Families of Abelian Varieties in Positive Characteristics

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§1. Modular varieties with Hecke symmetries

Consider (a special class of) Shimura varieties $(\tilde{M} = (M_i)_i, G)$:

G : a connected reductive group over $\mathbb{Q}$,

$\widetilde{M}$: a moduli space of abelian varieties with prescribed symmetries of a fixed type ($\longleftrightarrow$ a fixed faithful representation of G).

$\widetilde{M} = (M_i)_i \ni x \longleftrightarrow$

- an abelian variety A, with (full) level structure
- Hodge cycles/algebraic cycles $(s_\alpha)_\alpha$ in the tensors constructions of $\mathrm{H}_1(A)$
- $G =$ the fixer subgroup of the s_α's in $\mathrm{GL}(\mathrm{H}_1(A))$

Hecke symmetries

The group of finite adelic points of G operates on the tower $\widetilde{M}$:

$$\curvearrowleft G(\mathbb{A}_f) \quad (\,\mathrm{or,}\ G(\mathbb{A}_f^{(p)})\,)$$
$$\widetilde{M} = (\cdots \to \underbrace{M_n \to \cdots M_0 = M}_{G(\mathbb{Z}/n\mathbb{Z})})$$

On the "bottom level" $M = M_0$, the symmetries from $G(\mathbb{A}_f)$ induces **Hecke correspondences**, parametrized by $G(\widehat{\mathbb{Z}})\backslash G(\mathbb{A}_f)/G(\widehat{\mathbb{Z}})$.

Remark. (i) On a finite level $M_n \to M_0 = M$, the symmetry subgroup preserving the covering map is $G(\mathbb{Z}/n\mathbb{Z})$.

(ii) One uses $\mathbb{A}_f^{(p)} := \prod_{\ell, \ell \neq p} \otimes_{\mathbb{Z}} \mathbb{Q}$ instead of $\mathbb{A}_f := \prod_{\ell} \otimes_{\mathbb{Z}} \mathbb{Q}$ if the base ring is over $\mathbb{Z}_{(p)}$ or $\mathbb{F}_p$.

Examples:

1. The modular curve

(a) $G = \mathrm{GL}_2$.

(b) $M_0 =: \mathcal{A}_1$ parameterizes elliptic curves.

(c) $\mathcal{A}_1(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \cong \mathrm{GL}_2(\mathbb{Z})\backslash(\mathbb{C} - \mathbb{R})$.

2. The Siegel modular variety $\mathcal{A}_g$

(a) $G = \mathrm{GSp}_{2g}$.

(b) $M_0 =: \mathcal{A}_g$ parametrizes g-dimensional principally polarized abelian varieties.

(c) $\mathcal{A}_g(\mathbb{C})$ is isomorphic to the quotient of the Siegel upper-half space $\mathbb{H}_g$ by $\mathrm{Sp}_{2g}(\mathbb{Z})$.

3. The Hilbert-Blumenthal varieties

(a) $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$, where F is a (product of) totally real field(s).

(b) M_0 parametrizes $[F : \mathbb{Q}]$-dimensional abelian varieties with endomorphism by $\mathcal{O}_F$.

4. The Picard varieties

(a) G is a quasi-split group of type $\mathrm{U}(m, n)$, split by an imaginary quadratic field K.

(b) M_0 parametrizes $(m + n)$-dimensional abelian varieties A with endomorphism by $\mathcal{O}_K$, such that the action of $\mathcal{O}_K$ on $\mathrm{Lie}(A)$ has type (m, n).

§2. Fine structures in char. p

The base field is $\mathbb{F}_p$ or $\overline{\mathbb{F}_p}$ from now on. We only consider the good reduction case: M is smooth over $\overline{\mathbb{F}_p}$.

Source of fine structure on M (or $\widetilde{M}$): Every family of abelian varieties $A \to S$ gives rise to a *Barsotti-Tate group*

$$A[p^\infty]_S := \varinjlim_n A[p^n]_S \,,$$

an inductive system of finite locally free group schemes $A[p^n] := \mathrm{Ker}([p^n] : A \to A)$; the *height* of $A[p^\infty]$ is $2g = 2\dim(A/S)$.

