CM LIFTINGS

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§1 CM lifting questions

Book on CM liftings with B. Conrad and F. Oort,
`www.math.upenn.edu/~chai/papers_pdf/CMlifting_book_ver0615_2011.pdf`

Notation

- p is a prime number, $\overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$.
- B is an *isotypic* abelian variety over a finite field $\mathbb{F}_q$, $q \in p^{\mathbb{N}}$, $g := \dim(B)$,
- K is a CM field, $[K:\mathbb{Q}] = 2g$, K_0 is the maximal totally real subfield of L.
- $\beta : K \to \mathrm{End}^0(B) := \mathrm{End}(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a ring homomorphism.

We are interested in several version of lifting problems for the CM structure (B,β) to characteristic 0.

Background and motivation: Honda/Tate

Honda/Tate

- An abelian variety B over a finite field $\mathbb{F}_q$ admits a CM structure $K \to \mathrm{End}^0(B)$ for some CM field K if and only if $B_{/\overline{\mathbb{F}}_q}$ is isotypic.
- If $B \sim C^r$ with C simple over $\mathbb{F}_q$, then $\mathrm{End}^0(B) \cong \mathrm{M}_r(D)$, where $D = \mathrm{End}^0(B)$ is a division ring; C is determined up to $\mathbb{F}_q$-isogeny by the Weil q-number Fr_C.
- Every Weil q-number comes from a simple abelian variety over $\mathbb{F}_q$.

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Question (CML)

(CML) CM *lifting*: $\exists$ a lifting of (B, β) to char. 0.

Spelled out: there exists

- a local domain R with characteristic 0 and residue field $\mathbb{F}_q$,
- an abelian scheme A over R with relative dimension g,
- a ring hom. $\alpha : K \to \mathrm{End}^0(A) := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(A)$, and
- an isomorphism $\phi : (A, \alpha)_{\mathbb{F}_q} \simeq (B, \beta)$ of CM structures over $\mathbb{F}_q$.

CML holds after base extension and isogeny

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Honda/Tate + Shimura/Taniyama:

Given a CM structure $(B, K \to \mathrm{End}^0(B))_{/\mathbb{F}_q}$, $\exists n \in \mathbb{N}$ and an $\mathbb{F}_{q^n}$-isogeny $B_{/\mathbb{F}_{q^n}} \to B_1$ such that $(B_1, K \to \mathrm{End}^0(B_1))_{/\mathbb{F}_{q^n}}$ admits a CM lifting to characteristic 0.

Question (I)

(I) CM *lifting up to isogeny*: there exists an $\mathbb{F}_q$-isogeny $(B,\beta) \to (B_1,\beta_1)$ such that (B_1,β_1) lifts to char. 0.

Spelled out: there exists

- an $\mathbb{F}_q$-isogeny $\delta : (B,\beta) \to (B_1,\beta_1)$, where $\beta_1 : K \to \mathrm{End}^0(B_1)$ is the CM structure induced by the isogeny δ from (B,β)
- a local domain R with characteristic 0 and residue field $\mathbb{F}_q$,
- an abelian scheme A over R with relative dimension g
- a ring homomorphism $\alpha : K \to \mathrm{End}^0(A) := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(A)$,
- an isomorphism $\phi : (A,\alpha)_{\mathbb{F}_q} \simeq (B_1,\beta_1)$ of CM structures over $\mathbb{F}_q$.

Question (IN)

(IN) CM *lifting to normal domains up to isogeny*:

Spelled out: there exists a *normal* local domain R with characteristic 0 and residue field $\mathbb{F}_q$ such that (I) is satisfied for (B,β) using R.

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Answers

1. (CML) is false. (The size of fields of definition is an obstruction, leading to ubiquitous counterexamples; Oort 1992) ▸ Failure of CML at the level of p-divisible groups
2. (I) CM lifting up to isogeny holds.
3. (IN) CM lifting over normal domain up to isogeny:
   - There is an obstruction, from the sizes of the residue fields above p of reflex fields for CM types compatible with (B,β). ▸ statement of the residual reflex condition
   - The above *residual reflex condition* is the only obstruction.

Basic method:
Localize to CM lifting questions for p-divisible groups.

