HECKE ORBITS

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Unattributed results are due to suitable subsets of

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$\S$1. Moduli spaces and Hecke symmetries in char. p

$\widetilde{X}$: prime-to-p tower of modular variety of PEL-type, associated to a reductive group G over $\mathbb{Q}$, defined over a base field $k = \overline{k} \supset \mathbb{F}_p$.

The group $G(\mathbb{A}_f^{(p)})$ operates on the tower $\widetilde{X}$:

$$\begin{array}{l} \curvearrowleft G(\mathbb{A}_f^{(p)}) \\ \widetilde{X} = (\cdots \to \underbrace{X_n \to \cdots X_0 = X}_{G(\mathbb{Z}/n\mathbb{Z})})_{(n,p)=1} \end{array}$$

On the "bottom level" $X = X_0$, the symmetries from $G(\mathbb{A}_f^{(p)})$ induces *Hecke correspondences*.

Notation. $\mathcal{H}(x)$ = the prime-to-p Hecke orbits of x, $x \in X(k)$; it is a countable subset of $X(k)$.

Example 1. Siegel modular variety $\mathcal{A}_g$.

For $x = ([A_x, \lambda_x]) \in \mathcal{A}_g(k)$, $\mathcal{H}(x)$ consists of all $[A_y, \lambda_y]$ such that there exists a prime-to-p quasi-isogeny from A_x to A_y which preserves the polarizations.

Example 2. Hilbert modular varieties $\mathcal{M}_E$, E = a product of totally real fields.

For $x = [(A_x, \lambda)] \in \mathcal{M}_E(k)$, A_x an $\mathcal{O}_E$-abelian variety, $\mathcal{H}(x)$ consists of all $[(A_y, \lambda_y)]$ such that there exists a prime-to-p $\mathcal{O}_E$-linear quasi-isogeny between A_x and A_y which preserves the polarizations.

$\S$2. Leaves in the foliation of $\mathcal{A}_g$

Def. (Oort) For $x \in \mathcal{A}_g(k)$, denote by $\mathcal{C}(x)$ the constructible subset of $\mathcal{A}_g$, characterized by the following property: a geometric point $y \in \mathcal{A}_g(k)$, is in $\mathcal{C}(x)$ iff $\exists$ an isomorphism between $A_x[p^\infty]$ and $A_y[p^\infty]$ respecting the polarizations. We call $\mathcal{C}(x)$ the *leaf* on $\mathcal{A}_g$ passing through x.

Facts. $\mathcal{C}(x)$ is a locally closed subscheme of $\mathcal{A}_g$, and is smooth over k. Moreover, $\mathcal{C}(x)$ is stable under all prime-to-p Hecke correspondences

Rmk. (1) Similarly, one can define leaves for other PEL-type modular varieties.

(2) Can define leaves for any (polarized) Barsotti-Tate group over a noetherian reduced base scheme over k.

Prop. The Barsotti-Tate group $A[p^\infty]$ over $\mathcal{C}(x)$ admits a slope filtration

$$A[p^\infty] = G_0 \supset G_1 \supset \cdots \supset G_m \supset G_{m+1} = (0)$$

such that each graded piece $H_i = G_i/G_{i+1}$ is a Barsotti-Tate group over $\mathcal{C}(x)$ with a single slope λ_i, and $\lambda_0 < \ldots < \lambda_m$.

Rmk. Due to Zink when $C(x)$ is a "central stream".

§3. The Hecke orbit conjecture

Conj. (HO): $\mathcal{H}(x)$ is Zariski dense in $\mathcal{C}(x)$.

Conj. (HO)$_{\mathrm{ct}}$: $\dim(\overline{\mathcal{H}(x)}) = \dim(\mathcal{C}(x))$; equiv. $\overline{\mathcal{H}(x)}$ contains the irreducible component of $\mathcal{C}(x)$ passing through x.

Conj. (HO)$_{\mathrm{dc}}$: The prime-to-p Hecke correspondences operate transitively on the set of geometrically irreducible components of $\mathcal{C}(x)$.

