Moduli of Abelian Varieties

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Goal: survey the geometry of the moduli space of (principally polarized) abelian varieties, with emphasis in the case of positive characteristic $p>0$, and the related rigidity phenomena

- history: elliptic curves $\nearrow$ curves of higher genera $\dashrightarrow$ moduli space of curves $\mathcal{M}_g$
  $\searrow$ abelian varieties $\rightsquigarrow$ moduli space of abelian varieties $\mathcal{A}_g$ $\circlearrowleft$ Hecke symmetry
- definitions, p-divisible groups
- phenomena and structures in characteristic $p>0$. predictions (= conjectures)
- new tools/methods applicable to other problems

§1 From elliptic curves to abelian varieties and their moduli

1.1 What is an elliptic curve? several approaches

(a) algebra $E = \{ y^2 = 4x^3 - g_2 x - g_3 \}$, $\Delta := g_2^3 - 27 g_3^2 \neq 0$, $j := 1728 \frac{g_2^3}{\Delta}$

(b) geometry

$$\begin{array}{ccc} E(\mathbb{C}) & \xleftarrow{\sim} & \mathrm{Lie}(E)/H_1(E(\mathbb{C});\mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}\tau+\mathbb{Z} \\ \Downarrow & & \Downarrow \\ P & \longmapsto & \int_\infty^P \frac{dx}{y} \end{array}$$

$\tau \in \mathfrak{h} = \{ \tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0 \}$

$\Lambda_\tau := \mathbb{Z}\tau + \mathbb{Z}$

(c) analysis

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{\gamma \in \Lambda_\tau \setminus \{0\}} \left[\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right]$$

$$\rightsquigarrow \left(\frac{d}{dz} \wp(z,\tau) \right)^2 = 4\, \wp(z,\tau)^3 - g_2(\Lambda_\tau)\, \wp(z,\tau) - g_3(\Lambda_\tau)$$

where $g_2(\Lambda_\tau) = 60 \sum_{\gamma \in \Lambda_\tau \setminus \{0\}} \frac{1}{\gamma^4}$, $g_3(\Lambda_\tau) = 140 \sum_{\gamma \in \Lambda_\tau \setminus \{0\}} \frac{1}{\gamma^6}$

1.2. The origin of elliptic curves: Diophantine equations, elliptic integrals

- Fermat $x^4 - y^4 = z^2$ has no non-trivial rational solution (infinite descent)

- Gauss (1814) $E_{aff} := \{1 = x^2 + y^2 + x^2y^2\}$ $(a+\sqrt{-1}b)\cdot\mathbb{Z}[\sqrt{-1}]$: prime ideal s.t. $a+\sqrt{-1}b \equiv 1 \bmod (1+\sqrt{-1})^3$

$$\# E(\mathbb{Z}[\sqrt{-1}]/(a+\sqrt{-1}b)\cdot\mathbb{Z}[\sqrt{-1}]) = (a-1)^2 + b^2$$

- December 1751, paper by Fagnano reached Euler in Berlin

$\dfrac{dx}{\sqrt{1-x^4}} = \dfrac{dy}{\sqrt{1-y^4}}$ has rational solutions, i.e. $\displaystyle\int_0^x \frac{d\rho}{\sqrt{1-\rho^4}} = \int_0^y \frac{d\psi}{\sqrt{1-\psi^4}}$ admits solutions where $y =$ a rational function of x

Note: The elliptic curves involved in the above are all twists of "the same" curve with CM by $\mathbb{Z}[\sqrt{-1}]$, $j = 1728$

1.3. Periods of compact Riemann surfaces / smooth projective algebraic curves

$S = C(\mathbb{C})$ compact Riemann surface; $\gamma_1, \dots, \gamma_{2g}$: $\mathbb{Z}$-basis of $H_1(S;\mathbb{Z})$

$\omega_1, \dots, \omega_g$: $\mathbb{C}$-basis of $\Gamma(S, \Omega^1_S)$. $\Delta = (\gamma_i \cdot \gamma_j) \in M_{2g}(\mathbb{Z})$

$$P = \left(\int_{\gamma_j} \omega_r\right)_{\substack{1 \le r \le g \\ 1 \le j \le 2g}} = (P_{r,j}) \in M_{g \times 2g}(\mathbb{C})$$

