The abstract Artin problem for a global function field

Ching-Li Chai

February 20, 2004, version 0.1

§1. The abstract Artin problem

(1.1) Let K be a global field, and let Σ'_K be the set of all finite places of K. Let $\mathcal{P}$ be a countable indexing set. For each $\alpha \in \mathcal{P}$, let K_α be a finite Galois extension of K, and let Z_α be a union of conjugacy classes of $\mathrm{Gal}(K_\alpha/K)$. Let $G_\alpha := \mathrm{Gal}(K_\alpha/K)$, $n_\alpha := |\mathrm{Gal}(K_\alpha/K)|$, $s_\alpha := |Z_\alpha|$.

For each $v \in \Sigma'_K$, denote by $\left(\frac{K_\alpha/K}{\wp_v}\right)$ the set of all elements $\sigma \in \mathrm{Gal}(K_\alpha/K)$ such that there exists a place $\tilde{v}$ in K_α above v such that σ belongs to the decomposition group $D_{\tilde{v}}$ and induces the geometric Frobenii for the residue field extension $\kappa_{\tilde{v}}/\kappa_v$. K_α.

(1.1.1) Define a subset $S \subset \Sigma'_K$ by

$$S = \left\{ v \in \Sigma'_K \;:\; \left(\frac{K_\alpha/K}{\wp_v}\right) \not\subseteq Z_\alpha \;\; \forall\, \alpha \in \mathcal{P} \right\} .$$

For every finite subset $I \subset \mathcal{P}$, define a subset $S_I \subset \Sigma'_K$ by

$$S_I = \left\{ v \in \Sigma'_K \;:\; \left(\frac{K_\alpha/K}{\wp_v}\right) \not\subseteq Z_\alpha \;\; \forall\, \alpha \in I \right\} .$$

Also, define $M_I \subset \Sigma'_K$ by

$$M_I = \left\{ v \in \Sigma'_K \;:\; \left(\frac{K_\alpha/K}{\wp_v}\right) \subseteq Z_\alpha \;\; \forall\, \alpha \in I \right\} .$$

It is clear that $S_{I_1} \subseteq S_{I_2}$ if $I_1 \subset I_2$, and

$$S = \bigcap_I S_I \,.$$

(1.1.2) It is easy to see from Chebotarev's density theorem that the Dirichlet density of S_I exists for each finite $I \subset \mathcal{P}$; the same holds for each subset M_J of Σ'_K, J finite. The density of the subsets S_I and M_J of Σ'_K are related by

$$d(S_I) = \sum_{J \subseteq I} (-1)^{|J|}\, d(M_J) \,,$$

as follows from the inclusion-exclusion principle.

(1.2) The abstract Artin problem in the present context is to find a set of conditions on $(K, \{K_\alpha\}_{\alpha\in\mathcal{P}}, Z_\alpha)$ so that the Dirichlet density of S exists, and is equal to $\lim_{I\to\infty} d(S_I)$. Notice that the limit $\lim_{I\to\infty} S_I$ exists because $d(S_{I_1}) \geq d(S_{I_2})$ if $I_1 \subseteq I_2$.

(1.2.1) Remark Murty considered the case when K is a number field and each Z_α consists of the identity element of G_α; see [Mur2]. Clark and Kuwata treated, in [CK], the case when K is a global function field and Z_α consists of the identity element of G_α for each α. They also stated, without proof, a result for more general Z_α's in [CK, Thm. 4.1]. That statement does not seem to follow from the line of argument in [CK] though; see 3.7.1 for more comments.

(1.2.2) We recall that, for any subset $Z \subseteq \Sigma'_K$, the upper and lower Dirichlet density, $d^+(Z)$ and $d^-(Z)$ are defined by

$$d^+(Z) = \limsup_{s\to 1^+} \frac{\sum_{v\in Z} N_v^{-s}}{\sum_{v\in\Sigma'_K} N_v^{-s}} = \limsup_{s\to 1^+} \frac{\sum_{v\in Z} N_v^{-s}}{\log \zeta_K(s)}$$

and

$$d^-(Z) = \liminf_{s\to 1^+} \frac{\sum_{v\in Z} N_v^{-s}}{\sum_{v\in\Sigma'_K} N_v^{-s}} = \liminf_{s\to 1^+} \frac{\sum_{v\in Z} N_v^{-s}}{\log \zeta_K(s)}$$

respectively. The Dirichlet density of Z exists if and only if $d^+(Z) = d^-(Z)$, and the common value is the Dirichlet density $d(Z)$ of Z. Since $S \subseteq S_I$ for each finite subset $I \subset \mathcal{P}$, if follows that $d^+(S) \leq \lim_{I\to\infty} S_I$.

