HECKE ORBITS AS SHIMURA VARIETIES IN POSITIVE CHARACTERISTIC

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§1. Introduction

p: a fixed prime number,

$\mathcal{M}$: a modular variety of PEL type over $\mathbb{F} := \mathbb{F}_p^{\mathrm{alg}}$,

G: reductive group over $\mathbb{Q}$ attached to the PEL type

Want to study:

- HECKE SYMMETRIES: prime-to-p Hecke correspondences on $\mathcal{M}$ coming from the action of $G(\mathbb{A}_f^{(p)})$ on the prime-to-p tower of $\mathcal{M}$,
- LEAVES: subvarieties of $\mathcal{M}$, defined by fixing an *isomorphism type* of p-divisible group with prescribed symmetries, i.e. fix all p-adic invariants.

THEMES:

(1) Hecke symmetries on $\mathcal{M}$ determines the "foliation structure" by leaves:

Every Hecke orbit is dense in the leaf containing it.

(Oort's Hecke orbit conjecture)

(2) Tools for the Hecke orbit problems:

monodromy, canonical coordinates, local rigidity,

may be useful elsewhere.

(3) Leaves $\approx$ Shimura varieties in char. p

$\rightsquigarrow$ the reduction mod p of a Shimura variety X

$$= \coprod_{\text{infinite}} \text{char. } p \text{ renderings of } X.$$

(c.f. *Indra's Pearls*)

§2 Hecke symmetry on modular varieties

Unramified PEL data $(B, *, \mathcal{O}_B, V, V_{\mathbb{Z}_p}, \langle\cdot,\cdot\rangle, h)$:

- B – a finite dim. semisimple $\mathbb{Q}$-algebra

  unramified at p ,

- $\mathcal{O}_B$ – a maximal order of B maximal at p ,

- $*$ – a positive involution on B preserving $\mathcal{O}_B$,

- V – a B-module of finite dimension over $\mathbb{Q}$,

- $\langle\cdot,\cdot\rangle$ – a $\mathbb{Q}$-valued nondegen. alternating form

  on V compatible with $(B, *)$,

- $V_{\mathbb{Z}_p}$ – a self-dual $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$ stable

  under $\mathcal{O}_B$

- $h : \mathbb{C} \to \mathrm{End}_{B_{\mathbb{R}}}(V_{\mathbb{R}})$, a $*$-homomorphism s.t.

$$(v, w) \ \mapsto \ \langle v, h(\sqrt{-1})w\rangle$$

  is a pos. definite symmetric form on $V_{\mathbb{R}}$

MODULAR VARIETIES OF PEL TYPE

Given an unramified PEL data $\rightsquigarrow$

- G = unitary group attached to $(\mathrm{End}_B(V), *)$,
- $\widetilde{\mathcal{M}} = \left(\mathcal{M}_{K^p}\right)$, a tower of modular varieties over $\mathbb{F}$ indexed by the set of all compact open subgroups K^p of $G(\mathbb{A}_f^p)$, where
  - $\mathbb{A}_f^p = \prod'_{\ell \neq p} \mathbb{Q}_\ell$
  - $\mathcal{M}_{K^p}$ classifies abelian varieties with endomorphisms by $\mathcal{O}_B$, plus prime-to-p polarization and level structure, whose H_1 is modeled on the given PEL datum.

HECKE SYMMETRIES

(1) The group $G(\mathbb{A}_f^p)$ operates on the projective system $\widetilde{\mathcal{M}}$.

(2) If a level subgroup K_0^p is fixed, then on $\mathcal{M}_{K_0^p}$ the remnant from the action of $G(\mathbb{A}_f^p)$ takes the form of a family of finite étale algebraic correspondences on $\mathcal{M}_{K_0^p}$; they are known as *Hecke correspondences*.

(3) Given a point $x \in \mathcal{M}_{K_0^p}(\mathbb{F})$, let $\tilde{x}$ be a lift of x in $\widetilde{\mathcal{M}}(\mathbb{F})$. Define the *prime-to-$p$ Hecke orbit* $\mathcal{H}^p \cdot x$ of x to be the image in $\mathcal{M}_{K_0^p}(\mathbb{F})$ of the $G(\mathbb{A}_f^p)$-orbit of $\tilde{x}$; it is a countable set.

