k) = \sum_{i=0}^{k} \frac{d(x_i, x_{i+1})^2}{t_{i+1} - t_i}.$$

This is defined so that γ the piecewise p-geodesic curve defined by $(x_0, x_1, ..., x_k)$ satisfies $E(\gamma) = E_{(t_0, t_1, t_2, ..., t_k)}(x_0, x_1, ..., x_k)$. To be precise γ is built of minimizing p-geodesics from x_i to x_{i+1} parameterized proportional to arclength on $[t_1, t_{i+1}]$. Of course $E_{(t_0, t_1, t_2, ..., t_k)}(\gamma(t_0), \gamma(t_1), \gamma(t_2), ..., \gamma(t_k)) \leq E(\gamma)$ for any curve γ.

The technical tool that will replace second variation arguments is

Lemma 1. *Let M be a compact two dimensional manifold with smooth boundary, no conjugate points and the boundary has finite F_M. Let $\gamma : [0, 1] \to M$ be p-geodesic from x to y of length L parameterized proportional to arclength, $E > L^2$, and b defined as above. Then for any partition $0 = t_0 < t_1 < t_2 < ... < t_{k-1} < t_k = 1$ with $t_{i+1} - t_i < \frac{b^2}{E}$ there is a neighborhood $U \subset M^{k+1}$ of $(\gamma(t_0), \gamma(t_1), \gamma(t_2), ..., \gamma(t_k)) \in M^{k+1}$ such that for any $u \in U$ with $u \neq (\gamma(t_0), \gamma(t_1), \gamma(t_2), ..., \gamma(t_k))$ we have*

$$E_{(t_0, t_1, t_2, ..., t_k)}(\gamma(t_0), \gamma(t_1), \gamma(t_2), ..., \gamma(t_k)) < E_{(t_0, t_1, t_2, ..., t_k)}(u).$$

Proof. The condition on F_M tells us that γ consists of a finite number of segments each of which is either a geodesic with only endpoints on the boundary, a geodesic segment that lies on the boundary, or a segment of the boundary where the boundary is strictly concave. We can assume that $\gamma(0)$ and $\gamma(1)$ are not in the finite set F_M since we can handle that case by taking limits of the more general case.

We next point out that that if the proposition is true for any partition $0 = t_0 < t_1 < t_2 < ... < t_{k-1} < t_k = 1$ with $t_{i+1} - t_i < \frac{b^2}{E}$ then it is true for all such partitions. To see this let $0 = s_0 < s_1 < s_2 < ... < s_{l-1} < s_l = 1$ be such a partition where the proposition does not hold. That means there is a sequence $u^i \in M^{l+1}$ representing piecewise p-geodesic paths γ^i from x to y which converges to $(\gamma(s_0), \gamma(s_1), \gamma(s_2), ..., \gamma(s_l))$ such that $E_{(s_0, s_1, s_2, ..., s_l)}(u^i) \leq E(\gamma)$. This means that $E(\gamma^i) \leq E(\gamma)$. Thus $E_{(t_0, t_1, t_2, ..., t_k)}(\gamma^i(t_0), \gamma^i(t_1), \gamma^i(t_2), ..., \gamma^i(t_k)) \leq E(\gamma^i) \leq E(\gamma)$ while $(\gamma^i(t_0), \gamma^i(t_1), \gamma^i(t_2), ..., \gamma^i(t_k))$ converges to $(\gamma(t_0), \gamma(t_1), \gamma(t_2), ..., \gamma(t_k))$ which says that the proposition does not hold for the partition $0 = t_0 < t_1 < t_2 < ... < t_{k-1} < t_k = 1$ either.