(1.3) A sufficient condition, perhaps too strong, for an affirmative answer to the abstract Artin problem, is that there exists a positive real number $s_0 > 1$ such that the infinite series

$$\sum_{\alpha\in\mathcal{P}} \sum_{\substack{v\in\Sigma'_K \\ \left(\frac{K_\alpha/K}{\wp_v}\right)\subseteq Z_\alpha}} \frac{\mathrm{N}_v^{-s}}{\log \zeta_K(s)}$$

converges uniformly for $1 < s \leq s_0$. This statement is left as an exercise.

§2. The function field case: the setup

(2.1) In the rest of this talk, K is a global function field of the form $K = \kappa(C)$, where $\kappa \cong \mathbb{F}_q$ is a finite field, and C is a smooth projective geometrically irreducible curve over κ. Let $\mathcal{P}$ be a countable indexing set. For each $\alpha \in \mathcal{P}$, let K_α be a finite Galois extension of K, with Galois group G_α. Let κ_α be the subfield of constants in K_α, so that $K_\alpha = \kappa_\alpha(C_\alpha)$, where C_α is a smooth projective geometrically irreducible curve over κ_α.

(2.1.1) For each $\alpha \in \mathcal{P}$, let Z_α be a subset of G_α stable under conjugation, so that Z_α is a disjoint union of conjugacy classes: $Z_\alpha = \coprod_{1\le r\le k_\alpha} W_{\alpha,r}$, where each $W_{\alpha,r}$ is a conjugacy class in G_α. Pick an element $\sigma_{\alpha,r}$ each $W_{\alpha,r}$.

(2.1.2) Let $\pi_\alpha : C_\alpha \to C$ be the natural morphism corresponding to $K \hookrightarrow K_\alpha$. Denote by U_α the largest open subscheme of C such that π_α is smooth over U_α, and let $\iota_\alpha : U_\alpha \hookrightarrow C$ be the inclusion map.

(2.1.3) Let ℓ be a prime number prime to q. Let $\rho_{\alpha,1}, \ldots, \rho_{\alpha,m_\alpha}$ be the set of all irreducible $\overline{\mathbb{Q}_\ell}$-linear representations of G_α. The character of $\rho_{\alpha,i}$ will be denoted by $\chi_{\alpha,i}$, $i = 1, \ldots, m_\alpha$. The orthogonality relation in representation theory of finite groups says that

$$\frac{|W_{\alpha,r}|}{|G|} \cdot \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma_{\alpha,r})} \cdot \chi_{\alpha,i}(\sigma) = \begin{cases} 1 & \sigma \in W_{\alpha,r} \\ 0 & \sigma \notin W_{\alpha,r} \end{cases}$$

(2.1.4) For any finite place $v \in \Sigma'_K$, any $\alpha \in \mathcal{P}$, a place $\tilde{v}$ in K_α above v, and any $m \in \mathbb{Z}$, let $\mathbf{Fr}^m_{\tilde{v}/v}$ be the coset modulo the inertia subgroup $\mathrm{I}_{\tilde{v}/v}$ consisting of all elements of the decomposition group $\mathrm{D}_{\tilde{v}/v}$ which induce the m-th power of the geometric Frobenius for the residue field extension $\kappa_{\tilde{v}}/\kappa_v$. An unspecified element of $\mathbf{Fr}^m_{\tilde{v}/v}$ is often written as $\mathrm{Fr}^m_{\tilde{v}/v}$. For any character χ of a a finite dimensional linear representation of G_α, let

$$\chi(\mathbf{Fr}^m_v) = \frac{1}{|\mathbf{Fr}^m_{\tilde{v}/v}|} \sum_{\sigma\in\mathbf{Fr}^m_{\tilde{v}/v}} \chi(\sigma) = \frac{1}{|\mathrm{I}_{\tilde{v}/v}|} \sum_{\sigma\in\mathbf{Fr}^m_{\tilde{v}/v}} \chi(\sigma)$$

be the average of the values of χ over the subset $\mathbf{Fr}_{\tilde{v}/v}$ of G_α. We wrote $\chi(\mathbf{Fr}^m_v)$ instead of $\chi(\mathbf{Fr}^m_{\tilde{v}/v})$, since the latter is independent of the choice of $\tilde{v}$.