EXAMPLE. Siegel modular varieties $\mathcal{A}_{g,n}$, $(n,p)=1$, $n \geq 3$

- $\mathcal{A}_{g,n}$ classifies g-dimensional principally polarized abelian varieties (A,λ) with a symplectic level-n structure η.
- Two $\mathbb{F}$-points $[(A_1,\lambda_1,\eta_1)]$, $[(A_2,\lambda_2,\eta_2)]$ in $\mathcal{A}_{g,n}$ are in the same prime-to-p Hecke orbit iff $\exists$ a prime-to-p quasi-isogeny β (="$\beta_2 \circ \beta_1^{-1}$")

$$\beta : A_1 \xleftarrow{\beta_1} A_3 \xrightarrow{\beta_2} A_2$$

  defined by prime-to-p isogenies β_1 and β_2 s.t. β respects the principal polarizations λ_1 and λ_2, i.e. $\beta_1^*(\lambda_1) = \beta_2^*(\lambda_2)$.

PEL datum:

$B = \mathbb{Q}$, V = $2g$-dim. v.s. over $\mathbb{Q}$, $G = \mathrm{Sp}_{2g}$.

EXAMPLE. Hilbert modular varieties $\mathcal{M}_{E,d,n}$

$F_1, \ldots, F_r$: totally real number fields,

$E = F_1 \times \cdots \times F_r,\ \mathcal{O}_E = \mathcal{O}_{F_1} \times \cdots \times \mathcal{O}_{F_r}$,

$d, n \geq 1$, integers, $\gcd(dn, p) = 1$.

Hilbert modular variety $\mathcal{M}_{E,d,n}$ over $\mathbb{F}$:
classifies quadruples $(A \to S, \iota, \lambda, \eta)$, where

- $A \to S$ is an abelian scheme,
  $\dim(A \to S) = [E : \mathbb{Q}]$,
- $\iota : \mathcal{O}_E \to \mathrm{End}(A)$ is a ring homomorphism,
- λ is an $\mathcal{O}_E$-linear polarization on A of degree d,
- η is a level-n structure.

PEL datum:
$B = E$, V = free E-module of rank two,
$G = \prod_{E/\mathbb{Q}} \mathrm{SL}_2$.

§3 Leaves and the Hecke orbit conjecture

$\mathcal{M} = \mathcal{M}_{K_0^p}$, a modular variety of PEL type over $\mathbb{F}$

$x_0 = [(A_0, \lambda_0, \iota_0, \eta_0)] \in \mathcal{M}(\mathbb{F})$

DEF 1. The *leaf* $\mathcal{C}_{\mathcal{M}}(x_0)$ in $\mathcal{M}$ passing through x_0 is the reduced locally closed subscheme of $\mathcal{M}$ *smooth* over $\mathbb{F}$ such that $\mathcal{C}_{\mathcal{M}}(x_0)(\mathbb{F})$ consists of all points $x = [(A, \lambda, \iota, \eta)] \in \mathcal{C}_{\mathcal{M}}(x_0)(\mathbb{F})$ s.t.

$$(A, \lambda, \iota)[p^\infty]) \cong (A_0, \lambda_0, \iota_0)[p^\infty]\,,$$

where $(A, \lambda, \iota)[p^\infty]$ is the $\mathcal{O}_B$-linear polarized p-divisible group attached to (A, λ, ι).

Oort's Hecke orbit conjecture

CONJ 1 (HO**).** *Every prime-to-p Hecke orbit in a modular variety of PEL type $\mathcal{M}$ over $\mathbb{F}$ is dense in the leaf in $\mathcal{M}$ containing it.*

CONJ ($\mathrm{HO}_{\mathrm{ct}}$). The closure of any prime-to-p Hecke orbit in the leaf $\mathcal{C}$ containing it is an open-and-closed subset of $\mathcal{C}$, i.e. it is a union of irreducible components of the smooth variety $\mathcal{C}$.

CONJ ($\mathrm{HO}_{\mathrm{dc}}$). Every prime-to-$p$ Hecke orbit in a leaf $\mathcal{C}$ meets every irreducible component of $\mathcal{C}$.

Clearly HO $\Longleftrightarrow$ $\mathrm{HO}_{\mathrm{ct}}$ + $\mathrm{HO}_{\mathrm{dc}}$.

Remark. Conj. $\mathrm{HO}_{\mathrm{dc}}$ is an irreducibility statement.