The previous paragraph of the proof now allows us to choose our t_i such that $t_i \notin F_M$, thus one of three things hold. In the first instance $\gamma(t_i)$ is an interior point of M. In this case we call t_i interior. In the second, for all t in some open interval about t_i, $\gamma(t) \in \partial M$ and is a geodesic (i.e. the geodesic curvature of ∂M is 0). In this case t_i is called boundary geodesic. In the final case, for all t in some open interval about t_i, $\gamma(t) \in \partial M$ and the boundary is strictly concave. In this case we call t_i boundary concave. The condition on F_M allows us to make sure that there is at least one $\gamma(t_i)$ for each concave boundary interval of γ. Consider a sequence $t_i, t_{i+1}, ..., t_{i+m}$ such that $m > 1$, t_i and t_{i+m} are concave boundary while $t_{i+1}, ..., t_{i+m-1}$ are interior or boundary geodesic. Then the curve $\gamma|_{[t_i, t_{i+m}]}$ is precisely of the following form: there are numbers s_0 and s_1 so that $t_i < s_0 < t_{i+1}$, $t_{i+m-1} < s_1 < t_{i+m}$, $\gamma|_{[s_0, s_1]}$ is a geodesic $\tau : [s_0, s_1] \to M$ while $\gamma|_{[t_i, s_0]}$ and

$\gamma|_{[s_1,t_{i+m}]}$ are concave segments of the boundary. By the concavity of the boundary at $\gamma(s_0)$ and $\gamma(s_1)$ there is an $\epsilon > 0$ such that τ can be extended to a geodesic on $[s_0 - \epsilon, s_1 + \epsilon]$. Now any curve $\sigma : [t_i, t_i + m] \to M$ from $\gamma(t_i)$ to $\gamma(t_{i+m})$ close enough to γ must intersect τ in at least two points $\sigma(a) = \tau(a')$ and $\sigma(b) = \tau(b')$ with $s_0 - \epsilon < a' \leq s_0$ and $s_1 \leq b' < s_1 + \epsilon$. (See Figure 18.)
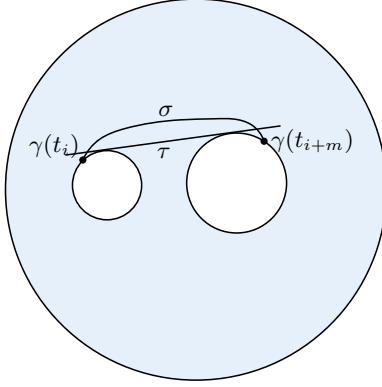

FIGURE 18. Any curve σ from $\gamma(t_i)$ to $\gamma(t_{i+m})$ close enough to γ must intersect τ in at least two points.

Note that the unique local minimality of τ, of $\gamma|_{[t_i,s_0]}$, and of $\gamma|_{[s_1,t_{i+m}]}$ tell us that for σ close enough to γ

$$L(\sigma|_{[t_i,t_{i+m}]}) \geq L(\sigma|_{[t_i,a]} \cup \tau|[a',b'] \cup \sigma|_{[b,t_{i+m}]}) =$$
$$= L(\sigma|_{[t_i,a]} \cup \tau|[a',s_0] \cup \tau|[s_0,s_1] \cup \tau|[s_1,b'] \cup \sigma|_{[b,t_{i+m}]})) \geq$$
$$\geq L(\gamma|[t_i,s_0] \cup \tau|[s_0,s_1] \cup \gamma|[s_1,t_{i+1}]) = L(\gamma|[t_i,t_{i+1}]),$$

and that equality can only hold if $\sigma|_{[t_i,t_{i+1}]}$ coincides with $\gamma|_{[t_i,t_{i+1}]}$ when parameterized proportional to arclength. Hence $\gamma|_{[t_i,t_{i+m}]}$ is a strict local minimum of length for paths between its endpoints. A similar (slightly easier) argument works when $i = 0$ (respectively $i + m = k$) and t_0 (respectively t_k) is interior or boundary geodesic.