Open question (sCML)

(sCML) *strong* CM lifing:

Every CM structure $(B, \mathscr{O}_K \to \mathrm{End}(B))$ over a finite field admits a CM lifting to characteristic 0.

(i.e. all obstructions to CML for $(B, K \to \mathrm{End}(B))$ disappear if the whole ring of integers $\mathscr{O}_L$ operates on B.)

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CM types

§2 CM theory over reflex fields

2.1 CM theory for abelian varieties

Notation

- K is a CM field; K_0 is the max. totally real subfield of K.
- ι denotes the complex multiplication.
- $\underline{K}^\times = \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$.
- Φ is a CM type for K, i.e. $\mathrm{Hom}(K,\overline{\mathbb{Q}}) = \Phi \sqcup \iota\cdot\Phi$.
- μ_Φ is the cocharacter of $\underline{K}^\times$ corresponding to Φ.
- $E(K,\Phi) \subset \overline{\mathbb{Q}}$ is the reflex field of (K,Φ); it is the field of definition of μ_Φ.

Reflex norm

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- $\mathrm{N}\mu_\Phi : \underline{E(K,\Phi)}^\times \to \underline{K}^\times$ is the reflex norm of (K,Φ).

  Recall: $\mathrm{N}\mu_\Phi$ is the unique $\mathbb{Q}$-homomorphism from $\underline{E(K,\Phi)}^\times$ to $\underline{K}^\times$ which sends the cocharacter of $\underline{E(K,\Phi)}^\times$ corresponding to the inclusion $E(K,\Phi) \hookrightarrow \overline{\mathbb{Q}}$ to the cocharacter μ_Φ of $\underline{K}^\times$.
- Let $M(K,\Phi)$ be the field of moduli for (K,Φ).

  Recall: $M(K,\Phi)$ is the unramified abelian extension of $E = E(K,\Phi)$ corresponding to the subgroup of $\mathbb{A}_E^\times$ consisting of all ideles $s = (s_\infty, s_f) \in \mathbb{A}_E^\times$ such that $\exists x \in K^\times$ satisfying

$$\mathrm{N}\mu_\Phi(s_f)\cdot\mathscr{O}_K = x\cdot\mathscr{O}_K \ \text{ and } \ \mathrm{Nm}_{K/K_0}(x) = |s_f|_{\mathbb{A}_E}^{-1}$$

Main theorem of complex multiplication

Theorem

1. (Shimura-Taniyama) *Let* $(A_{/F}, K \to \mathrm{End}^0(A))$ *be a CM abelian variety over a number field* $F \subset \overline{\mathbb{Q}}$ *of CM type* Φ. *Then* $F \supset E(K,\Phi) =: E$. *Moreover there exists an algebraic Hecke character* $\varepsilon : \mathbb{A}_F^\times \to K^\times$ *such that* $\varepsilon|_{F^\times} = (\mathrm{N}\mu_\Phi \circ \mathrm{Nm}_{F/E})|_{E^\times}$ *and the idele class character*

$$\varepsilon \cdot (\mathrm{N}\mu_\Phi^{-1} \circ \mathrm{Nm}_{F/E}) : \mathbb{A}_F^\times / F^\times \to (K \otimes \mathbb{Q}_\ell)^\times$$

corresponds to the ℓ*-adic representation attached to* $\mathrm{T}_\ell(A)$ *for every prime number* ℓ *(via* arithmetically *normalized CFT).*

2. (Casselman) *Every algebraic Hecke character* $\varepsilon : \mathbb{A}_F^\times \to K^\times$ *with algebraic part* $\mathrm{N}\mu_\Phi \circ \mathrm{Nm}_{F/E}$ *comes from a* K*-linear CM abelian variety with CM type* Φ*, unique up to* K*-linear isogeny.*

Existence over the field of moduli

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Let K be a CM field. Let Φ be a CM type for K and let M be the field of moduli for (K,Φ).

Theorem

1. (Shimura) *There exists a CM abelian variety* $(A, \mathscr{O}_K \to \mathrm{End}^0(A))$ *of CM type* Φ *over* M.
2. *Given a finite set* S *of prime numbers, there exists a prime number* ℓ *prime to* S *and a CM abelian variety* $(A, \mathscr{O}_K \to \mathrm{End}^0(A))$ *of CM type* Φ *over* M *which*
   - *has good reduction outside* ℓ*, and*
   - *at most tamely ramified at all places above* ℓ.

