p-adic Modular Forms and Arithmetic
A conference in honor of
Haruzo Hida
on his 60th birthday

六十而耳順

(I knew the truth in all I heard when I turned sixty. Confucious)

HECKE SYMMETRY, RIGIDITY AND THE LUBIN-TATE ACTION

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HECKE SYMMETRY, RIGIDITY AND THE LUBIN-TATE ACTION

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Hecke symmetry on PEL moduli varieties
Local Hecke symmetry
The global rigidity problem
Local rigidity problems
Known results, obstacles and hope
A glympse of a new approach
The Lubin-Tate action
Universal isomorphism
p-typical formal group laws
Sketch of the steps
The first test

Outline

PEL type modular varieties

I. An overview

A PEL type modular variety $\mathscr{M}$ is the moduli space attached to a PEL input datum $\mathscr{D} = (D, *, \mathscr{O}_D, H, \langle\cdot,\cdot\rangle, h)$, whose points corresponds to abelian varieties with imposed symmetry $(A, \rho : A \to A^t, \iota : \mathcal{O}_D \to \mathrm{End}(A), \text{level structure})$ whose H_1 are modeled on the linear algebra structure $\mathscr{D}$.

Fix a prime number p, *unramified* for the PEL datum $\mathscr{D}$. We will focus on the equal characteristic p situation unless otherwise specified: $\mathscr{M}$ is a moduli space over $\overline{\mathbb{F}}_p$.

Let $B = \mathrm{End}_D(H)$, with involution $*_B$ induced by $*$. Let $G = \mathrm{SU}(B, *_B)$ (or $\mathrm{GU}(B, *_B)$).

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Hecke symmetry

Let $\widetilde{\mathscr{M}}$ be the prime-to-p tower for $\mathscr{M}$; it is a profinite etale Galois cover of $\mathscr{M}$ with group $G(\hat{\mathbb{Z}}^{(p)})$. The group $G(\mathbb{A}_f^{(p)})$ operates on $\widetilde{M}$, inducing Hecke correspondences on $\mathscr{M}$.

Example: $\mathscr{M} = \mathscr{A}_g$ = the moduli space classifying g-dimensional principally polarized abelian varieties, $G = \mathrm{Sp}_{2g}$ (or GSp_{2g})).

Local Hecke symmetry

Given a point $x \in \mathscr{M}(\overline{\mathbb{F}}_p)$, corresponding to a quadruple $(A_x, \rho_x : A_x \to A_x^t, \iota_x : \mathcal{O}_D \to \mathrm{End}(A_x), \text{level structure})$.

Let $\mathscr{M}^{/x}$ be the formal completion of $\mathscr{M}$ at x.

Let $H_x := \mathrm{U}(\mathrm{End}_D^0(A_x), *_{\mathrm{Ros}})(\mathbb{Z}_{(p)})$, and let $G_x := \mathrm{U}(\mathrm{End}_D^0(A_x[p^\infty]), *_{\mathrm{Ros}})(\mathbb{Z}_p)$.

The Serre-Tate deformation theorem implies that there is a natural action of the compact p-adic group G_x on $\mathscr{M}^{/x}$, by "changing the marking".

This action can be regarded as a *local version* of the global Hecke symmetries.

Local stabilizer subgroups

We call G_x the *local stabilizer subgroup* at x. The group H_x can be thought of as the “intersection” of G_x with the global Hecke symmetries on $\mathscr{M}$.

Lemma. If a closed subvariety $Z \subset \mathscr{M}$ is stable under all Hecke symmetries, then $Z^{/x} \subset \mathscr{M}^{/x}$ is stable under the action of the p-adic closure of H_x in G_x.

Examples. For a “general” $x \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$” (in particular x is ordinary), the Zariski closure of H_x is a g-dimensional torus, while the Zariski closure of G_x is GL_g.

For a supersingular point $x \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$, H_x is p-adically dense in G_x, and the Zariski closure of G_x is a twist of Sp_{2g}.

