Chapter 5
Moduli of abelian varieties

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Introduction

We discuss three questions listed in the 1995 collection [74] and (partial) answers to them. We will also describe ideas connected with other problems on related topics. Throughout this article p denotes a fixed prime number.

0.1. [74, Conjecture 8B] (irreducibility of Newton polygon strata). Let β be a non-supersingular symmetric Newton polygon. Denote by $W_{\beta,1}$ the Newton polygon stratum in the moduli space $\mathcal{A}_{g,1}\otimes\mathbb{F}_p$ of g-dimensional principally polarized abelian varieties in characteristic p. We made the following conjectured in 1995.

> *Every non-supersingular* NP *stratum* $W_{\beta,1}$ *in* $\mathcal{A}_{g,1}\otimes\mathbb{F}_p$ *is geometrically irreducible.*

We knew at that time that (for large g) the supersingular locus W_σ in $\mathcal{A}_{g,1}\otimes\mathbb{F}_p$ is geometrically reducible; see 6.9 for references. And we had good reasons to believe that every non-supersingular Newton polygon strata in $\mathcal{A}_{g,1}\otimes\mathbb{F}_p$ should be *geometrically irreducible*. This conjecture was proved in [4, Thm. 3.1]. See Section 6 for a sketch of a proof, its ingredients and other details.

0.2. [74, § 12] (CM liftings). Tate proved that every abelian variety A defined over a finite κ field is of CM type, that is its endomorphism algebra $\mathrm{End}^0_\kappa(A) := \mathrm{End}_\kappa(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ contains a commutative semisimple algebra over $\mathbb{Q}$ whose dimension over $\mathbb{Q}$ is twice the dimension of A ; see [98], Th. 2 on page 140. Honda and Tate proved that every abelian variety over $\overline{\mathbb{F}_p}$ is isogenous to an abelian variety which admits a CM lifting to characteristic zero; see [33, Main Theorem, p. 83] and [99, Thm. 2, p. 102]. More precisely:

> *For any abelian variety A over a finite field $\kappa=\mathbb{F}_q$, there exists a finite extension field $K\supset\kappa$, an abelian variety B_0 over K, a K-isogeny between $A\otimes_\kappa K$ and B_0, and a* CM *lift of B_0 to characteristic zero.*

Two follow-up questions arise naturally.

> Is a field extension K/κ necessary (for the existence of a CM lift)?
> Is (a modification by) an isogeny necessary (for the existence of a CM lift)?

The answer to the second question above is "yes, *an isogeny is necessary in general*": we knew in 1992 [73] that there exist abelian varieties defined over $\overline{\mathbb{F}_p}$ which do not admit CM lifting to characteristic zero. Later, more precise questions were formulated, in 0.2.1 and 0.2.2 below.

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0.2.1. Suppose that A is an abelian variety over a finite field κ. Does there exists a κ-isogeny $A \to B_0$ such that B_0 admits a CM lift to characteristic zero? In other words, *is a field extension necessary in the Honda-Tate theorem*?

0.2.2. Suppose that A is an abelian variety over a finite field κ. Does there exist a κ-isogeny $A \to B_0$ and a CM lifting of B_0 to a *normal* mixed characteristics local domain with residue field κ?

Complete answers to 0.2.1, 0.2.2 and some further CM lifting questions are given in [2]. Below, in Section 7 we will describe the questions, methods and results.

0.3. [74, § 13] (generalized canonical coordinates). Around a moduli point $x_0 = [X_0, \lambda_0]$ of an *ordinary* abelian variety over a perfect field $k \supset \mathbb{F}_p$, the formal completion

$$\mathcal{A}_{g,1}^{/x_0} \quad \cong \quad \left((\mathbb{G}_m)^\wedge\right)^{g(g+1)/2}$$

of the moduli space $\mathcal{A}_{g,1}$ over $W(k)$ is a formal torus, where the origin is chosen to be the canonical lift of (X_0, λ_0); see [49] and [43]. Also see [55].

In [74, 13A] we asked for "*canonical coordinates*" around any moduli point in $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$, with a note that this question "should be made much more precise before it can be taken seriously." We will show results and limitations:

(a) We do not know how to produce "canonical coordinates" in *mixed characteristics* $(0, p)$ for the formal completion of $\mathcal{A}_{g,1}$ at a non-ordinary point, nor do we know a how to produce good or useful "canonical coordinates" for the formal completion of $\mathcal{A}_{g,1} \otimes \overline{\mathbb{F}_p}$ at a non-ordinary point.

(b) We (think we) can construct the right concept around an $\overline{\mathbb{F}_p}$-point of a *central leaf* in $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$; see [5].

Note that the central leaf through the moduli point of an *ordinary abelian variety* is the dense open subset of the moduli space consisting of all ordinary points. So this statement (b) can be regarded as a generalization of the Serre–Tate canonical coordinates. This is a weak generalization; its limitation is given in (a).

- The central leaf passing through an *almost ordinary* abelian variety (i.e. the p-rank equals $\dim(A) - 1$) is a dense open subset of the Newton polygon stratum in characteristic p attached to the given abelian variety.

  The notion "generalized canonical coordinates" we produce is only for equi-characteristic p, in the sense that it gives a natural structure of the formal completion of the leaf $\mathcal{C}(x_0)^{/x_0}$ at the given almost ordinary point x_0.

  One can show that $\mathcal{C}(x_0)$ does not admit a functorial lift to characteristic 0. Similarly the locus of all almost ordinaries in $\mathcal{A}_g$ does not admit a lift to a flat scheme over the ring of integers of a finite extension field of $\mathbb{Q}_p$ which is invariant under all prime-to-p Hecke correspondences.

- For other moduli points in positive characteristic central leaves are smaller than the related Newton polygon stratum; we do construct a generalization (analogue) of canonical coordinates on (the formal completion of) central leaves in characteristic p. But we do not construct something like "canonical coordinates" on the whole Newton polygon stratum if the p-rank is smaller than $\dim(A) - 1$. We will explain why and provide references. For more information see [5].

1 p-divisible groups

In this section we briefly recall notations we are going to use.