   In particular A *has good reduction at all places of* M *above* S.

p-adic CM types and reflex norms

2.2 CM theory for p-divisible groups

Notation

- L is a finite extension field of $\mathbb{Q}_p$.
- $\Psi \subset \mathrm{Hom}_{\mathbb{Q}_p}(L, \overline{\mathbb{Q}}_p)$ is a p-adic CM type for L.
- μ_Ψ is the cocharacter of the $\mathbb{Q}_p$-torus $\underline{L}^\times := \mathrm{Res}_{L/\mathbb{Q}_p}\mathbb{G}_m$ corresponding to Ψ.
- $\mathrm{N}\mu_\Psi : \underline{E(\Psi)}^\times \to \underline{L}^\times$ is the *reflex norm* for (L, Ψ).

  It is the unique $\mathbb{Q}_p$-homomorphism which sends the cocharacter of $\underline{E(\Psi)}^\times$ corresponding to the embedding $E(\Psi) \hookrightarrow \overline{\mathbb{Q}}_p$ to μ_Ψ.

CM theory for p-divisible groups

Theorem

1. *There exists an $\mathscr{O}_L$-linear CM p-divisible group $(X, \mathscr{O}_L \to \mathrm{End}(X))$ of CM type (L, Ψ) over the reflex field $E(L, \Psi) =: E$.*
2. $\mathrm{N}\mu_\Psi^{-1}|_{\mathscr{O}_E^\times} \to \mathscr{O}_L^\times$ *corresponds to the restriction to the inertia subgroup of the Galois representation attached to X (via* arithmetically *normalized local CFT).*
3. *If $(Y, L \to \mathrm{End}^0(X))$ is an L-linear CM p-divisible group of p-adic CM type (L, Ψ), then $L \supset E$ and the base change of X and Y to $\mathscr{O}_{L^{\mathrm{ur}}}$ are L-linearly isogenous.*

(I) holds

§3 Existence of CM lifting up to isogeny

Theorem

Let K be a CM field. Given a finite field $\mathbb{F}_q$, and a CM structure $(B, K \to \mathrm{End}^0(B))$ over $\mathbb{F}_q$ with $[K:\mathbb{Q}] = 2\dim(B)$, there exists an $\mathbb{F}_q$-isogeny $B \to B_1$ and a lifting $(A, K \to \mathrm{End}^0(A))$ of the induced CM structure $(B_1, K \to \mathrm{End}^0(B_1))$ to a local integral domain of characteristics $(0,p)$ and residue field $\mathbb{F}_q$.

A toy model

- p is a prime number with $p \equiv 2,3 \pmod 5$.
- C_0 is a $\mathbb{Z}[\zeta_5]$-linear abelian surface over $\mathbb{F}_{p^2}$ such that $\mathrm{Fr}_{C_0} = p\cdot\zeta_5$.

First properties:

- $(C_0, \mathbb{Z}[\zeta_5] \to \mathrm{End}(C_0))$ cannot be lifted to char. 0 (The CM type Φ of such a lifting $(A, \mathbb{Z}[\zeta_5] \to \mathrm{End}(A))$ is determined by the action of $(\mathbb{Z}[\zeta_5]/p) \cong \mathbb{F}_{p^4}$ on $\mathrm{Lie}(A)$, which forces Φ to be stable under cpx. conj .)
- $C_0[p] \cong \alpha_p \times \alpha_p$.

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How to construct of CM lifting of the toy model