The global rigidity problem

(Oort’s Hecke orbit conjecture)

Prediction. Let $Z \subset \mathscr{M}_{/\overline{\mathbb{F}}_p}$ be a reduced closed subset of $\mathscr{M}$ stable under all prime-to-p Hecke correspondences.
Then Z contains the leaf $C(x)$ passing through x for every point $x \in Z(\overline{\mathbb{F}}_p)$.

(Every Hecke-invariant closed subset of $\mathscr{M}_{/\overline{\mathbb{F}}_p}$ is a union of leaves; the latter can be regarded as “generalized Shimura subvarieties in char. p”.)

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Definition and examples of leaves

- A *leaf* $C(x)$ in $\mathscr{M}_{/\overline{\mathbb{F}}_p}$ is the locus in $\mathscr{M}_{/\overline{\mathbb{F}}_p}$ where *all* p-adic invariants have the same "value" as those of x.
- The *ordinary* locus $\mathscr{A}_g^{\mathrm{ord}} \subset \mathscr{A}_{g/\overline{\mathbb{F}}_p}$ is a leaf in $\mathscr{A}_{g\overline{\mathbb{F}}_p}$.
- The leaf passing through a *supersingular* point in $\mathscr{A}_g$ is finite.
- The leaf passing through a point in $\mathscr{A}_3$ corresponding to a 3-dimensional abelian variety with slopes $\{1/3, 2/3\}$ is two-dimensional. Such leaves form a one-dimensional family in the slopes $\{1/3, 2/3\}$ locus of $\mathscr{A}_3$.

  (The latter locus has dimension three.)

Strong forms of global rigidity problem

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Remark. In application(s) to Iwasawa theory pioneered by Hida, certain strong versions of the global rigidity problem appear naturally:

- The assumption on Z is weakened to:
  Z is stable under the action of a "not-to-small" subset of Hecke correspondences.
- The desired conclusion is that Z is a union of leaves in the reduction of certain Shimura subvarieties.

Local rigidity problems

Set-up. $Z \subset \mathscr{M}^{/x}$ is a reduced closed formal subscheme of $\mathscr{M}^{/x}$, stable under the action of a "not-too-small" subgroup of G_x.

Restricted local rigidity problem (to make it easier): Assume in addition that $Z \subset C(x)^{/x}$.

Desired conclusion. Z has a (very) special form (e.g. defined by a finite collection of Tate cycles.)

Results on the restricted local rigidity problem

II. Known results, obstacles and hope

Propostion. Restricted local rigidity holds for $\mathscr{A}_g$, in the case when A_x has only two slopes.

- $C(x)^{/x}$ has a natural structure as a torsor for an isoclinic p-divisible formal group X_x.
- If $Z \subset C(x)^{/x}$ is stable under a not-too-small subgroup of G_x, then Z_x is a torsor for a p-divisible subgroup of X_x.

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Restricted local rigidity: an example and consequences

An example. Let Z be an irreducible forma subscheme of a formal torus $\hat{\mathbb{G}}_m^r$ over $\overline{\mathbb{F}}_p$. Suppose that Z is closed under the action of $[1+p^m]$ for some $m \geq 2$. Then Z is a formal subtorus of $\hat{\mathbb{G}}_m^r$. (exercise)

Consequence of restricted local rigidity:
linearization of the global rigidity problem, helped by considerations of local and global p-adic monodromy.

Results on global rigidity using the Hilbert trick

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Theorem. Global rigidity holds for $\mathscr{A}_g$.

Remarks. (1) Besides the restricted local rigidity and monodromy arguments, the proof uses a trick:
Every point $x \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$ is contained in a Hilbert modular subvariety of $\mathscr{A}_g$.
(Global rigidity is substantially easier for these “small” modular varieties; see below.)

(2) This “Hilbert trick” fails for PEL modular varieties of type A or D.

(3) A strong global rigidity statement holds for Hilbert modular varieties (and many othother modular varieties assciated to semisimple groups of $\mathbb{Q}$-rank one).

