Baton Rouge, April 14, 2014

A scheme-theoretic definition of leaves and Serre-Tate coordinates

§1 Hom schemes for Barsotti-Tate groups and their stabilization

1.1 A teaser: $k = \bar{k} \supseteq \mathbb{F}_p$. $Y_{/k}$: slope $1/3$, dim $=1$

$Z_{/k}$: slope $4/5$, dim $=4$

$\underline{\mathrm{Hom}}(Y, Z) = \operatorname{Spec} \tilde{R}$.

$$\tilde{R} = k[t_0, t_1, t_2, \dots] \Big/ \left(\begin{matrix} t_0^{p}, t_1^{p}, t_2^{p^2}, t_3^{p^2}, t_4^{p^3}, t_5^{p^3}, t_6^{p^5}, \\ t_{i+7}^{p^5} - t_i \quad \text{for } i \geq 0 \end{matrix} \right)$$

One feels that $\tilde{R}$ is 7-dimensional in some sense.

Question: How to make this precise.

1.2. Y, Z: p-divisible groups over a field $\kappa \supseteq \mathbb{F}_p$

$H_n := \underline{\mathrm{Hom}}(Y[p^n], Z[p^n])$ scheme of finite type over κ

Have: $r_{i,n+i}: H_{n+i} \longrightarrow H_n$ (restriction homom.)

$\iota_{n+i,i}: H_i \longrightarrow H_{n+i}$ (inclusion homom. induced by $[p^n]_{H_{n+i}}$)

$\rightsquigarrow$

$$\begin{tikzcd} H_{n+i}/\iota_{n+i,i}(H_i) \arrow[r, "\nu_{n+i,i}"] & H_n \\ H_{n+i+j}/\iota_{n+i+j,i+j}(H_{i+j}) \arrow[u, "\nu_{n;i,i+j}"] \arrow[ur, "\nu_{n+i+j,i+j}"'] & \end{tikzcd}$$

Lemma 1.3 (stabilization) $\exists\, n_0$ s.t.

$$\nu_{n;1,2}: H_{n+2}/\iota_{n+2,n+1}(H_{n+1}) \xrightarrow{\sim} H_{n+1}/\iota_{n+1,n}(H_n) \qquad \forall n \geq n_0$$

Definition 1.4 $G_n = G_n(Y, Z) := H_{n+n_0} / \iota_{n+n_0, n_0}(H_n)$

$\rightsquigarrow \nu_n : G_n \rightarrowtail H_n$

$\rightsquigarrow G_n \xhookrightarrow[j_{n+i,i}]{} G_{n+i}$, $\quad G_{n+i} \xrightarrow[\pi_{n,n+i}]{} G_n$

$\mathcal{H}om'_{div}(Y, Z) := (G_n, j_{n+i,i}, \pi_{n,n+i})$ inductive system of finite group schemes$/\kappa$

$\cap\!\!|$

$\mathcal{H}om'(Y, Z) := (H_n, \iota_{n+i,i})$ inductive system of group schemes of finite type over κ

Proposition 1.5. (1) $\mathcal{H}om'_{div}(Y, Z) =$ the maximal p-divisible subgroup of $\mathcal{H}om'(X, Y)$

(2) $\mathcal{H}om'(Y, Z)^\wedge =$ smooth formal group $/\kappa$,

$\dim = \dim(Z) \cdot \dim(Y^t)$

Proposition 1.6. Suppose that Y, Z are isoclinic of slopes λ_Y and λ_Z.

(1) If $\lambda_Y > \lambda_Z$, then $\mathcal{H}om'_{div}(Y, Z) = 0$

(2) If $\lambda_Y \leq \lambda_Z$, then $\mathcal{H}om'_{div}(Y, Z)$ is isoclinic of slope $\lambda_Z - \lambda_Y$ and height $\mathrm{heigh}(Z) \cdot \mathrm{heigh}(Y)$

Definition 1.7. $\mathcal{E}xt'(Y, Z) : \begin{pmatrix} \text{augmented} \\ \text{Artinian local} \\ \text{rings } /\kappa \end{pmatrix} \longrightarrow \begin{pmatrix} \text{abelian} \\ \text{groups} \end{pmatrix}$

$R \rightsquigarrow \mathrm{Ker}\left(\mathrm{Ext}(Y_R, Z_R) \longrightarrow \mathrm{Ext}_\kappa(Y, Z)\right)$

Prop. 1.8. $\mathcal{E}xt'(Y, Z)$ is formally smooth of dimension $\dim(Z) \cdot \dim(Y^t)$

Prop 1.9. $\exists$ natural isomom.