1.1. For given integers $d \in \mathbb{Z}_{>0}$ and $h \in \mathbb{Z}_{\geq d}$, a *Newton polygon*[1] ζ of *height* h and *dimension* d is a lower convex polygon on $\mathbb{R}^2$ starting at $(0,0)$ and ending at (h,d) and having break points in $\mathbb{Z} \times \mathbb{Z}$; lower convex means that the whole polygon lies on or above every line spanned by a segment of the polygon. Equivalently: there exist rational numbers $s_1, \ldots, s_h$ with $1 \geq s_1 \geq \cdots \geq s_h \geq 0$, called the *slopes* of ζ such that

- $s_1 + \cdots + s_h = d$, and $s_h + \cdots + s_b \in \mathbb{Z}$ for all b for which $s_{b+1} < s_b$, and
- the slopes of the polygon ζ in the open intervals $(0,1), \ldots, (h-1,h)$ are $s_h, s_{h-1}, \ldots, s_1$ respectively. Thus ζ is the graph of the convex piecewise linear function $\underline{\zeta}$ on $[0,h]$ whose derivative is equal to s_{h-i+1} on the open interval $(i-1,i)$ for $i = 1, \ldots, h$. Explicitly,

$$\underline{\zeta}(t) = (t+b-h)s_b + \sum_{b+1 \leq i \leq h} s_i$$

  for all integer $b \in [1,h]$ and all real numbers $t \in [h-b, h-b+1]$.

The reason to list the slopes in the "reverse order" comes from geometry: the first non-trivial sub-p-divisible subgroup in the slope filtration of a p-divisible group X over a field of characteristic p with Newton polygon ζ has the largest (Frobenius) slope among the slopes of ζ; see [114], [88]. We explain and fix the notation.

(i) Suppose that $(m_1,n_1), \ldots, (m_a,n_a)$ are a pairs of relatively prime non-negative integers such that
$$\frac{m_1}{m_1+n_1} > \frac{m_2}{m_2+n_2} > \cdots > \frac{m_a}{m_a+n_a},$$
and $\mu_1, \ldots, \mu_a$ are positive integers. We introduce the notation
$$\mathrm{NP}\big(\mu_1 * (m_1,n_1) + \cdots + \mu_a * (m_a,n_a)\big)$$
for the Newton polygon with slopes $m_i/(m_i+n_i)$ and where this slope has multiplicity $h_i = \mu_i(m_i+n_i)$. This is the Newton polygon attached to the p-divisible group $(G_{m_1,n_1})^{\mu_1} \times \cdots \times (G_{m_a,n_a})^{\mu_a}$; see the paragraph after the statement of Theorem 1.4; the dimension of $G_{m,n}$ is m, and the height is $m+n$.

(ii) For any Newton polygon ζ with slopes $1 \geq s_1 \geq \cdots \geq s_h$ of height h, the *p-rank* of ζ, denoted by $f(\zeta)$, is $f(\zeta) := \{i \mid s_i = 0,\ 1 \leq i \leq h\}$, the number of slopes of ζ which are equal to 0. Clearly $f(\zeta) \leq \mathrm{ht}(\zeta) - \dim(\zeta)$.

(iii) A Newton polygon is called *isoclinic* if all slopes are equal; in this case $s_j = d/h$ for every j.

(iv) For a given Newton polygon ζ we define the *opposite polygon* ζ^* of ζ as the graph of the piecewise linear continuous function on $[0,h]$ whose derivative on the open interval $(i-1,i)$ is equal to s_i for $i = 1, \cdots, h$. Note that, while ζ is lower convex, ζ^* is upper convex.

[1] Here we only consider Newton polygons whose slopes are between 0 and 1, suitable for p-divisible groups. Under this restriction the notion of *dimension* and *height* of a Newton polygon corresponds to the dimension and height of p-divisible groups.

(v) A *symmetric Newton polygon* is a Newton polygon ξ with $d = g$, and $h = 2g$ whose slopes $1 \geq s_1 \geq \cdots \geq s_{2g}$ satisfy the self-duality condition $s_i + s_{2g-i+1} = 1$ for all $i = 1, \ldots, 2g$ for all $i = 1, \ldots, 2g$.

(vi) There is exactly one isoclinic symmetric Newton polygon σ_g of height $2g$; all slopes of σ_g are equal to $1/2$. It is called the *supersingular* Newton polygon of dimension g.

(vii) A Newton polygon ζ is said to be *ordinary* if all of its slopes are either 0 or 1; equivalently $f(\zeta) = \mathrm{ht}(\zeta) - \dim(\zeta)$. For any pair of natural numbers d, h with $d \leq h$, there is exactly one ordinary Newton polygon $\rho_{d,h}$ with dimension d and height h; it is symmetric if $h = 2d$

(viii) Let $\zeta = \mathrm{NP}(s_1, \ldots, s_h)$ be a Newton polygon of height h. The *dimension* of ζ is

$$\dim(\zeta) := s_1 + \cdots + s_h;$$

the *codimension* of ζ is defined by

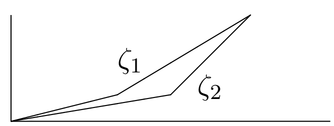
$$\mathrm{codim}(\zeta) := h - \dim(\zeta).$$

(ix) Define a partial ordering on the set of all Newton polygons as follows. For Newton polygons ζ_1, ζ_2

$$\zeta_1 \preceq \zeta_2 \iff \mathrm{ht}(\zeta_1) = \mathrm{ht}(\zeta_2),\ \dim(\zeta_1) = \dim(\zeta_2) \text{ and no point of } \zeta_1 \text{ is strictly below } \zeta_2$$

We will say that "ζ_1 is *on or above* ζ_2" when $\zeta_1 \preceq \zeta_2$.

Illustration: $\zeta_1 \precneqq \zeta_2$



(This terminology may seem a bit odd at first sight. The reason is that if ζ_1 is the Newton polygon of a specialization of a p-divisible group with Newton polygon ζ_2, then $\zeta_1 \preceq \zeta_2$ holds. This implies that for a family of p-divisible groups over a base scheme S in characteristic p, if the Newton polygon stratum in S indexed by ζ_1 is contained in the Newton polygon stratum in S indexed by ζ_2, then $\zeta_1 \preceq \zeta_2$.)

- Every Newton polygon ζ with slopes between 0 and 1 lies above the ordinary Newton polygon, denoted by ρ, with the same height and dimension, i.e. $\zeta \preceq \rho_{\dim\zeta, \mathrm{ht}(\zeta)}$.

1.2. For any scheme S, a p-divisible group X of height h over S is an inductive system

$$\{G_i \hookrightarrow G_{i+1}\}_{i \in \mathbb{N}}$$

of finite locally free group schemes G_i over S killed by p^i, such that the transition homomorphisms are closed embeddings, and the homomorphism $[p]_{G_{i+1}} : G_{i+1} \to G_{i+1}$ factors as the composition of a faithfully flat homomorphism $G_{i+1} \to G_i$ and the embedding $G_i \hookrightarrow G_{i+1}$ for every i. Here $[p]_{G_{i+1}}$ denotes the endomorphism of G_{i+1} induced by "multiplication by p".