The Barsotti-Tate group $A[p^\infty]_S$ is closely related to the first crystalline homology of A. The Frobenius $F_A : A \to A^{(p)}$ and Verschiebung $V_A : A^{(p)} \to A$ passes to $A[p^\infty]$.

A. The slope stratification

The slopes of a Barsotti-Tate group $A[p^\infty]$ over a field $k/\mathbb{F}_p$ is a sequence $2g$ of rational numbers

$$\lambda = (\lambda_j)\,, \quad 0 \leq \lambda_1 \leq \cdots \leq \lambda_{2g} \leq 1,$$

such that $\lambda_j + \lambda_{2g+1-j} = 1$. The denominator of each λ_j divides its multiplicity. The slopes are defined using divisibility properties of iterations of the Frobenius.

Kottwitz extended the above notion of slopes when there are prescribed symmetries present. Grothendieck's theorem that the slope polygon "goes up" under specialization generalizes.

The notion of slopes defines a finite stratification

$$M = \coprod_{\lambda} M_{\lambda}$$

The Zariski closure of each stratum M_{α} is equal to a union of (smaller) strata.

Examples:

(a) $\mathcal{A}_1$ is the union of two strata. The open stratum has slopes $(0, 1)$, corresponding to *ordinary* elliptic curves. The closed (0-dimensional) stratum has slopes $(\frac{1}{2}, \frac{1}{2})$, corresponding to the supersingular abelian varieties.

(b) The open dense stratum of $\mathcal{A}_g$ corresponds to *ordinary* abelian varieties, with slopes $(0, \ldots, 0, 1, \ldots, 1)$. The minimal stratum of $\mathcal{A}_g$ corresponds to *supersingular* abelian varieties, with slopes $(\frac{1}{2}, \ldots, \frac{1}{2})$, and has dimension $\lfloor g^2/4 \rfloor$ (Li-Oort).

B. Oort's foliation structure on M

Replacing numerical invariants (such as slopes) by the **isomorphism types** of the geometric fibers of the universal family $(A \to M, G)$, one arrives at a much finer decomposition of the modular variety M. The locus of M with a fixed isomorphism type of $(A + \mathrm{symmetry})$ is called a *leaf*.

- Each leaf is a locally closed subset of M, smooth over $\overline{\mathbb{F}_p}$. (consequence of a general constructibility result of Zink.)

- (With a few exceptions) there are infinitely many leaves on M. For instance the leaf containing a supersingular point in $\mathcal{A}_g$ is *finite*.

- The dense open slope stratum of M is a leaf. For instance if there exists an ordinary fiber A_x, then the ordinary locus in M is a leaf.

§3. Characterize leaves by Hecke symmetries

Clearly the foliation structure of M is stable under all prime-to-p Hecke correspondences; this leads to the following conjecture on Hecke orbits (Oort):

CONJ (HO). The foliation structure is characterized by the prime-to-p Hecke symmetries: For each point $x \in M$, the prim-to-p Hecke orbit of x is dense in the leaf containing x.

Note: Each Hecke orbit is a countable subset of M.

Evidence for Conj. (HO) (and motivation for the foliation structure historically):

THM 1. *Conj. (HO) holds for the ordinary locus of $\mathcal{A}_g$. Every ordinary symplectic prime-to-p isogeny class is dense in $\mathcal{A}_g$. The same holds for modular varieties of PEL-type C. (CLC)*

Other Evidence:

- the stratum of $\mathcal{A}_g$ which is dense in the non-ordinary locus

- $\mathcal{A}_g$ for $g = 1, 2, 3$ (with Oort)

- the HB varieties (work in progress with C.-F. Yu). Write $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \oplus_i F_{\mathfrak{p}_i}$, $A_x[p^\infty] = \oplus A_x[\mathfrak{p}_i^\infty] =: B_i$. Each B_i has two slopes $\frac{r_i}{g_i}, \frac{s_i}{g_i}$ with multiplicity $g_i = [F_{\mathfrak{p}_i} : \mathbb{Q}_p]$. Then the dimension of the leaf passing through x is $\sum_i |r_i - s_i|$.

- the Hecke orbit of a "very symmetric" ordinary point of M (uses Tate-linear subvarieties, to be discussed later)

$\S$4. The ordinary locus:

(A) **Tate-linear subvarieties**: Let $x \in \mathcal{A}_g^{\mathrm{ord}}$, i.e. A_x is an ordinary abelian variety.