Assume that γ does not satisfy the statement of the Lemma. Then there is a sequence $u^j \in M^{k+1}$ never equal to $(\gamma(t_0), \gamma(t_1), \gamma(t_2), ..., \gamma(t_k))$ but converging to it with corresponding piecewise geodesics γ^j converging to γ and with $E(\gamma^j) \leq E_{(t_0,t_1,t_2,...,t_k)}(u^i) \leq E_{(t_0,t_1,t_2,...,t_k)}(\gamma) = E(\gamma)$. In particular $L(\gamma^j) \leq L(\gamma)$. Thus if $\bar\gamma^i$ is the reparametrization of γ^i proportional to arclength then $w^j = (w_0^j, w_1^j, ..., w_k^j) \equiv (\bar\gamma^j(t_0), \bar\gamma^j(t_1), ..., \bar\gamma^j(t_k))$ also converges to $(\gamma(t_0), \gamma(t_1), \gamma(t_2), ..., \gamma(t_k))$ and for each i, $d(w_i^j, w_{i+1}^j) \leq L(\gamma|_{[t_i,t_{i+1}]})$. Further if t_i is concave boundary then the strict concavity of the boundary near $\gamma(t_i)$ says that for large j we can replace w_i^j with a point on the boundary so as to decrease both $d(w_{i-1}^j, w_i^j)$ and $d(w_i^j, w_{i+1}^j)$ (unless w_i^j lies on the boundary to begin with). Thus we will assume that w_i^j lies on the boundary when t_i is concave boundary. Define ϵ_i^j by $w_i^j = \gamma(t_i + \epsilon_i^j)$ which is well defined for all large j. By the previous paragraph if t_i and t_{i+m} are consecutive concave boundary then $\epsilon_i^j - \epsilon_{i+m}^i \geq 0$, while if t_{i_0} is the first concave boundary and t_{i_1} the last concave boundary we see $\epsilon_{i_0}^j \leq 0$ and $\epsilon_{i_1}^j \geq 0$. This implies that for all

concave boundary t_i, $\epsilon_i^j = 0$ and $w_i^j = \gamma(t_i)$. Now the uniqueness of the previous paragraph forces $w_i^j = \gamma(t_i)$ for all i. Since each step in moving from u_i to w_i strictly decreased energy (unless no change was made) we see that the original u^j had to be simply $u_i^j = \gamma(t_i)$ yielding the Lemma. $\qquad\square$

Proof. (of Proposition 1)

By passing to the universal cover we can assume that M is simply connected and that p-geodesic segments satisfy Lemma 1. We will show that p-geodesic segments minimize. The idea is to mimic a standard minimax argument using Lemma 1 in place of saying that all p-geodesics are critical points of index 0 for E.

Let $\gamma : [0,1] \to M$ be a p-geodesic segment (parameterized proportional to arclength) from x to y and let $\tau : [0,1] \to M$ be a length minimizing p-geodesic from x to y. We may assume that $E(\gamma) = L^2(\gamma) > L^2(\tau) = E(\tau)$. By assumption there is a homotopy from γ to τ in the space $\mathcal{C}_{(x,y)}$ of C^1 rectifiable curves from x to y. If E is the maximum energy of a curve in this homotopy and b is the uniqueness radius of the closed ball of radius $E^{1/2}$ then by using the partition with $t_i = \frac{i}{k}$ where $\frac{1}{k} < \frac{b^2}{E}$ we can replace each of the curves in the homotopy with piecewise p-geodesics (each piece of length less than b parameterized on an interval of length $\frac{1}{k}$). This defines a curve $u(s) : [0,1] \to M^{k+1}$ from $(x = \gamma(0), \gamma(\frac{1}{k}), \gamma(\frac{2}{k}), ..., \gamma(1) = y)$ to $(x = \tau(0), \tau(\frac{1}{k}), \tau(\frac{2}{k}), ..., \tau(1) = y)$ such that $E_{0, \frac{1}{k}, \frac{2}{k}, ..., 1}(u(s)) \leq E$. In fact $u(s)$ lies in the compact space $B(x, E^{1/2})^{k+1}$. We can take the neighborhood $U(\gamma)$ of $(x = \gamma(0), \gamma(\frac{1}{k}), \gamma(\frac{2}{k}), ..., \gamma(1) = y)$ promised by Lemma 1 to be a small metric ball in M^{k+1} (in the product metric) since the boundary $\partial U(\gamma)$ is compact $E_{0, \frac{1}{k}, \frac{2}{k}, ..., 1}(u) \geq L^2(\gamma) + \epsilon$ for some $\epsilon > 0$ and all $u \in \partial U(\gamma)$.