A tantalizing dream

The **holy grail** for the rigidity problems (don't have better leads):

To pry actionable intelligence out of the action of the local stabilizing subgroup.

Main obstacle: Our poor understanding of this action (so cannot deploy enhanced interrogation techniques).

- Don't have helpful (exact or approximate) formulas (have tried Norman's algorithm).
- Linearization via crystalline techniques leads to formulas with high powers of p in denominators.

A glympse of a new approach

We will explain a method to obtain an approximate (or even asymptotic) formula for the action of the local stabilizer subgroup, in the first non-trivial case,

where $\mathscr{M}^{/x} = \mathrm{Def}(G_0)$ is the Lubin-Tate moduli deformation space for a one-dimensional formal group G_0 of finite height h over $\overline{\mathbb{F}}_p$.

Notation

Let h be a positive integer.

Let G_1 be the one-dimensional formal group over $\mathbb{Z}_{(p)}$ with logarithm

$$\sum_{j\in\mathbb{N}} p^{-j}x^{p^{jh}} = x + \frac{x^{p^h}}{p} + \frac{x^{p^{2h}}}{p^2} + \cdots$$

(so it is a Lubin-Tate formal group for $W(\mathbb{F}_{p^h})$.)

Let G_0 be the base extension to $\overline{\mathbb{F}}_p$ of the closed fiber of G_1; it is a one-dimensional formal group over $\mathbb{F}_p$ of height h.

It is well-known that $\mathrm{End}(G_0)$ is the maximal order of $\mathrm{End}^0(G_0)$ = a central division algebra over $\mathbb{Q}_p$ of dimension h^2.

The Lubin-Tate action

Let $\mathscr{M}_h := \mathrm{Def}(G_0)$; it is a smooth formal scheme over $W(\overline{\mathbb{F}}_p)$ of relative dimension $h-1$.
Let $G_{\mathrm{univ}} \to \mathscr{M}_h$ be the universal formal group over $\mathscr{M}_h$.

The compact p-adic group $\mathrm{Aut}(G_0) = \mathrm{End}(G_0)^\times$ operates on $\mathscr{M}_h$ by functoriality, as follows.

$\forall\, \gamma \in \mathrm{Aut}(G_0)$, $\exists!$ formal scheme automorphism $\rho(\gamma)$ of $\mathscr{M}_h$ and a formal group isomorphism

$$\tilde{\rho}(\gamma) : G_{\mathrm{univ}} \to \rho(\gamma)^* G_{\mathrm{univ}}$$

such that $\tilde{\rho}(\gamma)|_{G_0} = \gamma$

Remark. This action $\gamma \mapsto \rho(\gamma)$ of $\mathrm{Aut}(G_0)$ on the Lubin-Tate moduli space $\mathscr{M}_h$ was first studied by Lubin and Tate in 1966. It is also known as (the essential part of) the *Morava stabilizer subgroup* action in chromatic homotopy theory.

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The universal p-typical formal group law

Let $\tilde{R} = \mathbb{Z}_{(p)}[\underline{v}] = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$, and let $\sigma : \tilde{R} \to \tilde{R}$ be the ring homomorphism such that $\sigma(v_j) = v_j^p$ for all $j \geq 1$

Let $G_{\underline{v}}(x) \in \tilde{R}[[x, y]]$ be the one-dimensional p-typical formal group law over $\tilde{R}$ whose logarithm

$$g_{\underline{v}}(x) \in \tilde{R}[1/p][[x]] = \sum_{n \geq 1} a_n(\underline{v}) \cdot x^{p^n}$$

satisfies

$$g_{\underline{v}}(x) = x + \sum_{i=1}^{\infty} \frac{v_i}{p} \cdot g_{\underline{v}}^{(\sigma^i)}(x^{p^i})$$

Remarks on the formal group law $G_{\underline{v}}$

Remarks. (1) The above "functional equation" is a recursive formula for the coefficients $a_n(\underline{v}) \in p^{-n} \cdot \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n]$ of $g_{\underline{v}}(x)$.