$\rightsquigarrow \mathcal{A}_g^{/x}$ has a natural structure as a formal torus (Serre-Tate).

(i) A closed subscheme $Z \subseteq \mathcal{A}_g^{\mathrm{ord}}$ is *Tate-linear* at a point $x \in Z$ if $Z^{/x} \subseteq \mathcal{A}_g^{/x}$ is a formal subtorus of $\mathcal{A}_g^{/x}$.

(ii) Z is a *Tate-linear subvariety* of $\mathcal{A}_g^{\mathrm{ord}}$ if it is Tate-linear at every point.

(iii) The same definition makes sense over $W(\overline{\mathbb{F}_p})$.

PROP 2. *Suppose that $Z \subseteq \mathcal{A}_g^{\mathrm{ord}}$ is Tate-linear at one point $x \in Z$. Let $f : Y \to Z$ be the normalization map of Z. Then f maps $Y^{/y}$ isomorphically to a formal subtorus of $\mathcal{A}_g^{/f(y)}$ for every $y \in Y$. In particular Z is Tate-linear if it is normal.*

Remark. The Zariski closure (in M^{ord}) of the Hecke orbit of any ordinary point of M is Tate-linear. ($\S$5, Cor. 6)

PROP 3 (Noot). *The (reduction of) each special (or, Shimura) subvariety of $\mathcal{A}_g^{\mathrm{ord}} \to \operatorname{Spec} W(\overline{\mathbb{F}_p})$ is Tate-linear.*

PROP 4. *Let $Z \subseteq \mathcal{A}_g^{\mathrm{ord}}$ be a Tate-linear subvariety over $\overline{\mathbb{F}_p}$. Then there exists a unique lifting of Z to a Tate-linear subscheme $Z^\wedge$, formally smooth over $W(\overline{\mathbb{F}_p})$, of the p-adic completion of $\mathcal{A}_g^{\mathrm{ord}} \to \operatorname{Spec} W(\overline{\mathbb{F}_p})$.*

Remark. The proof uses *canonical coordinates* of A/S (next).

(B) **canonical coordinates** This is a global version of the Serre-Tate coordinates for the local moduli.

$A \to S$: a family of ordinary abelian varieties $\rightsquigarrow$

$$q_A : \mathrm{T}_p(A[p^\infty]_{\text{ét}}) \otimes_{\mathbb{Z}_p} \mathrm{T}_p(A^t[p^\infty]_{\text{ét}}) \longrightarrow \nu_{S,p^\infty}$$

called the *canonical coordinates* of $A \to S$, where

- $\nu_{S,p^\infty} := \varprojlim_n \nu_{S,p^n}$, a projective limit of sheaves on $S_{\text{ét}}$
- the sheaf ν_{S,p^n} fits into an exact sequence of sheaves on $S_{\text{ét}}$

$$0 \to \mu_{p^n} \to \mathbb{G}_m \xrightarrow{[p^n]} \mathbb{G}_m \to \nu_{S,p^n} \to 0$$

Why ν_{S,p^n}: from Kummer theory (for the flat topology)

(C) p**-adic monodromy**

$A \to S$: a family of ordinary abelian varieties,
$s \in S(\overline{\mathbb{F}_p})$

$\rightsquigarrow$ a fiber functor $\omega_{s,\mathbb{Q}_p}$ over $\mathbb{Q}_p$, on the $\otimes$-category of convergent F-isocrystals over S generated by
$\mathbb{D}(A/S) \otimes_{W(\overline{\mathbb{F}_p})} B(\overline{\mathbb{F}_p})$

$\rightsquigarrow$ a subgroup $\Gamma_s \subseteq \mathrm{GSp}(\mathrm{H}_1(A_s))$ over $\mathbb{Q}_p$, the ("sophisticated") p-adic monodromy group for A/S with base point s

Explicit description of monodromy:

$\Gamma_s \subseteq \mathrm{GSp}(\mathrm{H}_1(A_s))$ consists of all matrices in block form

$$\begin{pmatrix} aA & B \\ 0 & D \end{pmatrix}, \qquad \text{with}$$

- $a \in \mathbb{Q}_p^\times$,
- (A, D) belongs to the “naive” p-adic monodromy attached to the toric and étale part of $A[p^\infty]$ respectively
- B belongs to the $\mathbb{Z}_p$-direct summand “cut out” by $\mathrm{Ker}(q_A)$.