(2.1.5) For each conjugacy class $W_{\alpha,r}$ in Z_α, consider the following linear combination

$$\frac{|W_{\alpha,r}|}{|G_\alpha|} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,r}(\sigma_{\alpha,r})} \log L(s, \chi_{\sigma,i}, K_\alpha/K)$$

of logarithms of Artin L-functions. We have, for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$,

$$
\begin{aligned}
&\frac{|W_{\alpha,r}|}{|G_\alpha|} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,r}(\sigma_{\alpha,r})} \log L(s, \chi_{\sigma,i}, K_\alpha/K) \\
&= \frac{|W_{\alpha,r}|}{|G_\alpha|} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,r}(\sigma_{\alpha,r})} \sum_{v \in \Sigma'_K} \sum_{m \geq 1} \frac{\chi_{\sigma,i}(\mathbf{Fr}_v^m)}{m} \cdot \mathrm{N}_v^{-ms} \\
&= \frac{|W_{\alpha,r}|}{|G_\alpha|} \sum_{i,v} \overline{\chi_{\alpha,r}(\sigma_{\alpha,r})} \cdot \chi_{\alpha,i}(\mathbf{Fr}_v) \cdot \mathrm{N}_v^{-s} + \frac{|W_{\alpha,r}|}{|G_\alpha|} \sum_{i} \overline{\chi_{\alpha,r}(\sigma_{\alpha,r})} \cdot \sum_{m \geq 2} \frac{\chi_{\alpha,i}(\mathbf{Fr}_v^m)}{m} \cdot \mathrm{N}_v^{-s} \\
&= \sum_v \frac{1}{\left|\mathrm{I}_{v,K_\alpha/K}\right|} \cdot \# \left\{ u \in \mathrm{I}_{v,K_\alpha/K} \,:\, \mathrm{Fr}_{v,K_\alpha/K} \cdot u \in W_{\alpha,r} \right\} \cdot \mathrm{N}_v^{-s} \\
&\quad + \sum_v \sum_{m \geq 2} \frac{1}{\left|\mathrm{I}_{v,K_\alpha/K}\right|} \cdot \frac{1}{m} \cdot \# \left\{ u \in \mathrm{I}_{v,K_\alpha/K} \,:\, \mathrm{Fr}_{v,K_\alpha/K}^m \cdot u \in W_{\alpha,r} \right\} \cdot \mathrm{N}_v^{-ms}
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{r=1}^{k_\alpha} \frac{|W_{\alpha,r}|}{|G_\alpha|} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,r}(\sigma_{\alpha,r})} \log L(s, \chi_{\sigma,i}, K_\alpha/K) \\
&= \sum_v \frac{1}{\left|\mathrm{I}_{v,K_\alpha/K}\right|} \cdot \# \left\{ u \in \mathrm{I}_{v,K_\alpha/K} \,:\, \mathrm{Fr}_{v,K_\alpha/K} \cdot u \in Z_\alpha \right\} \cdot \mathrm{N}_v^{-s} \\
&\quad + \sum_v \frac{1}{\left|\mathrm{I}_{v,K_\alpha/K}\right|} \cdot \sum_{m \geq 2} \frac{1}{m} \cdot \# \left\{ u \in \mathrm{I}_{v,K_\alpha/K} \,:\, \mathrm{Fr}_{v,K_\alpha/K}^m \cdot u \in Z_\alpha \right\} \cdot \mathrm{N}_v^{-ms}
\end{aligned}
$$

Note that the term in the last line of the displayed formula above should be regarded as an "error term", but we do not give an explicit estimate of it here.

(2.2) We will use Grothendieck's theory of ℓ-adic cohomology and Weil's conjectures proved by Deligne in [Weil II]. Denote by $\mathcal{F}_{\alpha,i}$ the smooth $\overline{\mathbb{Q}_\ell}$-sheaf over $U_\alpha \subseteq C$ attached to the irreducible representation $\rho_{\alpha,i}$ of G_α. It has the property that for every closed point x of U_α, corresponding to a finite place v of K, the trace of the action of the geometric Frobenius of x on the fiber $\mathcal{F}_{\bar{x}}$ is equal to $\chi_{\alpha,i}(\mathrm{Fr}_v)$. Let $L(T, C, \iota_{\alpha*}\mathcal{F}_{\alpha,i})$ be the L-function attached to the $\overline{\mathbb{Q}_\ell}$-sheaf $\iota_{\alpha*}\mathcal{F}_{\alpha,i}$ on C; it is a formal power series in the variable T such that $L(q^{-s}, C, \iota_{\alpha*}\mathcal{F}_{\alpha,i})$ is formally identical to the Artin L-function $L(s, \chi_{\alpha,i})$. The Lefschetz trace formula in ℓ-adic cohomology implies that

