HOW TO COMPUTE THE LUBIN-TATE ACTION

Ching-Li Chai

Department of Mathematics
University of Pennsylvania
and
Institute of Mathematics
Academia Sinica

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Outline

One dimensional formal group *laws*

a one-dimensional formal group law over a comm. ring R

= a one-dimensional comm. smooth formal group G over R
+ a rigidification $\mathrm{Spf}(R[[x]]) \xrightarrow{\sim} G$

= a formal power series $G(x,y) \in R[[x,y]]$ such that

- $G(x,y) = G(y,x)$
- $G(x,y) \equiv x + y \pmod{\text{degree } \geq 2}$
- $G(x, G(y,z)) = G(G(x,y),z)$

A homomorphism from $G(x,y)$ to $F(x,y)$ over R is (represented by) a formal power series $\phi(x) \in R[[x]]$ such that

$$F(\phi(x), \phi(y)) = \phi(G(x,y))$$

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The *height* of a one-dimensional formal group

Let $k \supset \mathbb{F}_p$ be a field of char. $p > 0$. Let $G(x,y)$ be a one-dim. formal group law over k.

$$[p]_G(x) = \begin{cases} 0 & & \text{height} = \infty \\ x^{p^h} & \pmod{x^{p^h+1}} & \text{height} = h \end{cases}$$

If $k = k^{\text{alg}}$, then $G(x,y)$ is determined by its height up to non-unique isomorphisms.

Examples.

- $\mathbb{G}_a(x,y) = x + y$, height = ∞ in char. $p > 0$
- $\mathbb{G}_m(x,y) = x + y + xy$, height = 1 in char. p.

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Notation: a Lubin-Tate formal group

Let h be a positive integer, fixed from now on. Let $\kappa_s = \mathbb{F}_{p^h}$.

- $g_h(x) := \sum_{j\in\mathbb{N}} p^{-j} x^{p^{jh}} = x + \frac{x^{p^h}}{p} + \frac{x^{p^{2h}}}{p^2} + \cdots$
- Define $G_{W(\kappa_s)} \in \mathbb{Z}_{(p)}[[x,y]] \subset W(\kappa_s)[[x,y]]$ by

$$G_{W(\kappa_s)}(x,y) := g_h^{-1}(g_h(x) + g_h(y))$$

Remark. $G_{W(\kappa_s)}$ is a Lubin-Tate formal group for $W(\mathbb{F}_{p^h})$: $\mathrm{End}_{W(\kappa_s)}(G_{W(\kappa_s)}) \simeq W(\mathbb{F}_{p^h})$

- Let G_s be the closed fiber of $G_{W(\kappa_s)}$; it is a one-dimensional formal group (law) over $\mathbb{F}_p$ of height h.
- It is well-known that $\mathrm{End}_{\kappa_s}(G_s)$ is the maximal order of $\mathrm{End}^0_{\kappa_s}(G_s)$ = a central division algebra over $\mathbb{Q}_p$ of dimension h^2. So $\mathrm{Aut}_{\kappa_s}(G_s) = \mathrm{End}_{\kappa_s}(G_s)^\times$ is a compact h^2-dimensional p-adic group with center $\mathbb{Z}_p{}^\times$.

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The Lubin-Tate deformation functor

Let $\mathbf{Art}_{\kappa_s}$ be the category whose objects consists of pairs $(R, \varepsilon : \kappa_s \to \kappa)$, where

- R is an Artinian commutative local ring,
- $\kappa = R/\mathfrak{m}_R$,
- ε is a ring homomorphism.

The deformation functor

$$\mathscr{D}ef(G_s) : \mathbf{Art}_{\kappa_s} \longrightarrow \mathbf{Sets}$$

sends each object $(R, \varepsilon : R \to \kappa)$ of $\mathbf{Art}_{\kappa_s}$ to the set of all isomorphism classes of pairs

$$\left(\Phi, \psi : \Phi \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa) \xrightarrow{\sim} \Phi_s \times_{\mathrm{Spec}(\kappa_s),\varepsilon} \mathrm{Spec}(\kappa)\right),$$

where Φ is a one-dimensional formal group over R.