Let $Z(x_0)$ be the Zariski closure of the prime-to-p Hecke orbit of x_0 for the group $G_{\mathrm{der}}^{\mathrm{sc}}$ in the leaf $\mathcal{C}(x_0)$.

PROP 2. *Assume that the prime-to-p Hecke orbit of x_0 with respect to every simple factor of $G_{\mathrm{der}}^{\mathrm{sc}}$ is infinite. Then $Z(x_0)$ is irreducible, and the Zariski closure of the ℓ-adic monodromy group of $Z(x_0)$ is $G_{\mathrm{der}}(\mathbb{Q}_\ell)$ for every prime number $\ell \neq p$.*

Note. Irreducibility of $Z(x_0)$ uses: $G_{\mathrm{der}}^{\mathrm{sc}}(\mathbb{Q}_\ell)$ has no proper subgroup of finite index.

EVIDENCE OF THE HECKE ORBIT CONJECTURE

THM 3 ($\mathrm{HO}^{\mathrm{siegel}}$). *Conjecture* HO *holds for Siegel modular varieties.*

THM 4 ($\mathrm{HO}^{\mathrm{hilbert}}$). *Conjecture* HO *holds for the Hilbert modular varieties attached to a finite product* $F_1 \times \cdots \times F_r$ *of totally real fields. Here the prime* p *may be ramified in any or all of* $F_1, \cdots, F_r$*.*

Remark. (i) Thm. 3 is joint work with F. Oort.

(ii) Thm. 4 is joint work with C.-F. Yu. The proof of $\mathrm{HO}_{\mathrm{dc}}^{\mathrm{hilbert}}$ is due to C.-F. Yu.

(iii) The proof of $\mathrm{HO}_{\mathrm{ct}}^{\mathrm{siegelel}}$ uses $\mathrm{HO}^{\mathrm{hilbert}}$.

§4 Canonical coordinates on leaves

Recall classical Serre-Tate coordinates:

A_0 : ordinary abelian variety over $k \supset \mathbb{F}_p$,

$$k = k^{\mathrm{alg}},\ g = \dim(A_0).$$

$$\rightsquigarrow \quad \mathrm{DEF}(A_0) = \mathrm{DEF}(A_0[p^\infty]) \cong \mathbb{G}_m^{g^2}.$$

Phenomenon:

$\exists$ generalization to leaves, so that every local jet space of a leaf $\mathcal{C}$ in $\mathcal{M}$ is “built up” from p-divisible formal groups via a family of fibrations.

$\mathcal{C}$ = a leaf in $\mathcal{M}$, $\tilde{X} = A_{\mathrm{univ}}[p^\infty]|_{\mathcal{C}} \to \mathcal{C}$.

PROP 5. (i) $\exists$ *p-divisible groups*

$$0 = \tilde{X}_0 \subset \tilde{X}_1 \subset \tilde{X}_2 \subset \cdots \subset \tilde{X}_m = \tilde{X}$$

over $\mathcal{C}$ s.t. $\tilde{Y}_i := \tilde{X}_i/\tilde{X}_{i-1}$ is a non-trivial isoclinic p-divisible group over $\mathcal{C}$ of slope μ_i, $\forall$ $i = 1, \ldots, m$, $1 \geq \mu_1 > \cdots > \mu_m \geq 0$.

(ii) *The filtration*

$$0 = \tilde{X}_0 \subset \tilde{X}_1 \subset \tilde{X}_2 \subset \cdots \tilde{X}_m = \tilde{X}$$

is uniquely determined by $\tilde{X} \to \mathcal{C}$. Each subgroup $\tilde{X}_i \subseteq \tilde{X}$ is stable under the natural action of $\mathcal{O}_B$.