For a p-divisible group $\{G_i \hookrightarrow G_{i+1}\}$ as above, let $G = \cup_i G_i$ be the inductive limit of the G_i's as a sheaf in the fppf topology over S. Then G_i is naturally isomorphic to the kernel $G[p^i]$ of $[p^i]_G$ for each i. Moreover there exists an $\mathbb{N}$-valued function h on S, which is locally constant for the Zariski topology of S, such that $\mathrm{rk}(G_i) = p^{ih}$ for each i.

For any abelian scheme A over a base scheme S, the system of finite locally free group schemes $A[p^i]$ over S is a p-divisible group over S, denoted by $A[p^\infty]$. This is an important class of examples of p-divisible groups.

If $S = \mathrm{Spec}(K)$ and $p \in K^\times$, the notion of p-adic Tate module modules induces an equivalence of categories between the category of p-divisible groups over K and the category of p-adic Tate modules over K; the equivalence sends a p-divisible group $G = \{G_i \hookrightarrow G_{i+1}\}$ over K to the smooth $\mathbb{Z}_p$-sheaf $\mathrm{T}_p(G) = \varprojlim_i G_i$ over K. In the case when K is a perfect K and $p \in K^\times$, $\mathrm{T}_p(G)$ is determined by the $\mathrm{Gal}(\overline{K}/K)$-module $\varprojlim_i G_i(\overline{K})$.

However if p is the characteristic of the base field K we should avoid using the notion of a "p-adic Tate module", but the notion of a p-divisible group remains useful. In this case the notion of a p-divisible group is much more than "just a Galois representation".

Over an arbitrary base scheme S one can define the notion of a p-divisible group over S; for details see [37]. For an abelian scheme $A \to S$ indeed $X = A[p^\infty]$ is a p-divisible group over S. This enables us to use this notion in mixed characteristic, even when the residue characteristic is p.

An isogeny of p-divisible groups $\psi : X \to Y$ over a base scheme S is a faithfully flat S-homomorphism with finite locally free kernel; such an isogeny ψ induces an isomorphism $X/\mathrm{Ker}(\psi) \xrightarrow{\sim} Y$. A non-trivial p-divisible group X over a field K is said to be *simple* if for every p-divisible group subgroup $Y \subset X$ over K we have either $Y = 0$ or $Y = X$.[2]

1.3. A basic tool for studying p-divisible groups over a perfect field κ of characteristic p is the theory of Dieudonné modules, which gives an equivalence of categories from the exact category of p-divisible groups over K to a suitable category of p-linear algebra data; the latter is collectively known as "Dieudonné modules". There are many Diedonné theories, each with a distinct flavor and range of validity. We refer to [50], [66], [9] and [18] for contravariant variant Dieudonné theories, to [46], [52], [112], [115], [113] for covariant Dieudonné theories, and to [2, App. B3] for a discussion of the various approaches.

We use the covariant theory in this article; see [3, 479–485] for a summary.. For a *perfect* field $\kappa \supset \mathbb{F}_p$, let $\Lambda = \Lambda_\infty(\kappa)$ be the ring of p-adic Witt vectors[3] with entries in κ, and let $\sigma \in \mathrm{Aut}_{\mathrm{ring}}(\Lambda_\infty(\kappa))$ be the natural lift to Λ of the Frobenius automorphism $x_0 \mapsto x_0^p$ on κ on κ. We have the Dieudonné ring R_κ, which contains the ring Λ and two elements $\mathcal{F}$, $\mathcal{V}$, such that the follows relations are satisfied:

$$\mathcal{F}{\cdot}\mathcal{V} = p = \mathcal{V}{\cdot}\mathcal{F}, \quad \mathcal{F}{\cdot}x = x^\sigma{\cdot}\mathcal{F}, \quad x{\cdot}\mathcal{V} = \mathcal{V}{\cdot}x^\sigma, \quad \forall x \in \Lambda.$$

We define R_κ to be the non-commutative ring generated by $\Lambda(\kappa)$, $\mathcal{F}$ and $\mathcal{V}$ and the relations given in the previous sentence. (The completion of R for the linear topology attached to the filtration by right ideals $\mathcal{V}^n{\cdot}R$, $n \in \mathbb{N}$, is the Cartier ring $\mathrm{Cart}_p(\kappa)$ for κ used in Cartier theory.) The ring R_κ is commutative if and only if $\kappa = \mathbb{F}_p$.

[2] A simple p-divisible group X has many non-trivial subgroup schemes (such as $X[p^n]$); here "simple" refers to either the category of p-divisible groups, or the category of p-divisible groups up to isogeny.

[3] The usual notation for the Witt ring is W. However we have used the letter W to denote Newton polygon strata (see below), hence the notation Λ for the Witt ring.

To every finite commutative κ-group scheme N (respectively every p-divisible group G) over κ, there is a left R_κ-module $\mathbb{D}(N)$ (respectively a left R_κ-module $\mathbb{D}(G)$) functorially attached to N (respectively X).. The Λ-module underlying the Dieudonné module $\mathbb{D}(G)$ attached to a p-divisible group G over κ is a free Λ-module whose rank is equal to the height $\mathrm{ht}(G)$ of G. Conversely every left R_κ-module whose underlying Λ-module is free of finite rank is the Dieudonné module of a p-divisible group over κ. The Λ-module underlying the Dieudonné module $\mathbb{D}(N)$ of a commutative finite group scheme over κ is of finite length over Λ, and the length of $\mathbb{D}(N)$ is equal to the order of N. Every R_κ-module whose underlying Λ-module is of finite length is the Diedonné module of a commutative finite group scheme N over κ. Moreover the map from homomorphisms between p-divisible groups over κ (respectively commutative finite group schemes over κ) to R_κ-module homomorphisms between Dieudonné modules is bijective, so that the Dieudonné functor defines equivalence of categories both for p-divisible groups and for commutative finite group schemes over κ.

In [50] the contravariant theory is defined, used and developed. The covariant theory is easier in a number of situations, especially with respect to Cartier theory and the theory of displays. Up to duality the covariant and contravariant theories are essentially equivalent, so it does not make much difference which is used to prove desired results.

