# Leaves and local Hecke symmetry

Ching-Li Chai

[1] Institute of Mathematics
Academia Sinica

[2] Department of Mathematics
University Pennsylvania

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# Outline

# PEL type modular varieties

A PEL type modular variety $\mathscr{M}$ is the moduli space attached to a PEL input datum $\mathscr{D} = (D, *, \mathscr{O}_D, H, \langle\cdot,\cdot\rangle, h)$; points of $\mathscr{M}$ correspond to abelian varieties with imposed symmetry $(A, \rho : A \to A^t, \iota : \mathcal{O}_D \to \mathrm{End}(A), \text{level structure})$ whose $\mathrm{H}_1$ are modeled on the linear algebra structure $\mathscr{D}$.

Fix a prime number $p$, *unramified* for the PEL datum $\mathscr{D}$. We will focus on the equal characteristic $p$ situation unless otherwise specified: $\mathscr{M}$ is a moduli space over $\overline{\mathbb{F}}_p$.

Let $B = \mathrm{End}_D(H)$, with involution $*_B$ induced by $*$. Let $G = \mathrm{SU}(B, *_B)$ (or $\mathrm{GU}(B, *_B)$).

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# Hecke symmetry

Let $\widetilde{\mathscr{M}}$ be the prime-to-$p$ tower for $\mathscr{M}$; it is a profinite etale Galois cover of $\mathscr{M}$ with group $G(\hat{\mathbb{Z}}^{(p)})$.

The group $G(\mathbb{A}_f^{(p)})$ operates on $\widetilde{M}$, inducing Hecke correspondences on $\mathscr{M}$.
($\mathscr{M} = \widetilde{\mathscr{M}}/G(\hat{\mathbb{A}}^{(p)})$; the Hecke correspondences on $\mathscr{M}$ is the *remnant* of the $G(\mathbb{A}_f^{(p)})$-action on $\widetilde{\mathscr{M}}$.)

**Example**: $\mathscr{M} = \mathscr{A}_g$ = the moduli space classifying $g$-dimensional principally polarized abelian varieties, $G = \mathrm{Sp}_{2g}$ (or $\mathrm{GSp}_{2g}$)).

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# Local Hecke symmetry

Given a point $x \in \mathscr{M}(\overline{\mathbb{F}}_p)$, corresponding to a quadruple $(A_x, \rho_x : A_x \to A_x^t, \iota_x : \mathcal{O}_D \to \mathrm{End}(A_x), \text{level structure})$.

Let $\mathscr{M}^{/x}$ be the formal completion of $\mathscr{M}$ at $x$.

Let $H_x := \mathrm{U}(\mathrm{End}_D^0(A_x), *_{\mathrm{Ros}})(\mathbb{Z}_{(p)})$, and let $G_x := \mathrm{U}(\mathrm{End}_D^0(A_x[p^\infty]), *_{\mathrm{Ros}})(\mathbb{Z}_p)$.

The Serre-Tate deformation theorem implies that there is a natural action of the compact $p$-adic group $G_x$ on $\mathscr{M}^{/x}$, by "changing the marking".

This action can be regarded as a *local version* of the global Hecke symmetries.

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# Local stabilizer subgroups





We call $G_x$ the *local stabilizer subgroup* at $x$. The group $H_x$ can be thought of as the "intersection" of $G_x$ with the global Hecke symmetries on $\mathscr{M}$.

**Lema** (Local stabilizer principle) If a closed subvariety $Z \subset \mathscr{M}$ is stable under all Hecke symmetries, then $Z^{/x} \subset \mathscr{M}^{/x}$ is stable under the action of the $p$-adic closure of $H_x$ in $G_x$.

**Examples.** For a "general" $x \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$" (in particular $x$ is ordinary), the Zariski closure of $H_x$ is a $g$-dimensional torus, while the Zariski closure of $G_x$ is $\mathrm{GL}_g$.

For a supersingular point $x \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$, $H_x$ is $p$-adically dense in $G_x$, and the Zariski closure of $G_x$ is a twist of $\mathrm{Sp}_{2g}$.

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# The global rigidity problem

(Oort's Hecke orbit conjecture)

**Prediction.** Let $Z \subset \mathscr{M}_{/\overline{\mathbb{F}}_p}$ be a reduced closed subset of $\mathscr{M}$ stable under all prime-to-$p$ Hecke correspondences.
Then $Z$ contains the leaf $C(x)$ passing through $x$ for every point $x \in Z(\overline{\mathbb{F}}_p)$.

(Every Hecke-invariant closed subset of $\mathscr{M}_{/\overline{\mathbb{F}}_p}$ is a union of leaves; the latter can be regarded as "generalized Shimura subvarieties in char. $p$".)