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The Lubin-Tate moduli space

Equivalently, $\mathscr{D}ef(G_s)(R,\varepsilon)$ is the set of all $*$-isomorphism classes of one-dimensional formal group laws Φ over R whose closed fiber is $\varepsilon_* G_s$.

Recall: An isomorphism $\phi(x)$ from G_1 to G_2 over R is a $*$-isomorphism if $\phi(x) \equiv x \pmod{\mathfrak{m}_R}$.

Fact. $\mathscr{D}ef(G_s)$ is representable by a formal scheme $\mathscr{M}_h$ which is formally smooth over $W(\kappa_s)$ of relative dimension $h-1$. In other words there is a universal one-dimensional deformation Φ_{univ} over $\mathscr{M}_h$ such that every deformation of G_s over (R,ε) is the pull-back of Φ_{univ} via a unique morphism $\mathrm{Spf}(R) \to \mathscr{M}_h$.

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The Lubin-Tate action

The compact p-adic group $\mathrm{Aut}(G_0) = \mathrm{End}(G_0)^{\times}$ operates on $\mathscr{M}_h$ by "change of marking", as follows:

$$\gamma\colon\ [(\Phi,\psi)] \ \mapsto\ [(\Phi,\gamma\circ\psi)] \qquad \forall [(\Phi,\psi)] \in \mathscr{D}(G_s)((R,\varepsilon))$$

for any $\gamma \in \mathrm{Aut}(G_s)$ and any object (R,ε) in $\mathbf{Art}_{\kappa_s}$.

Equivalently, applying the above to the universal deformation Φ_{univ}: $\forall\, \gamma \in \mathrm{Aut}(G_s)$, we have a commutative diagram

$$\begin{array}{ccc}
\Phi_{\mathrm{univ}} & \xrightarrow{\ \tilde{\xi}(\gamma)\ } & \Phi_{\mathrm{univ}} \\
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where $\xi(\gamma)$ is an automorphism of $\mathscr{M}_h$ and $\tilde{\xi}(\gamma)$ is a formal group isomorphism with $\tilde{\xi}(\gamma)|_{G_s} = \gamma$.

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The Lubin-Tate action, continued

Remark. 1. This action $\gamma \mapsto \rho(\gamma)$ of $\mathrm{Aut}(G_0)$ on the Lubin-Tate moduli space $\mathscr{M}_h$ was first studied by Lubin and Tate in 1966. It is also known as (the essential part of) the *Morava stabilizer subgroup* action in chromatic homotopy theory.

2. If one passes to the divided power envelope

$$W(\kappa_s)[[w_1,\ldots,w_{h-1}]][w_i^n/n!]_{n\in\mathbb{N},i\leq h-1}$$

of the coordinate ring $W(\kappa_s)[[w_1,\ldots,w_{h-1}]]$, one can "linearize" the action by crystalline theory.
However this is not very useful for studying the action of $\mathrm{Aut}(G_s)$ on the characteristic p fiber of $\mathscr{M}_h$.

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Set-up with a Frobenius lifting

Fix a prime number p. Our base ring A is cast in a quadruple $(A, K, \mathfrak{a}, \sigma)$, where

- K is a *commutative* ring with 1,
- A is a subring of K containing 1,
- $\mathfrak{a} \subset A$ is an ideal of A, and
- $\sigma : K \to K$ is a ring endomorphism,

such that conditions (a)–(c) below are satisfied.

(a) $p \in \mathfrak{a}$,

(b) $\sigma(A) \subset A$,

(c) $\sigma(a) \equiv a^q \pmod{\mathfrak{a}}$ for all $a \in A$,

(d) $\sigma((A : \mathfrak{a})) \subset (A : \mathfrak{a})$, where $(A : \mathfrak{a}) := \{y \in K \mid y \cdot \mathfrak{a} \subset A\}$.