Riemann bilinear relations

$$\boxed{\begin{aligned} &P \cdot \Delta^{-1} \cdot {}^tP = 0 \\ &-\sqrt{-1} \cdot P \cdot \Delta^{-1} \cdot {}^t\overline{P} \gg 0_g \end{aligned}}$$

$$C \hookrightarrow \mathrm{Pic}^1(C)$$

$$\circlearrowleft$$

$$\mathrm{Pic}^0(C) =: \mathrm{Jac}(C) = \Gamma(C, \Omega^1)^\vee / H_1(C(\mathbb{C}); \mathbb{Z})$$

1.4. Abelian varieties

<u>Def^n</u> (i) (over $\mathbb{C}$) a compact complex torus $\mathbb{C}^g / Q\cdot\mathbb{Z}^{2g}$, $Q \in M_{g\times 2g}(\mathbb{C})$ is a complex abelian variety iff $\exists$ a skew-symmetric $E \in M_{2g}(\mathbb{Z})$ with $\det(E) \neq 0$ s.t.

$$\begin{cases} Q\cdot E^{-1}\cdot {}^tQ = 0 \\ \sqrt{-1}\cdot Q\cdot E^{-1}\cdot {}^t\overline{Q} >> 0 \end{cases}$$

(i)$'$ (equivalent to (i)) a compact complex torus is a complex abelian variety iff it admits a holomorphic embedding into $\mathbb{P}^N(\mathbb{C})$ for some N.

(ii) (Weil 1948) An irreducible algebraic group over a field k which is a complete (i.e. proper) over k is an abelian variety.

Def^n (polarization of abelian varieties)

(i) A **polarization** of an abelian variety A is an algebraic equivalence class of an ample divisor on A.

(ii) A polarization of an abelian variety A represented by an ample divisor D on A is a **principal polarization** if $D^g = g!$

self intersection g times

Fact :(1) The polarization attached to an ample divisor D on A is uniquely determined by the algebraic homomorphism

$$\varphi_{[D]} : A \longrightarrow {}^tA = \mathrm{Pic}^0(A)$$

= dual abelian variety, classifying line bundles on A which are algebraically equivalent to 0.

$$x \longmapsto \mathcal{O}_A((D-x)-D)$$

(2) Over $\mathbb{C}$, $\underset{\text{fund'l class}}{c(D)} \in H^2(A(\mathbb{C}); \mathbb{Z}(1))$ corresponds to a non-singular

$\mathbb{Z}(1)$: $H^2(\mathbb{P}^1(\mathbb{C}); \mathbb{Z})^\vee$

skew-symmetric pairing $H_1(A(\mathbb{C}); \mathbb{Z}) \times H_1(A(\mathbb{C}); \mathbb{Z}) \longrightarrow \mathbb{Z}(1)$

$[D]$ is a principal polarization $\iff$ the above is a perfect pairing over $\mathbb{Z}$

D: effective divisor on A

Over $\mathbb{C}$

1) Every principally polarized abelian variety of dimension g over $\mathbb{C}$ is isomorphic to

$$A_\Omega := \mathbb{C}^g / \Omega\cdot\mathbb{Z}^g + \mathbb{Z}^g$$

for some $\Omega \in \mathfrak{h}_g := \{\Omega \in M_g(\mathbb{C}) \mid {}^t\Omega = \Omega,\ \mathrm{Im}(\Omega) \gg 0_g\}$

2) $(A_{\Omega_1}, \lambda_{\Omega_1}) \cong (A_{\Omega_2}, \lambda_{\Omega_2})$ iff $\exists \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$

polarization

i.e. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix} = \begin{pmatrix} I_g & 0 \\ 0 & I_g \end{pmatrix}$

such that $(A\Omega_1 + B)\cdot(C\Omega_1 + D^{-1}) = \Omega_2$

<u>Note</u>: $Sp_{2g}(\mathbb{R})$ acts on $\mathfrak{h}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \Omega \longmapsto (A\Omega + B)\cdot(C\Omega + D)^{-1}$$

This is a transitive action

1.5. The moduli space of g-dimensional principally polarized abelian varieties $\mathcal{A}_g$

Idea / phenomenon

- The set of all isomorphism classes of g-dimensional abelian varieties (with level-n structure) has a natural structure as an algebraic variety
- A subvariety of $\mathcal{A}_g$ corresponds to a family of abelian varieties

  or more generally, a morphism $S \to \mathcal{A}_g$

Ex. $g = 1$. The set of all isomorphism classes of elliptic curves is parametrized by $\mathbb{A}^1$: $E \longmapsto j(E)$

