Conference on Arithmetic Algebraic Geometry

In honor of

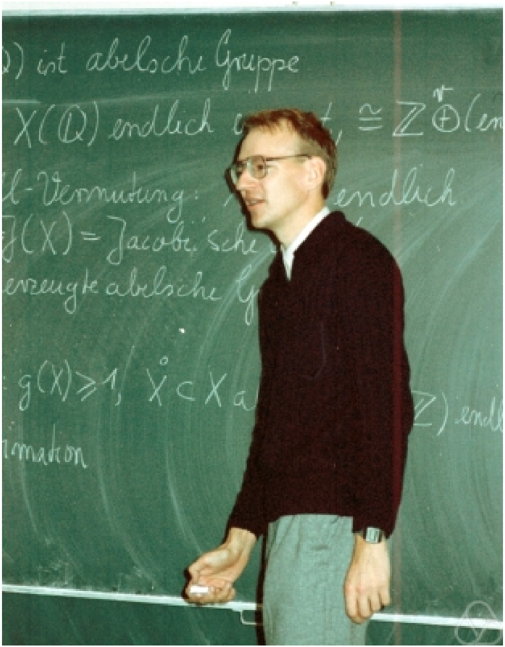
Gerd Faltings

on the occasion of his 60th birthday

六十而耳順

(I knew the truth in all I heard when I turned sixty. Confucious)

(Your hearing gets better when you turn 60. modern translation, Shouwu Zhang.)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SUSTAINED p-DIVISIBLE GROUPS

Ching-Li Chai

Department of Mathematics
University Pennsylvania

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Outline

Leaves in PEL moduli spaces

The notion of leaves, due to F. Oort, was announced in 1999 (Texel).

p= a prime number

$\mathscr{M}$ = a moduli space of PEL type over a field $k = k^{\mathrm{a}} \supset \mathbb{F}_p$, assumed (for simplicity) to have good reduction at p.

$\mathscr{A} \to \mathscr{M}$: universal abelian scheme with additional structure

Definition. The leaf $\mathscr{C}(x_0) \subset \mathscr{M}$ through a point $x_0 \in \mathscr{M}(k)$ is the locus (with reduced structure) in $\mathscr{M}$ where the p-divisible group $\mathscr{A}[p^\infty]$ (with additional structure) is isomorphic to the fiber over x_0.

Fact: $\mathscr{C}(x_0)$ is a *locally closed smooth* subscheme of $\mathscr{M}$.

Remark. (a) Leaves are "complementary" to Rapoport-Zink spaces. (b) They are stable under prime-to-p Hecke corresp.

Geometrically fiberwise constant p-divisible groups

The definition of leaves was based on:

Definition. A p-divisible group X over a scheme $S/\mathbb{F}_p$ is geometrically fiberwise constant if any two fibers X_{s_1}, X_{s_2} are isomorphic when based-changed to a common algebraically closed field K which contains both $\kappa(s_1)$ and $\kappa(s_2)$.

Remark: (i) The above notion results in reasonable properties when the base scheme S is *normal*. (Zink, Oort-Zink)

(ii) A disadvantage: cannot use it to study differential properties of leaves.

An update (with Frans Oort)

Goals.

1. Find a *scheme-theoretic substitute* of the notion of "geometrically fiberwise constant" p-divisible groups.

2. Perform "differential analysis" to study the local structure of leaves. The big picture that emerged is:

> *locally every leaf is "built up from p-divisible groups" via dévissage.*

(These are "local Serre-Tate coordinates on leaves"—but only in equi-characteristic $p > 0$ in general.)

Goals, continued

3. The above local structure on leaves vary algebraically over the base (e.g. a moduli space).

The resulting "global Serre-Tate coordinates" allows one to prove some weak rigidity statement of the following sort:

(weak rigidity) Let $Z \subset \mathscr{C}$ be a connected irreducible closed subscheme of a leaf $\mathscr{C}$. Suppose that $Z^{/z_0}$ is Tate-linear at a closed point z_0 of a leaf $\mathscr{C}$, in the sense that it is built up from p-divisible subgroups of those involved in the local structure of $\mathscr{C}^{/z_0}$. Then Z is Tate-linear at every point of Z.