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# Definition and examples of leaves



- A *leaf* $C(x)$ in $\mathscr{M}_{/\overline{\mathbb{F}}_p}$ is the locus in $\mathscr{M}_{/\overline{\mathbb{F}}_p}$ where *all* $p$-adic invariants have the same "value" as those of $x$.[1]
- The *ordinary* locus $\mathscr{A}_g^{\mathrm{ord}} \subset \mathscr{A}_{g/\overline{\mathbb{F}}_p}$ is a leaf in $\mathscr{A}_{g\overline{\mathbb{F}}_p}$.
- The leaf passing through a *supersingular* point in $\mathscr{A}_g$ is finite.
- The leaf passing through a point in $\mathscr{A}_3$ corresponding to a 3-dimensional abelian variety with slopes $\{1/3, 2/3\}$ is two-dimensional. Such leaves form a one-dimensional family in the slopes $\{1/3, 2/3\}$ locus of $\mathscr{A}_3$.

  (The latter locus has dimension three.)

[1] The universal $p$-divisible group $A[p^\infty]$ (with imposed symmetries) over $C(x)$ is a "twist" of the constant $p$-divisible group $A_x[p^\infty]$.

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# Strong forms of global rigidity problem

**Remark.** In application(s) to Iwasawa theory pioneered by Hida, certain strong versions of the global rigidity problem appear naturally:

- The assumption on $Z$ is weakened to:
  $Z$ is stable under the action of a "not-to-small" subset of Hecke correspondences.
- The desired conclusion is that $Z$ is a union of leaves in the reduction of certain Shimura subvarieties.

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# Local rigidity problems

**Set-up**. $Z \subset \mathscr{M}^{/x}$ is a reduced closed formal subscheme of $\mathscr{M}^{/x}$, stable under the action of a "not-too-small" subgroup of $G_x$.

**Restricted** local rigidity problem (to make it easier): Assume in addition that $Z \subset C(x)^{/x}$.

**Desired conclusion**. $Z$ has a (very) special form (e.g. defined by a finite collection of Tate cycles.)

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# Irreducibility criterion through Hecke symmetry

(a biproduct)

$$\text{Hecke transitive} \implies \text{irreduciblity}$$

for Hecke-invariant subvarieties which are not generically supersingular/basic.

Sample application.[2]

**Proposition.** (i) Every non-supersingular Newton polygon stratum in $\mathscr{A}_g$ is (geometrically) irreducible.

(ii) Every non-supersingular leaf in $\mathscr{A}_g$ is irreducible.

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## Results on the restricted local rigidity problem

**Proposition**. Restricted local rigidity holds for $\mathscr{A}_g$, in the case when $A_x$ has only two slopes.

(Aside/Fact) $C(x)^{/x}$ has a natural structure as a torsor for an isoclinic $p$-divisible formal group $X_x$.

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# Restricted local rigidity: an example and consequences





**An example.** Let $Z$ be an irreducible forma subscheme of a formal torus $\hat{\mathbb{G}}_m^r$ over $\overline{\mathbb{F}}_p$. Suppose that $Z$ is closed under the action of $[1+p^m]$ for some $m \geq 2$. Then $Z$ is a formal subtorus of $\hat{\mathbb{G}}_m^r$. (exercise)

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# Results on global rigidity using the Hilbert trick

**Theorem**. Global rigidity holds for $\mathscr{A}_g$.

**Remarks.** (1) Besides the restricted local rigidity and monodromy arguments, the proof uses a trick:
Every point $x \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$ is contained in a Hilbert modular subvariety of $\mathscr{A}_g$.[3]

(2) This "Hilbert trick" fails for PEL modular varieties of type A or D.

(3) A strong global rigidity statement holds for Hilbert modular varieties (and many othother modular varieties associated to semisimple groups of $\mathbb{Q}$-rank one).[4]

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**Remarks.** (1) Besides the restricted local rigidity and monodromy arguments, the proof uses a trick:
Every point $x \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$ is contained in a Hilbert modular subvariety of $\mathscr{A}_g$.[3]

(2) This "Hilbert trick" fails for PEL modular varieties of type A or D.

(3) A strong global rigidity statement holds for Hilbert modular varieties (and many othother modular varieties associated to semisimple groups of $\mathbb{Q}$-rank one).[4]

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# Leaves of $p$-divisible groups

The notion of leaves was announced by F. Oort in 2001 (Texel).

(old) **Definition.** A $p$-divisible group $X$ over a scheme $S/\mathbb{F}_p$ is *geometrically fiberwise constant* if any two fibers $X_{s_1}$, $X_{s_2}$ are isomorphic when based-changed to a common algebraically closed field $K$ which contains both $\kappa(s_1)$ and $\kappa(s_2)$.

**Remark:** (i) This definition, though awkward, is reasonable when the base scheme $S$ is normal.
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# An update 12 years after

**Definition.** (with F. Oort) Let $X$ be a $p$-divisible group over an algebraically closed field $k \supset \mathbb{F}_p$.

1. A $\mathrm{BT}_n$-group $\mathscr{X}_n$ over a $k$-scheme $U$ is $X$-sustained if there exists an fppf morphism $f : V \to U$ and an isomorphism $f^* \mathscr{X}_n \simeq X[p^n] \times_{\mathrm{Spec}\, k} V$.
2. A $p$-divisible group $\mathscr{X}$ over a $k$-scheme is $X$-sustained if the $\mathrm{BT}_n$-group $\mathscr{X}[p^n]$ is sustained for every $n \geq 1$.