Ex. $\mathcal{A}_g(\mathbb{C}) \cong Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{h}_g$

- Existence of $\mathcal{A}_g$ over $\mathbb{Z}$: Mumford 1965

- Fact : $\mathcal{A}_{g/k}$ is irreducible $\forall$ field k
  - case $k = \mathbb{C}$: immediate from complex uniformization $\mathcal{A}_g(\mathbb{C}) \cong {}_{Sp_{2g}(\mathbb{Z})}\backslash \mathfrak{H}_g$
  - char $(k) = 0$: follows from the case $k = \mathbb{C}$
  - char $(k) = p > 0$ Faltings-C. 1984

§2 Hecke symmetry on $\mathcal{A}_g$

2.1 Definition

complex version (transcendental) : $\forall \gamma \in Sp_{2g}(\mathbb{Q})$, $\quad Sp_{2g}(\mathbb{Z}) \cdot \gamma \cdot \overbrace{Sp_{2g}(\mathbb{Z})}^{\Gamma(1) \,=}$ induces an algebraic correspondence on $Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{h}_g$:

$$\begin{tikzcd}
\Gamma(1)\backslash \mathfrak{h}_g & \mathfrak{h}_g \arrow[l, two heads, "\mathrm{pr}"'] \arrow[r, "\gamma\cdot"] \arrow[d] & \mathfrak{h}_g \arrow[d] \\
& (\Gamma(1) \cap \gamma^{-1}\Gamma(1)\gamma)\backslash \mathfrak{h}_g \arrow[ul, two heads, "\mathrm{pr}"] \arrow[r, "\gamma\cdot"'] & \Gamma(1)\backslash \mathfrak{h}_g
\end{tikzcd}$$

(Think of
$$\begin{tikzcd}
\Gamma(1)\backslash \mathfrak{h}_g & & \\
(\Gamma(1) \cap \gamma^{-1}\Gamma(1)\gamma)\backslash \mathfrak{h}_g \arrow[u, two heads] \arrow[r, "\gamma"] & \Gamma(1)\backslash \mathfrak{h}_g
\end{tikzcd}$$
as a "multi-valued map" from $\Gamma(1)\backslash \mathfrak{h}_g$ to $\Gamma(1)\backslash \mathfrak{h}_g$)

algebraic version:

$p = \mathrm{char}(k)$

* $[(A_1,\lambda_1)], [(A_2,\lambda_2)] \in \mathcal{A}_g(k)$ are in the same (prime-to-p) <u>Hecke orbit</u> if $\exists$ an isogeny $\alpha: A_1 \to A_2$ and $n \in \mathbb{Z}_{>0}$ (with $\gcd(n,p)=1$) such that $\alpha^*(\lambda_2) = n\cdot\lambda_1$

Adelic picture:

$$\mathbb{A}_f^{(p)} = \Big(\underbrace{\prod_{\ell \neq p} \mathbb{Z}_\ell}_{\hat{\mathbb{Z}}^{(p)}}\Big) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\begin{array}{c} Sp_{2g}(\mathbb{A}_f^{(p)}) \circlearrowright \tilde{\mathcal{A}}_{g/k}^{(p)} = \varprojlim_{\gcd(n,p)=1} \mathcal{A}_{g,n} \\ \Big\downarrow \text{Galois with group } Sp_{2g} \\ \mathcal{A}_g \end{array}$$

$\Gamma(n)\backslash \mathfrak{h}_g$ when $k = \mathbb{C}$

prime-to-p Hecke orbits on $\mathcal{A}_g$ $\longleftrightarrow$ $Sp_{2g}(\mathbb{A}_f^{(p)})$-orbits on $\tilde{\mathcal{A}}_g^{(p)}$

phenomenon: Every Hecke orbit is large ("as large as possible")