The covariant Dieudonné theory commutes with extension of perfect base fields: if $\kappa \to \kappa_1$ is a ring homomorphism between perfect fields of characteristic p, then there is a functorial isomorphism $\mathbb{D}(X) \otimes_{\Lambda_\infty(\kappa)} \Lambda_\infty(\kappa_1) \xrightarrow{\sim} \mathbb{D}(X_{\kappa_1})$ for every p-divisible group X over κ, where $X_{\kappa_1} := X \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa_1)$. In particular we have a functorial isomorphism $\mathbb{D}(X^{(p)}) \xleftarrow{\sim} \mathbb{D}(X) \otimes_{\Lambda,\sigma} \Lambda$, where $X^{(p)}$ is the pull-back of X by the absolute Frobenius morphism $\mathrm{Fr}_\kappa : \mathrm{Spec}(\kappa) \to \mathrm{Spec}(\kappa)$. Note that the relative Frobenius homomorphisms $\mathrm{Fr}_{G/\kappa} : G \to G^{(p)}$ and the canonical isomorphism $\mathbb{D}(X^{(p)}) \xleftarrow{\sim} \mathbb{D}(X) \otimes_{\Lambda,\sigma} \Lambda$ induces the σ^{-1}-linear operator $\mathcal{V}$ on the covariant Dieudonné module $\mathbb{D}(G)$. Similarly the Verschiebung $\mathrm{Ver}_{G/\kappa} : G^{(p)} \to G$ induces the σ-linear operator $\mathcal{F}$ on $\mathbb{D}(G)$. For this reason we use different fonts to distinguish the morphisms F and V (on group schemes) from the operator $\mathcal{V}$ and $\mathcal{F}$ (on covariant Dieudonné modules). For an explanation see [76, 15.3].

Some examples. (i) The Dieudonné module of the p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ over a perfect field $\kappa \supset \mathbb{F}_p$ is a rank-one free Λ-module with a free generator e such that $\mathcal{V}{\cdot}e = e$ and $\mathcal{F}{\cdot}e = pe$.

(ii) The Dieudonné module of the p-divisible group μ_{p^∞} over a perfect field $\kappa \supset \mathbb{F}_p$ corresponds to a free rank-one free Λ-module with a free generator e such that $\mathcal{V}{\cdot}e = pe$ and $\mathcal{F}{\cdot}e = e$.

(iii) For a p-divisible group X of dimension d over κ, the quotient $\mathbb{D}(X)/(\mathcal{V} \cdot \mathbb{D}(X))$ is a d-dimensional vector space over κ:

$$\mathrm{Lie}(X) \cong \mathbb{D}(X)/(\mathcal{V} \cdot \mathbb{D}(X)).$$

In particular

$$\dim(X) = \dim_\kappa(\mathbb{D}(X)/(\mathcal{V} \cdot \mathbb{D}(X)))$$

.

For coprime non-negative integers $m, n \in \mathbb{Z}_{\geq 0}$ we denote by $G_{m,n}$ the p-divisible group over $\mathbb{F}_p$ whose Dieudonné module is

$$\mathbb{D}(G_{m,n}) = R_{\mathbb{F}_p}/R_{\mathbb{F}_p}{\cdot}(\mathcal{V}^n - \mathcal{F}^m)$$

It is easily checked that $\dim_{\mathbb{F}_p}(R_{\mathbb{F}_p}/(\mathcal{V}\cdot R_{\mathbb{F}_p}+R_{\mathbb{F}_p}\cdot(\mathcal{V}^n-\mathcal{F}^m)))=m$ and that $\mathbb{D}(G_{m,n})$ is a free Λ-module with the image of $1,\mathcal{V}^1,\cdots,\mathcal{V}^n,\mathcal{F}^1,\cdots\mathcal{F}^{m-1}$ as a set of free Λ-generators. In other words $\dim(G_{m,n})=m$ and $\mathrm{ht}(G_{m,n})=m+n$. One can check that p-divisible group $G_{m,n}$ remains simple when base changed to any field of characteristic p, or equivalently to any algebraically closed field of characteristic p. We will write $G_{m,n}$ instead of $G_{m,n}\otimes K$ if the base field $K\supset\mathbb{F}_p$ is understood.

1.4. Theorem (Dieudonné, Manin [50, p. 35]). *Suppose that $k=\overline{k}\supset\mathbb{F}_p$ is an algebraically closed field. For any p-divisible group X over k there exist coprime pairs of non-negative integers $(m_1,n_1),\ldots,(m_r,n_r)$ with*

$$\frac{m_1}{m_1+n_1}\le\cdots\le\frac{m_r}{m_1+n_r}$$

and a k-isogeny

$$X\longrightarrow\bigoplus_{1\le i\le r}\ G_{m_i,n_i}.$$

Moreover the sequence of coprime pairs $(m_1,n_1),\ldots,(m_r,n_r)$ is uniquely determined by X.

As a corollary, one obtains a bijection from the k-isogeny classes of p-divisible groups over an algebraically closed field $k\supset\mathbb{F}_p$ to the set of all Newton polygons. If X is k-isogenous to $\oplus_{1\le i\le r}G_{m_i,n_i}$, the Newton polygon $\mathcal{N}(X)$ attached to X is defined as the Newton polygon such that for every non-negative rational number λ, the multiplicity of λ as a possible slope of $\mathcal{N}(X)$ is

$$\sum_{1\le i\le r,\ m_i/(m_i+n_i)=\lambda}(m_i+n_i).$$

In other words, $\mathcal{N}(G_{m,n})$ is the isoclinic Newton polygon with slope $m/(m+n)$ repeated $m+n$ times for every coprime pair (m,n), and for a p-divisible group X as above the Newton polygon $\mathcal{N}(X)$ is formed from the isoclinic Newton polygons $\mathcal{N}(G_{m_1,n_1}),\ldots,\mathcal{N}(G_{m_r,n_r})$ by joining suitable translates of $\mathcal{N}(G_{m_1,n_1}),\ldots,\mathcal{N}(G_{m_r,n_r})$ in succession. In this situation we say that $\mathcal{N}(X)$ is the *concatenation* of the Newton polygons $\mathcal{N}(G_{m_1,n_1}),\ldots,\mathcal{N}(G_{m_r,n_r})$. In the notation introduced in 1.1 (ii), $\mathcal{N}(X)=\mathrm{NP}\big((m_1,n_1)+\cdots+(m_r,n_r)\big)$.

For a p-divisible group over an arbitrary field $K\supset\mathbb{F}_p$, the Newton polygon $\mathcal{N}(X)$ of X is defined to be $\mathcal{N}(X\otimes_K L)$, where L is any algebraically closed field containing K. Clearly this definition is independent of the choice of L. The slopes of $\mathcal{N}(X)$ are also called the slopes of the p-divisible group X.