(2) Explicitly:

$$\begin{aligned} a_n(\underline{v}) &= \sum_{\substack{i_1, i_2, \ldots, i_r \geq 1 \\ i_1 + \cdots + i_r = n}} p^{-r} \cdot \prod_{s=1}^{r} v_{i_s}^{p^{i_1 + i_2 + \cdots + i_{s-1}}} \\ &= \sum_{\substack{i_1, i_2, \ldots, i_r \geq 1 \\ i_1 + \cdots + i_r = n}} p^{-r} \cdot v_{i_1} \cdot v_{i_2}^{p^{i_1}} \cdot v_{i_3}^{p^{i_1 + i_2}} \cdots v_{i_r}^{p^{i_1 + \cdots + i_{r-1}}} \end{aligned}$$

Note that $a_n(\underline{v})$ is a homogeneous polynomial in $v_1, \ldots, v_n$ of weight $p^n - 1$ when v_j is given the weight $p^j - 1$ $\forall j \geq 1$.

(3) The formal group law $G_{\underline{v}}$ over $\tilde{R}$ is "the" universal one-dimensional p-typical formal group law.

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The universal formal group over $\mathscr{M}_h$ made explicit

Let $R = R_h = W(\overline{\mathbb{F}}_p)[[w_1, w_2, \ldots, w_{h-1}]]$.

Let $\pi = \pi_h : \tilde{R} \to R$ be the ring homomorphism such that

$$\pi(v_i) = \begin{cases} w_i & \text{if} \quad 1 \le i \le h-1 \\ 1 & \text{if} \quad i = h \\ 0 & \text{if} \quad i \ge h+1 \end{cases}$$

The classifying morphism $\mathrm{Spf}(R) \to \mathscr{M}_h$ for the deformation $\pi_* G_{\underline{v}}$ of G_0 is an isomorphism.

We will identify $\mathscr{M}_h$ with $\mathrm{Spf}(R)$ and the universal deformation G_{univ} of G_0 with the formal group underlying the formal group law $G_R := \pi_* G_{\underline{v}}$.

The universal strict isomorphism

Let $\mathbb{Z}_{(p)}[\underline{v}, \underline{t}] = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots; t_1, t_2, t_3, \ldots]$, and let $\sigma : \mathbb{Z}_{(p)}[\underline{v}, \underline{t}] \to \mathbb{Z}_{(p)}[\underline{v}, \underline{t}]$ be the obvious Frobenius lifting as before, with $\sigma(v_i) = v_i^p$ and $\sigma(t_i) = t_i^p$ $\forall i \ge 1$.

Let $G_{\underline{v},\underline{t}}(x, y)$ be the one-dimensional formal group law over $\mathbb{Z}_{(p)}[\underline{v}, \underline{t}]$ whose logarithm $g_{\underline{v},\underline{t}}(x)$ satisfies

$$g_{\underline{v},\underline{t}}(x) = x + \sum_{i=1}^{\infty} t_i \cdot x^{p^i} + \sum_{j=1}^{\infty} \frac{v_j}{p} \cdot g_{\underline{v},\underline{t}}^{(\sigma^j)}(x^{p^j})$$

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The universal strict isomorphism, continued

It is known that $\alpha_{\underline{v},\underline{t}} := g_{\underline{v},\underline{t}}^{-1} \circ g_{\underline{v}} \in \mathbb{Z}_{(p)}[\underline{v},\underline{t}][[x]]$, and defines a *strict isomorphism*

$$\alpha_{\underline{v},\underline{t}} : G_{\underline{v}} \to G_{\underline{v},\underline{t}}$$

between p-typical formal group laws over $\mathbb{Z}_{(p)}[\underline{v},\underline{t}]$.
(A *strict* isomorphism is an isomorphism between formal group laws which is $\equiv x$ modulo higher degree terms in x.)

Moreover $\alpha_{\underline{v},\underline{t}}$ is "the" universal strict isomorphism between one-dimensional p-typical formal group laws.