We now consider Let $E_0 = inf\{max\{E_{0, \frac{1}{k}, \frac{2}{k}, ..., 1}(u(s)) | s \in [0,1]\}\}$ where the infimum is taken over the collection of all such curves $u(s)$. Since any $u(s)$ must cross $\partial U(\gamma)$ we see that $E_0 \geq L(\gamma)^2 + \epsilon$ By an earlier argument we know that all such curves lie in the compact space $B(x, E^{1/2})^{k+1}$. Usual compactness arguments show that there is a minimax p-geodesic σ from x to y. That is:

i) $E(\sigma) = E_{0, \frac{1}{k}, \frac{2}{k}, ..., 1}(\sigma(0), \sigma(\frac{1}{k}), \sigma(\frac{2}{k}), ..., \sigma(1)) = E_0$ and

ii) there are sequences u_i and s_i with $u_i(s_i) \to \sigma$ and

$E_{0, \frac{1}{k}, \frac{2}{k}, ..., 1}(u_i(s_i)) \geq E_{0, \frac{1}{k}, \frac{2}{k}, ..., 1}(u_i(s))$ for all s.

The argument is the same as the usual one - we sketch it. By compactnees there are convergent sequences as in i) and ii) converging to σ. The only thing to check is that we can assume σ is p-geodesic. If σ is any piecewise p-geodesic curve that is not a p-geodesic then it has a nonzero angle at some join then there is a tangent vector $V \in T_\sigma M^{k+1}$ which can be extended smoothly in a neighborhood such that $V(E) < 0$ at all points in the neighborhood. Thus (as usual) if our sequence $u_i(s_i)$ of curves with maximum energies approaching E_0 has no p-geodesic as a minimax point then we could "push the curves" down (using the above vector fields) to energies below E_0 which contradicts the definition of E_0.

The above contradicts Lemma 1. Choose our neighborhood $U(\sigma)$ to be a small metric ball centered at $(\sigma(0), \sigma(\frac{1}{k}), \sigma(\frac{2}{k}), ..., \sigma(1))$ and as before there is an ϵ_1 such that $E_{0, \frac{1}{k}, \frac{2}{k}, ..., 1}(u) \geq E_0 + \epsilon_1$ for all u in $\partial U(\sigma)$. However for all large i and all s $E_{0, \frac{1}{k}, \frac{2}{k}, ..., 1}(u_i(s)) \leq E_{0, \frac{1}{k}, \frac{2}{k}, ..., 1}(u_i(s_i)) \leq E_0 + \frac{\epsilon_1}{2}$ but since the curve $u_i(s)$ must intersect $\partial U(\sigma)$ we get the desired contradiction.

Note that γ is the unique length minimizing path between its endpoints in its homotopy class. This follows since if τ is another such then τ is also a p-geodesic and the above minimax argument for paths from γ to τ leads to the same contradiction.

$\square$

Remark. Proposition 1 is false in higher dimensions.

To see this let us first consider a metric g_0 on $R^2 - (0,0)$ defined in polar coordinates by $ds^2 = dr^2 + f^2(r)d\theta^2$ where $f : [0,\infty) \to R^+$ is a smooth function such that:

a) $f(r) = \sinh(r)$ for $r \geq 1$,

b) $f(0) = 0$, $f'(r) > 0$ and $f''(r) \geq 0$,

c) $f(r) = r/3$ for $r \leq 1/10$.

It is straightforward to check that such an f exists. For example, find a smooth function f' on $[0,1]$ such that $f'(r) = 1/3$ for $r \in [0,1/10]$, $f'(r) = \cosh(r)$ for $r \in [9/10,1]$, $f'(r)$ is increasing, and $\int_0^1 f'(r) = \sinh(1)$. Then define $f(r)$ as $\int_0^r f'(t)dt$. Such an f' exists since $\frac{1}{30} + \frac{8}{10}\cosh(\frac{9}{10}) \approx 1.179802441$ which is larger than $\sinh(\frac{9}{10}) \approx 1.026516726$

Condition b) tells us that the metric has nonpositive curvature and hence no conjugate points.