A twisted power series ring

Let $K[[\partial]]_\sigma$ be the ring of formal power series in ∂ with coefficients in K such that $\partial b = \sigma(b)\partial$ for all $b \in K$, i.e.

$$\left(\sum_{j\in\mathbb{N}} b_j\cdot\partial^j\right)\cdot\left(\sum_{i\in\mathbb{N}} c_i\cdot\partial^i\right) = \sum_{k\in\mathbb{N}}\left(\sum_{j+i=k} b_j\cdot\sigma^j(c_i)\right)\cdot\partial^k .$$

Define a left action of the ring $K[[\partial]]_\sigma$ on power series rings $K[[\underline{t}]]^n$ by

$$\left(\Big(\sum_{j\in\mathbb{N}} b_j\cdot\partial^j\Big)*g\right)(\underline{t}) = \sum_{j\in\mathbb{N}} b_j\cdot(\sigma_*^j g)(t_1^q,\cdots,t_m^q)$$

Functional equations

- An element $u \in K[[\partial]]_\sigma$ is a *special element* if u has the form
$$u = 1 - \sum_{j \geq 1} s_j \cdot \partial^j, \quad s_j \in K \ \ \forall j \geq 1$$
such that $\mathfrak{a} \cdot s_j \subset A \ \forall j \geq 1$.
- An element $h(x) \in K[[\underline{t}]]$ is *u-integral* if $u * h \in A[[\underline{t}]]$.
- An element $f(x) \in K[[x]]_0$ (i.e. $f(0) = 0$) is said to be *regular u-integral* if $u * f \in A[[x]]$ and $f'(0) \in A^\times$. In other words $f(x)$ satisfies a "functional equation" of the form
$$f(x) = g(x) + \sum_{j \geq 1} s_j \cdot (\sigma^j_* f)(x^{q^j})$$
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Functional equation lemma

Proposition. Let $u \in K[[\partial]]_\sigma$ be a special element. Let $f(\underline{x})$ be a *regular* u-integral element in $K[[x]]_0$

(1) The element $F(\underline{x},\underline{y}) := f^{-1}(f(x)+f(y)) \in K[[x,y]]$ belongs to $A[[x,y]]$. Hence $F(x,y)$ is a formal group law over A.

(2) For any u-integral element $h(\underline{t}) \in K[[\underline{t}]]_0 = K[[t_1,\ldots,t_m]]_0$, the element $f^{-1}(h(\underline{t})) \in K[[\underline{t}]]_0$ belongs to $A[[\underline{t}]]_0$.

(3) Suppose that $\alpha(\underline{z}) \in A[[\underline{z}]]_0 = A[[z_1,\ldots,z_k]]_0$, $\beta(\underline{z}) \in K[[\underline{z}]]_0 = K[[z_1,\ldots,z_k]]_0$. Then for all $r \geq 1$

$$\alpha(\underline{z}) \equiv \beta(\underline{z}) \pmod{\mathfrak{a}^r} \quad \text{iff} \quad f(\alpha(\underline{z})) \equiv f(\beta(\underline{z})) \pmod{\mathfrak{a}^r}.$$

(4) $\forall \psi \in A[[\underline{t}]]_0$, the element $f(\psi(\underline{t})) \in K[[\underline{t}]]_0$ is u-integral.

(5) $\forall v \in A[[\partial]]_\sigma$ and $\forall \psi \in A[[\underline{t}]]_0$ we have

$$v * (f \circ \psi) \equiv (v * f) \circ \psi \quad (\text{mod } \mathfrak{a} \cdot A[[\underline{t}]])$$

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Functional equation lemma

Proposition. Let $u \in K[[\partial]]_\sigma$ be a special element. Let $f(\underline{x})$ be a *regular* u-integral element in $K[[x]]_0$

(1) The element $F(\underline{x},\underline{y}) := f^{-1}(f(x)+f(y)) \in K[[x,y]]$ belongs to $A[[x,y]]$. Hence $F(x,y)$ is a formal group law over A.