Rmk. Clearly $(\mathrm{HO}) \Longleftrightarrow (\mathrm{HO})_{\mathrm{ct}} + (\mathrm{HO})_{\mathrm{dc}}$

Rmk. Conj. (HO) says that foliation structure on $\mathcal{A}_g$ is determined by the Hecke symmetries.

Rmk. Previously known case: When A_x is an ordinary abelian variety (CLC, 1995)

Rmk. Similarly, have Conj (HO) for other modular varieties. Known case: PEL-type C, A_x ordinary.

Main Thm. The Hecke orbit conjecture (HO) holds for the Siegel modular variety $\mathcal{A}_g$. In other words, every prime-to-p Hecke orbit is Zariski dense in the leaf containing it.

Thm. The Hecke orbit conjecture (HO) holds for Hilbert modular varieties.

Rmk. The latter result is used in the proof of the main Thm.

$\S$4. ℓ-adic monodromy of leaves

Thm. Let ℓ be a prime number, $(\ell, p) = 1$. Let Z be a smooth locally closed subvariety of $\mathcal{A}_g$ over k.
Assume:

(i) Z is stable under all ℓ-adic Hecke correspondences.

(ii) The ℓ-adic Hecke correspondences operate transitively on the set of irreducible components of Z.

(iii) $A \to Z$ is not supersingular.

Then (1) The image of the ℓ-adic monodromy representation of $A \to Z$ is $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$.

(2) Z is irreducible.

(3) Z is stable under all prime-to-p Hecke correspondences on $\mathcal{A}_g$.

§5. The Local Stabilizer Principle

Let $Z \subset \mathcal{A}_g$ be a reduced closed subscheme stable under all prime-to-p Hecke correspondences.

$x = ([A_x, \lambda_x]) \in Z(k)$, $E := \mathrm{End}_k(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}$.

$*$:= the involution of E induced by λ_x.

$H := \{u \in E^\times | u \cdot u^* = u^* \cdot u \in \mathbb{Q}_p^\times\}$

$U_x := H \cap \mathrm{End}_k(A_x)^\times$, called the local stabilizer subgroup at x.

There is a natural action of U_x on $\mathcal{A}_g^{/x}$.

Prop. (**local stabilizer principle**) Notation as above. Then the closed formal subscheme $Z^{/x}$ of $\mathcal{A}_g^{/x}$ is stable under the action of the local stabilizer subgroup U_x on $\mathcal{A}_g^{/x}$.

The local stabilizer principle is the foundation of our approach to (HO)$_{\mathrm{cont}}$.

A. Techniques to exploit the local stabilizer principle:

- canonical coordinates on leaves
- rigidity of subgroups of p-divisible formal groups
- hypersymmetric points

B. In addition, we have a **Hilbert trick**, special to $\mathcal{A}_g$. Its effects include

$\star$ existence of hypersymmetric points in $\overline{\mathcal{H}(x)} \cap C(x)$

- "splitting at supersingular points" (another trick, using also the local stabilizer principle)

§6. Logical interdependencies

(0) $(\mathrm{HO}) \iff (\mathrm{HO})_{\mathrm{ct}} + (\mathrm{HO})_{\mathrm{dc}}$

(1) If $x \in \mathcal{A}_g(k)$ is not supersingular, then

$$(\mathrm{HO})_{\mathrm{dc}} \text{ for x} \iff \mathcal{C}(x) \text{ is irreducible}$$

(2) Suppose that $x, y \in \mathcal{A}_g(k)$, and there is a quasi-isogeny from A_x to A_y which preserves the polarizations. Then

$$\mathrm{HO}_? \text{ for x} \iff \mathrm{HO}_? \text{ for y}$$

where the subscript $?$ is either empty, or an element of $\{\mathrm{ct}, \mathrm{dc}\}$.