**Remark.** The category of all $X$-sustained $p$-divisible groups is not an Artin stack. For instance an $X$-sustained $p$-divisible group $\mathscr{X}$ may not be locally isomorphic to $X$ for the fppf topology.

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# A differential criterion

Let $S$ be a scheme of finite type over $k$.
Let $S_1$ be the formal completion of $S \times S$ along the diagonal $\Delta_S \subset S_1 \times S_1$.

Let $\mathcal{X}$ be a $p$-divisible group over $S$.

Assume (for simplicity) that each fiber of $\mathcal{X}$ has two slopes.

**Proposition.** $\mathcal{X}$ is sustained iff the following conditions hold.

1. $\mathcal{X}$ admits a slope filtration $0 \to \mathcal{Z} \to \mathcal{X} \to \mathcal{Y} \to 0$ over $S$, where $\mathcal{Y}, \mathcal{Z}$ are isoclinic sustained $p$-divisible groups. (Therefore both $\mathcal{Y}$ and $\mathcal{Z}$ are Galois twists of constant $p$-divisible groups.) Moreover $\mathrm{slope}(Y) < \mathrm{slope}(Z)$.
2. The extension class $[\mathrm{pr}_1^*\mathcal{X}] - [\mathrm{pr}_1^*\mathcal{X}] \in \underline{\mathrm{Ext}}(\mathcal{Y}, \mathcal{Z})(S_1)$ is $p$-divisible in the sense that it belongs to $\underline{\mathrm{Ext}}(\mathcal{Y}, \mathcal{Z})_{\mathrm{pdiv}}(S_1)$.

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# Hom schemes for $p$-divisible groups

Let $Y, Z$ be $p$-divisible groups over $k$.

$G_n := \underline{\mathrm{Hom}}(Y[p^n], Z[p^n])$, a group scheme of finite type over $k$,

$i_{n+1,n} : G_n \hookrightarrow G_{n+1}$ is the natural "inclusion homomorphism",

$r_{n,n+1} : G_{n+1} \to G_n$ is the natural "restriction homomorphism".

**Remark.** $\varprojlim_n G_n = \underline{\mathrm{Hom}}(Y, Z)$ as sheaves on the flat big site of Spec $k$. It is representable by an affine scheme over $k$ but often not very useful.

For instance in the case when $Y$ is a one-dimensional $p$-divisible group of height 3 and $Z$ is the Serre dual of $Y$, one has $\underline{\mathrm{Hom}}(Y, Z) = \mathrm{Spec} R/I$, where $R = k[x_0, x_1, x_2, x_3, \ldots]$ and $I$ is the ideal of $R$ generated by $x_0, x_1, x_2$ and $x_{i+3}^p - x_i$, $i \geq 0$.

The maximal ideal of this local ring $R$ is the nil radical of $R$, however $R$ seems to "come from a 3-dimensional thing".

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# Stabilization for the $G_n$'s

**Proposition.** (i) There exists a constant $c$ such that

$$\begin{array}{ccc} G_{n+j}/i_{n+j,j}(G_j) & \xrightarrow{\sim} & \mathrm{im}(r_{n,n+j}: G_{n+j} \to G_n) \\ \downarrow \wr & & \downarrow \wr \\ G_{n+i}/i_{n+i,i}(G_i) & \xrightarrow{\sim} & \mathrm{im}(r_{n,n+i}: G_{n+i} \to G_n) \end{array}$$

for all $j \geq i \geq c$. Denote this finite group scheme over $k$ by $H_n$, which is naturally a subgroup scheme of $G_n$.

(ii) The homomorphisms $H_n \hookrightarrow H_{n+1}$ and $H_{n+1} \to H_n$ induced by $i_{n+1,n}$ and $r_{n,n+1}$ give $(H_n)_{n\geq 1}$ a structure of a $p$-divisible group $\mathcal{H}(Y,Z)$ over $k$.

(iii) $\mathcal{H}(Y,Z) = 0$ if $\lambda_Y > \lambda_Z$ for all slopes $\lambda_Y$ of $Y$ and all slopes $\lambda_Y$ of $Y$.

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# Serre-Tate coordinates on leaves

Assume that $Y, Z$ are isoclinic with slopes $\lambda_Y, \lambda_Z$, and $\lambda_Y < \lambda_Z$.

Let $\widetilde{\mathcal{G}}(Y,Z) := \varinjlim_n G_n$ w.r.t. $i_{n+1,n} : G_n \to G_{n+1}$

Let $\mathrm{T}_p(Y)$ be the formal $p$-adic Tate module, i.e. the projective system of the $Y[p^n]$'s. Let $\mathrm{V}_p(Y) = T_p(Y) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

The push-out construction, applied to the short exact sequence

$$0 \to \mathrm{T}_p(Y) \to \mathrm{V}_p \to Y \to 0,$$

gives a map

$$\delta_{Y,Z} : \widetilde{\mathcal{G}}(Y,Z) \to \underline{\mathrm{Ext}}(Y,Z)$$

between sheaves for the flat topology.

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The push-out construction, applied to the short exact sequence

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between sheaves for the flat topology.