(2) For any u-integral element $h(\underline{t}) \in K[[\underline{t}]]_0 = K[[t_1,\dots,t_m]]_0$, the element $f^{-1}(h(\underline{t})) \in K[[\underline{t}]]_0$ belongs to $A[[\underline{t}]]_0$.

(3) Suppose that $\alpha(\underline{z}) \in A[[\underline{z}]]_0 = A[[z_1,\dots,z_k]]_0$, $\beta(\underline{z}) \in K[[\underline{z}]]_0 = K[[z_1,\dots,z_k]]_0$. Then for all $r \geq 1$

$$\alpha(\underline{z}) \equiv \beta(\underline{z}) \pmod{\mathfrak{a}^r} \quad \text{iff} \quad f(\alpha(\underline{z})) \equiv f(\beta(\underline{z})) \pmod{\mathfrak{a}^r}.$$

(4) $\forall \psi \in A[[\underline{t}]]_0$, the element $f(\psi(\underline{t})) \in K[[\underline{t}]]_0$ is u-integral.

(5) $\forall v \in A[[\partial]]_\sigma$ and $\forall \psi \in A[[\underline{t}]]_0$ we have

$$v * (f \circ \psi) \equiv (v * f) \circ \psi \pmod{\mathfrak{a} \cdot A[[\underline{t}]]}$$

Essential uniqueness of functional equation

Proposition. Let u be a special element of $K[[\partial]]_\sigma$ and let $f(x) \in K[[x]]$ be regular u-integral. Suppose that

$$v = \sum_{j\in\mathbb{N}} \tilde{s}_j \cdot \partial^j \ \in K[[\partial]]_\sigma, \quad \tilde{s}_j \in (A:\mathfrak{a})\ \forall j$$

and f is v-integral. Then $\exists!\, c \in A[[\partial]]_\sigma$ such that $v = c \cdot u$ in $K[[\partial]]_\sigma$. In particular if v is also a special element, then $v \in A[[\partial]]_\sigma^\times \cdot u$.

How to construct homomorhisms over $A/\mathfrak{a}$

Let u, v be special elements in $K[[\partial]]_\sigma$. Let $f, g \in K[[x]]_0$ be regular u-integral and v-integral respectively. Let F and G be the formal group laws over A with logarithms f and g.

Proposition. For any $c \in A[[\partial]]_\sigma$, let $\phi_{g,c,f}(x) := g^{-1}(c * f)$.

(1) $\phi_{g,c,f}(x) \in A[[x]]_\sigma$ iff $\exists d \in A[[\partial]]_\sigma$ such that $v \cdot c = d \cdot u$.

(2) If $v \cdot c = d \cdot u$ and $d \in A[[\partial]]_\sigma$, then the image of $\phi_{f,g,c}$ in $(A/\mathfrak{a})[[x]]$ defines an $(A/\mathfrak{a})$-homomorphism

$$[\phi_{g,c,f}] : (F \mod \mathfrak{a}) \to (G \mod \mathfrak{a}).$$

(3) Suppose that w-is a special element in $K[[\partial]]_\sigma$, $h \in K[[x]]_0$ is w-regular and H is the formal group law over A with logarithm h. Suppose $c, d \in A[[\partial]]_\sigma$ and $w \cdot c' = d' \cdot v$. Then

$$[\phi_{h,c',g}] \circ [\phi_{g,c,f}] = [\phi_{h,c'c,f}].$$

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The universal p-typical formal group law

Let $\tilde{R} = \mathbb{Z}_{(p)}[\underline{v}] = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$, and let $\sigma : \tilde{R} \to \tilde{R}$ be the ring homomorphism such that $\sigma(v_j) = v_j^p$ for all $j \geq 1$