2.2. p-divisible groups   Tate 1967

Definition: A <u>p-divisible group</u> $X \to S$ is an inductive system of commutative finite locally free group schemes

$$\left((X_n \to S)_{n\in\mathbb{N}}, \quad i_{n+1,n} : X_n \hookrightarrow X_{n+1} \right)$$

together with faithfully flat homomorphisms

$$\pi_{n,n+1} : X_{n+1} \longrightarrow X_n$$

such that $i_{n+1,n} \circ \pi_{n,n+1} = [p]_{X_{n+1}}$, $\pi_{n,n+1} \circ i_{n+1,n} = [p]_{X_n}$ $\forall n$

Fact: $\exists !$ locally constant function $h : S \to \mathbb{N}$ such that $\mathrm{rk}(X_n/S) = p^{nh}$ $\forall n$

height of X/S

Main Example: $A \to S$ abelian scheme $\rightsquigarrow$ $A[p^\infty] := \left(\varinjlim_n A[p^n] \right)_{n\in\mathbb{N}}$ is a p-divisible group of rank $2\cdot\dim(A/S)$

a substitute for Lie algebra in char. $p>0$

2.3 p-adic invariants of abelian varieties

- All p-adic invariants of an abelian variety $A_{/k}$, $k \supseteq \overline{\mathbb{F}}_p$, come from the p-divisible group $A[p^\infty]$
- Every prime-to-p Hecke symmetry / symplectic isogeny between principally polarized abelian varieties over $k \supseteq \overline{\mathbb{F}}_p$ preserves all p-adic invariants

correspondence

Examples of p-adic invariants

(a) slopes / Newton polygon of a p-divisible group $X_{/k}$

<u>idea</u> :- compare $\mathrm{Fr}^{(p^n)}_{X/k} : X \to X^{(p^n)}$ with $X[p^i]$

- slopes = p-adic valuation of "eigenvalues" of $\mathrm{Fr}^{(p)}_{X/k}$

↑ does not quite make sense unless k is finite

Every g-dim^l abelian variety has 2g slopes

$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{2g} \le 1$, $\lambda_i \in \mathbb{Q}$ $\forall i$, $\lambda_i + \lambda_{g+1-i} = 1$ $\forall i$

denominator(λ_i) | multiplicity(λ_i) $\forall i$

- A is <u>ordinary</u> $\iff$ slopes are 0 or 1

  A is <u>supersingular</u> $\iff$ all slopes are $\frac{1}{2}$

  ordinary abelian varieties in $\mathcal{A}_g$ form an open dense subset of $\mathcal{A}_{g/k}$

Example (b) $(A,\lambda) \rightsquigarrow$ isom. class of $(A[p], \lambda[p])$

(c) $(A,\lambda) \rightsquigarrow$ isom class of $(A[p^\infty], \lambda[p^\infty])$

Thm (C. 1995) $\forall x = [(A,\lambda)] \in \mathcal{A}_g(k)$ with A ordinary, the prime-to-p Hecke orbit of x is Zariski dense in $\mathcal{A}_g$

§3 Leaves in $\mathcal{A}_g / k \supseteq \mathbb{F}_p$, $\bar{k} = k$

<u>Def^n</u> (Oort, 1999) Given $x = [(A, \lambda)] \in \mathcal{A}(k)$, the <u>leaf</u> $C(x)$ through x is the locally closed subvariety of $\mathcal{A}_g$ s.t.

$$C(x)(k) = \left\{ [(B,\mu)] \in \mathcal{A}_g(k) \;\middle|\; (B[p^\infty], \mu[p^\infty]) \cong (A[p^\infty], \lambda[p^\infty]) \right\}$$

Fact: Every leaf in $\mathcal{A}_g$ is smooth, and stable under all prime-to-p Hecke correspondences

Conjecture (Oort) Given a leaf $C \subset \mathcal{A}_g$ and $x \in C(k)$, the prime-to-p Hecke orbit of x is Zariski dense in C

<u>Ans</u>. True (Oort + C. 2006)

proof uses a special property of $\mathcal{A}_g$ ("Hilbert trick").

The conjecture for general PEL-type moduli spaces remains open

§3 New tools, structures and conjectures/predictions/phenomena related to Hecke symmetry

3.1 Monodromy and irreducibility results

Prop. A. (reducing irreducibility to Hecke transitivity)
(C.)
Let $Z \subset \mathcal{A}_g$ be a positive dimensional locally closed subvariety stable under all prime-to-p Hecke correspondences. If Hecke symmetries are transitive on $\pi_0(Z)$, then Z is irreducible

Prop. B. Let $C \subseteq \mathcal{A}_g$ be a positive-dimensional leaf on $\mathcal{A}_g$
(Oort + C.)
(equivalently, C is not supersingular), then the naive p-adic monodromy of C is maximal, so is the ℓ-adic monodromy of C $\forall \ell \neq p$.