$$\begin{array}{ccc} \mathcal{H}om'(Y,Z)^{\wedge} & \xrightarrow[\sim]{\delta} & \mathcal{E}xt'(Y,Z) \\ \cup & & \cup\vert \\ \mathcal{H}om'_{div}(Y;Z) & \longrightarrow & \mathcal{E}xt'_{div}(Y,Z) \end{array}$$

$\mathcal{E}xt'_{div}(Y,Z)$ = max. p-divisible subgroup of the smooth formal group $\mathcal{E}xt'(Y,Z)$

Rmk 1.10. interpretation in terms of biextension :

$\exists$ a canonical / tautological biextension of $Y \times \mathcal{H}om'_{div}(Y,Z)$ by Z

Prop. 1.10. Suppose that κ is perfect. $M(Y)$, $M(Z)$ = covariant (Cartier) Dieudonné modules for Y. Z

On $\mathrm{Hom}_{W(k)}(M(X), M(Y)) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$, have F, V given by

$(Vh)(y) = V \cdot h(V^{-1}y)$, $(Fh)(y) = F \cdot h(Vy)$ $\quad \forall h \in \mathrm{Hom}(M(Y), M(Z))[\frac{1}{p}]$, $\forall y \in M(Y)$

(1) The covariant Dieudonné module for $\mathcal{H}om'_{div}(Y,Z) \simeq \mathcal{E}xt'_{div}(Y,Z)$

= the largest $W(k)$-submodule of $\mathrm{Hom}_{W(k)}(M(X), M(Y))$ which is stable under F and V.

(2) The Cartier module of the smooth formal group $\mathcal{E}xt'(Y,Z) \simeq \mathcal{H}om'(Y,Z)$ is (canonically isomorphic to)

$$\mathrm{Ext}^1_{\mathrm{Cart}_p(\kappa)}\left(M(Y),\ BC_p(\kappa) \otimes_{\mathrm{Cart}_p(\kappa)} M(Z)\right),$$

where

- $BC_p(\kappa)$ = the Cartier module of the infinite dim'l comm. smooth formal group $R \rightsquigarrow \mathrm{Cart}_p(R)$ over κ
  = the group of all p-typical formal curves in $\mathrm{Cart}_p(\kappa[\![t]\!]_+)$
  $\left(= \mathrm{Ker}\left(\mathrm{Cart}_p(\kappa[\![t]\!]) \to \mathrm{Cart}_p(\kappa)\right)\right)$
- $BC_p(\kappa)$ has a natural $\mathrm{Cart}_p(\kappa)$-$\mathrm{Cart}_p(\kappa)$ bimodule structure, used in the formula $\left(\mathrm{Ext}^1_{\mathrm{Cart}_p(\kappa)}\left(M(Y), BC_p(\kappa) \otimes_{\mathrm{Cart}_p(\kappa)} M(Z)\right)\right)$
- $BC_p(\kappa)$ has a natural left action by $\mathrm{Cart}_p(\kappa)$, coming from the fact that it is the Cartier module of a comm. smooth formal group over κ. This "third" $\mathrm{Cart}_p(\kappa)$-module structure is compatible with the above $\mathrm{Cart}_p(\kappa)$-$\mathrm{Cart}_p(\kappa)$ bimodule structure, and induces a left action of $\mathrm{Cart}_p(\kappa)$ on $\mathrm{Ext}^1_{\mathrm{Cart}_p(\kappa)}\left(M(Y), BC_p(\kappa) \otimes_{\mathrm{Cart}_p(\kappa)} M(Z)\right)$

§2 Sustained p-divisible groups

2.1. $\kappa \supseteq \mathbb{F}_p$: a field of char. $p>0$; S/κ : a κ-scheme

X_0/κ : a p-divisible group

Definition: A p-divisible group $X \to S$ is <u>strongly κ-sustained modeled on X_0/κ</u> if $\forall n>0$,

$$\underline{\mathrm{Isom}}\left(X_0[p^n] \times_{\mathrm{Spec}\,\kappa} S,\ X[p^n]\right) \longrightarrow S \quad \text{is faithfully flat}$$

<u>Definition</u> A p-divisible group $X \to S$ is <u>κ-sustained</u> if either of the two equiv conditions hold

(1) $\exists$ ext^n field L/κ and a p-divisible group Y_0/L s.t. $X \times_{\operatorname{Spec}\kappa} \operatorname{Spec} L \longrightarrow S \times_{\operatorname{Spec}\kappa} \operatorname{Spec} L$ is strongly L-sustained modeled on Y_0/L

(2) $\forall n > 0$, $\underline{\mathrm{Isom}}_{S \times_{\operatorname{Spec}\kappa} S}(pr_1^* X[p^n], pr_2^* X[p^n]) \longrightarrow S \times_{\operatorname{Spec}\kappa} S$ is faithfully flat

2.2 <u>Prop.</u> $X \longrightarrow S/\kappa$: κ-sustaind.
There exists a unique filtration $X = X_0 \supset X_1 \supset \cdots \supset X_{m-1} \supseteq X_m = (0)$ s.t.