We mention that the slopes of the Newton polygon of a p-divisible group X over a finite field $\mathbb{F}_{p^a}$ are given by the p-adic valuations of the Frobenius of X. More precisely the p^a-Frobenius $\mathrm{Fr}_{p^a,X}$ is an endomorphism of X over $\mathbb{F}_{p^a}$, which induces the Λ-linear endomorphism $\mathcal{V}^a$ on $\mathbb{D}(X)$. Let $\chi_1,\ldots,\chi_h$ be the eigenvalues of $\mathcal{V}^a$, where $h=\mathrm{ht}(X)$. Then $(1/a)\mathrm{ord}_p(\chi_1),\ldots,(1/a)\mathrm{ord}_p(\chi_h)$ are the slopes of X, where ord_p is the p-adic valuation on $\overline{\mathbb{Q}_p}$ with $\mathrm{ord}_p(p)=1$. As an example, for $X=G_{m,n}$ over $\mathbb{F}_p$, the operators $\mathcal{F},\mathcal{V}$ are Λ-linear on $\mathbb{D}(G_{m,n})$, with $\mathcal{V}^n=\mathcal{F}^m$. From $\mathcal{V}^{n+m}=\mathcal{V}^n\mathcal{V}^m=\mathcal{F}^m\mathcal{V}^m=p^m$, we see that the p-adic valuations of each of the $m+n$ eigenvalues of $\mathcal{V}$ is equal to $m/(m+n)$.

1.5. Minimal p-divisible groups. One can ask, when is a p-divisible group X determined by $X[p]$, the truncated Barsotti-Tate group of level one attached to X. This is answered in [80]. Also see [81].

For coprime $m,n\in\mathbb{Z}_{\ge0}$ define a p-divisible group $H_{m,n}$ to be the p-divisible group over $\mathbb{F}_p$ whose Dieudonné module $M=\mathbb{D}(H_{m,n})$ is the left $R_{\mathbb{F}_p}$-module whose underlying Λ-module is free with basis $\{e_0, e_1, \cdots, e_{m+n-1}\} \subset M$, such that if we define elements $e_i \in M$, $i \in \mathbb{N}$, by $e_{i+b(m+n)} := p^b e_i$ for all $i = 0, 1, \cdots, m+n-1$ and all $b \in \mathbb{N}$, then the following equalities

$$\mathcal{F}{\cdot}e_i = e_{i+n}, \quad \mathcal{V}{\cdot}e_i = e_{i+m}$$

hold for all $i \in \mathbb{N}$. As $(\mathcal{F}^m - \mathcal{V}^n)(e_i) = 0$ for every $0 \le i$, we see that

$$\mathbb{D}(G_{m,n}) = R_{\mathbb{F}_p}{\cdot}e_0 \subset \mathbb{D}(H_{m,n}), \quad \text{hence} \quad G_{m,n} \quad \text{is isogenous to} \quad H_{m,n}.$$

This p-divisible group $H_{m,n}$ is simple over any extension field. We write $H_{m,n}$ instead of $H_{m,n} \otimes K$ for any $K \supset \mathbb{F}_p$ if no confusion is possible, and we know that $G_{m,n}$ is K-isogenous to $H_{m,n}$ over any extension field K of $\mathbb{F}_p$. Note that $G_{1,0} \cong H_{1,0} \cong \mathbb{G}_m[p^\infty]$ and $G_{0,1} \cong H_{0,1}$ is the constant etale p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$.

Note that there exists an endomorphism $u \in \mathrm{End}_{\mathbb{F}_p}(\mathbb{D}(H_{m,n})) = \mathrm{End}_{\mathbb{F}_p}(H_{m,n})$ with the property that $u(e_i) = e_{i+1}$ for all $i \in \mathbb{Z}$. Clearly $\mathrm{End}^0((H_{m,n})_K) = \mathrm{End}^0((G_{m,n})_K)$ for any $K \supset \mathbb{F}_p$. If $K \supset \mathbb{F}_{p^{m+n}}$ we know that $\mathrm{End}^0((H_{m,n})_K)$ is a central simple division algebra over $\mathbb{Q}_p$ of rank $(m+n)^2$, non-commutative if $m+n > 1$, and $\mathrm{End}^0_K(H_{m,n} \otimes K) = \mathrm{End}^0_{\mathbb{F}_{p^{m+n}}}(H_{m,n} \otimes \mathbb{F}_{p^{m+n}})$ for every field $K \supset \mathbb{F}_p$. The Brauer invariant

$$\mathrm{inv}_{\mathbb{Q}_p}(\mathrm{End}^0_{\overline{\mathbb{F}_p}}(H_{m,n} \otimes \overline{\mathbb{F}_p})) = n/(m+n),$$

see [39, 5.4]. For any algebraically closed field $k \supset \mathbb{F}_p$, we have an embedding $\Lambda(\mathbb{F}_{p^{m+n}}) \hookrightarrow \mathrm{End}(H_{m,n} \otimes k)$, and $\mathrm{End}(H_{m,n} \otimes k)$ is generated by $\Lambda(\mathbb{F}_{p^{m+n}})$ and u. Moreover $\mathrm{End}(H_{m,n} \otimes k)$ is the maximal order of the central division algebra $\mathrm{End}^0(H_{m,n} \otimes k) = \mathrm{End}^0(G_{m,n} \otimes k)$ over $\mathbb{Q}_p$ for every algebraically closed field $k \supset \mathbb{F}_p$. The last property characterizes $H_{m,n} \otimes k$ up to (non-unique) isomorphism.

Suppose that a Newton polygon ζ is the concatenation of the Newton polygons $\mathcal{N}(G_{m_1,n_1}), \ldots, \mathcal{N}(G_{m_r,n_r})$, where $(m_1, n_1), \ldots, (m_r, n_r)$ are coprime pairs with $m_1/(m_1 + n_1) \le \cdots \le m_r/(m_r + n_r)$, we define a p-divisible group H_ζ by

$$H_\zeta := \oplus_{1 \le i \le r} \quad H_{m_i,n_i}.$$

We call H_ζ the *minimal p-divisible group* in the isogeny class given by ζ.