# Serre-Tate coordinates on leaves



Assume that $Y, Z$ are isoclinic with slopes $\lambda_Y, \lambda_Z$, and $\lambda_Y < \lambda_Z$.

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# Serre-Tate coordinates on leaves, statement

**Theorem.** (1) The restriction of the sheaf $\underline{\mathrm{Ext}}(Y,Z)$ (respectively $\widetilde{\mathcal{G}}(Y,Z)$ ) to the category of artinian local $k$-algebras is a smooth formal group $\mathcal{E}(Y,Z)$ (respectively $\mathcal{G}(Y,Z)$ ) over $k$.

(2) $\dim(\mathcal{E}(Y,Z)) = \dim(\mathcal{G}(Y,Z)) = \dim(Z)\cdot\dim(Y^t)$.

(3) $\delta_{Y,Z} : \mathcal{G}(Y,Z) \xrightarrow{\sim} \mathcal{E}(Y,Z)$, and induces an isomorphism $\mathcal{H}(Y,Z) \xrightarrow{\sim} \mathcal{E}(Y,Z)_{\mathrm{pdiv}}$, where

$\mathcal{E}(Y,Z)_{\mathrm{pdiv}}$ = the maximal $p$-divisible subgroup of $\mathcal{E}(Y,Z)$.

(4) $\mathcal{H}(Y,Z) \simeq \mathcal{E}(Y,Z)_{\mathrm{pdiv}}$ is naturally identified with the $X$-sustained locus in the universal $p$-divisible group over the deformation space $\mathcal{D}ef(Y\times Z)$.

(5) The $p$-divisible group $\mathcal{H}(Y,Z)$ is isoclinic of slope $\lambda_Z-\lambda_Y$, height $\mathrm{ht}(Y)\cdot\mathrm{ht}(Z)$ and dimension

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# The Cartier module of $\mathcal{H}(Y,Z)$

Let $Y,Z$ be isoclinic $p$-divisible groups over $k$ with $\lambda_Y < \lambda_Z$.

Let $M,N$ be the Cartier modules of $Y$ and $Z$; both are finite free $W(k)$-modules, with semi-linear operators $F$ and $V$.

Let $H := \mathrm{Hom}_{W(k)}(M,N)$. We have natural semi-linear actions on $H \otimes_{W(k)} W(k)[1/p]$,

$$F_H : h \mapsto F_N \circ h \circ V_M, \quad V_H : h \mapsto p^{-1} V_N \circ h \circ F_M.$$

Let $H_1 :=$ the largest submodule of $H$ stable under $F_H$ and $V_H$

**Proposition.** The Cartier module of the $p$-divisible group $\mathcal{H}(Y,Z)$ is naturally isomorphic to $H_1$.

**Remark.** The smallest submodule of $H \otimes_{W(k)} W(k)[1/p]$ which contains $H$ and stable under both $F_H$ and $V_H$ is the Cartier module of the maximal $p$-divisible quotient of $\mathcal{G}(Y,Z)$.

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Let $\mathrm{Cart}_{p,k}$ be the Cartier ring functor on commutative $k$-algebras; it is an infinite dimensional commutative smooth formal group over $k$.

Let $\mathrm{BC}_p(k)$ be the set of all $p$-typical formal curves in $\mathrm{Cart}_{p,k}$.

- The ring structure of $\mathrm{Cart}_{p,k}$ gives $\mathrm{BC}_p(k)$ a $\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k)$-bimodule structure.
- $\mathrm{BC}_p(k)$, being the set of all $p$-typical curves in a commutative smooth formal group, has a natural $\mathrm{Cart}_p(k)$-module structure which commutes with the above bimodule structure.

**Proposition.** The Cartier module of the smooth formal group $\mathcal{G}(Y,Z)$ is naturally isomorphic to

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# Global Serre-Tate coordinates

We illustrate the idea in the case of a leaf $\mathscr{C} = \mathscr{C}(x) \subset \mathscr{A}_g$, where $x$ corresponds to a $g$-dimensional principally polarized abelian variety $A_x$ over $k$ with two slopes $\lambda, 1-\lambda$, $\lambda < 1/2$.

Let $\mathscr{C}_1 = (\mathscr{C} \times \mathscr{C})^{/\Delta\mathscr{C}}$, the formal completion along the diagonal. $\mathrm{pr}_1 : \mathscr{C}_1 \to \mathscr{C}$ is formally smooth of relative dimension $(1-2\lambda)g(g+1)/2$.

**Proposition.** $\mathrm{pr}_1 : \mathscr{C}_1 \to \mathscr{C}$ has a natural structure as an isoclinic *sustained* $p$-divisible group over $\mathscr{C}$, with slope $1-2\lambda$ and height $g(g+1)/2$.

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# A tantalizing dream

The **holy grail** for the rigidity problems:[5]

To pry actionable intelligence out of the action of the local stabilizing subgroup.

**Main obstacle**: Our poor understanding of this action (so cannot deploy enhanced interrogation techniques).

- Don't have helpful (exact or approximate) formulas (have tried Norman's algorithm).
- Linearization via crystalline techniques leads to formulas with high powers of $p$ in denominators.