§5. A local characterization of Tate-linear subvarieties

- k : an algebraically closed field of characteristic p.
- N: a free $\mathbb{Z}_p$-module of finite rank.
- T: a formal torus over k with cocharacter group N.
- L: a linear algebraic group over $\mathbb{Q}_p$
- $\rho : L \to \mathrm{GL}(N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$, a linear representation of L with no trivial subquotient

$\rightsquigarrow$ There exists an open subgroup U of $G(\mathbb{Q}_p)$ such that N is stable under $\rho(U)$, hence U operates on T.

THM 5. *Let $Z \subset T$ be an irreducible closed formal subscheme of T which is stable under the action of an open subgroup of $G(\mathbb{Q}_p)$. Then Z is a formal subtorus of T, i.e. $Z := N_1 \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}_m}$ for a $\mathbb{Z}_p$-direct summand N_1 of N.*

COR 6. *Let x be an ordinary point of M, i.e. A_x is an ordinary abelian variety. Then the Zariski closure of the prime-to-p Hecke orbit of x in M^{ord} is a Tate-linear subvariety of M^{ord}.*

Remark. Only a much weaker version of Thm. 5 is needed for Cor. 6: assume in addition that $k = \overline{\mathbb{F}_p}$ and Z is formally smooth over k.

§6. Conjectures for the ordinary locus

CONJ (HO$_{\mathrm{ord}}$). Let x be an ordinary point of M. Then the prime-to-p Hecke orbit of x is dense in M.

CONJ (TL). Every Tate-linear subvariety of $\mathcal{A}_g^{\mathrm{ord}}$ is the reduction of a special (or, Shimura) subvariety.

Remark. In the context of Cor. 4, Conj. (TL) means that the Tate-linear lifting $Z^\wedge$ is algebraic. (Moonen)

CONJ (SS). Let S be a subvariety of $\mathcal{A}_g^{\mathrm{ord}}$, let $A \to S$ be the restriction to S of the universal abelian scheme. Let $\Gamma_s^0 \subset \mathrm{GSp}(\mathrm{H}_1(A_s))$ be the neutral component of the p-adic monodromy group of A/S with base point $s \in S(\overline{\mathbb{F}_p})$. Then there exists a reductive subgroup $G_{/\mathbb{Q}_p}$ of $\mathrm{GSp}(\mathrm{H}_1(A_s))$ such that Γ_s^0 is equal to the intersection of $G_{/\mathbb{Q}_p}$ with the "Siegel parabolic subgroup" of $\mathrm{GSp}(\mathrm{H}_1(A_s))$.

Remark. Conj. (SS) implies the semi-simplicity of the naive p-adic monodromy for A/S.

CONJ (Ratl). The group $G_{/\mathbb{Q}_p}$ in Conj. (SS) is defined by (absolute) Hodge cycles of the canonical lifting of A_s; in particular G has a natural $\mathbb{Q}$-structure. For every prime $\ell \neq p$, $G(\mathbb{Q}_\ell)$ is equal to the neutral component of the ℓ-adic monodromy group of $A \to S$.

Remark. Conj. (Ratl) is a generalization of the Mumford-Tate conjecture for ordinary abelian varieties over function fields in characteristic p.

Relation between conjectures:

$$(\mathrm{Ratl}) \implies (\mathrm{SS})$$

$$(\mathrm{Ratl}) \implies (\mathrm{TL}) \implies (\mathrm{HO_{ord}})$$

§7. Results and Applications

(A) The conjectures (about the ordinary locus) hold for (product of) HB varieties and "fake" HB varieties.

(B) Thm. 5 and (A) gives the vanishing of Iwasawa's μ-invariant for Katz p-adic L-functions (Hida).

(C) If x is a "very symmetric point" ordinary point of M (i.e. the stabilizer subgroup of x is a *large* Levi subgroup L of G), then the prime-to-p Hecke orbit of x is dense in M. (The point is that the action of L on the character group of the Serre-Tate formal torus is *irreducible*. This easy result has application to L-values.)