(3) Let W_ξ^0 be a non-supersingular "open" Newton polygon stratum on $\mathcal{A}_g$, and let $\mathcal{C}$ be a leaf in W_ξ^0. Then

$$W_\xi^0 \text{ is irreducible } \Longrightarrow \mathcal{C} \text{ is irreducible}\,.$$

(4) (HO) for Hilbert modular varieties

$\Longrightarrow$ $\exists$ hypersymmetric points on every leaf of a Hilbert modular variety $\mathcal{M}_F$ s.t. $F \otimes \mathbb{Q}_p$ is a field

$\Longrightarrow$ $\exists$ hypersymmetric points on every leaf in $\mathcal{A}_g$

§7. Canonical coordinates for leaves

Will describe the local structure of a leaf in two "essential cases".

Notation.

- X, Y: Barsotti-Tate groups over k, purely of Frobenius slopes μ_X, μ_Y, $\mu_X < \mu_Y$.
- $\mathrm{Spf}(R) =$ the local deformation space of $X \times Y$.

In the unpolarized case:
$\mathrm{Spf}(R) \supset \mathcal{C}_{\mathrm{up}}^{\wedge} :=$ the leaf through the closed point, for the universal deformation of $X \times Y$.

In the polarized case:
$\lambda =$ a principal polarization of $X \times Y$,
$\mathrm{Spf}(R/I) =$ local deform. space of $(X \times Y, \lambda)$,
$\mathrm{Spf}(R/I) \supset \mathcal{C}^{\wedge} :=$ the leaf in $\mathrm{Spf}(R/I)$ through the closed point.

Phenomenon: The slope filtration "determines" the local completions $\mathcal{C}^{\wedge}_{\mathrm{up}}$ and $\mathcal{C}^{\wedge}$.

$\mathcal{D}\mathcal{E}(X,Y) :=$ the extension part of the deformation space $\mathrm{Spf}(R)$, with a natural structure as a smooth formal group.

Thm. (1) (unpolarized case) $\mathcal{C}^{\wedge}_{\mathrm{up}}$ is the maximal p-divisible formal subgroup of $\mathcal{D}\mathcal{E}(X,Y)$. The p-divisible group $\mathcal{C}^{\wedge}_{\mathrm{up}}$ has slope $\mu_Y - \mu_X$.

(2) (polarized case) The polarization λ induces an involution on $\mathcal{C}^{\wedge}_{\mathrm{up}}$, whose fixer subscheme in $\mathcal{C}^{\wedge}_{\mathrm{up}}$ is $\mathcal{C}^{\wedge}$.

Let $\mathrm{M}(X), \mathrm{M}(Y)$ be the covariant Dieudonné module of X, Y respectively.

$B(k) :=$ the fraction field of $W(k)$.

$\mathrm{Hom}_{W(k)}(\mathrm{M}(X), \mathrm{M}(Y)) \otimes_{W(k)} B(k)$ has a natural structure as a V-isocrystal.

Thm. (1) There exists a natural isomorphism of V-isocrystals from $\mathrm{M}(\mathcal{C}_{\mathrm{up}}^{\wedge}) \otimes_{W(k)} B(k)$ to $\mathrm{Hom}_{W(k)}(\mathrm{M}(X), \mathrm{M}(Y)) \otimes_{W(k)} B(k)$

(2) In the polarized case, the polarization induces an isomorphism from $\mathrm{M}(\mathcal{C}^{\wedge}) \otimes_{W(k)} B(k)$ to the symmetric part of the isocrystal $\mathrm{Hom}_{W(k)}(\mathrm{M}(X), \mathrm{M}(Y)) \otimes_{W(k)} B(k)$

Cor. (1) In the unpolarized case, we have
$h(\mathcal{C}^{\wedge}_{\mathrm{up}}) = h(X) \cdot h(Y)$, and
$\dim(\mathcal{C}^{\wedge}_{\mathrm{up}}) = (\mu_Y - \mu_X) \cdot h(X) \cdot h(Y)$.

(2) In the polarized case, we have
$h(X) = h(Y)$, $h(\mathcal{C}^{\wedge}) = \frac{h(X)(h(X)+1)}{2}$, and
$\dim(\mathcal{C}^{\wedge}) = \frac{1}{2}(\mu_Y - \mu_X)h(X)(h(X)+1)$.