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# The Lubin-Tate case

We will explain a method to obtain an approximate (or even asymptotic) formula for the action of the local stabilizer subgroup, in the first non-trivial case,

where $\mathscr{M}^{/x} = \mathrm{Def}(G_0)$ is the Lubin-Tate moduli deformation space for a one-dimensional formal group $G_0$ of finite height $h$ over $\overline{\mathbb{F}}_p$.

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# How to compute the Lubin-Tate action

1. Go from the category of 1-dimensional formal groups to the category of *p-typical formal group laws*. (This is a "rigidification process".)

- The latter category is a groupoid with infinitely many affine coordinates $t_1, t_2, t_3, \ldots$
- Honda's formalism[6] provides many objects and arrows in the groupoid.

One such object, a one-dimensional $p$-typical groupoid over the $h-1$-dimensional base $W(\overline{\mathbb{F}}_p)[[t_1, \ldots, t_{h-1}]]$ with

$t_h = 1, 0 = t_{h+1} = t_{h+2} = \cdots$ represents universal one-dimensional formal group $\tilde{G}$ over $\mathrm{Def}(G_0)$.

**Goal:** Given an automorphism $\gamma$ of $G_0$, compute an automorphism $\tilde{\gamma}$ of $\tilde{G}$ over an automorphism $\rho_\gamma$ of the base (formal) scheme of $\tilde{G}$ which induces $\gamma$ over the closed fiber $G_0$.

[6] or Hazewinkel's "functional equation lemma"

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- Honda's formalism[6] provides many objects and arrows in the groupoid.

One such object, a one-dimensional $p$-typical groupoid over the $h-1$-dimensional base $W(\overline{\mathbb{F}}_p)[[t_1, \ldots, t_{h-1}]]$ with

$t_h = 1, 0 = t_{h+1} = t_{h+2} = \cdots$ represents universal one-dimensional formal group $\tilde{G}$ over $\mathrm{Def}(G_0)$.

**Goal:** Given an automorphism $\gamma$ of $G_0$, compute an automorphism $\tilde{\gamma}$ of $\tilde{G}$ over an automorphism $\rho_\gamma$ of the base (formal) scheme of $\tilde{G}$ which induces $\gamma$ over the closed fiber $G_0$.

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PROCEDURE: Perform the computation of $(\tilde{\gamma}, \rho_\gamma)$ in several steps (corresponding to a factorization of $(\tilde{\gamma}, \rho_\gamma)$), in the groupoid of $p$-typical formal group *laws*.

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# Notation

‣ universal p-typical group laws ‣ formula for coeff ‣ universal strict isom
‣ integral recursive formula for universal strict isom ‣ param for Fc

Let $h$ be a positive integer.

Let $G_1$ be the one-dimensional formal group over $\mathbb{Z}_{(p)}$ with logarithm

$$\sum_{j\in\mathbb{N}} p^{-j} x^{p^{jh}} = x + \frac{x^{p^h}}{p} + \frac{x^{p^{2h}}}{p^2} + \cdots$$

(so it is a Lubin-Tate formal group for $W(\mathbb{F}_{p^h})$.)

Let $G_0$ be the base extension to $\overline{\mathbb{F}}_p$ of the closed fiber of $G_1$; it is a one-dimensional formal group over $\mathbb{F}_p$ of height $h$.

It is well-known that $\mathrm{End}(G_0)$ is the maximal order of $\mathrm{End}^0(G_0)$ = a central division algebra over $\mathbb{Q}_p$ of dimension $h^2$.

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Let $\mathscr{M}_h := \mathrm{Def}(G_0)$; it is a smooth formal scheme over $W(\overline{\mathbb{F}}_p)$ of relative dimension $h-1$.

Let $G_{\mathrm{univ}} \to \mathscr{M}_h$ be the universal formal group over $\mathscr{M}_h$.

The compact $p$-adic group $\mathrm{Aut}(G_0) = \mathrm{End}(G_0)^\times$ operates on $\mathscr{M}_h$ by functoriality, as follows.

$\forall \gamma \in \mathrm{Aut}(G_0)$, $\exists !$ formal scheme automorphism $\rho(\gamma)$ of $\mathscr{M}_h$ and a formal group isomorphism

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such that $\tilde{\rho}(\gamma)|_{G_0} = \gamma$

**Remark.** This action $\gamma \mapsto \rho(\gamma)$ of $\mathrm{Aut}(G_0)$ on the Lubin-Tate moduli space $\mathscr{M}_h$ was first studied by Lubin and Tate in 1966. It is also known as (the essential part of) the *Morava stabilizer subgroup* action in chromatic homotopy theory.