Parameters of $G_{\underline{v},\underline{t}}$

By the universality $G_{\underline{v}}$ for p-typical formal group laws, there exists a unique ring homomorphism

$$\eta : \mathbb{Z}_{(p)}[\underline{v}] \to \mathbb{Z}_{(p)}[\underline{v},\underline{t}]$$

such that

$$\eta_* G_{\underline{v}} = G_{\underline{v},\underline{t}}.$$

The elements

$$\overline{v}_n = \overline{v}_n(\underline{v},\underline{t}) \in \mathbb{Z}_{(p)}[\underline{v},\underline{t}], \quad n \in \mathbb{N}_{\geq 1}$$

are the *parameters* of the p-typical formal group law $G_{\underline{v},\underline{t}}$.

A known recursive formula for the parameters of $G_{\underline{v},\underline{t}}$

$$
\begin{aligned}
\bar{v}_n = v_n + p\,t_n &+ \sum_{\substack{i+j=n\\ i,j\geq 1}} \left(v_j\, t_i^{p^j} - t_i\, \bar{v}_j^{p^i}\right) \\
&+ \sum_{j=1}^{n-1} a_{n-j}(\underline{v}) \cdot \left(v_j^{p^{n-j}} - \bar{v}_j^{p^{n-j}}\right) \\
&+ \sum_{k=2}^{n-1} a_{n-k}(\underline{v}) \cdot \sum_{\substack{i+j=k\\ i,j\geq 1}} \left(v_j^{p^{n-k}}\, t_i^{p^{n-i}} - t_i^{p^{n-k}}\, \bar{v}_j^{p^{n-j}}\right)
\end{aligned}
$$

(This formula contains high power of p in the denominators. Consequently it is not very useful for our purpose.)

An *integral* recursion formula for $\bar{v}_n(\underline{v},\underline{t})$

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(useful for computing the Lubin-Tate action)

$$
\begin{aligned}
\bar{v}_n = v_n + p\,t_n &- \sum_{j=1}^{n-1} t_j \cdot \bar{v}_{n-j}^{p^j} + \\
&+ \sum_{l=1}^{n-1} v_l \sum_{k=1}^{n-l-1} \frac{1}{p} \cdot a_{n-k-l}(\underline{v})^{(p^l)} \cdot \Big\{ \big(\bar{v}_k^{(p^l)}\big)^{p^{n-l-k}} - \big(\bar{v}_k^{p^l}\big)^{p^{n-l-k}} \\
&\qquad\qquad + \sum_{\substack{i+j=k\\ i,j\geq 1}} t_j^{p^{n-k}} \left[\big(\bar{v}_i^{(p^l)}\big)^{p^{n-l-i}} - \big(\bar{v}_i^{p^l}\big)^{p^{n-l-i}} \right] \Big\} \\
&+ \sum_{l=1}^{n-1} v_l \cdot \Big\{ \frac{1}{p}\big(\bar{v}_{n-l}^{(p^l)} - \bar{v}_{n-l}^{p^l}\big) + \sum_{\substack{i+j=n-l\\ i,j\geq 1}} t_j^{p^l} \cdot \frac{1}{p} \cdot \left[\big(\bar{v}_i^{(p^l)}\big)^{p^j} - \big(\bar{v}_i^{p^l}\big)^{p^j} \right] \Big\}
\end{aligned}
$$

for every $n \geq 1$.

Step 1

Given an element $\gamma \in \mathrm{Aut}(G_0)$, construct

- a p-typical one-dimensional formal group *law* $F = F_\gamma$ over R whose closed fiber is equal to G_0, and
- an isomorphism

$$\overline{\psi} = \overline{\psi}_\gamma : F_{\overline{R}} \to G_{\overline{R}}$$

over $\overline{R} := R/pR = \overline{\mathbb{F}}_p[[w_1, \ldots, w_{h-1}]]$ whose restriction to the closed fibers is

$$(\psi|_{G_0} : G_0 \to G_0) = \gamma.$$

Here $F_{\overline{R}} = F \otimes_R \overline{R}$, $G_{\overline{R}} = G_R \otimes_R \overline{R}$.