Cor. Let A_x be a g-dimensional principally polarized abelian variety over k. Suppose that $A_x[p^\infty]$ has Frobenius slopes $\mu_1 < \mu_2 < \cdots < \mu_m$, Let h_i be the multiplicity of μ_i. (So we have
$\mu_i + \mu_{m-i+1} = 1, h_i = h_{m-i+1}, \sum_{i=1}^m h_i = 2g$,
$\sum_{i=1}^m h_i\mu_i = g$.)
Then
$\dim(C(x)) = \frac{1}{2}\sum_{i<j,\, i+j\neq 1} (\mu_j - \mu_i) \cdot h_i \cdot h_j$
$+\frac{1}{2}\sum_{2i\leq m} (1 - 2\mu_i) \cdot h_i(h_i + 1)$.

§8. Rigidity of subgroups of p-divisible formal groups

Notation.

- X is a p-divisible formal group over k.
- $\mathbf{r}_X$:= the regular repr. of $\mathrm{End}(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$
- H is a connected reductive group over $\mathbb{Q}_p$.
- $\rho : H(\mathbb{Q}_p) \to (\mathrm{End}(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\times$ is a $\mathbb{Q}_p$-rational representation of H
- $U \subset H(\mathbb{Q}_p)$ is an open subgroup of $H(\mathbb{Q}_p)$ such that $\rho(U) \subseteq \mathrm{End}_k(X)^\times$, so that U operates on X via ρ.
- Z is an irreducible closed formal subscheme of X which is stable under the action of U.

Thm. Assume that $\mathbf{r}_X \circ \rho$ does not contain the trivial representation as a subquotient. Then Z is a p-divisible formal subgroup of X.

$\S$9. Hypersymmetric points

Def. An abelian variety A over k is *hypersymmetric* if $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} \mathrm{End}_k(A[p^\infty])$.

Thm. Let $[(A_x, \lambda_x)]$ be a point of $\mathcal{A}_g(k)$ such that A_x is hypersymmetric and A_x is isomorphic to a product $B_1 \times \cdots \times B_r$, where each B_i is an abelian variety over k, and each B_i has at most two slopes. Then the the prime-to-p Hecke orbit $\mathcal{H}(x)$ is Zariski dense in the leaf $\mathcal{C}(x) \subset \mathcal{A}_g$.

Rem. The trick “splitting at supersingular points” can be used to simplify the proof of the Thm.

§10. The Hilbert trick

Notation.

• $x \in \mathcal{A}_g(\overline{\mathbb{F}_p})$

• $E =$ a maximal commutative semisimple subalgebra of $\mathrm{End}(A_x) \otimes \mathbb{Q}$ fixed by the Rosati involution

• $\mathcal{M} =$ the Hilbert modular variety attached to E

Prop. There exists a non-empty open-and-closed subscheme $\mathcal{M}_0$ of $\mathcal{M}$ and a finite morphism $f : \mathcal{M}_0 \to \mathcal{A}_g$ with the following properties.

(1) $\mathcal{M}_0$ is stable under all prime-to-p Hecke correspondences coming from from $\mathrm{SL}(2, E)$.

(2) $\exists$ a point $y_0 \in \mathcal{M}_0(k)$ s.t. $f(y) = x$.

(3) f is equivariant w.r.t. the prime-to-p Hecke correspondences on $\mathcal{M}_0$ and $\mathcal{A}_g$.

(4) f comes from an isogeny correspondence between the universal abelian schemes.

§11. Splitting at supersingular points

Prop. Let $\mathcal{C}$ be a leaf in $\mathcal{A}_g$ over an algebraically closed field k of characteristic p. Then there exists a point z_0 in the Zariski closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ such that A_{z_0} is a supersingular abelian variety over k.

Prop. Let $\mathcal{C}$ be a leaf in $\mathcal{A}_g$ over $\overline{\mathbb{F}_p}$. Then there exists a point $y_1 \in \mathcal{C}(\overline{\mathbb{F}_p})$ and abelian varieties $B_1, \ldots, B_r$ over $\overline{\mathbb{F}_p}$ such that A_y is isogenous to $B_1 \times \cdots \times B_r$, and each B_j has at most two slopes, $j = 1, \ldots, r$.