Prop. C (Oort + C.) Every non-supersingular Newton stratum in $\mathcal{A}_g$ is irreducible

Prop. D. Every non-supersingular leaf in $\mathcal{A}_g$ is irreducible

<u>Note</u> Prop. A is used in the proof of B–D.
i.e. the maximality / irreducibility results in B–D are proved using Hecke symmetry.

3.2. Local structure of leaves (the 2-slope case) $k = \bar{k} \supseteq \mathbb{F}_p$

$\mathcal{A}_g \supseteq \underset{\substack{\uparrow \\ \text{a leaf}}}{C} \ni x_0 = [(A_0, \lambda_0)]$ slope of $A_0 = \{\lambda, 1-\lambda\}$ $\lambda < \frac{1}{2}$

<u>Prop.</u> $C^{/x_0}$ = the formal completion of C at x_0 has a natural structure as a (trivial torsor for) an isoclinic p-divisible formal group with slope $\underset{\substack{\uparrow \\ (1-\lambda) - \lambda}}{1-2\lambda}$ and height $g \cdot \underset{\substack{\uparrow \\ \dim \mathcal{A}_g}}{(g+1)/2}$

3.3. Local stabilizer principle $\quad k = \bar{k} \supseteq \mathbb{F}_p$

Prop. Let $Z \subseteq \mathcal{A}_g$ be a locally closed subvariety stable under all prime-to-p Hecke correspondences, $x_0 = [(A_0, \lambda_0)] \in Z(k)$.
Then $Z^{/x_0} \subseteq \mathcal{A}_g^{/x_0}$ is stable under the natural action of an open subgroup of $\mathrm{U}(\mathrm{End}(A_0), *_{\lambda_0})(\mathbb{Z}_p)$ on $\mathcal{A}_g^{/x_0}$.

$$\underset{\substack{\uparrow \\ \text{unitary group}}}{\mathrm{U}}\Big(\underbrace{\mathrm{End}(A_0),\ *_{\lambda_0}}_{\text{semisimple algebra with involution}}\Big)(\mathbb{Z}_p)$$

Explanation:

$$\mathrm{Aut}(A_0[p^\infty], \lambda_0[p^\infty]) \quad \text{acts on} \quad \mathcal{A}_g^{/x_0} = \mathrm{Def}(A_0, \lambda_0) \xrightarrow[\sim]{\text{Serre-Tate}} \mathrm{Def}(A_0[p^\infty], \lambda_0[p^\infty])$$

$$\begin{array}{c} \mathrm{Aut}(A_0[p^\infty], \lambda_0[p^\infty]) \\ \cup \\ \mathrm{U}(\mathrm{End}(A_0), *_{\lambda_0}) \end{array}$$

"locally stabilizer subgroup at x_0", corresponding to Hecke symmetries fixing x_0

3.4. Rigidity phenomena

$k = \bar{k} \supseteq \mathbb{F}_p$, $X_{/k}$: p-divisible formal group

Theorem (C., local rigidity for p-divisible groups)

$Z \subseteq X$ irreducible formal subvariety.

$G \subseteq \mathrm{Aut}(X)$, a p-adic Lie group acting on X

Assume :
- No open subgroup of G operates trivially on any non-zero subquotient of X (G operates "strongly non-trivially" on X)
- Z is stable under G

Then Z is a p-divisible formal subgroup of X

Remark: (recent progress: local rigidity holds for bi-extensions of p-divisible groups)

"Exercise" Case $X = \hat{\mathbb{G}}_m^h = \mathrm{Spf}(\overline{\mathbb{F}_p}[[T_1, \dots, T_h]])$

group law: $\psi : \overline{\mathbb{F}_p}[[T_1, \dots, T_h]] \longrightarrow \overline{\mathbb{F}_p}[[u_1, \dots, u_h, v_1, \dots, v_h]]$

$$T_i \longmapsto u_i + v_i + u_i v_i$$

$(1+u)\cdot(1+v) = 1 + u + v + uv$

$G = 1 + p^2\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ operates on $\overline{\mathbb{F}_p}[[T_1, \dots, T_h]]$
$= \langle 1+p^2 \rangle$

$$[1+p^2]^* : f(T_1, \dots, T_h) \mapsto f((1+T_1)^{1+p^2} - 1, \dots, (1+T_h)^{1+p^2} - 1)$$