Condition c) tells us that ds^2 defines a flat (cone like) metric for $r \leq 1/10$ gotten by taking a sector of the flat disc of radius $1/10$ subtending an angle of $\frac{2\pi}{3}$ and gluing the edge radii together. Thus there is a geodesic segment τ that self intersects (e.g. the geodesic that corresponds to the straight line between the points $1/20$ along the edge radii). Choose $\epsilon > 0$ so that $\tau \subset R^2 - B(\epsilon)$ where we let $B(r)$ represent the open ball of radius r centered at $(0,0)$. So the metric on $R^2 - B(\epsilon)$ has no conjugate points and geodesics that do not minimize. Of course it is not simply connected.

We will consider a metric on $R^3 - U$ where we think of R^3 as $R \times R^2$ parameterized by x, r, and θ. The metric will be $dx^2 + dr^2 + sinh(r)d\theta^2$ (i.e. a line cross with the hyperbolic metric on the plane) when $|x| > 1$ and it will be $dx^2 + dr^2 + f^2(r)d\theta^2$ for $|x| \leq 1$. The open set $U = \{(x,r,\theta)| -1 < x < 1, \ and \ r < r(x)\}$ where $r(x)$ is a smooth positive function such that $r(0) = \epsilon$, $r(1) > 1$ and $r(-1) > 1$. In particular U is homeomorphic to a 3-ball and hence $R^3 - U$ is simply connected. We note that condition a) tells us that g_0 is just the hyperbolic metric when $r > 1$ and hence the metric g is smooth on $R^3 - U$ and has no conjugate points since it has nonpositive curvature. The curve τ on the totally geodesic $\{0\} \times (R^2 - B(\epsilon))$ is a self intersecting geodesic in g and hence not minimizing.

To make this example compact simply use a large closed ball in R^3 (with an extra boundary component) rather than all of R^3.

REFERENCES

[ABB1] Stephanie B Alexander, I David Berg, and Richard L Bishop. The riemannian obstacle problem. *Illinois journal of mathematics*, 31(1):167–184, 1987.

[ABB2] Stephanie B. Alexander, I. David Berg, and Richard L. Bishop. Cut loci, minimizers, and wavefronts in riemannian manifolds with boundary. *Michigan Math. J.*, 40(2):229–237, 1993.

[ABB3] Stephanie B Alexander, I David Berg, and Richard L Bishop. Geometric curvature bounds in riemannian manifolds with boundary. *Transactions of the American Mathematical Society*, 339(2):703–716, 1993.

[BCG]   G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geom. Funct. Anal.*, 5(5):731–799, 1995.

[BI1]   Dmitri Burago and Sergei Ivanov. Boundary rigidity and filling volume minimality of metrics close to a flat one. *Annals of mathematics*, 171(2):1183–1211, 2010.

[BI2]   Dmitri Burago and Sergei Ivanov. Area minimizers and boundary rigidity of almost hyperbolic metrics. *Duke Mathematical Journal*, 162(7):1205–1248, 2013.

[CH]   Christopher B. Croke and Pilar Herreros. Lens rigidity with trapped geodesics in two dimensions, 2011.

[Cro1] C.B. Croke. Rigidity and the distance between boundary points. *J. Diff. Geom.*, 33(2):445–464, 1991.

[Cro2] Christopher Croke. Scattering rigidity with trapped geodesics. *Ergodic Theory and Dynamical Systems*, 34(03):826–836, 2014.

[dC]   M.P. do Carmo. *Riemannian Geometry*. Mathematics (Boston, Mass.). Birkhäuser, 1992.

[Gro]   Mikhael Gromov. Filling riemannian manifolds. *J. Differential Geom*, 18(1):1–147, 1983.

[Mic]   René Michel. Sur la rigidité imposée par la longueur des géodésiques. *Invent. Math.*, 65(1):71–83, 1981/82.

[PU]   Leonid Pestov and Gunther Uhlmann. Two dimensional compact simple Riemannian manifolds are boundary distance rigid. *Ann. of Math. (2)*, 161(2):1093–1110, 2005.

[UW]   Gunther Uhlmann and Jenn-Nan Wang. Boundary determination of a riemannian metric by the localized boundary distance function. *Advances in applied mathematics*, 31(2):379–387, 2003.

[Wen]   Haomin Wen. Simple riemannian surfaces are scattering rigid. *Geom. Topol.*, 19:2329–2357, 2015.

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