Theorem ([80, Theorem 1.2]). *Let $k \supset \mathbb{F}_p$ be an algebraically closed field, let X be a p-divisible group over k with $\mathcal{N}(X) = \zeta$; let H_ζ be the minimal p-divisible group in the isogeny class corresponding to ζ. If $X[p]$ is isomorphic to $(H_\zeta \otimes k)[p]$, then X is isomorphic to $H_\zeta \otimes k$.*

A historical remark. For a p-divisible group $X = \cup_i \ G_i$ with $G_i = X[p^i]$, the building blocks G_{i+1}/G_i are all isomorphic to the BT_1 group scheme G_1. In January 1970 Grothendieck asked Mumford

$$\text{whether } X[p] \cong Y[p] \text{ implies the existence of an isomorphism } X \xrightarrow{\sim} Y.$$

Mumford explained that this is not the case in general. Judging from the next letter it seems that Grothendieck did not understand Mumford's argument; see the three letters in [61, pp. 744–747]. Indeed mathematics may not be as simple as it can be on a first guess. As far as we know neither Grothendieck nor Mumford pursued this question further. One can reformulate the question, and ask *which p-divisible groups over an algebraically closed field $k \supset \mathbb{F}_p$ have the property that the isomorphism class of its BT_1 group scheme determines the isomorphism class of the p-divisible group itself?*

We considered this reformulated question, and came up with the precise answer consisting of the notion of "minimal p-divisible groups" and the above theorem. Since then this notion has played a crucial role in studying fine structures of moduli spaces of abelian varieties in positive characteristics.

1.6. Duality. Recall first that the Cartier dual G^D of a finite locally free commutative group scheme G over a base scheme S is the commutative locally free group scheme which represents the fppf sheaf $T \mapsto \mathrm{Hom}_T(G \times_S T, \mathbb{G}_m)$ on the category of S-schemes. The contravariant exact functor $G \rightsquigarrow G^D$ is an auto-equivalence on the category of finite locally free commutative group schemes over a base scheme S, and there is a functorial isomorphism $G \xrightarrow{\sim} (G^D)^D$.

Suppose that X is a p-divisible group over a base scheme S, with p-power torsion subgroup schemes $X[p^i] =: G_i$. The Cartier duality for finite locally free group scheme applied to short exact sequences

$$0 \to G_i \longrightarrow G_{i+j} \longrightarrow G_j \to 0,$$

yield short exact sequences

$$0 \leftarrow (G_i)^D \leftarrow (G_{i+j})^D \leftarrow (G_j)^D \leftarrow 0.$$

Define a p-divisible group G^t over S, call the *Serre-dual of X,* by

$$G^t := \varinjlim_i \; G_i^D,$$

where the transition homomorphisms $(G_j)^D \hookrightarrow (G_{j+1})^D$ *are the Cartier dual* $(G_{j+1} \twoheadrightarrow G_j)^D$ *of the epimorphisms* $G_{j+1} \twoheadrightarrow G_j$ *induced by* $[p]_{G_{j+1}}$.

Example/Remark.

$$(G_{m,n})^t \quad \cong \quad G_{n,m}.$$

1.6.1. Theorem (duality theorem), see [68, Theorem 19.1]. *Let $\psi : A \longrightarrow B$ be an isogeny between abelian varieties over a base scheme S, and let $N := \mathrm{Ker}(\psi)$. The dual*

$$B^t \xrightarrow{\psi^t} A^t$$

of ψ is again an isogeny. From the exact sequence

$$0 \to N \longrightarrow A \xrightarrow{\psi} B \to 0.$$

we obtain a natural isomorphism

$$N^D \quad \cong \quad \mathrm{Ker}(\psi^t)$$

i.e. we obtain an exact sequence

$$0 \to N^D \longrightarrow B^t \xrightarrow{\psi^t} A^t \to 0$$

in a functorial way.

1.6.2. Corollary. (i) *For an abelian scheme A over a base scheme S there is a natural isomorphism*
$$A^t[p^\infty] \cong (A[p^\infty])^t \,.$$
(ii) *For any abelian variety A over a field $K \supset \mathbb{F}_p$, the associated Newton polygon $\mathcal{N}(A) := \mathcal{N}(A[p^\infty])$ is symmetric.*

2 Stratifications and foliations

In the world of algebraic geometry over a field of positive characteristic, there are properties and structures that are not present in characteristic-zero algebraic geometry. It is reasonable to expect that a "characteristic-p invariant" induces a decomposition of moduli spaces into a disjoint union of subsets or subvarieties, with reasonable properties. If the characteristic-p invariant is discrete, the corresponding subvarieties often form a stratification. On the other hand if the characteristic-p invariant itself vary in a continuous manner (or "have moduli"), then the corresponding subvarieties may be intuitively thought of as a "foliation" of some sort.

In this section we describe and study stratifications and foliations, with the moduli space $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ of polarized abelian varieties in characteristic p as the main example. This moduli space $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ is the disjoint union of subspaces $\mathcal{A}_{g,d}$'s, where $d \in \mathbb{N}_{\geq 1}$ denotes the degree of polarization. To be more precise, we choose a positive integer $n \geq 3$ prime to p, and $\mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}$ denotes the fine moduli space of triples (A, μ, α), where A is a g-dimensional abelian variety, μ is a polarization on A of degree d, and α is a simplectic level-n structure on (A, μ). The choice of n is unimportant for our purpose, therefore it is suppressed. Note also that for the geometry of these moduli space considered in this chapter, there is really no difference between properties of $\mathcal{A}_{g,d_1}$ and $\mathcal{A}_{g,d_1 d_2}$ if d_2 is prime to p.

The terminology "stratification" and "foliation" will be used in a loose sense:

- A *stratification* will be a way of writing a variety as a finite disjoint union of locally closed subvarieties. These subvarieties are called *strata*.

  In some nice case an incidence relation holds among the strata, that the boundary of every stratum is a union of "lower strata"; such an incidence relation then defines a partial ordering on the finite set of strata.

- A *foliation* will be a way of writing a variety as a disjoint union of locally closed subvarieties; these subvarieties are called *leaves*.

  Leaves can have moduli: often they can be parametrized by an algebraic variety. This tends to happen when the foliation structure on a moduli space arises from a "continuous" characteristic-p invariant, and each leaf is the locus where the characteristic-p invariant takes a fixed value.

  Note that the word "foliation" is used in a different way when compared with the standard usage in differential geometry.

Over a perfect base field we study these locally closed subvarieties (strata or leaves) with their induced reduced scheme structures. In some cases a better functorial definition can be given, so that the subvariety in question, be it a stratum or a leave, has a natural structure as a (possibly non-reduced) scheme. Whether there is a good definition that yields the "correct" scheme structure can a topic for further investigation.