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# The universal $p$-typical formal group law

Let $\tilde{R} = \mathbb{Z}_{(p)}[\underline{v}] = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$, and let $\sigma : \tilde{R} \to \tilde{R}$ be the ring homomorphism such that $\sigma(v_j) = v_j^p$ for all $j \geq 1$

Let $G_{\underline{v}}(x) \in \tilde{R}[[x, y]]$ be the one-dimensional $p$-typical formal group law over $\tilde{R}$ whose logarithm

$$g_{\underline{v}}(x) \in \tilde{R}[1/p][[x]] = \sum_{n \geq 1} a_n(\underline{v}) \cdot x^{p^n}$$

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# Remarks on the formal group law $G_{\underline{v}}$

**Remarks.** (1) The above "functional equation" is a recursive formula for the coefficients $a_n(\underline{v}) \in p^{-n} \cdot \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n]$ of $g_{\underline{v}}(x)$. ▸ univ strict isom p-typical

(2) Explicitly:

$$a_n(\underline{v}) = \sum_{\substack{i_1, i_2, \ldots, i_r \geq 1 \\ i_1 + \cdots + i_r = n}} p^{-r} \cdot \prod_{s=1}^{r} v_{i_s}^{p^{i_1 + i_2 + \cdots + i_{s-1}}}$$

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Note that $a_n(\underline{v})$ is a homogeneous polynomial in $v_1, \ldots, v_n$ of weight $p^n - 1$ when $v_j$ is given the weight $p^j - 1$ $\forall j \geq 1$.

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# The universal formal group over $\mathscr{M}_h$ made explicit

Let $R = R_h = W(\overline{\mathbb{F}}_p)[[w_1, w_2, \ldots, w_{h-1}]]$.

Let $\pi = \pi_h : \tilde{R} \to R$ be the ring homomorphism such that

$$\pi(v_i) = \begin{cases} w_i & \text{if} \quad 1 \le i \le h-1 \\ 1 & \text{if} \quad i = h \\ 0 & \text{if} \quad i \ge h+1 \end{cases}$$

The classifying morphism $\mathrm{Spf}(R) \to \mathscr{M}_h$ for the deformation $\pi_* G_{\underline{v}}$ of $G_0$ is an isomorphism.

We will identify $\mathscr{M}_h$ with $\mathrm{Spf}(R)$ and the universal deformation $G_{\mathrm{univ}}$ of $G_0$ with the formal group underlying the formal group law $G_R := \pi_* G_{\underline{v}}$.

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# The universal strict isomorphism

Let $\mathbb{Z}_{(p)}[\underline{v},\underline{t}] = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots; t_1, t_2, t_3, \ldots]$, and let $\sigma : \mathbb{Z}_{(p)}[\underline{v},\underline{t}] \to \mathbb{Z}_{(p)}[\underline{v},\underline{t}]$ be the obvious Frobenius lifting as before, with $\sigma(v_i) = v_i^p$ and $\sigma(t_i) = t_i^p$ $\forall\, i \geq 1$.

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# The universal strict isomorphism, continued





It is known that $\alpha_{\underline{v},\underline{t}} := g_{\underline{v},\underline{t}}^{-1} \circ g_{\underline{v}} \in \mathbb{Z}_{(p)}[\underline{v},\underline{t}][[x]]$, and defines a *strict isomorphism*

$$\alpha_{\underline{v},\underline{t}} : G_{\underline{v}} \to G_{\underline{v},\underline{t}}$$

between $p$-typical formal group laws over $\mathbb{Z}_{(p)}[\underline{v},\underline{t}]$.

(A *strict* isomorphism is an isomorphism between formal group laws which is $\equiv x$ modulo higher degree terms in $x$.)

Moreover $\alpha_{\underline{v},\underline{t}}$ is "the" universal strict isomorphism between one-dimensional $p$-typical formal group laws.

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Moreover $\alpha_{\underline{v},\underline{t}}$ is "the" universal strict isomorphism between one-dimensional $p$-typical formal group laws.

# Parameters of $G_{\underline{v},\underline{t}}$

By the universality $G_{\underline{v}}$ for $p$-typical formal group laws, there exists a unique ring homomorphism

$$\eta : \mathbb{Z}_{(p)}[\underline{v}] \to \mathbb{Z}_{(p)}[\underline{v}, \underline{t}]$$

such that

$$\eta_* G_{\underline{v}} = G_{\underline{v},\underline{t}}.$$

The elements

$$\overline{v}_n = \overline{v}_n(\underline{v}, \underline{t}) \in \mathbb{Z}_{(p)}[\underline{v}, \underline{t}], \quad n \in \mathbb{N}_{\geq 1}$$

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# A known recursive formula for the parameters of $G_{\underline{v},\underline{t}}$

$$
\begin{aligned}
\bar{v}_n = v_n + p\,t_n + \sum_{\substack{i+j=n\\ i,j\geq 1}} \left(v_j\, t_i^{p^j} - t_i\, \bar{v}_j^{p^i}\right) \\
+ \sum_{j=1}^{n-1} a_{n-j}(\underline{v}) \cdot \left(v_j^{p^{n-j}} - \bar{v}_j^{p^{n-j}}\right) \\
+ \sum_{k=2}^{n-1} a_{n-k}(\underline{v}) \cdot \sum_{\substack{i+j=k\\ i,j\geq 1}} \left(v_j^{p^{n-k}} t_i^{p^{n-i}} - t_i^{p^{n-k}} \bar{v}_j^{p^{n-j}}\right)
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(This formula contains high power of $p$ in the denominators. Consequently it is not very useful for our purpose.)