(i) X_i / X_{i+1} is a κ-sustained p-divisible group, isoclinic of slope λ_i. $\forall i = 0, 1, \dots, m-1$

(ii) $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_{m-1} \leq 1$

2.3. <u>Prop.</u> Suppose that S/κ is reduced and locally of finite type$/\kappa$.
$X \longrightarrow S$ p-divisible group. X_0/κ : p-divisible group
Assume : $\forall s \in S$, X_s is strongly κ-sustained modeled on X_0/κ

Then $X \longrightarrow S$ is strongly κ-sustained modeled on X_0/κ

2.4. Prop. $S_{/\kappa}$ = locally noetherian, $\kappa \supseteq \mathbb{F}_p$. $X_{0/\kappa}$: p-divisible group
$X \to S$: p-divisible group

There exists a locally closed subscheme $T \hookrightarrow S$ s.t.

(a) $X \times_S T$ is κ-sustained modeled on $X_{0/\kappa}$

(b) If $T_1 \hookrightarrow S$ is a locally closed subscheme of S and $X \times_S T_1$ is κ sustained modeled on $X_{0/\kappa}$, then $T_1 \subseteq T$.

2.5. κ-Firm p-divisible group; 2-slope case

Setup : Let $\kappa \supseteq \mathbb{F}_p$ be a field, $S_{/\kappa}$ a κ-scheme, let

$0 \to Z \to X \to Y \to 0$ be a short exact sequence of p-divisible groups over S such that $Y \to S$ and $Z \to S$ are κ-sustained isoclinic p-divisible groups with slopes $\lambda_Y < \lambda_Z$.

Let $S_1 := S \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}\, S$, $pr_1, pr_2 : S_1 \to S$

Let

$$\mathcal{I}_{Y,n} := \mathcal{I}som(pr_1^* Y[n], pr_2^* Y[n])$$
$$\mathcal{I}_{Z,n} := \mathcal{I}som(pr_1^* Z[n], pr_2^* Z[n])$$

: stabilized Isom schemes $\rightsquigarrow$ $q_n : \mathcal{I}_{Y,n} \times_{S_1} \mathcal{I}_{Z,n} \to S_1$

$\rightsquigarrow$ For $N \geq n$, have tautological isomorphism

$$\tau_{N,n} : \underset{\substack{\Downarrow \\ [q_N^* pr_1^*(0\to Z\to X\to Y\to 0)]}}{\mathrm{Ext}\,(q_N^* pr_1^* Y[p^n], q_N^* pr_1^* Z[p^n])} \xrightarrow{\sim} \underset{\substack{\Downarrow \\ [q_N^* pr_2^*(0\to Z\to X\to Y\to 0)]}}{\mathrm{Ext}\,(q_N^* pr_2^* Y[p^n], q_N^* pr_2^* Z[p^n])}$$

Definition We say that $X \to S$ is κ-firm with 2 slopes if

$\forall N \geq n$, $\exists$ an fppf morphism $T \xrightarrow{g} \mathcal{I}_{Y,N} \times_{S_1} \mathcal{I}_{Z,N}$ s.t

$$g^*\Big(\tau_{N,n}\big([q_N^* pr_1^*(0\to Z\to X\to Y\to 0)] - [q_N^* pr_2^*(0\to Z\to X\to Y\to 0)]\big)\Big) = 0$$

(Roughly: the difference between the two extension classes corresponding to pr_1^* and pr_2^* of $0\to Z\to X\to Y\to 0$ is p-divisible)

2.6. Prop. A p-divisible group $X_{/S_{/\kappa}}$ is κ-firm with 2 slopes iff it is κ-sustained with 2 slopes

2.7. Definition Notation as in Definition 2.5.
Replace S_1 by $S_1^{/\Delta(S)}$ = formal completion of $S \overset{\Delta}{\hookrightarrow} S_1$ along the diagonal

$\rightsquigarrow$ get a weaker notion of infinitesimally κ-firm p-divisible group with 2 slopes

2.8. Prop. Suppose that $S_{/\kappa}$ is a scheme of finite type over $\kappa \supseteq \mathbb{F}_p$ field
Then infinitesimally κ-firm with 2 slopes $\iff$ κ-sustained with 2 slopes $\iff$ κ-firm with 2-slopes