Strongly κ-sustained p-divisible groups

$\kappa \supset \mathbb{F}_p$: a field of char. $p > 0$
S/κ: a κ-scheme
X_0/κ: a p-divisible group

Definition. A p-divisible group $X \to S$ is strongly κ-sustained modeled on X_0 if $\forall n > 0$, the Isom-scheme

$$\mathscr{I}\mathrm{SOM}_S(X_0[p^n] \times_{\mathrm{Spec}(\kappa)} S, X[p^n]) \longrightarrow S$$

is faithfully flat.[1]

Motivation: Manin's thesis (1963)

[1] A *sustained note* feels constant, but isn't really so.

κ-sustained p-divisible groups

S/κ: as before

Definition. A p-divisible group $X \to S$ is κ-sustained if either of the following two equivalent conditions hold.

(1) $\exists$ extension field L/κ and a p-divisible group Y_0/L such that

$$X \times_{\operatorname{Spec}\kappa} \operatorname{Spec} L \longrightarrow S \times_{\operatorname{Spec}\kappa} \operatorname{Spec} L$$

is strongly L-sustained modeled on Y_0/L.

(2) $\forall n > 0$, the Isom-scheme

$$\mathscr{I}\mathrm{SOM}_{S\times_{\operatorname{Spec}\kappa}S}(\mathrm{pr}_1^* X[p^n], \mathrm{pr}_2^* X[p^n]) \longrightarrow S \times_{\operatorname{Spec}\kappa} S$$

is faithfully flat.

Existence of slope filtration

(an indication that the definition is reasonable)

Proposition. Let $X \to S/\kappa$ be a κ-sustained p-divisible group. There exists a unique slope filtration[2]

$$X = X_0 \supset X_1 \supset \cdots \supset X_{m-1} \supset X_m = (0)$$

of X be p-divisible subgroups such that

(1) X_i/X_{i+1} is a κ-sustained p-divisible group, isoclinic of slope λ_i $\forall i = 0, 1, \ldots, m-1$

(2) $0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_{m-1} \le 1$.

[2] due to Zink and Oort-Zink over reduced base

Compatibility with the notion "geom. fiberwise constant"

Proposition. Suppose that S is reduced and locally of finite type over κ. Let $X \to S/\kappa$ and X_0/κ be p-divisible groups. If X_s is strongly κ-sustained modeled on X_0 $\forall s \in S$, then $X \to S$ is strongly κ-sustained modeled on X_0/κ.

Proposition. Let $X \to S/\kappa$ and X_0/κ be p-divisible groups. Assume that S/κ is locally noetherian. There exists a locally closed subscheme $T \hookrightarrow S$ satisfying

(a) $X \times_S T$ is κ-sustained modeled on X_0.

(b) If $T_1 \hookrightarrow S$ is a locally closed subschema of S and $X \times_S T_1$ is κ-sustained modeled on X_0, then $T_1 \subseteq T$.

Hom-schemes for p-divisible groups

Interlude: certain (systems of) $\mathscr{H}\mathrm{om}$-schemes

Given Y, Z: p-divisible groups over a field $\kappa \supset \mathbb{F}_p$.
Define group schemes of finite type over κ

$$H_n := \mathscr{H}om(Y[p^n], Z[p^n])$$

We have arrows

- $r_{i,n+i} \colon H_{n+i} \to H_n$ (restriction homomorphism)
- $\iota_{n+i,i} \colon H_i \hookrightarrow H_{n+i}$ (induced by $[p^n]_{H_{n+i}}$, $= \mathrm{Ker}(r_{i,n+i})$)
- $\nu_{n;i} \colon H_{n+i}/\iota_{n+i,i}(H_i) \rightarrowtail H_n$ ($r_{i,n+i}$ mod its kernel)
- $\nu_{n;i,i+j} \colon H_{n+i+j}/\iota_{n+i+j,i+j}(H_{i+j}) \rightarrowtail H_{n+i}/\iota_{n+i,i}(H_i)$ (induced by $r_{n+i+j,n+i}$)