Let $G_{\underline{v}}(x) \in \tilde{R}[[x, y]]$ be the one-dimensional p-typical formal group law over $\tilde{R}$ whose logarithm

$$g_{\underline{v}}(x) \in \tilde{R}[1/p][[x]] = \sum_{n \geq 1} a_n(\underline{v}) \cdot x^{p^n}$$

satisfies

$$g_{\underline{v}}(x) = x + \sum_{i=1}^{\infty} \frac{v_i}{p} \cdot g_{\underline{v}}^{(\sigma^i)}(x^{p^i})$$

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Remarks on the formal group law $G_{\underline{v}}$

Remarks. (1) The above "functional equation" is a recursive formula for the coefficients $a_n(\underline{v}) \in p^{-n} \cdot \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n]$ of $g_{\underline{v}}(x)$.

(2) Explicitly:

$$a_n(\underline{v}) = \sum_{\substack{i_1, i_2, \ldots, i_r \geq 1 \\ i_1 + \cdots + i_r = n}} p^{-r} \cdot \prod_{s=1}^{r} v_{i_s}^{p^{i_1 + i_2 + \cdots + i_{s-1}}}$$

$$= \sum_{\substack{i_1, i_2, \ldots, i_r \geq 1 \\ i_1 + \cdots + i_r = n}} p^{-r} \cdot v_{i_1} \cdot v_{i_2}^{p^{i_1}} \cdot v_{i_3}^{p^{i_1 + i_2}} \cdots v_{i_r}^{p^{i_1 + \cdots + i_{r-1}}}$$

Note that $a_n(\underline{v})$ is a homogeneous polynomial in $v_1, \ldots, v_n$ of weight $p^n - 1$ when v_j is given the weight $p^j - 1$ $\forall j \geq 1$.

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The universal formal group over $\mathscr{M}_h$ made explicit

Let $R = R_h = W(\overline{\mathbb{F}}_p)[[w_1, w_2, \ldots, w_{h-1}]]$.

Let $\pi = \pi_h : \tilde{R} \to R$ be the ring homomorphism such that

$$\pi(v_i) = \begin{cases} w_i & \text{if} \quad 1 \le i \le h-1 \\ 1 & \text{if} \quad i = h \\ 0 & \text{if} \quad i \ge h+1 \end{cases}$$

The classifying morphism $\mathrm{Spf}(R) \to \mathscr{M}_h$ for the deformation $\pi_* G_{\underline{v}}$ of G_0 is an isomorphism.

We will identify $\mathscr{M}_h$ with $\mathrm{Spf}(R)$ and the universal deformation G_{univ} of G_0 with the formal group underlying the formal group law $G_R := \pi_* G_{\underline{v}}$.

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The universal strict isomorphism

Let $\mathbb{Z}_{(p)}[\underline{v},\underline{t}] = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots; t_1, t_2, t_3, \ldots]$, and let $\sigma : \mathbb{Z}_{(p)}[\underline{v},\underline{t}] \to \mathbb{Z}_{(p)}[\underline{v},\underline{t}]$ be the obvious Frobenius lifting as before, with $\sigma(v_i) = v_i^p$ and $\sigma(t_i) = t_i^p \; \forall i \geq 1$.

Let $G_{\underline{v},\underline{t}}(x,y)$ be the one-dimensional formal group law over $\mathbb{Z}_{(p)}[\underline{v},\underline{t}]$ whose logarithm $g_{\underline{v},\underline{t}}(x)$ satisfies

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The universal strict isomorphism, continued

It is known that $\alpha_{\underline{v},\underline{t}} := g_{\underline{v},\underline{t}}^{-1} \circ g_{\underline{v}} \in \mathbb{Z}_{(p)}[\underline{v},\underline{t}][[x]]$, and defines a *strict isomorphism*

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between p-typical formal group laws over $\mathbb{Z}_{(p)}[\underline{v},\underline{t}]$.

(A *strict* isomorphism is an isomorphism between formal group laws which is $\equiv x$ modulo higher degree terms in x.)

Moreover $\alpha_{\underline{v},\underline{t}}$ is "the" universal strict isomorphism between one-dimensional p-typical formal group laws.