2.1. NP - EO - Fol. Recall that a point of $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ "is" an isomorphism class $[(A,\mu)]$ of a g-dimensional abelian variety A in characteristic p together with a polarization $\mu : A \to A^t$ on A. To a point $[(A,\mu)]$ of $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ we associate the isomorphism class $[(A[p^\infty], \mu[p^\infty])]$ of polarized p-divisible group. Here $A[p^\infty]$ is the inductive limit of p-power torsion subgroup schemes of the abelian variety A, and $\mu[p^\infty] : A[p^\infty] \to A[p^\infty]^t$ is the isogeny from $A[p^\infty]$ to its Serre-dual $A[p^\infty]^t$, which is defined as the composition of the isogeny $A[p^\infty] \to A^t[p^\infty]$ induced by the polarization $\mu : A \to A^t$ with the natural isomorphism $A^t[p^\infty] \xrightarrow{\sim} A[p^\infty]^t$. The fact that $\mu^t = \mu$ implies that the isogeny $\mu[p^\infty]$ is symmetric, in the sense that its Serre-dual $\mu[p^\infty]^t$ is equal to itself.

p-adic invariants. To a polarized p-divisible group (X,λ) over an algebraically closed base field of characteristic p we can attach various "invariants". An invariant $(X,\lambda) \mapsto \mathrm{inv}(X,\lambda)$ for polarized p-divisible groups gives rise to an invariant $(A,\mu) \mapsto \mathrm{inv}(A[p^\infty], \mu[p^\infty])$ for polarized abelian varieties. Invariants for polarized abelian varieties which arise from invariants of polarized p-divisible groups in the above manner are call *p-adic invariants* for polarized abelian varieties.

Three p-adic invariants for polarized abelian varieties and the resulting stratification or foliation on $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ are listed in the first column of the following table.

the isogeny class of X	ξ	NP	W_ξ
the isomorphism class of $(X[p], \lambda[p])$	φ	EO	$\mathcal{S}_\varphi$
the isomorphism class of (X,λ)	$[(X,\lambda)]$	CFol	$\mathcal{C}(x)$

The first two entries in the second column are our notation for elements of two convenient finite sets, which are explicitly defined and in bijection with the set of all possible values of the respective invariants. For instance the Greek letter ξ denotes a symmetric Newton polygon. The set of all possible values of the third invariant is indexed by itself, because an explicit description of this infinite set is unavailable. The entries the third column are abbreviations for the Newton polygon stratification, the Ekedahl–Oort stratification and the central foliation, associated to the three invariants respectively. The fourth column gives the notation for the strata or leaves. For instance $\mathcal{C}(x)$ denotes the *central leaf* passing through a point $x = [(A_x, \lambda_x)] \in \mathcal{A}_g(k)$ where $k \supset \mathbb{F}_p$ is an algebraically closed field. In other words $\mathcal{C}(x)$ is the locally closed subset of $\mathcal{A}_g \otimes k$ over the algebraically closed field k whose k-points consists of all k-points $[(A_y, \lambda_y)]$ of $\mathcal{A}_g \otimes k$ such that $(A_y[p^\infty], \lambda_y[p^\infty])$ is isomorphic to $(A_x[p^\infty], \lambda_x[p^\infty])$ as polarized p-divisible groups. We explain these notions and notations in the three rows of this table.

NP The invariant used is

$$[(A,\lambda)] \mapsto \text{ the Newton polygon } \mathcal{N}(A[p^\infty]).$$

According to the Dieudonné–Manin theorem, p-divisible groups over algebraically closed fields $k \supset \mathbb{F}_p$ are classified up to isogeny by their Newton polygons. See §3 for more information about the Newton polygon stratification of $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$

EO The EO-stratification of $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ is defined by the invariant

$$(A,\mu) \;\mapsto\; \text{the isomorphism class of } (A[p], \lambda[p]).$$

This stratification was first considered in [76] for the moduli space $\mathcal{A}_{g,1} \otimes \overline{\mathbb{F}_p}$ of g-dimensional principally polarized abelian varieties in characteristic p. An important phenomenon, due to

Kraft and Oort, which underlies the construction of the EO-stratification is that the number of geometric isomorphism classes of commutative finite group schemes over fields of characteristic p of a given rank and annihilated by p is *finite*. See §5.

CFol Clearly the "finest" among all p-adic invariants for polarized abelian varieties is the one which to every polarized abelian variety (A, μ) over an algebraically closed field $k \supset \mathbb{F}_p$ assigns the isomorphism class of the polarized p-divisible group $(A[p^\infty], \mu[p^\infty])$. This invariant gives rise to a foliation structure on $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$, called the *central foliation*.

For an algebraically closed field $k \supset \mathbb{F}_p$ and a k-point $x_0 = [(A_0, \mu_0)] \in \mathcal{A}_g(k)$, the *central leaf* $C(x_0)$ with reduced structure is a locally closed reduced subscheme of $\mathcal{A}_g \otimes k$ such that for every algebraically closed field k' containing k, the set of all k'-points of $C(x_0)$ is the set of all isomorphism classes $[(A, \mu)] \in \mathcal{A}_g(k')$ such that the polarized p-divisible group $(A[p^\infty], \mu[p^\infty])$ is isomorphic to $(A_0[p^\infty], \mu_0[p^\infty]) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$. The existence of a locally closed subscheme $C(x_0)$ of $\mathcal{A}_g \otimes_{\overline{\mathbb{F}_p}} k$ with the above properties is a basic fact for the central foliation structure of $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$. See [79] and also [82].

We mention several facts. See §4 for more information.

- The whole *ordinary locus* $W_{\rho_{g,2g}}$ in the moduli space $\mathcal{A}_{g,1} \otimes \overline{\mathbb{F}_p}$ of principally g-dimensional abelian varieties is a single central leaf.
- Let $\xi_1 = \mathrm{NP}((g-1) * (0,1) + (1,1))$ be the symmetric Newton polygon of dimension g and f-rank $g-1$. Then the Newton polygon stratum W_{ξ_1}, sometimes called the "*almost ordinary locus*", is a single leaf.
- An open Newton polygon stratum
$$\mathcal{W}_\xi(\mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}) \text{ in } \mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p} \text{ with } f(\xi) \leq g-2$$
contains an infinite number of central leaves.
- Every central leaf in the *supersingular Newton polygon stratum* $\mathcal{W}_{\sigma_g}(\mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p})$ of the moduli space $\mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}$ is finite.

2.2. Remarks. (i) Isogeny correspondences in characteristic p over a Newton polygon stratum $\mathcal{W}_\xi(\mathcal{A}_g \otimes \overline{\mathbb{F}_p})$ involves blow-up and blow-down in a rather wild pattern in general. When restricted to central leaves such isogeny correspondences become finite-to-finite above the source central leaf and the target central leaf.

(ii) Unlike the case of central leaves, the dimension of Newton polygon strata in $\mathcal{A}_{g,d} \otimes \mathbb{F}_p$ associated to the same Newton polygon ξ in general depends on the degree of the polarization in consideration. Similarly the dimension in $\mathcal{A}_{g,d} \otimes \mathbb{F}_p$ of EO strata associated to a fixed isomorphism class of BT_1 group generally depends on the polarization degree.