Stabilization

$$\begin{array}{ccccccc}
 & & & & H_{n+i+j}/\iota_{n+i+j,i+j}(H_i) & \xhookrightarrow{\nu_{n;i+j}} & H_n \\
 & & & & \big\downarrow{\scriptstyle \nu_{n;i,i+j}} & & \big\downarrow{\scriptstyle =} \\
 & & & & H_{n+i}/\iota_{n+i,i}(H_i) & \xhookrightarrow{\nu_{n;i}} & H_n \\
 & & & & \big\uparrow & & \big\downarrow{\scriptstyle =} \\
0 & \longrightarrow & H_i & \xrightarrow{\iota_{n+i,i}} & H_{n+i} & \xrightarrow{r_{i,n+i}} & H_n
\end{array}$$

Lemma. There exists a positive integer n_0 such that the monomorphism

$$\nu_{n;1,2} \colon H_{n+2}/\iota_{n+2,n+1}(H_{n+1}) \rightarrowtail H_{n+1}/\iota_{n+1,n}(H_n)$$

is an isomorphism $\forall n \geq n_0$.

The stabilized $\mathscr{H}\mathrm{om}'_{\mathrm{div}}$ for p-divisible group

Definition. We define a family of commutative finite group schemes G_n over κ and and κ homormorphisms.

- $G_n := H_{n+n_0}/\iota_{n+n_0,n_0}(H_{n_0})$
- $\nu_n : G_n \rightarrowtail H_n$, defined by $\nu_n = \nu_{n;i_0}$
- $j_{n+i,n} : G_n \hookrightarrow G_{n+i}$, induced by ι_{n+i+n_0}
- $\pi_{n,n+i} : G_{n+i} \twoheadrightarrow G_n$, induced by $r_{n+n_0,n+n_0+i}$

Proposition. The system $(G_n, j_{n+i,n}, \pi_{n,n+i})$ is p-divisible group over κ, denoted by

$$\mathscr{H}\mathrm{om}'_{\mathrm{div}}(Y,Z)$$

$\mathscr{H}om'_{\mathrm{div}}(Y,Z)$ and $\mathscr{H}om'(Y,Z)$

Definition.

- $\mathscr{H}om'(Y,Z)$ = the inductive system $(H_n, \iota_{n+i,n})$
- The monomorphisms $\nu_n : G_n \rightarrowtail H_n$ define a mono $\nu : \mathscr{H}om'_{\mathrm{div}}(Y,Z) \rightarrowtail \mathscr{H}om'(Y,Z)$.

Proposition. (1) $\mathscr{H}om'_{\mathrm{div}}(Y,Z)$= the maximal p-divisible subgroup of $\mathscr{H}om'(Y,Z)$.

(2) The formal completion $\mathscr{H}om'(Y,Z)^\wedge$ of $\mathscr{H}om'(Y,Z)$ is a smooth formal group over κ.

(3) $\dim(\mathscr{H}om'(Y,Z)^\wedge) = \dim(Z) \cdot \dim(Y^t)$.

Slopes and dimension of $\mathscr{H}om'_{\mathrm{div}}(Y,Z)$

Proposition. Suppose that Y,Z are isoclinic over κ, with slopes λ_Y and λ_Z respectively.

- If $\lambda_Y > \lambda_Z$, then $\mathscr{H}om'_{\mathrm{div}}(Y,Z) = (0)$.
- If $\lambda_Y \leq \lambda_Z$, then $\mathscr{H}om'_{\mathrm{div}}(Y,Z)$ is isoclinic of slope $\lambda_Z - \lambda_Y$ and height $\mathrm{ht}(Z)\cdot\mathrm{ht}(Y)$.