Rmk. The last Prop. is an offshoot of the stabilizer principal and the Hilbert trick; it uses the action of the local stabilizer subgroup at a supersingular point in the closure of $\mathcal{C}$.

§12. Outline of the proof

Thm. (Oort) Every non-supersingular Newton polygon stratum in $\mathcal{A}_{g,n}$ is irreducible.

Cor. (HO)$_{\mathrm{dc}}$ holds for $\mathcal{A}_g$. (By (4) of §6.)

Rem. The above Thm. is uses Oort's results on the Grothendieck conjecture on the Newton polygon stratification, the Thm. on ℓ-adic monodromy of leaves in §4, and a standard degeneration argument.

Proof of (HO)$_{\mathrm{ct}}$.

1. We have seen that it suffices to show that there are hypersymmetric points on the closure of any given Hecke orbit in the leaf containing it.

2. Since there are hypersymmetric points on every leaf in a Hilbert modular variety attached to a totally real field F such that $F \otimes \mathbb{Q}_p$ is a field, we only have to prove (HO) for the Hilbert modular varieties, as we saw in $\S$6 (4).

3. (HO)$_{\mathrm{ct}}$ for Hilbert modular varieties follows from canonical coordinates, rigidity, and "known arguments". So it remains to prove (HO)$_{\mathrm{dc}}$ for Hilbert modular varieties.

4. The proof of (HO)$_{\mathrm{dc}}$ for Hilbert modular varieties uses the Lie-alpha stratification on Hilbert modular varieties, which has the following properties: $\forall$ Newton polygon stratum W^0_ξ on $\mathcal{M}_F$, $\exists$ a leaf $\mathcal{C} \subset W^0_\xi$ s.t. $\mathcal{C}$ is an open subset of some Lie-alpha stratum of $\mathcal{M}_F$.

5. The last and crucial step was done by C.-F. Yu, who constructed enough deformations to facilitate an induction on the partial ordering coming from the incidence relation among the Lie-alpha strata.

§13. Maximality of p-adic monodromy of leaves

Let S be an irreducible normal scheme of finite type over $\overline{\mathbb{F}_p}$. Let $A \to S$ be an abelian variety over S which is fiberwise geometrically constant. One can defined p-adic monodromy groups of different flavors, G_{naive}, G_{oconv}, and G_{conv}. The last two uses the $\otimes$-category of overconvergent and convergent F-isocrystals over S generated by $\mathbb{D}(A[p^\infty] \to S)$.

Thm. The naive p-adic monodromy of any leaf in $\mathcal{A}_g$ is "as large as possible". The same holds for the overconvergent and convergent monodromy groups.

Idea of proof: Use a hypersymmetric point as base point, and the fact that the naive p-adic monodromy group for Hilbert modular varieties are "as large as possible".

§14. Conjectures and outlook

Monodromy Conjecture. Notation as before. (1) G_{oconv} is reductive, and is “the same” as the ℓ-adic monodromy group for $A \to S$.

(2) G_{conv} is a parabolic subgroup of A.

(3) G_{naive} is a Levi subgroup of G_{conv}

Rem. (a) There exists a global version of the theory of canonical coordinates on leaves.

(b) G_{conv} is a semi-direct product of G_{naive} with a unipotent group defined by the theory of canonical coordinates.

Tate linear subvarieties. Over the ordinary locus of $\mathcal{A}_g$, one can define a notion of *Tate-linear subvarieties*; Tate-linear means being a formal subtorus in the Serre-Tate formal torus.

Analytic propagation: Tate-linear at one point implies Tate-linear everywhere.

Conj. (T) A Tate-linear subvariety is the reduction of a Shimura subvariety.

Conj. (AO) Every irreducible component of the Zariski closure of a collection of *ordinary* hypersymmetric points in $\mathcal{A}_g$ is the reduction of a Shimura subvariety.

Expectation: Can generalize the notion of Tate-linear subvarieties to subvarieties in a leaf, and formulate an analogue of the above conjectures.