1. $\exists$ a $\mathbb{Z}[\zeta_5]$-linear isogeny $\delta\colon A_0 \to C_0 \times_{\mathrm{Spec}(\mathbb{F}_{p^2})} \mathrm{Spec}(\mathbb{F}_{p^4})$ s.t. $\mathrm{Ker}(\delta) \cong \alpha_p$ is the unique subgroup scheme of order p of A_0.
2. The action of $\mathbb{Z}[\zeta_5]$ on the Lie algebra of the $\mathbb{Z}[\zeta_5]$-linear formal lifting $\mathfrak{A}$ of A_0 over $W(\mathbb{F}_{p^4})$ is a CM type of $\mathbb{Q}(\zeta_5)$. So $\mathfrak{A}$ algebraizes to a $\mathbb{Z}[\zeta_5]$-linear abelian scheme A over $W(\mathbb{F}_{p^4})$.
3. Over a suitable finite flat local ring R over $W(\mathbb{F}_{p^4})$, we have a finite flat subgroup $\mathfrak{G}$ of $A \times_{\mathrm{Spec}\, W(\mathbb{F}_p^4)} \mathrm{Spec}\, R$ of degree p. Then $A/\mathfrak{G}$ is a $(\mathbb{Z}+p\mathbb{Z}[\zeta_5])$-linear lifting of a base field extension of the toy model C_0 (because the closed fiber of $\mathfrak{G}$ has to be $\mathrm{Ker}(\delta)$).
4. Conclude by an argument using the deformation theory.

Skip proof of (I)

Localize the CM lifting problem (I)

Sketch of a proof of (I)

Reduction Step 1. By the Serre-Tate theorm, an algebraicity criterion for CM formal abelian schemes and the deformation theory argument used for the toy model, reduce (I) to the following problem:

Given an $\mathscr{O}_K$-linear abelian variety B over $\mathbb{F}_q$ with $2\dim(B) = [K:\mathbb{Q}]$ and a p-adic place v of K_0; write K_v for $K \otimes_{K_0} K_{0,v}$. Construct

1. an $\mathscr{O}_{K,v}$-linear $\mathbb{F}_q$-isogeny $B[v^\infty] \to X$ of p-divisible groups over $\mathbb{F}_q$,
2. a K_v-linear p-divisible group $(\mathfrak{X}, K_v \to \mathrm{End}^0(\mathfrak{X}))$ over a characteristic 0 local domain R with residue field $\overline{\mathbb{F}}_p$ whose closed fiber is (the base extension to $\overline{\mathbb{F}}_p$ of) $(X, K_v \to \mathrm{End}^0(X))$.

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Lie types of $\mathscr{O}_{K_v}$-linear p-div. groups in char. p

Definition

The Lie type of an $\mathscr{O}_{K_v}$-linear p-divisible group Z of height $[K_v : \mathbb{Q}_p]$ over a finite field $\kappa \supset \mathbb{F}_p$ is the class $[\mathrm{Lie}(Z)]$ in the Grothendieck group $\mathrm{R}(\mathscr{O}_{K,v} \otimes \overline{\mathbb{F}}_p)$ of all finitely generated $(\mathscr{O}_{K,v} \otimes \overline{\mathbb{F}}_p)$-modules.

- $\exists$ an explicit combinatorial description of $\mathrm{R}(\mathscr{O}_{K,v} \otimes \overline{\mathbb{F}}_p)$ with a natural action by $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ s.t. $[\mathrm{Lie}(Z)]$ is κ-rational.
- $\exists$ a combinatorial notion of slopes for elements of $\mathrm{R}(\mathscr{O}_{K,v} \otimes \overline{\mathbb{F}}_p)$, compatible with the definition of slopes of an K_v-linear p-divisible group.
- From the action of the "complex conjugation" for K/K_0 on $\mathrm{R}(\mathscr{O}_{K,v} \otimes \overline{\mathbb{F}}_p)$ we have a notion of *duality* for Lie types.
- $\exists$ a reduction map from the set of all p-adic CM types for K_v to $\mathrm{R}(\mathscr{O}_{K,v} \otimes \mathbb{F}_q)$. The reduction of the v-component Φ_v of a CM type Φ for K is self-dual.

Classification and weak descent with Lie types

Proposition

1. *(Chia-Fu Yu) An $\mathscr{O}_{K,v}$-linear p-divisible group of height $[K_v : \mathbb{Q}_p]$ over $\overline{\mathbb{F}}_p$ is determined up to (non-unique) isomorphism by its Lie type.*
2. *If Z is an $\mathscr{O}_{K,v}$-linear p-divisible group of height $[K_v : \mathbb{Q}_p]$ over a finite field $\kappa \subset \overline{\mathbb{F}}_p$ and ξ is a κ-rational Lie type for K_v with the same slopes as Z, then there exists an $\mathscr{O}_{K,v}$-linear κ-isogeny $Z \to Y$ such that ξ is the slope of the $\mathscr{O}_{K,v}$-linear p-divisible group Y over κ.*

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Tougher places for the CM lifting problem (I)

Recall that K is a CM field, v is a p-adic place of K_0 and $\kappa \supset \mathbb{F}_p$ is a finite field

Definition

A p-adic place v of K_0 is bad for (K, κ) if the following conditions hold.