Formal groups defined by Ext

Definition. Given p-divisible groups Y, Z over a field $\kappa \supset \mathbb{F}_p$, define a group-valued formal functor $\mathscr{E}\mathfrak{x}\mathfrak{T}(Y,Z)$, which sends every augmented Artinian local κ-algebra $(R, \varepsilon: R \to \kappa)$ to

$$\mathscr{E}\mathfrak{x}\mathfrak{T}(Y,Z) := \mathrm{Ker}\left(\mathrm{Ext}_R(Y_R, Z_R) \to \mathrm{Ext}_\kappa(Y_\kappa, Z_\kappa)\right)$$

Proposition. $\mathscr{E}\mathfrak{x}\mathfrak{T}(Y,Z)$ is formally smooth of dimension $\dim(Z) \cdot \dim(Y^t)$ over κ.

Definition. $\mathscr{E}\mathfrak{x}\mathfrak{T}_{\mathrm{div}}(Y,Z)$:= the maximal p-divisible subgroup of the smooth formal group $\mathscr{E}\mathfrak{x}\mathfrak{T}(Y,Z)$.

Relating $\mathscr{H}\mathrm{om}'(Y,Z)$ to $\mathscr{E}\mathrm{xt}(Y,Z)$

Proposition. There is a natural isomorphism

$$\delta : \mathscr{H}\mathrm{om}'(Y,Z)^{\wedge} \xrightarrow{\sim} \mathscr{E}\mathrm{xt}(Y,Z),$$

which induces an isomorphism

$$\mathscr{H}\mathrm{om}'_{\mathrm{div}}(Y,Z) \xrightarrow{\sim} \mathscr{E}\mathrm{xt}_{\mathrm{div}}(Y,Z).$$

Remark. (a) δ is a coboundary map coming from
$0 \to \varprojlim_n Y[p^n] \to \left(\varprojlim_n Y[p^n]\right) \otimes \mathbb{Z}[1/p] \to Y \to 0$
(b) There is a canonical/tautological biextension of $(Y, \mathscr{H}\mathrm{om}'_{\mathrm{div}}(Y,Z))$ by Z.

The Dieudonné module of $\mathscr{H}om'_{\mathrm{div}}(Y,Z)$

Assume: $\lambda_Y < \lambda_Z$ and κ is perfect.
Let M, N be the Cartier modules of Y and Z.

Let $H := \mathrm{Hom}_{W(\kappa)}(M,N)$.
Have semi-linear actions on $H \otimes_{W(k)} W(\kappa)[1/p]$

$$F_H : h \mapsto F_N \circ h \circ V_M, \quad V_H : h \mapsto p^{-1} V_N \circ h \circ F_M.$$

Let $H_1 :=$ the largest $W(\kappa)$-submodule of H stable under F_H and V_H

Proposition. The Cartier module of the p-divisible group $\mathscr{H}om'_{\mathrm{div}}(Y,Z)$ is naturally isomorphic to H_1.

Remark. The smallest submodule of $H \otimes_{W(\kappa)} W(k)[1/p]$ which contains H and stable under both F_H and V_H is the Cartier module of the maximal p-divisible quotient of $\mathscr{H}om'(Y,Z)^\wedge$.

The Cartier module of $\mathscr{H}\mathrm{om}'(Y,Z)$

Definition. Let $\mathrm{BC}_p(\kappa)$ = the Cartier module of the infinite dimensional connected smooth formal group

$$R \mapsto \mathrm{Ker}(\mathrm{Cart}_p(R) \to \mathrm{Cart}_p(\kappa))$$

$\forall$ augmented Artinian commutative κ-algebra R.

$\mathrm{Cart}_p(\kappa)$ is a triple $\mathrm{Cart}_p(\kappa)$-module: two from the Cartier ring structure; the third because it is the Cartier module of a smooth formal group.

Proposition. The Cartier module of $\mathscr{H}\mathrm{om}'(Y,Z)^\wedge \simeq \mathscr{E}\mathrm{xt}(Y,Z)$ is

$$\mathrm{Ext}^1_{\mathrm{Cart}_p(\kappa)}(M, \mathrm{BC}_p(\kappa) \otimes_{\mathrm{Cart}_p(\kappa)} N)$$

with action by $\mathrm{Cart}_p(\kappa)$ via the "third" $\mathrm{Cart}_p(\kappa)$-module structure on $\mathrm{BC}_p(\kappa)$.