$$
L(T, C, \iota_{\alpha*}\mathcal{F}_{\alpha,i}) = \frac{\det(\mathrm{Id} - T \cdot \mathrm{Fr}_q \,|\, \mathrm{H}^1(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i})}{\det(\mathrm{Id} - T \cdot \mathrm{Fr}_q \,|\, \mathrm{H}^0(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i}) \cdot \det(\mathrm{Id} - T \cdot \mathrm{Fr}_q \,|\, \mathrm{H}^2(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i})}
$$

for all $\alpha \in \mathcal{P}$ and all i, $1 \leq i \leq m_\alpha$. where $\overline{C} = C \times_{\mathrm{Spec}\,\kappa} \mathrm{Spec}\,\overline{\kappa}$, and Fr_q denotes the geometric Frobenius acting on sheaves over $\mathrm{Spec}\,\kappa$. Choose and fix an embedding $\overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$, then we can substitute T by q^{-s} in the above displayed formula, and get an expression of the $\mathbb{C}$-valued Artin L-function $L(s, \chi_{\alpha,i}, K_\alpha/K)$ as a rational function in q^{-s}.

(2.3) Combining our previous discussions, we obtain the following inequality.

$$\sum_{\alpha\in\mathcal{P}} \sum_{\substack{v\in\Sigma'_K \\ \left(\frac{K_\alpha/K}{\wp v}\right)\subseteq Z_\alpha}} \frac{\mathrm{N}_v^{-s}}{\zeta_K(s)}$$

$$\begin{aligned}
&\leq \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)} \cdot \log\left(\det\left(\mathrm{Id} - \mathrm{Fr}_q \cdot q^{-s} \,|\, \mathrm{H}^0(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i})\right)^{-1} \right| \right) \cdot \log\zeta_K(s)^{-1} \\
&\quad + \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)} \cdot \log\left(\det\left(\mathrm{Id} - \mathrm{Fr}_q \cdot q^{-s} \,|\, \mathrm{H}^2(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i})\right)^{-1} \right) \right| \cdot \zeta_K(s)^{-1} \\
&\quad + \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)} \cdot \log\left(\det\left(\mathrm{Id} - \mathrm{Fr}_q \cdot q^{-s} \,|\, \mathrm{H}^1(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i})\right)\right) \right| \cdot \zeta_K(s)^{-1}
\end{aligned}$$

Therefore we get an upper bound of

$$\sum_{\alpha\in\mathcal{P}} \sum_{\substack{v\in\Sigma'_K \\ \left(\frac{K_\alpha/K}{\wp v}\right)\subseteq Z_\alpha}} \frac{\mathrm{N}_v^{-s}}{\log\zeta_K(s)}$$

as a sum of three terms, coming from contributions from H^0, H^2 and H^1 respectively. According to 1.3, if we can get an upper bound for each of the three terms, uniformly for $1 < s \leq s_0$ for some $s_0 > 1$, we will get a solution, surely not optimal, to the abstract Artin problem.

(2.3.1) By [Weil II], every eigenvalue of Fr_q on $\mathrm{H}^1(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i})$ is an algebraic number in $\overline{\mathbb{Q}_\ell}$, all of whose complex absolute values are equal to $\sqrt{q}$. This is the *only* deep result used in our estimates in the next section.

§3. Estimates

(3.1) We first estimate the contributions from H^0 and H^2. It is easy to see that the cohomology group $\mathrm{H}^0(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i}) \neq (0)$ if and only if $\mathcal{F}_{\alpha,i}$ is geometrically constant on U_α, i.e. the restriction of $\rho_{\alpha,i}$ to the subgroup $\mathrm{Gal}(K_\alpha/K\cdot\kappa_\alpha)$ of G_α is trivial. The above statement also holds with $\mathrm{H}^0(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i})$ replaced by $\mathrm{H}^0(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i})$. Put it in another way, the sheaves $\iota_{\alpha*}\mathcal{F}_{\alpha,i}$ with non-vanishing H^0 or H^2 over $\overline{C}$ are exactly those of the form $\mathcal{G}_\xi$, where ξ is a one-dimensional $\overline{\mathbb{Q}_\ell}$-valued character of $\mathrm{Gal}(\kappa_\alpha/\kappa)$, and $\mathcal{G}_\xi$ denotes the pull-back, from $\mathrm{Spec}\,\kappa$ to C, of the $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathrm{Spec}\,\kappa$ attached to ξ.