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Parameters of $G_{\underline{v},\underline{t}}$

By the universality $G_{\underline{v}}$ for p-typical formal group laws, there exists a unique ring homomorphism

$$\eta : \mathbb{Z}_{(p)}[\underline{v}] \to \mathbb{Z}_{(p)}[\underline{v}, \underline{t}]$$

such that

$$\eta_* G_{\underline{v}} = G_{\underline{v},\underline{t}}.$$

The elements

$$\bar{v}_n = \bar{v}_n(\underline{v}, \underline{t}) \in \mathbb{Z}_{(p)}[\underline{v}, \underline{t}], \quad n \in \mathbb{N}_{\geq 1}$$

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A known recursive formula for the parameters of $G_{\underline{v},\underline{t}}$

$$
\begin{aligned}
\bar{v}_n = v_n + p\, t_n &+ \sum_{\substack{i+j=n\\ i,j\geq 1}} (v_j\, t_i^{p^j} - t_i\, \bar{v}_j^{p^i}) \\
&+ \sum_{j=1}^{n-1} a_{n-j}(\underline{v}) \cdot \left(v_j^{p^{n-j}} - \bar{v}_j^{p^{n-j}} \right) \\
&+ \sum_{k=2}^{n-1} a_{n-k}(\underline{v}) \cdot \sum_{\substack{i+j=k\\ i,j\geq 1}} \left(v_j^{p^{n-k}} t_i^{p^{n-i}} - t_i^{p^{n-k}} \bar{v}_j^{p^{n-j}} \right)
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(This formula contains high power of p in the denominators. Consequently it is not very useful for our purpose.)

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An *integral* recursion formula for $\bar{v}_n(\underline{v},\underline{t})$

(useful for computing the Lubin-Tate action)

$$
\begin{aligned}
\bar{v}_n = {} & v_n + p\,t_n - \sum_{j=1}^{n-1} t_j \cdot \bar{v}_{n-j}^{p^j} + \\
& + \sum_{l=1}^{n-1} v_l \sum_{k=1}^{n-l-1} \frac{1}{p} \cdot a_{n-k-l}(\underline{v})^{(p^l)} \cdot \Big\{ (\bar{v}_k^{(p^l)})^{p^{n-l-k}} - (\bar{v}_k^{p^l})^{p^{n-l-k}} \\
& \qquad\qquad + \sum_{\substack{i+j=k \\ i,j\geq 1}} t_j^{p^{n-k}} \left[(\bar{v}_i^{(p^l)})^{p^{n-l-i}} - (\bar{v}_i^{p^l})^{p^{n-l-i}} \right] \Big\} \\
& + \sum_{l=1}^{n-1} v_l \cdot \Big\{ \frac{1}{p} (\bar{v}_{n-l}^{(p^l)} - \bar{v}_{n-l}^{p^l}) + \sum_{\substack{i+j=n-l \\ i,j\geq 1}} t_j^{p^l} \cdot \frac{1}{p} \cdot \left[(\bar{v}_i^{(p^l)})^{p^j} - (\bar{v}_i^{p^l})^{p^j} \right] \Big\}
\end{aligned}
$$

for every $n \geq 1$.

The groupoid underlying the universal strict isomorphism

Let $X_0 := \mathrm{Spec}(\tilde{R})$, $X_1 := \mathrm{Spec}(\tilde{\Gamma})$. Consider the diagram

$$X_0 \xleftarrow{\text{source}} X_1 \xrightarrow{\text{target}} X_0,$$

where the *source* arrow corresponds to $\tilde{R} \hookrightarrow \tilde{\Gamma}$ and the *target* arrow corresponds to the ring homomorphism $\eta : \tilde{R} \to \tilde{\Gamma}$. The above diagram is part of a natural groupoid structure such that the (partial) product

$$\mu : X_1 \times_{s,X_0,t} X_1 \to X_1$$

corresponds to composition of strict isomorphisms between p-typical formal group laws.