§3. Serre-Tate coordinates

3.1 Local Serre-Tate coordinates : 2-slope case

$\kappa \supseteq \mathbb{F}_p$ perfect field, $0 \to Z_0 \to X_0 \to Y_0 \to 0$ short exact sequence of p-divisible groups
Y_0, Z_0 : isoclinic, $\lambda_{Y_0} < \lambda_{Z_0}$

Let $\mathcal{C}(\mathrm{Def}(X_0), X_0)$ = "the leaf in $\mathrm{Def}(X_0)$ passing through X_0"
local deformation space of X_0 in equi-characteristic p
= the maximal strongly κ-sustained locus in $\mathrm{Def}(X_0)$ modeled on X_0

Prop. $\mathcal{C}(\mathrm{Def}(X_0), X_0)$ has a natural structure as a torsor for $\mathcal{H}om'_{\mathrm{div}}(Y_0, Z_0) \simeq \mathcal{E}xt'_{\mathrm{div}}(Y_0, Z_0)$

3.2. Global Serre-Tate coordinates: 2-slope case

$\mathcal{A}_{g/\overline{\mathbb{F}}_p} \ni [(A_0, \lambda_0)] = x_0 \qquad 0 \to Z_0 \to A_0[p^\infty] \to Y_0 \to 0$

$\cup$ $\qquad Y_0, Z_0$: isoclinic, $\lambda_{Y_0} < \lambda_{Z_0}$

$\mathcal{C} = \mathcal{C}(x_0)$ = the leaf in $\mathcal{A}_{g/\overline{\mathbb{F}}_p}$ through x_0 $\qquad \lambda_0$ induces: $Y_0 \xrightarrow{\sim} Z_0^t$

= the maximal strong $\overline{\mathbb{F}}_p$-sustained locus modeled on $(A_0[p^\infty], \lambda_0[p^\infty])$

$\rightsquigarrow$
$$\mathcal{C}_1 = \mathcal{C} \times_{\mathrm{Spec}(\overline{\mathbb{F}}_p)} \mathcal{C} \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} \mathcal{C}$$

Over $\mathcal{C}$, have a short exact sequence

$$0 \to Z \to A_0[p^\infty] \to Y \to 0. \qquad Y, Z: \text{isoclinic}$$

strongly κ-sustained

$Y_0 \xrightarrow{\sim} Z_0^t$

Over $\mathcal{C}_1$, have canonical isom. $pr_1^* Y \xrightarrow{\sim} pr_2^* Y$

$pr_1^* Z \xrightarrow{\sim} pr_2^* Z$

$\rightsquigarrow$ Have stabilized $\mathcal{H}om'(Y[p^n], Z[p^n])$

and

$$\mathcal{H}om'_{\mathrm{div}}(Y, Z) \supseteq \mathcal{H}om'_{\mathrm{div}}(Y, Z)_{\mathrm{sym}}$$

w.r.t. the involution given by the principal polarization

<u>Prop</u>: $\mathcal{C}_1 \xrightarrow{pr_1} \mathcal{C}$ has a natural structure as a $\mathcal{H}om'_{\mathrm{div}}(Y, Z)_{\mathrm{sym}}$-torsor

Baton Rouge April 14, 2014

A scheme-theoretic definition of leaves and Serre-Tate coordinates

§0. A teaser: Hom from slope 1/3 to slope 4/5

§1. Hom schemes for Barsotti-Tate groups and their stabilization

§2. Sustained p-divisible groups

- basic definitions
- relation with the previous concep "geometrically fiberwise constant p-divisible groups" (over reduced scheme of finite type over κ); existence of maximal κ-sustained locus modeled on X_0/κ (scheme-theoretic)
- Basic properties
  - slope filtration
  - $X_{/L}$, L an ext^n field of κ: When is $X_{/L}$ κ-sustained.
  - Related question: Existence of a p-divisible group $X_{/L}$ s.t. $X[p^n]$ is isomorphic to a given BT_n-group X_n/L. (Known when L is perfect.)
  - Equivalent definition via the notion of "κ-firm p-divisible groups"

§3. - Local Serre-Tate coordinates

- 2-slope case
- sketch general case

- Global Serre-Tate coordinates: sketch only the 2-slope case.

Initial plan: Total 45 min

- teaser + stabilized Hom schemes: 12 min
- def^n of sustained p-divisible groups + relation with old def^n + max. κ-sustained locus: 10 min
- basic properties 8 min
- Serre-Tate coord. 10 min