(3.2) For each one-dimensional $\overline{\mathbb{Q}_\ell}$-valued character ξ of $\mathrm{Gal}(\kappa_\alpha/\kappa)$, $\mathrm{H}^0(\overline{C}, \mathcal{G}_\xi)$ is a $\overline{\mathbb{Q}_\ell}$ vector space of dimension one; the action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ factors through $\mathrm{Gal}(\kappa_\alpha/\kappa)$ and acts by the character ξ. For every $\alpha \in \mathcal{P}$ and every $\sigma \in Z_\alpha$, define a positive integer a_σ by

$$a_\sigma = \mathrm{Min}\left\{ r \in \mathbb{N}_{>0} \,:\, \sigma|_{\kappa_\alpha} = \ (\mathrm{Fr}_q|_{\kappa_\alpha})^r \right\} .$$

Notice that $0 < a_\sigma \le c_\alpha$. Recall from 2.3 that the contribution, from H^0, to the upper bound of

$$\sum_{\alpha\in\mathcal{P}} \sum_{\substack{v\in\Sigma'_K \\ \left(\frac{K_\alpha/K}{\wp_v}\right)\subseteq Z_\alpha}} \frac{\mathrm{N}_v^{-s}}{\log\zeta_K(s)}$$

is

$$\sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)} \cdot \log\left(\det\left(\mathrm{Id} - \mathrm{Fr}_q \cdot q^{-s} \,|\, \mathrm{H}^0(\overline{C}, \iota_{\alpha *}\mathcal{F}_{\alpha,i})\right)^{-1}\right)\right| \cdot \log\zeta_K(s)^{-1}\,.$$

For $s > 1$, we have the following inequality:

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)} \cdot \log\left(\det\left(\mathrm{Id} - \mathrm{Fr}_q \cdot q^{-s} \,|\, \mathrm{H}^0(\overline{C}, \iota_{\alpha *}\mathcal{F}_{\alpha,i})\right)^{-1}\right)\right| \cdot \log\zeta_K(s)^{-1} \\
&= \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{\xi} \overline{\xi}(\sigma) \cdot \log\left((1-\xi(\mathrm{Fr}_q)q^{-s})^{-1}\right)\right| \cdot \log\zeta_K(s)^{-1} \\
&= \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \sum_{\sigma\in Z_\alpha} \sum_{n\ge 1} \frac{1}{n} q^{-ns} \sum_{\xi} \overline{\xi}(\sigma) \cdot \xi(\mathrm{Fr}_q^n) \cdot \log\zeta_K(s)^{-1} \\
&= \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \sum_{\sigma\in Z_\alpha} \sum_{\substack{n\ge 1,\ n\equiv a_\sigma \\ (\mathrm{mod}\ c_\alpha)}} \frac{c_\alpha}{n} \cdot q^{-ns} \cdot \log\zeta_K(s)^{-1} \\
&= \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \sum_{\sigma\in Z_\alpha} \frac{c_\alpha}{a_\alpha} q^{-a_\sigma s} \cdot \log\zeta_K(s)^{-1} \\
&\qquad + \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \sum_{\sigma\in Z_\alpha} \sum_{m\ge 1} \frac{c_\alpha}{a_\sigma + mc_\alpha} q^{-(a_\sigma + mc_\alpha)s} \cdot \log\zeta_K(s)^{-1}
\end{aligned}$$

(3.2.1) Remark The above argument shows that

$$\sum_{\sigma\in Z_\alpha} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)} \cdot \log\left(\det\left(\mathrm{Id} - \mathrm{Fr}_q \cdot q^{-s} \,|\, \mathrm{H}^0(\overline{C}, \iota_{\alpha *}\mathcal{F}_{\alpha,i})\right)^{-1}\right) \ge 0\,,$$

hence the absolute value symbols in the displayed formulae above can be omitted.

(3.2.2) Remark (i) Because

$$\lim_{s\to 1^+} \frac{\log\zeta_K(s)}{\log\left((1-q^{1-s})^{-1}\right)} = 1\,,$$

one can replace $\log\zeta_K(s)$ by $\log((1-q^{1-s})^{-1})$ in our considerations.

(ii) We have

$$\sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \sum_{\sigma\in Z_\alpha} \sum_{m\ge 1} \frac{c_\alpha}{a_\sigma + mc_\alpha} q^{-(a_\sigma + mc_\alpha)s} \le \sum_{\alpha\in\mathcal{P}} \frac{|Z_\alpha|}{|G_\alpha|} \cdot \log\left((1-q^{-s})^{-1}\right) \quad \forall\, s \ge 1\,.$$