The statement is: If $P \subseteq \overline{\mathbb{F}_p}[[T_1, \dots, T_h]]$ is a prime ideal s.t. $[1+p^2]^*(P) \subseteq P$, then $\psi(P) \subseteq (pr_1^*(P), pr_2^*(P))$

where $pr_1^*(f(T_1, \dots, T_h)) := f(u_1, \dots, u_h)$
$pr_2^*(f(T_1, \dots, T_h)) := f(v_1, \dots, u_h)$

Application (Exercise + local stabilizer principle)

E_0 : an ordinary elliptic curve over $\overline{\mathbb{F}_p}$

$A_0 = \underbrace{E_0 \times \cdots \times E_0}_{g\text{-times}}$, λ_0 = product polarization on A_0 $\quad x_0 = [(A_0, \lambda_0)] \in \mathcal{A}_g(\overline{\mathbb{F}_p})$

Then the prime-to-p Hecke orbit of x_0 is Zariski dense in $\mathcal{A}_g$

Pf. $\mathcal{A}_g^{/x_0} \cong \widehat{\mathbb{G}_m}^{g(g+1)/2}$, $U(\mathrm{End}(A_0), *_{\lambda_0})(\mathbb{Z}_p) \cong GL_g(\mathbb{Z}_p)$

The action of $GL_g(\mathbb{Z}_p)$ on $X^*(\widehat{\mathbb{G}_m}^{g(g+1)/2}) \cong \mathbb{Z}_p^{g(g+1)/2}$

$\cong S^2$(standard representation of $GL_g(\mathbb{Z}_p)$ on $\mathbb{Z}_p^g$)

3.5 Global rigidity conjecture

Conj. Suppose $Z \subseteq \mathcal{A}_g^{\mathrm{ord}}$, $x_0 = [(A_0, \lambda_0)] \in \mathcal{A}_g^{\mathrm{ord}}(\overline{\mathbb{F}}_p)$.

Assume that $Z^{/x_0} \subseteq \mathcal{A}_g^{/x_0}$ (= Serre-Tate formal torus) is a formal subtorus.

Then Z is the reduction of a Shimura subvariety of $\mathcal{A}_g$

Remark. True if $Z \subset$ a Hilbert modular subvariety (C.)

This is the main geometric ingredient of Hida's proof (together with the local rigidity for p-divisible groups) of the non-vanishing of the μ-invariant for Katz p-adic L-functions (Ann. Math. 2012)

3.6 A (special case of a) local rigidity conjecture

G_0 = a 1-dim^l smooth formal group over $\overline{\mathbb{F}_p}$, $ht(G_0) = h$

(so slope$(G_0) = \frac{1}{h}$)

$\mathcal{M}$ = equi-characteristic p deformation space of G_0

$\underset{\text{Lubin-Tate 1966}}{\cong} \mathrm{Spf}(\overline{\mathbb{F}_p}[[x_1, \dots, x_{h-1}]])$ $\qquad x_1$ = Hasse invariant

Conj. Suppose $Z \subseteq \mathcal{M}$ is an irreducible formal subvariety such that $x_1|_Z \neq 0$ (i.e. Z is generically ordinary), and Z is stable under the action of an open subgroup of $\mathrm{Aut}(G_0)$,

then $Z = \mathcal{M}$

↑ group of units in a central division algebra over $\mathbb{Q}_p$, $\dim_{\mathbb{Q}_p} = h^2$, with Brauer invariant $1/h$

3.7. New/better definition of leaves

Definition. Let κ = a field of characteristic $p>0$

$X_0/_\kappa$: a p-divisible group over κ

$S/_\kappa$: a κ-scheme

A p-divisible group $X \to S$ is <u>strongly κ-sustained modeled on X_0</u> if

$$\underline{\mathrm{Isom}}_S\left(X_0[p^n]\times_{\mathrm{Spec}\,\kappa} S,\ X[p^n]\right) \to S \quad \text{is faithfully flat } \forall n\in S$$