Three ways to think about the moduli space $\mathscr{M}_h$

1. All p-typical deformations of G_s whose parameters satisfy $v_h = 1, v_{h+1} = v_{h+2} = \cdots = 0$.

2. All p-typical deformations of G_s whose parameters satisfy $v_{h+1} = v_{h+2} = \cdots = 0$, up to/modulo *scaling by units*

3. All p-typical deformations of G_s, modulo the equivalence relations generated by

- strict $*$-isomorphisms, and
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Rough idea

Start with an element $\gamma \in \mathrm{Aut}(G_s)$.

1. Use Honda's formalism to construct an isomorphism $\psi_\gamma : F_\gamma \to \phi_* G_{\underline{v}}$ in equi-characteristic p whose closed fiber is equal to γ.

2. Compute the parameters $v_1, v_2, v_3, \ldots$ of F_γ. (By recursive relations).

3. Change F_γ by a strict isomorphism with suitable parameters $t_1, t_2, t_3, \ldots$, to a new p-typical formal group law whose (new) parameters satisfy $v_{h+1} = v_{h+2} = \cdots = 0$.
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Rough idea

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Given an element $\gamma \in \mathrm{Aut}(G_0)$, construct

- a p-typical one-dimensional formal group *law* $F = F_\gamma$ over R whose closed fiber is equal to G_0, and
- an isomorphism

$$\overline{\psi} = \overline{\psi}_\gamma : F_{\overline{R}} \to G_{\overline{R}}$$

over $\overline{R} := R/pR = \overline{\mathbb{F}}_p[[w_1, \ldots, w_{h-1}]]$ whose restriction to the closed fibers is

$$(\psi|_{G_0} : G_0 \to G_0) = \gamma.$$

Here $F_{\overline{R}} = F \otimes_R \overline{R}$, $G_{\overline{R}} = G_R \otimes_R \overline{R}$.

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The formal group law F_c, $c \in W(\mathbb{F}_{p^h})^\times$

For $\gamma = [c] \in W(\mathbb{F}_{p^h})^\times = \mathrm{Aut}(G_1)$, we can take F_c to be the formal group over R whose logarithm $g_c(x)$ satisfies

$$f_c(x) = x + \sum_{i=1}^{h} \frac{c^{-1+\sigma^i} \cdot w_i}{p} \cdot f_c^{(\sigma^i)}(x^{p^i})$$

(w_h=1 by convention).

Let

$$\psi_c(x) = \log_{G_R}^{-1} \circ (c \cdot f_c)$$

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Step 2

Compute the parameters

$$(u_i = u_i(w_1, \ldots, w_{h-1}))_{i \in \mathbb{N}_{\geq 1}}$$

for the p-typical group law $F = F_\gamma$ over R.

The above condition means that

$$\xi_* G_{\tilde{v}} = F,$$

where

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Parameters for F_c, $c \in W(\mathbb{F}_{p^h})^\times$

In the case when $\gamma \in \mathrm{Aut}(G_0)$ lifts to an element [c] with $c \in W(\mathbb{F}_{p^h})^\times \simeq \mathrm{Aut}(G_1)$, we have the following integral recursive formula for the parameters $u_n = u_n(c;\underline{w})$.

$$
\begin{aligned}
u_n(c;\underline{w}) &= c^{-1+\sigma^n} w_n \\
&\quad + \sum_{j=1}^{n-1} c^{-1+\sigma^j} \cdot \frac{1}{p} \left[u_{n-j}(c;\underline{w})^{(p^j)} - u_{n-j}(c;\underline{w})^{p^j} \right] \cdot w_j \\
&\quad + \sum_{j=1ve}^{n-1} \sum_{i=1}^{n-j-1} \frac{1}{p} a_{n-i-j}(\underline{w})^{(p^j)} \cdot c^{-1+\sigma^{n-i}} \cdot \\
&\qquad \left[(u_i(c;\underline{w})^{(p^j)})^{p^{n-i-j}} - (u_i(c;\underline{w})^{p^j})^{p^{n-i-j}} \right] \cdot w_j
\end{aligned}
$$

where $w_h = 1$, $w_m = 0\ \forall m \geq h+1$ by convention.