- K/K_0 is unramified and inert above v; denote by w the p-adic place of L above v.
- $e(K/\mathbb{Q}, w) = e(K_0/\mathbb{Q}, v)$ is odd,
- $f_w \equiv 0 \pmod 4$,
- $[\kappa_w : (\kappa_w \cap \kappa)]$ is even.

Good places: change to a self-dual Lie type

Step 2: case when v is a good place for $(K, \mathbb{F}_q)$

Proposition

Suppose that v is a good place for $(K_v, \mathbb{F}_q)$.

- *$\exists$ an $\mathbb{F}_q$-rational self-dual Lie type ξ for K_v with the same slopes as $B[v^\infty]$. Hence $\exists$ an $\mathscr{O}_{K,v}$-linear p-divisible group X_v over $\mathbb{F}_q$ which is $\mathscr{O}_{K,v}$-linearly $\mathbb{F}_q$-isogenous to $B[v^\infty]$ with Lie type ξ.*
- *$\exists$ a p-adic CM type Ψ_v for K_v whose reduction modulo p is ξ such that $\mathrm{Hom}_{\mathbb{Q}_p}(K_v, \overline{\mathbb{Q}}_p) = \Psi_v \sqcup \iota \cdot \Psi_v$. Let $(\mathfrak{X}_v, \mathscr{O}_{K,v} \to \mathrm{End}(\mathfrak{X}_v))$ be an $\mathscr{O}_{K,v}$-linear p-divisible group over a char. $(0,p)$ discrete valuation ring R with residue field $\overline{\mathbb{F}}_p$ with CM type Ψ_v.*
- *The closed fiber of $(\mathfrak{X}_v, \mathscr{O}_{K,v} \to \mathrm{End}(\mathfrak{X}_v))$ is $\mathscr{O}_{K,v}$-linearly isomorphic to (the base extension of) $(X_v, \mathscr{O}_{K,v} \to \mathrm{End}(X_v))$.*

Bad places: move to a maximally symm. Lie type

Step 3: Reduction to the toy model

Proposition

Suppose that v is a bad place for $(K, \mathbb{F}_q)$.

- *$\exists$ an $\mathbb{F}_{p^2}$-rational Lie type ξ_s for K_v with the same slope as $B[v^\infty]$.*
- *$\exists$ an $\mathscr{O}_{K_v}$-linear p-divisible group Y_v over $\mathbb{F}_q$ with Lie type ξ_s and an $\mathscr{O}_{K,v}$-linear $\mathbb{F}_q$-isogeny $B[v^\infty] \to Y_v$.*
- *$\exists$ an $\mathscr{O}_{K_v}$-linear isomorphism from $Y_v \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_p$ to $\mathscr{O}_{K,v} \otimes_{\mathbb{F}_{p^4}} (C_0[p^\infty] \times_{\operatorname{Spec}\mathbb{F}_{p^2}} \operatorname{Spec}\overline{\mathbb{F}}_p)$. (tensor product of fpqc sheaves)*

From the CM lifting for the toy model we obtain a K_v-linear p-divisible group $\mathfrak{Y}_{\mathfrak{v}}$ over a char. $(0,p)$ d.v.r. R with residue field $\overline{\mathbb{F}}_p$ whose closed fiber is isomorphic to $(Y_v \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_p, K_v \to \operatorname{End}^0(Y_v))$.

(CML) fails if all fields of definition are large

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§4 Size of fields of definition as obstruction to (CML)

- Let L be a finite extension field of $\mathbb{Q}_p$,
- let $\Psi \subset \operatorname{Hom}_{\mathbb{Q}_p}(L, \overline{\mathbb{Q}}_p)$ be a p-adic CM type,
- $E = E(L, \Psi) \subset \overline{\mathbb{Q}}_p$ is the reflex field of (L, Ψ),
- κ_E is the residue field of $\mathscr{O}_E$,

Theorem

Let $(Z, L \to \operatorname{End}^0(Z))_{/\overline{\mathbb{F}}_p}$ be an L-linear p-divisible group over $\overline{\mathbb{F}}_p$ with $[L:\mathbb{Q}_p] = ht(Z)$. If $(Z, L \to \operatorname{End}^0(Z))$ admits a CM lifting $(Y, L \to \operatorname{End}^0(Y))$ to characteristic 0 of CM type Ψ, then it is isomorphic to the base extension of an CM p-divisible group $(Z_1, L \to \operatorname{End}^0(Z_1))_{\kappa_E}$ over κ_E.