Note that both the formal group law F over R and the isomorphism ψ over $\overline{R}$ depends on the given element $\gamma \in \mathrm{Aut}(G_0)$.

The formal group law F_c, $c \in W(\mathbb{F}_{p^h})^\times$

For $\gamma = [c] \in W(\mathbb{F}_{p^h})^\times = \mathrm{Aut}(G_1)$, we can take F_c to be the formal group over R whose logarithm $g_c(x)$ satisfies

$$f_c(x) = x + \sum_{i=1}^{h} \frac{c^{-1+\sigma^i} \cdot w_i}{p} \cdot f_c^{(\sigma^i)}(x^{p^i})$$

(w_h=1 by convention).

Let

$$\psi_c(x) = \log_{G_R}^{-1} \circ (c \cdot f_c)$$

We have $\psi_c(x) \in R[[x]]$ and ψ_c defines an isomorphism from F_c to G_R over R (not just over $\overline{R}$!) with $\psi_c|_{G_0} = [c]$.

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Step 2

Compute the parameters

$$(u_i = u_i(w_1, \ldots, w_{h-1}))_{i \in \mathbb{N}_{\geq 1}}$$

for the p-typical group law $F = F_\gamma$ over R.

The above condition means that

$$\xi_* G_{\tilde{v}} = F,$$

where

$$\xi = \xi_\gamma : \tilde{R} \to R$$

is the ring homomorphism such that

$$\xi(v_i) = u_i \quad \forall i \geq 1.$$

Parameters for F_c, $c \in W(\mathbb{F}_{p^h})^\times$

In the case when $\gamma \in \mathrm{Aut}(G_0)$ lifts to an element [c] with $c \in W(\mathbb{F}_{p^h})^\times \simeq \mathrm{Aut}(G_1)$, we have the following integral recursive formula for the parameters $u_n = u_n(c; \underline{w})$.

$$\begin{aligned} u_n(c;\underline{w}) = c^{-1+\sigma^n} w_n & \\ &+ \sum_{j=1}^{n-1} c^{-1+\sigma^j} \cdot \frac{1}{p} \left[u_{n-j}(c;\underline{w})^{(p^j)} - u_{n-j}(c;\underline{w})^{p^j} \right] \cdot w_j \\ &+ \sum_{j=1ve}^{n-1} \sum_{i=1}^{n-j-1} \frac{1}{p} a_{n-i-j}(\underline{w})^{(p^j)} \cdot c^{-1+\sigma^{n-i}} \cdot \\ &\qquad \left[(u_i(c;\underline{w})^{(p^j)})^{p^{n-i-j}} - (u_i(c;\underline{w})^{p^j})^{p^{n-i-j}} \right] \cdot w_j \end{aligned}$$

where $w_h = 1$, $w_m = 0 \ \forall m \geq h+1$ by convention.

Parameters for F_c, continued

Remark. The above recursive formula for the parameters $u_n(c;\underline{w})$ can be turned into an explicit "path sum" formula for $u_n(c,\underline{w})$, with terms indexed by "paths".

Step 3

Find/compute the uniquely determined element

$$\tau_n \in \mathfrak{m}_R, \quad n \in \mathbb{N}_{\geq 1}$$

and

$$\hat{u}_1 \in \mathfrak{m}_R, \ldots, \hat{u}_{h-1} \in \mathfrak{m}_R, \hat{u}_h \in 1 + \mathfrak{m}_R$$

such that

$$\overline{v}_n(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h, 0, 0, \ldots; \underline{\tau}) = u_n \quad \forall n \geq 1.$$

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Remark. (1) The existence and uniqueness statement above is an application the implicit function theorem for an infinite dimensional space over $\tilde{R}$, applied to the "vector-valued" function with components $\bar{v}_n$ in the integral recursion formula discussed before.

(2) This step is a substitute for the operation *taking the quotient of the group "changes of coordinates"* in a space of formal group laws.

(3) The approximate solution coming from the linear term in the τ_j variables is often good enough for our application.