Parameters for F_c, continued

Remark. The above recursive formula for the parameters $u_n(c;\underline{w})$ can be turned into an explicit "path sum" formula for $u_n(c,\underline{w})$, with terms indexed by "paths".

Step 3

Find/compute the uniquely determined element

$$\tau_n \in \mathfrak{m}_R, \quad n \in \mathbb{N}_{\geq 1}$$

and

$$\hat{u}_1 \in \mathfrak{m}_R, \ldots, \hat{u}_{h-1} \in \mathfrak{m}_R, \hat{u}_h \in 1 + \mathfrak{m}_R$$

such that

$$\bar{v}_n(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h, 0, 0, \ldots; \underline{\tau}) = u_n \quad \forall n \geq 1.$$

Remark. (1) The existence and uniqueness statement above is an application the implicit function theorem for an infinite dimensional space over $\tilde{R}$, applied to the "vector-valued" function with components $\overline{v}_n$ in the integral recursion formula discussed before.

(2) This step is a substitute for the operation *taking the quotient of the group "changes of coordinates"* in a space of formal group laws.

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A congruence formula for $\bar{v}_n$

The follow formula helps to explain the last remark.

$$\bar{v}_n \equiv v_n - \sum_{j=1}^{n} t_j \cdot v_{n-j}^{p^j} + \sum_{\substack{i,j,t,s_1,s_2,\dots,s_t \geq 1 \\ s_1+\dots+s_t+i+j=n}} (-1)^{t-1}\, t_i \cdot v_j^{p^i} \cdot v_1^{(p^{s_1}+p^{s_2}+\cdots+p^{s_t}-t)/(p-1)} \cdot v_{n-s_1}^{p^{s_1}-1} \cdot v_{n-s_1-s_2}^{p^{s_2}-1} \cdots v_{n-s_1-\cdots-s_t}^{p^{s_t}-1} \mod (pt_a, t_a \cdot t_b)_{a,b\geq 1}\mathbb{Z}[\underline{v},\underline{t}]$$

Step 4

Rescale $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h$ as follows:

$\exists! \; \tau_0 \in \mathfrak{m}_R$ such that

$$(1+\tau_0)^{p^h-1} \cdot \hat{u}_h = 1.$$

Let

$$\hat{v}_i := (1+\tau_0)^{p^i-1} \cdot \hat{u}_i \quad \text{for } i = 1, \ldots, h-1.$$

Let $\omega : \tilde{R} \to R$ be the ring homomorphism such that

$$\omega(v_i) = \hat{u}_i \quad \forall i \geq 1.$$

Let $\rho : R \to R$ be the $W(\overline{\mathbb{F}}_p)$-linear ring homomorphism such that

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The meaning of Steps 3 and 4

The universal strict isomorphism $\alpha_{\underline{v},\underline{t}}$ specializes to a strict isomorphism

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with $\alpha|_{G_0} = \mathrm{Id}_{G_0}$.

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Combined with $\overline{\psi}$, we obtain an isomorphism

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Conclusion. The given element $\gamma \in \mathrm{Aut}(G_0)$ operates on the equi-characteristic deformation space $\mathrm{Spf}(\overline{R})$ of G_0 via the ring automorphism ρ.
(Notice that $\overline{\psi}$, α and β all depend on γ.)

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Local rigidity for the Lubin-Tate moduli space: the first non-trivial case

Proposition. Let $Z \subset \mathscr{M}_{3\overline{\mathbb{F}}_p} = \mathrm{Spf}(\overline{\mathbb{F}}_p[[w_1, w_2]])$ be an irreducible closed formal subscheme of $\mathscr{M}_3$ over $\overline{\mathbb{F}}_p$ corresponding to a hight one prime ideal of $\overline{\mathbb{F}}_p[[w_1, w_2]]$.
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