An example of slopes $1/3$ and $4/5$

Suppose that $\lambda_Y = 1/3$, $\dim(Y) = 1$ $\lambda_Z = 4/5$, $\dim(Z) = 4$, and κ is alg. closed.

A computation with Cartier modules shows that:

$$\mathscr{H}om_{\text{big site}}(Y,Z) = \mathrm{Spec}(\tilde{R}),$$

where

$$\tilde{R} = \kappa[t_0,t_1,t_2,\ldots] \Big/ \left((t_0^p, t_1^p, t_2^{p^2}, t_3^{p^2}, t_4^{p^3}, t_5^{p^3}, t_6^{p^5}) + (t_{i+7}^{p^5} - t_i)_{i\geq 0} \right)$$

Remark. $\tilde{R}$ looks like being 7-dimensional in some sense ..., but why 7?

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answer

$$\mathrm{Spec}(\tilde{R}) = \varprojlim_{n} \mathscr{H}om'_{\mathrm{div}}(Y,Z)[p^n]$$

and 7 is the numerator of $\frac{4}{5} - \frac{1}{3}$. (This computation also serves as a reality check.)

How to build a sustained p-divisible group?

Sketchy answer:

1. A purported κ-sustained p-divisible group $X \to S$ must satisfy:

(a) $\exists$ a slope filtration

$$X = X_0 \supset X_1 \supset \cdots \supset X_{m-1} \supset X_m = (0)$$

such that each X_i/X_{i+1} is κ-sustained and isoclinic, and

$$\text{slope}(X_i/X_{i+1}) < \text{slope}(X_{i+1}/X_{i+2})\ \forall i.$$

(b) All extension classes involved are of the form

constant + p-divisible

in the flat topology (suitably interpreted).

2. The above conditions (a), (b) are also sufficient: a successive extension of κ-sustained isoclinic p-divisible groups by extension classes satisfying condition (b) is a κ-sustained p-divisible group.

The two-slope case explained

Notation

- S/κ: a κ-scheme
- $0 \to Z \to X \to Y \to 0$: a short exact sequence of p-divisible groups over κ, such that $Y \to S$ and $Z \to S$ are κ-sustained, $\lambda_Y < \lambda_Z$
- $S_1 := S \times_{\mathrm{Spec}\,\kappa} S$, $\mathrm{pr}_1, \mathrm{pr}_2 : S_1 \to S$
- $\mathscr{I}\mathrm{som}_{Y,n} = \mathscr{I}\mathrm{som}(\mathrm{pr}_1^* Y[p^n], \mathrm{pr}_2^* Y[p^n])$ = stabilized Isom-scheme over S_1; similarly for $\mathscr{I}\mathrm{som}_{Z,n}$
- $q_n : \mathscr{I}\mathrm{som}_{Y,n} \times_{S_1} \mathscr{I}\mathrm{som}_{Z,n} \to S_1$: structural map
- Over $\mathscr{I}\mathrm{som}_{Y,n} \times_{S_1} \mathscr{I}\mathrm{som}_{Z,n}$, have a tautological isomorphism $\tau_{N,n} : \mathrm{Ext}(q_N^* \mathrm{pr}_1^* Y[p^n], q_N^* \mathrm{pr}_1^* Z[p^n]) \xrightarrow{\sim} \mathrm{Ext}(q_N^* \mathrm{pr}_2^* Y[p^n], q_N^* \mathrm{pr}_2^* Z[p^n])$

The two slope case, continued

Definition. We say that $X \to S$ is κ-firm with two slopes if $\forall N \geq n$, $\exists$ an fppf morphism $g : T \to \mathscr{I}\mathrm{som}_{Y,n} \times_{S_1} \mathscr{I}\mathrm{som}_{Z,n}$ such that

$$g^* \begin{pmatrix} \tau_{N,n}([q_N^* \mathrm{pr}_1^*(0 \to Z \to X \to Y \to 0)]) \\ - [q_N^* \mathrm{pr}_2^*(0 \to Z \to X \to Y \to 0)] \end{pmatrix} = 0$$

(i.e. the difference between the two extension classes, pulled back by pr_1^* and pr_2^*, is p-divisible fppf-locally on the base.)