A congruence formula for $\bar{v}_n$

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The follow formula helps to explain the last remark.

$$\begin{aligned}\bar{v}_n \equiv v_n &- \sum_{j=1}^{n} t_j \cdot v_{n-j}^{p^j}\\ &+ \sum_{\substack{i,j,t,s_1,s_2,\ldots,s_t\geq 1\\ s_1+\ldots+s_t+i+j=n}} (-1)^{t-1}\, t_i\cdot v_j^{p^i}\cdot v_1^{(p^{s_1}+p^{s_2}+\cdots+p^{s_t}-t)/(p-1)}\\ &\qquad\qquad \cdot v_{n-s_1}^{p^{s_1}-1}\cdot v_{n-s_1-s_2}^{p^{s_2}-1}\cdots v_{n-s_1-\cdots-s_t}^{p^{s_t}-1}\\ &\qquad\qquad\qquad \mod (pt_a, t_a\cdot t_b)_{a,b\geq 1}\mathbb{Z}[\underline{v},\underline{t}]\end{aligned}$$

Step 4

Rescale $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h$ as follows:
$\exists !\ \tau_0 \in \mathfrak{m}_R$ such that

$$(1+\tau_0)^{p^h-1} \cdot \hat{u}_h = 1.$$

Let

$$\hat{v}_i := (1+\tau_0)^{p^i-1} \cdot \hat{u}_i \ \text{ for } i = 1, \ldots, h-1.$$

Let $\omega : \tilde{R} \to R$ be the ring homomorphism such that

$$\omega(v_i) = \hat{u}_i \quad \forall i \geq 1.$$

Let $\rho : R \to R$ be the $W(\overline{\mathbb{F}}_p)$-linear ring homomorphism such that

$$\rho(w_i) = \hat{v}_i \quad \forall i \geq 1.$$

The meaning of Steps 3 and 4

The universal strict isomorphism $\alpha_{\underline{v},\underline{t}}$ specializes to a strict isomorphism

$$\alpha = \alpha_{\underline{\hat{u}},\underline{\tau}} : F \to \omega_* G_{\underline{v}}$$

with $\alpha|_{G_0} = \mathrm{Id}_{G_0}$.

The rescaling in step 4 gives an isomorphism (not necessarily a strict isomorphism)

$$\beta : \omega_* G_{\underline{v}} \to \rho_* G_R$$

with $\beta|_{G_0} = \mathrm{Id}_{G_0}$.

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Conclusion

Combined with $\overline{\psi}$, we obtain an isomorphism

$$\overline{\psi}\circ\overline{\alpha}^{-1}\circ\overline{\beta}^{-1}:\overline{\rho}_{*}G_{\overline{R}}\to G_{\overline{R}}$$

whose restriction to the closed fiber G_0 is equal to the given element $\gamma\in\mathrm{Aut}(G_0)$.
(Here $\overline{\alpha}=\alpha\otimes_R\overline{R}$ and $\overline{\beta}=\beta\otimes_R\overline{R}$.)

Conclusion. The given element $\gamma\in\mathrm{Aut}(G_0)$ operates on the equi-characteristic deformation space $\mathrm{Spf}(\overline{R})$ of G_0 via the ring automorphism ρ.
(Notice that $\overline{\psi}$, α and β all depend on γ.)

Local rigidity for the Lubin-Tate moduli space: the first non-trivial case

Proposition. Let $Z\subset\mathscr{M}_{3_{\overline{\mathbb{F}}_p}}=\mathrm{Spf}(\overline{\mathbb{F}}_p[[w_1,w_2]])$ be an irreducible closed formal subscheme of $\mathscr{M}_3$ over $\overline{\mathbb{F}}_p$ corresponding to a hight one prime ideal of $\overline{\mathbb{F}}_p[[w_1,w_2]]$.
If Z is stable under the action of an open subgroup of $W(\mathbb{F}_{p^3})^{\times}$, then $Z=\mathrm{Spf}(\overline{\mathbb{F}}_p[[w_1,w_2]]/(w_1))$.