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&+ \sum_{j=1}^{n-1} a_{n-j}(\underline{v}) \cdot \left( v_j^{p^{n-j}} - \bar{v}_j^{p^{n-j}} \right) \\
&+ \sum_{k=2}^{n-1} a_{n-k}(\underline{v}) \cdot \sum_{\substack{i+j=k \\ i,j\geq 1}} \left( v_j^{p^{n-k}} t_i^{p^{n-i}} - t_i^{p^{n-k}} \bar{v}_j^{p^{n-j}} \right)
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# An *integral* recursion formula for $\bar{v}_n(\underline{v},\underline{t})$

(useful for computing the Lubin-Tate action)

$$
\begin{aligned}
\bar{v}_n = v_n + p\,t_n - \sum_{j=1}^{n-1} t_j \cdot \bar{v}_{n-j}^{p^j} + \\
+ \sum_{l=1}^{n-1} v_l \sum_{k=1}^{n-l-1} \frac{1}{p} \cdot a_{n-k-l}(\underline{v})^{(p^l)} \cdot \Big\{ (\bar{v}_k^{(p^l)})^{p^{n-l-k}} - (\bar{v}_k^{p^l})^{p^{n-l-k}} \\
+ \sum_{\substack{i+j=k \\ i,j\geq 1}} t_j^{p^{n-k}} \Big[ (\bar{v}_i^{(p^l)})^{p^{n-l-i}} - (\bar{v}_i^{p^l})^{p^{n-l-i}} \Big] \Big\} \\
+ \sum_{l=1}^{n-1} v_l \cdot \Big\{ \frac{1}{p}(\bar{v}_{n-l}^{(p^l)} - \bar{v}_{n-l}^{p^l}) + \sum_{\substack{i+j=n-l \\ i,j\geq 1}} t_j^{p^l} \cdot \frac{1}{p} \cdot \Big[ (\bar{v}_i^{(p^l)})^{p^j} - (\bar{v}_i^{p^l})^{p^j} \Big] \Big\}
\end{aligned}
$$

for every $n \geq 1$. parameters for Fc back to key observation

# Step 1

Given an element $\gamma \in \mathrm{Aut}(G_0)$, construct

- a $p$-typical one-dimensional formal group *law* $F = F_\gamma$ over $R$ whose closed fiber is equal to $G_0$, and
- an isomorphism

$$\overline{\psi} = \overline{\psi}_\gamma : F_{\overline{R}} \to G_{\overline{R}}$$

over $\overline{R} := R/pR = \overline{\mathbb{F}}_p[[w_1, \ldots, w_{h-1}]]$ whose restriction to the closed fibers is

$$(\psi|_{G_0} : G_0 \to G_0) = \gamma.$$

Here $F_{\overline{R}} = F \otimes_R \overline{R}$, $G_{\overline{R}} = G_R \otimes_R \overline{R}$.

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# The formal group law $F_c$, $c \in W(\mathbb{F}_{p^h})^\times$

For $\gamma = [c] \in W(\mathbb{F}_{p^h})^\times = \mathrm{Aut}(G_1)$, we can take $F_c$ to be the formal group over $R$ whose logarithm $g_c(x)$ satisfies

$$f_c(x) = x + \sum_{i=1}^{h} \frac{c^{-1+\sigma^i} \cdot w_i}{p} \cdot f_c^{(\sigma^i)}(x^{p^i})$$

($w_h$=1 by convention).

Let

$$\psi_c(x) = \log_{G_R}^{-1} \circ (c \cdot f_c)$$

We have $\psi_c(x) \in R[[x]]$ and $\psi_c$ defines an isomorphism from $F_c$ to $G_R$ over $R$ (not just over $\overline{R}$!) with $\psi_c|_{G_0} = [c]$.

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# Step 2

Compute the parameters

$$(u_i = u_i(w_1, \ldots, w_{h-1}))_{i \in \mathbb{N}_{\geq 1}}$$

for the $p$-typical group law $F = F_\gamma$ over $R$.

The above condition means that

$$\xi_* G_{\tilde{v}} = F,$$

where

$$\xi = \xi_\gamma : \tilde{R} \to R$$

is the ring homomorphism such that

$$\xi(v_i) = u_i \quad \forall i \geq 1.$$

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# Parameters for $F_c$, $c \in W(\mathbb{F}_{p^h})^\times$

In the case when $\gamma \in \mathrm{Aut}(G_0)$ lifts to an element [c] with $c \in W(\mathbb{F}_{p^h})^\times \simeq \mathrm{Aut}(G_1)$, we have the following integral recursive formula for the parameters $u_n = u_n(c;\underline{w})$.

$$
\begin{aligned}
u_n(c;\underline{w}) = c^{-1+\sigma^n} w_n \\
&+ \sum_{j=1}^{n-1} c^{-1+\sigma^j} \cdot \frac{1}{p} \left[ u_{n-j}(c;\underline{w})^{(p^j)} - u_{n-j}(c;\underline{w})^{p^j} \right] \cdot w_j \\
&+ \sum_{j=1ve}^{n-1} \sum_{i=1}^{n-j-1} \frac{1}{p} a_{n-i-j}(\underline{w})^{(p^j)} \cdot c^{-1+\sigma^{n-i}} \cdot \\
&\qquad \left[ (u_i(c;\underline{w})^{(p^j)})^{p^{n-i-j}} - (u_i(c;\underline{w})^{p^j})^{p^{n-i-j}} \right] \cdot w_j
\end{aligned}
$$

where $w_h = 1$, $w_m = 0 \; \forall m \geq h+1$ by convention.