Idea:

- $\exists$ an $\mathscr{O}_L$-linear CM p-divisible group $X_{/E}$ over $\mathscr{O}_E$ of CM type Ψ such that every torsion point of X is defined over a totally ramified extension field of E.
- Y and X are isogenous after suitable base change.

Failure of (CML)

- Let $L = L_1 \times \cdots L_r$, L_i be a finite extension field of $\mathbb{Q}_p$ for each $i = 1, \ldots, r$.
- Let $(Z, L \to \mathrm{End}^0(Z))_{/\overline{\mathbb{F}}_p}$ be an F-linear CM p-divisible group over $\overline{\mathbb{F}}_p$.

Proposition

Assume that the ***non-ordinary*** *part of Z is neither a one-dimensional p-divisible formal group nor the dual of a one-dimensional p-divisible formal group. For any given finite field $\kappa \supset \mathbb{F}_p$, there exists a p-divisible group Z' isogenous to Z which cannot be defined over κ.*

Corollary

There exists a p-divisible group Z' over $\overline{\mathbb{F}}_p$ isogenous to Z which does not admit a CM lifting to char. 0.

Back to answers of CM lifting questions

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How to produce a Z' "with large moduli"

Idea: In a $\mathbb{P}^1$-family of p-divisible groups isogenous to Z, $\exists Z'$ which cannot be defined over the composite of all $\kappa_{E(L_i,\Psi)}$'s, where $i = 1,\ldots,r$ and Ψ runs through all possible p-adic CM types for L_i.

Shimura-Taniyama formula for slopes

§5 CM lifting over normal domains up to isogeny

(Shimura-Taniyama formula for slopes)

If $(A, K \to \mathrm{End}^0(A))_{/R}$ is a CM abelian scheme over a noetherian normal local domain $R \subset \overline{\mathbb{Q}}_p$ of generic characteristic 0 and residue field $\mathbb{F}_q$, with p-adic CM type $\Phi \subset \mathrm{Hom}(K, \overline{\mathbb{Q}}_p)$, then

$$\frac{\mathrm{ord}_v(\mathrm{Fr}_{B,q})}{\mathrm{ord}_v(q)} = \frac{\#\{\phi \in \Phi \mid \phi \text{ induces } v \text{ on } K\}}{[K_v : \mathbb{Q}_p]}$$

Residual reflex condition

Necessary condition for (IN)

(Residual reflex condition) If (IN) holds for a given CM abelian variety $(B, K \to \mathrm{End}^0(B))_{/\mathbb{F}_q}$ over $\mathbb{F}_q$, then there exists a p-adic CM type $\Phi \subset \mathrm{Hom}(K, \overline{\mathbb{Q}}_p)$ for K such that

- the slopes of B are given by the Shimura-Taniyama formula, and
- $\kappa_{E,v} \subset \mathbb{F}_q$, where
  - v is the p-adic place of $E(K,\Phi)$ induced by $E(K,\Phi) \subset \overline{\mathbb{Q}}_p$,
  - $\kappa_{E(K,\Phi),v}$ is the residue field of $E(K,\Phi)$ at v.

Return to answers of CM lifting questions

Sufficiency of the residual reflex condition

CM LIFTINGS
Ching-Li Chai

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Theorem

If $(B, K \to \mathrm{End}^0(B))_{/\mathbb{F}_q}$ is a CM structure over a finite field $\mathbb{F}_q$ which satisfies the residual reflex condition, then (IN) holds for $(B, K \to \mathrm{End}^0(B))_{/\mathbb{F}_q}$.

- global proof: Casselman's theorem + a "surgery procedure" for algebraic Hecke characters.
- local proof: Serre-Tate theorem + CM theory for p-divisible groups.