(3.3) The contribution from H^2 can be estimated by the same method used for H^0. For each one-dimensional $\overline{\mathbb{Q}_\ell}$-valued character ξ of $\mathrm{Gal}(\kappa_\alpha/\kappa)$, $\mathrm{H}^2(\overline{C}, \mathcal{G}_\xi)$ is one-dimensional; the geometric Frobenius Fr_q operates on it as $q\xi(\mathrm{Fr}_q$. The contribution from H^2, to the upper bound of

$$\sum_{\alpha\in\mathcal{P}} \sum_{\substack{v\in\Sigma'_K \\ \left(\frac{K_\alpha/K}{\wp v}\right)\subseteq Z_\alpha}} \frac{\mathrm{N}_v^{-s}}{\log\zeta_K(s)}$$

is

$$\sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)} \cdot \log\left(\det\left(\mathrm{Id} - \mathrm{Fr}_q\cdot q^{-s} \,|\, \mathrm{H}^2(\overline{C}, \iota_{\alpha *}\mathcal{F}_{\alpha,i})\right)^{-1}\right)\right| \cdot \log\zeta_K(s)^{-1}\,.$$

The same method used in 3.2 gives the following inequality:

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)} \cdot \log\left(\det\left(\mathrm{Id} - \mathrm{Fr}_q\cdot q^{-s} \,|\, \mathrm{H}^2(\overline{C}, \iota_{\alpha *}\mathcal{F}_{\alpha,i})\right)^{-1}\right)\right| \cdot \log\zeta_K(s)^{-1} \\
&= \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{\xi} \bar{\xi}(\sigma)\cdot\log\left((1 - q\cdot\xi(\mathrm{Fr}_q)q^{-s})^{-1}\right)\right| \cdot \log\zeta_K(s)^{-1} \\
&= \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \sum_{\sigma\in Z_\alpha} \frac{c_\alpha}{a_\alpha}\, q^{a_\sigma(1-s)} \cdot \log\zeta_K(s)^{-1} \\
&\quad + \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \sum_{\sigma\in Z_\alpha} \sum_{m\geq 1} \frac{c_\alpha}{a_\sigma + mc_\alpha}\, q^{(a_\sigma + mc_\alpha)(1-s)} \cdot \log\zeta_K(s)^{-1}
\end{aligned}$$

(3.3.1) Remark (i) Just as in 3.2,

$$\sum_{\sigma\in Z_\alpha} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)} \cdot \log\left(\det\left(\mathrm{Id} - \mathrm{Fr}_q\cdot q^{-s} \,|\, \mathrm{H}^2(\overline{C}, \iota_{\alpha *}\mathcal{F}_{\alpha,i})\right)^{-1}\right) \geq 0\,,$$

so the absolute value symbols in the displayed formulae above can be omitted.

(ii) We have

$$\sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \sum_{\sigma\in Z_\alpha} \sum_{m\geq 1} \frac{c_\alpha}{a_\sigma + mc_\alpha}\, q^{(a_\sigma + mc_\alpha)(1-s)} \leq \sum_{\alpha\in\mathcal{P}} \frac{|Z_\alpha|}{|G_\alpha|} \cdot \log\left((1 - q^{1-s})^{-1}\right) \quad \forall\, s > 1\,.$$

(3.3.2) Remark (i) In 3.2 and 3.3, we used the basic character identity for finite abelian groups, which takes care of the cancellation effect in the contributions from H^0 and H^2 respectively.

(ii) The contribution from H^0 is dominated by the contribution from H^2. Each contribution was written as a sum of a "dominant term" and an "error term". The error term for H^0 is dominated by the error term for H^2.

(3.4) In this section we estimate the contribution from H^1 to the upper bound of

$$\sum_{\alpha\in\mathcal{P}} \sum_{\substack{v\in\Sigma'_K \\ \left(\frac{K_\alpha/K}{\wp_v}\right)\subseteq Z_\alpha}} \frac{\mathrm{N}_v^{-s}}{\log\zeta_K(s)}$$

in 2.3.

(3.5) A well-known fact from the character theory of finite groups tells us that

$$\sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(1)}\,\chi_{\alpha,i} = \chi_{\mathrm{reg},G_\alpha}$$

where $\chi_{\mathrm{reg},G_\alpha}$ denotes the character of the regular representation of G_α. Therefore

$$\oplus_{i=1}^{m_\alpha} \iota_{\alpha*}\mathcal{F}_{\alpha,i}^{\oplus\overline{\chi_{\alpha,i}(1)}} \cong \iota_{\alpha*}\mathcal{F}_{\mathrm{reg},G_\alpha}\,,$$

where $\mathcal{F}_{\mathrm{reg},G_\alpha}$ denotes the smooth $\overline{\mathbb{Q}_\ell}$-sheaf on U_α attached to the regular representation of G_α. Since $\mathcal{F}_{\mathrm{reg},G_\alpha}$ is canonically isomorphic to $(\pi_\alpha^{-1}(U_\alpha)\to U_\alpha)_*\overline{\mathbb{Q}_\ell}$, it follows that