# Parameters for $F_c$, continued

Remark. The above recursive formula for the parameters $u_n(c;\underline{w})$ can be turned into an explicit "path sum" formula for $u_n(c,\underline{w})$, with terms indexed by "paths".

# Step 3

Find/compute the uniquely determined element

$$\tau_n \in \mathfrak{m}_R, \quad n \in \mathbb{N}_{\geq 1}$$

and

$$\hat{u}_1 \in \mathfrak{m}_R, \ldots, \hat{u}_{h-1} \in \mathfrak{m}_R, \hat{u}_h \in 1 + \mathfrak{m}_R$$

such that

$$\bar{v}_n(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h, 0, 0, \ldots; \underline{\tau}) = u_n \quad \forall n \geq 1.$$

**Remark.** (1) The existence and uniqueness statement above is an application the implicit function theorem for an infinite dimensional space over $\tilde{R}$, applied to the "vector-valued" function with components $\bar{v}_n$ in the integral recursion formula discussed before.

(2) This step is a substitute for the operation *taking the quotient of the group "changes of coordinates"* in a space of formal group laws.

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# A congruence formula for $\bar{v}_n$

The follow formula helps to explain the last remark.

$$\bar{v}_n \equiv v_n - \sum_{j=1}^{n} t_j \cdot v_{n-j}^{p^j} + \sum_{\substack{i,j,t,s_1,s_2,\dots,s_t \geq 1 \\ s_1+\dots+s_t+i+j=n}} (-1)^{t-1}\, t_i \cdot v_j^{p^i} \cdot v_1^{(p^{s_1}+p^{s_2}+\cdots+p^{s_t}-t)/(p-1)} \cdot v_{n-s_1}^{p^{s_1}-1} \cdot v_{n-s_1-s_2}^{p^{s_2}-1} \cdots v_{n-s_1-\cdots-s_t}^{p^{s_t}-1} \mod (pt_a, t_a \cdot t_b)_{a,b\geq 1}\mathbb{Z}[\underline{v}, \underline{t}]$$

# Step 4

**Rescale** $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h$ as follows:

$\exists!\ \tau_0 \in \mathfrak{m}_R$ such that

$$(1+\tau_0)^{p^h-1} \cdot \hat{u}_h = 1.$$

Let

$$\hat{v}_i := (1+\tau_0)^{p^i-1} \cdot \hat{u}_i \quad \text{for } i = 1, \ldots, h-1.$$

Let $\omega : \tilde{R} \to R$ be the ring homomorphism such that

$$\omega(v_i) = \hat{u}_i \quad \forall i \geq 1.$$

Let $\rho : R \to R$ be the $W(\overline{\mathbb{F}}_p)$-linear ring homomorphism such that

$$\rho(w_i) = \hat{v}_i \quad \forall i \geq 1.$$

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# The meaning of Steps 3 and 4





The universal strict isomorphism $\alpha_{\underline{v},\underline{t}}$ specializes to a strict isomorphism

$$\alpha = \alpha_{\underline{\hat{u}},\underline{\tau}} : F \to \omega_* G_{\underline{v}}$$

with $\alpha|_{G_0} = \mathrm{Id}_{G_0}$.

The rescaling in step 4 gives an isomorphism (not necessarily a strict isomorphism)

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# Conclusion





Combined with $\overline{\psi}$, we obtain an isomorphism

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whose restriction to the closed fiber $G_0$ is equal to the given element $\gamma\in\mathrm{Aut}(G_0)$.
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**Conclusion**. The given element $\gamma\in\mathrm{Aut}(G_0)$ operates on the equi-characteristic deformation space $\mathrm{Spf}(\overline{R})$ of $G_0$ via the ring automorphism $\rho$.
(Notice that $\overline{\psi}$, $\alpha$ and $\beta$ all depend on $\gamma$.)

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# Local rigidity for the Lubin-Tate moduli space: the first non-trivial case

**Proposition**. Let $Z \subset \mathscr{M}_{3\overline{\mathbb{F}}_p} = \mathrm{Spf}(\overline{\mathbb{F}}_p[[w_1, w_2]])$ be an irreducible closed formal subscheme of $\mathscr{M}_3$ over $\overline{\mathbb{F}}_p$ corresponding to a hight one prime ideal of $\overline{\mathbb{F}}_p[[w_1, w_2]]$.
If $Z$ is stable under the action of an open subgroup of $W(\mathbb{F}_{p^3})^\times$, then $Z = \mathrm{Spf}(\overline{\mathbb{F}}_p[[w_1, w_2]]/(w_1))$.

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