(iii) *$\tilde{Y}_i \to \mathcal{C}$ is geometrically constant $\forall\, i$, hence it is the twist of a constant p-divisible group by a smooth étale $\mathbb{Z}_p$-sheaf over $\mathcal{C}$.*

Local moduli on a leaf comes from deformation of the slope filtration

Explain the two-slope case:

- X, Y : isoclinic p-div. group over $\mathbb{F}$ with slopes $\mu_X < \mu_Y$ and heights h_X, h_Y,
- $\mathrm{DE}(X, Y)$ = deformation functor of the filtration $0 = Y \subset X \times Y$, i.e. deform by extensions; it is a commutative smooth formal group over $\mathbb{F}$ via Baer sum construction,
- $\mathrm{DE}(X, Y)_{\mathrm{pdiv}}$ = the maximal p-divisible subgroup of $\mathrm{DE}(X, Y)$,
- $\mathrm{M}(X), \mathrm{M}(Y)$: Cartier module of X, Y,
- $H_{\mathbb{Q}} := \mathrm{Hom}_{W(k)}(\mathrm{M}(X), \mathrm{M}(Y)) \otimes \mathbb{Q}$ with operators F, V , defined by

$$(V \cdot h)(u) = V(h(V^{-1}u))$$
$$(F \cdot h)(u) = F(h(V(u))$$

THM 6. (i) *The p-divisible formal group $\mathrm{DE}(X,Y)_{\mathrm{pdiv}}$ is isoclinic of slope $\mu_Y - \mu_X$; its height is $h_X \cdot h_Y$.*

(ii) *The Cartier module of $\mathrm{DE}(X,Y)_{\mathrm{pdiv}}$ is naturally isomorphic to the maximal $W(k)$-submodule of*

$$\mathrm{Hom}_{W(k)}(\mathrm{M}(X), \mathrm{M}(Y))$$

stable under the F and V.

(iii) *Suppose that $Y = X^t$, $A_0[p^\infty] \cong X \times Y$. Then*
$\mathcal{C}^{/x_0} \cong \mathrm{DE}(X, X^t)^{\mathrm{sym}}_{\mathrm{pdiv}} \subset \mathrm{DE}(X, X^t)_{\mathrm{pdiv}}$.

(iv) *The Cartier module of $\mathrm{DE}(X, X^t)^{\mathrm{sym}}_{\mathrm{pdiv}}$ is the maximal $W(k)$-submodule of*

$$\mathrm{Hom}_{W(k)}(\mathrm{S}^2(\mathrm{M}(X)), W(k))$$

stable under F and V.

§5 Hypersymmetric points

B: a simple algebra over $\mathbb{Q}$, $\mathcal{O}_B$: an order of B.
$k \supset \mathbb{F}_p$, $k = k^{\mathrm{alg}}$.

DEF 2. (i) An $\mathcal{O}_B$-linear abelian variety (A, ι) over k is B-*hypersymmetric*, or *hypersymmetric* for short, if the canonical map

$$\mathrm{End}_{\mathcal{O}_B}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}_{\mathcal{O}_B}(A[p^\infty])$$

is an isomorphism.

(ii) A point $x \in \mathcal{M}(k)$ is *hypersymmetric* if the underlying $\mathcal{O}_B$-linear abelian variety (A_x, ι_x) is hypersymmetric.

Remark. (i) When $B = \mathbb{Q}$, an abelian variety A is hypersymmetric iff

- $A \sim B_1 \times \cdots \times B_r$, B_i defined over a finite field $\forall i$,
- Fr_{B_i} has at most two eigenvalues $\forall i$,
- B_i and B_j share no common slope if $i \neq j$.

(ii) Hypersymmetric points exist on every leaf on $\mathcal{A}_{g,n}$. This statement does not hold for Hilbert modular varieties $\mathcal{M}_{E,n}$.

§6 Action of stabilizer subgroup and rigidity

$\mathcal{M}$: modular variety of PEL type over $\mathbb{F}$

$x = [(A_x, \iota_x, \lambda_x, \eta_x)] \in \mathcal{M}(\mathbb{F})$

DEF. (i) $\mathrm{G_x}(\mathbb{Z}_p) := \mathrm{Aut}\,((A_x, \iota_x \lambda_x)[p^\infty])$, called the *local p-adic automorphism group* at x.

(ii) $\mathrm{H_x} := \mathrm{U}(\mathrm{End}_{\mathcal{O}_B}(A_x) \otimes \mathbb{Q}, *_x)$, the unitary group attached to above s.s. algebra with involution, $*$ = Rosati.

(iii) $\mathrm{H}_x(\mathbb{Z}_p) := \mathrm{H_x}(\mathbb{Q}_p) \cap \mathrm{G_x}(\mathbb{Z}_p)$, called the *local stabilizer subgroup* at x.

● Have a natural action of $\mathrm{G_x}(\mathbb{Z}_p)$ on $\mathcal{M}^{/x}$ from the Serre-Tate theorem.