$$\sum_{i=1}^{m_\alpha} \overline{\chi_{\sigma,i}(1)}\cdot \mathrm{H}^1(\overline{C},\iota_{\alpha,i_*}\mathcal{F}_{\alpha,i}) = [\kappa_\alpha:\kappa]\cdot g(C_\alpha)$$

where $g(C_\alpha)$ denotes the genus of $C_\alpha\times_{\mathrm{Spec}\,\kappa_\alpha}\mathrm{Spec}\,\kappa$.

(3.6) The contribution from H^1 to the upper bound of

$$\sum_{\alpha\in\mathcal{P}} \sum_{\substack{v\in\Sigma'_K \\ \left(\frac{K_\alpha/K}{\wp_v}\right)\subseteq Z_\alpha}} \frac{\mathrm{N}_v^{-s}}{\log\zeta_K(s)}$$

in 2.3 is

$$\sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|}\left|\sum_{\sigma\in Z_\alpha}\sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)}\cdot\log\Big(\det\Big(\mathrm{Id}-\mathrm{Fr}_q\cdot q^{-s}\,|\,\mathrm{H}^1(\overline{C},\iota_{\alpha*}\mathcal{F}_{\alpha,i})\Big)\Big)\right|\cdot\log\zeta_K(s)^{-1}\,.$$

We use the conclusion in 3.5 and the argument in 3.2, 3.3 to estimate the contribution from H^1. Let $\{\omega_{\alpha,i,j}\}_{1\le j\le\dim(\mathrm{H}^1(\overline{C},\iota_{\alpha*}\mathcal{F}_{\alpha,i}))}$ be the eigenvalues of Fr_q on $\mathrm{H}^1(\overline{C},\iota_{\alpha*}\mathcal{F}_{\alpha,i})$, with multiplicity. By [Weil II], the complex absolute values of any $\omega_{\alpha,i,j}$ is equal to $\sqrt{q}$. The same argument gives

$$\sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{i=1}^{m_\alpha} \overline{\chi_{\alpha,i}(\sigma)} \cdot \log\left(\det\left(\mathrm{Id} - \mathrm{Fr}_q \cdot q^{-s} \,|\, \mathrm{H}^1(\overline{C}, \iota_{\alpha*}\mathcal{F}_{\alpha,i})\right)\right) \right| \cdot \log\zeta_K(s)^{-1}$$

$$= \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \left| \sum_{\sigma\in Z_\alpha} \sum_{\substack{n\geq 1, n\equiv a_\sigma \\ (\mathrm{mod}\ c_\alpha)}} \sum_{i,j} \overline{\chi_{\alpha,i}(\sigma)} \cdot \frac{1}{n} \cdot \omega_{\alpha,i,j}^n \cdot q^{-ns} \right| \cdot \log\zeta_K(s)^{-1}$$

$$\leq \sum_{\alpha\in\mathcal{P}} \frac{1}{|G_\alpha|} \sum_{\sigma\in Z_\alpha} \sum_{\substack{n\geq 1, n\equiv a_\sigma \\ (\mathrm{mod}\ c_\alpha)}} g(C_\alpha) \cdot c_\alpha \cdot \frac{1}{n} \cdot q^{n(\frac{1}{2}-s)} \cdot \log\zeta_K(s)^{-1}$$

$$\leq \frac{1}{1-q^{-\frac{1}{2}}} \cdot \sum_{\alpha\in\mathcal{P}} \frac{g(C_\alpha)\cdot c_\alpha}{|G_\alpha|} \frac{1}{a_\sigma} \cdot q^{a_\sigma(\frac{1}{2}-s)} \cdot \log\zeta_K(s)^{-1}$$

for all $s > 1$. In the last inequality we used a geometric series to get a simple estimate.

Combining the estimates in 3.2, 3.3, 3.6, and Remarks in 3.2.2, 3.3.1, 1.3, we obtain the following solution to the abstract Artin problem.