Definition. Replacing S_1 by $S_1^{/\Delta(S)}$ = formal completion along the diagonal, we get a weaker notion of infinitesimally κ-firm p-divisible group with two slopes.

The two slope case, continued

Proposition. Let X be a p-divisible group over S/κ with a slope filtration $0 \to Z \to X \to Y \to 0$, where Y, Z are κ-sustained isoclinic with $\mathrm{slope}(Y) < \mathrm{slope}(Z)$.

(a) X is κ-sustained with if and only if it is κ-firm.

(b) Suppose S is of finite type over κ. Then X is κ-sustained with if and only if it is infinitesimally κ-firm.

local structure of leaves: two-slope case

Proposition. Let Y_0, Z_0 be isoclinic p-divisible groups over a perfect base field $\kappa \supset \mathbb{F}_p$. Let $X_0 = Y_0 \times Z_0$. Inside the equi-characteristic deformation space $\mathscr{D}\mathrm{ef}(X_0)$ of X_0, the maximal κ-sustained locus modeled on X_0 is naturally isomorphic to $\mathscr{H}\mathrm{om}'_{\mathrm{div}}(Y, Z)$.

Remark. Can/will also study global differential property of a leaf $\mathscr{C}$ using $(\mathscr{C} \times \mathscr{C})^{/\Delta(\mathscr{C})}$.

Application 0: leaves in moduli spaces

Let $\mathscr{M}$ be a PEL moduli space over an alg. closed field $\kappa \supset \mathbb{F}_p$, with good reduction.

Let $(\mathscr{A}, *) \to \mathscr{M}$ be the universal abelian scheme, plus extra (PEL) structure.

Let $x_0 \in \mathscr{M}(\kappa)$ be a closed point of $\mathscr{M}$, and let $(A_0, *_0)$ be the fiber over x_0.

Let $\mathscr{C}(x_0)$ be the leaf passing through x_0, i.e. $\mathscr{C}(x_0)$ is the maximal locus in $\mathscr{M}$ over which $(\mathscr{A}[p^\infty], *[p^\infty])$ is strongly sustained modeled on $(A_0[p^\infty], *_0[p^\infty])$

proposition. $\mathscr{C}(x_0)$ is reduced and smooth over κ.

Differential analysis of leaves: two-slope case

$\mathscr{A}_g$ = moduli space of principally polarized g-dimensional abelian varieties over κ.

$x_0 = [(A_0, \lambda_0)] \in \mathscr{A}_g(\kappa)$: a closed point

$\mathscr{C}(x_0)$ = the leaf through x_0

Assume: A_0 has exactly two slopes λ_0 and $1 - \lambda_0$, $\lambda_0 < \frac{1}{2}$

Proposition. (1) $\mathrm{pr}_2 : (C_0 \times C_0)^{/\Delta(C_0)} \to C_0$ has a natural structure as a neutral torsor for an isoclinic sustained p-divisible group $G \to C_0$.

(2) $G \to C_0$ has height $g(g+1)/2$ and slope $1 - 2\lambda_0$.

A weak rigidity statement and a question

$C_0 \subset \mathscr{A}_g$: as in the previous slide

Let $Z \subset \mathscr{C}$ be an irreducible closed subscheme of C_0.

proposition. (weak rigidity) Suppose that Z is Tate-linear at a closed point $z_0 \in \mathscr{C}$, in the sense that $Z^{/z_0} \subset C^{/z_0}$ is a torsor for a p-divisible subgroup of G_{z_0}. Then Z is Tate-linear at every point of Z.

Question. Is Z the reduction of a Shimura subvariety of $\mathscr{A}_g$?

THE END

Happy 60th, Gerd!