Example: Lubin-Tate moduli space, with action by $\mathcal{O}_D$, D= central division algebra over $\mathbb{Q}_p$ with inv. $1/h$.

PROP 7 (Local stabilizer principle)**.** *Let* Z *be a closed subvariety of* $\mathcal{M}$ *stable under all prime-to-*p *Hecke correspondence. Then* $\forall x \in Z(\mathbb{F})$*, the formal subscheme* $Z^{/x} \subset \mathcal{M}^{/x}$ *is stable under the action of the subgroup* $\mathrm{H}_x(\mathbb{Z}_p)$ *of* $\mathrm{G_x}(\mathbb{Z}_p)$*.*

Remark. (1) Prop. 7 is an analog of: If a Lie group G operates on a manifold M and N is a submanifold stable under G, then $\mathrm{T}_y(N)$ is stable under G_y $\forall\, y \in N$.

(2) The local stabilizer principle can be effectively deployed when combined with the local rigidity result below, resulting in a *linearization* of the Hecke orbit problem.

$k = k^{\mathrm{alg}} \supset \mathbb{F}$

X: a p-divisible formal group over k,

H: a connected reductive linear alg. group over $\mathbb{Q}_p$,

$\rho : H(\mathbb{Q}_p) \longrightarrow (\mathrm{End}_k(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{\times}$, a rational linear repr. of $H(\mathbb{Q}_p)$ s.t.

$$\bigl(\text{reg. repr. of } \mathrm{End}_k^0(X)\bigr) \circ \rho$$

does not contain $\mathbf{1}_{H(\mathbb{Q}_p)}$.

$H(\mathbb{Z}_p) := \rho^{-1}(\mathrm{End}_k(X))$,

Z: an irreducible closed formal subscheme of X.

THM 8 (Local rigidity)**.** *Assume that Z is stable under the natural action of an open subgroup U of $H(\mathbb{Z}_p)$ on X. Then Z is a p-divisible formal subgroup of X.*

§7 Open questions and outlook

A. Tate-linear subvarieties

DEF. An irreducible closed subscheme $Z \subset \mathcal{A}_{g,n}^{\mathrm{ord}}$ over $\mathbb{F}$ is *Tate-linear* if $\forall\, x \in Z(\mathbb{F})$ s.t. $Z^{/x}$ is a formal subtorus of the Serre-Tate formal torus $\mathcal{A}_{g,n}^{\mathrm{ord}}$

REM. The condition above is independent of the choice of $x \in Z(\mathbb{F})$.

Question. Is every Tate-linear subvariety $Z \subset \mathcal{A}_{g,n}^{\mathrm{ord}}$ over $\mathbb{F}$ the reduction of a Shimura subvariety of $\mathcal{A}_{g,n}^{\mathrm{ord}}$ over $\overline{\mathbb{Q}}$?

REM. One can formulate a similar question, replacing $\mathcal{A}_{g,n}^{\mathrm{ord}}$ by a leaf $\mathcal{C}$ in $\mathcal{A}_{g,n}$.

p-ADIC MONODROMY

$\mathcal{M}$: modular var. of PEL type s.t. G^{der} is $\mathbb{Q}$-simple,

Z = Zariski closure of $\mathcal{H}^p(x)$ in $\mathcal{C}(x) \subset \mathcal{M}$,

$\rho_{p,Z} : \pi_1(Z, x) \to \mathrm{Aut}\left((A_x, \iota_x, \lambda_x)[p^\infty]\right)$, the p-adic monodromy of Z,.

CONJ 9. *If* $\dim(Z) > 0$*, then the image of* $\rho_{p,Z}$ *is an open subgroup of* $\mathrm{Aut}\left((A_x, \iota_x, \lambda_x)[p^\infty]\right)$*.*

Remark. (1) Conj. 9 implies $\mathrm{HO}_{\mathrm{ct}}$ for Z.

(2) There is a stronger version of Conj. 9:

$\rho_{p,Z}$ is surjective.

(3) If x is a hypersymmetric point on a leaf $\mathcal{C}$, then

$$\rho_{p,\mathcal{C}} : \pi_1(\mathcal{C}, x) \twoheadrightarrow \mathrm{Aut}\left((A_x, \iota_x, \lambda_x)[p^\infty]\right) .$$