(3.7) Theorem *Notation as in 2.1. Recall that* $n_\alpha = |G_\alpha|$, $s_\alpha = |Z_\alpha|$, $c_\alpha = [\kappa_\alpha : \kappa]$, *for any* $\alpha \in \mathcal{P}$, *and* $a_\sigma = \mathrm{Min}\,\{\, r \in \mathbb{N}_{>0} : \sigma|_{\kappa_\alpha} = (\mathrm{Fr}_q)^r \,\}$, *for* $\sigma \in Z_\alpha$. *Suppose that the following conditions hold.*

(1) $\displaystyle\sum_{\alpha\in\mathcal{P}} \frac{s_\alpha}{n_\alpha} = \sum_{\alpha\in\mathcal{P}} \frac{|Z_\alpha|}{|G_\alpha|} < \infty$

(2) *There exists a positive constant* M *such that* $g(C_\alpha) \leq M_\alpha \cdot \dfrac{n_\alpha}{c_\alpha}$ *for all* $\alpha \in \mathcal{P}$.

(3) *There exists a real number* $s_0 > 1$ *such that the infinite series*

$$\sum_{\alpha\in\mathcal{P}} \sum_{\sigma\in Z_\alpha} \frac{1}{n_\alpha} \frac{c_\alpha}{a_\sigma} \frac{q^{a_\sigma(1-s)}}{|\log(1-q^{(1-s)})|}$$

is uniformly convergent for $1 < s \leq s_0$.

(4) $\displaystyle\sum_{\alpha\in\mathcal{P}} \sum_{\sigma\in Z_\alpha} \frac{1}{a_\sigma} q^{-\frac{a_\sigma}{2}} < \infty$

Then the infinite series

$$\sum_{\alpha\in\mathcal{P}} \sum_{\substack{v\in\Sigma'_K \\ \left(\frac{K_\alpha/K}{\wp v}\right)\cap Z_\alpha\neq\emptyset}} \mathrm{N}_v^{-s} \cdot |\log(1-q^{(1-s)})|^{-1}$$

is uniformly convergent for $1 < s \leq s_0$. *Consequently, the Dirichlet density* $d(S)$ *exists, and*

$$d(S) = \lim_{J\to\infty} |S_J| = \lim_{J\to\infty} \sum_{I\subseteq J} (-1)^{|I|}\, d(M_I)\,.$$

(3.7.1) Remark (i) Thm. 3.7 specializes to [CK, Thm. 2.1] when Z_α consists of the identity element of G_α for every α, because $a_\sigma = c_\alpha$ for every $\sigma \in Z_\alpha$.

(ii) Thm. 3.7 is weaker than the the statement of [CK, Thm. 4.1] in two aspects. First, condition (4) of 3.7 is stronger than condition (2) of [CK, Thm. 4.1], which in our notation requires that $\sum_{\alpha\in\mathcal{P}} \sum_{\sigma\in Z_\alpha} \frac{1}{c_\alpha} q^{-\frac{a_\sigma}{2}} < \infty$. Second, condition (3) of 3.7 does not appear in [CK, Thm. 4.1].

(iii) In [CK, 4.1], they defined $a(\nu)$, equal to a_σ in our notation, to be *any* positive integer r such that the restriction of Fr_q^r to κ_α is equal to the restriction of σ to κ_α, so that $a(\nu)$ is uniquely determined only modulo $c(\nu)$, equal to $c(\sigma)$ in our notation. That definition makes condition (2) of [CK, 4.1] void, for one can increase each $a(\nu)$ by sufficiently large multiples of $c(\nu)$ to make sure that convergence condition is satisfied.

References

[Bi] H. Bilharz. Primdivisoren mit vorgegebener Primitivwurzel. *Math. Ann.*, 114:476–492, 1937.

[CK] D. A. Clark and M. Kuwata. Generalized Artin's conjecture for primitive roots and cyclicity mod $\wp$ of elliptic curves over function fields. *Canadian Math. Bulletin*, 38:167–173, 1995.

[Weil I] P. Deligne. Le conjecture de Weil, I. *Publ. Math. IHES*, 44:5–47, 1975.

[Weil II] P. Deligne. Le conjecture de Weil, II. *Publ. Math. IHES*, 52:138–252, 1980.

[GM1] R. Gupta and M. R. Murty. A remark on Artin's conjecture. *Inv. Math.*, 78:127–130, 1984.

[Ho1] C. Hooley. On Artin's conjecure. *J. reine angew. Math.*, 225:209–220, 1967.

[Kz] N. M. Katz. *Gauss Sums, Kloosterman Sums and Monodromy Groups.* Annals of Math. Studies, vol. 116, Princeton Univ. Press, 1988.

[Mur1] M. R. Murty. Artin's conjecture for primitive roots. *Math. Intelligencer*, 10:59–67, 1988.

[Mur2] M. R. Murty. On Artin's conjecture. *J. Number Theory*, 16:147–168, 1983.

[Vi] A. I. Vinogradov. Artin L-series and his conjectures. *Proc. Steklov Inst. Math.*, 112:124-142, 1971.