(iii) The p-adic invariant $(A, \mu) \mapsto f(\mathcal{N}(A[p^\infty], \mu[p^\infty]))$ defines a stratification of $\mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}$ by the p-rank, which is coarser than the Newton polygon stratification. For any integer f with $0 \leq f \leq g$, Every irreducible component of the stratum of $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ with p-rank f is equal to $g(g-1)/2 + f$; see [63, Theorem 4.1].

(iv) The dimension of an irreducible component of a central leaf contained in a given Newton polygon stratum $W_{\xi,d} \subset \mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}$ is determined by ξ and does not depend on the polarization degree d; see [79], [82], 4.13 and 4.5 (ii).

(v) In [79] we also find the definition of *isogeny leaves*, where it is shown that every irreducible component of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ is the image of a finite morphism whose source is a finite locally free cover of an irreducible component of the product of a central leaf and isogeny leaf; see [79, 5.3].

A warning: The name "isogeny leaf" may seem to suggest that isogeny leaves in $\mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}$ are defined with the same paradigm in 2.1, using a suitable invariant for polarized p-divisible groups. This optimistic hope does not hold, but there is an invariant which is closely related to isogeny leaves. Namely, to every polarized abelian variety (A, μ) in characteristic p one associate its α-isogeny class; two polarized $(A_1, \mu_2), (A_2, \mu_2)$ are said to be in the same α-isogeny class if there exists an α-quasi-isogeny $\psi : A_1 \dashrightarrow A_2$ such that $\psi^*(\mu_2) = \mu_1$. See 4.7.1 for the definition of α-quasi-isogeny and 4.7.2 for the definition of isogeny leaves.

(vi) Clearly every central leaf $\mathcal{C}([(A_0, \mu_0)])$ in $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ is contained in the open Newton polygon stratum $\mathcal{W}_\xi$, where $\xi = \mathcal{N}(A_0[p^\infty])$. Similarly every central leaf in $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ is contained in a single EO-stratum. The question whether an EO-stratum is equal to a central leaf is answered by the theory of minimal p-divisible groups; see 1.5

The incidence relation between Newton polygon strata and EO-strata is more complicated. A NP-stratum may intersect several EO-strata, and an EO-stratum may intersect several NP-strata; see 10.3.

2.3. Generally the strata or leaves of supersingular abelian varieties in moduli spaces of abelian varieties behave quite differently from strata or leaves for non-supersingular abelian varieties. What has been proved now include:

> **supersingular** NP-*strata and* EO-*strata in* $\mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}$ *are* **reducible**, for $p \gg 0$ (Hasse, Deuring, Eichler, Igusa for elliptic curves, Katsura–Oort, Li–Oort, Hashimoto–Ibukiyama, Harashita for higher dimensional abelian varieties),

e.g. see [35], [31], [34], [40], [48] and many other references. But

> **non-supersingular** NP-*strata and* EO-*strata in* $\mathcal{A}_{g,1} \otimes \overline{\mathbb{F}_p}$*, and nonsupersingular central leaves in* $\mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}$ *are* **irreducible** (Oort, Ekedahl–Van der Geer, Chai–Oort).

Note that we have abuse the adjectives "supersingular" and "non-supersingular" above. Strictly speaking the stratum $\mathcal{W}_\sigma$ itself is not supersingular; rather it is indexed by the supersingular Newton polygon σ and is the locus in $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ corresponding to supersingular abelian varieties.

3 Newton polygon strata

Notation and convention for the rest of this article.

- We fix a prime number p. Base fields and base schemes will be in characteristic p unless specified otherwise.
- The letters k and Ω will denote algebraically closed fields of characteristic p.
- We write A for an abelian variety and X for a p-divisible group. The notation $\mathcal{A}_{g,d}$ will stand for $\mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}$ or $\mathcal{A}_{g,d} \otimes k$ for some algebraically closed field $k \supset \mathbb{F}_p$.

- For an separated scheme T of finite type over a field K we write $\Pi_0(T)$ for the set of irreducible components of T_k, where k is an algebraically closed field containing K. This definition is independent of the choice of the algebraically closed field $k \supset K$.

.

3.1. Let S be a scheme in characteristic p. For any abelian scheme $A \to S$ (or respectively any p-divisible group $X \to S$) and a symmetric Newton polygon ξ (respectively a Newton polygon ζ), we define

$$\mathcal{W}_\xi(A \to S) := \{x \in S \mid \mathcal{N}(A[p^\infty]) = \xi\},\ \mathcal{W}_{\preceq\xi}(A \to S) := \{x \in S \mid \mathcal{N}(A[p^\infty]) \preceq \xi\}$$

and

$$\mathcal{W}_\xi(X \to S) := \{x \in S \mid \mathcal{N}(X) = \xi\},\ \mathcal{W}_{\preceq\zeta}(X \to S) := \{x \in S \mid \mathcal{N}(X) \preceq \zeta\}$$

The subsets

$$\mathcal{W}_\xi(A \to S) \text{ and } \mathcal{W}_\zeta(X \to S) \text{ are locally closed in } S,$$

while the subsets

$$\mathcal{W}_{\preceq\xi}(A \to S) \text{ and } \mathcal{W}_{\preceq\zeta}(X \to S) \text{ are closed in } S;$$

see [25] and [42, 2.3.2]. Inside $\mathcal{A}_{g,d}$ we have locally closed subsets $W_{\xi,d} := \mathcal{W}_\xi(\mathcal{A}_{g,d})$ and a closed subset $\mathcal{W}_{\preceq\xi}(\mathcal{A}_{g,d})$. For $d = 1$ we have locally closed subsets

$$W_\xi = W_{\xi,1} := \mathcal{W}_\xi(\mathcal{A}_{g,1})$$

in the moduli space $\mathcal{A}_{g,1}$ of g-dimensional principally polarized abelian varieties in characteristic p.

Discussion. (i) This is a "point-wise" definition. We do not have a good functorial approach that will produce for instance, a reasonable notion of a p-divisible group with constant Newton polygon ξ over the dual numbers $k[\epsilon]/(\epsilon^2)$.

(ii) We will consider $W_{\xi,d}$ as a reduced locally closed subscheme of $\mathcal{A}_{g,d}$. For every symmetric Newton polygon ξ with $\dim(\xi) = g$, the proof of Grothendieck and Katz does produce many locally closed subschemes of $\mathcal{A}_{g,d}$ with $W_{\xi,d}$ as the underlying reduced subscheme. However their proof gives (non-canonical) scheme structures: equations whose zero locus coincides with $W_{\xi,d}$ as sets are given in the proof, depending on choices. Among all possible choices none looks "more natural" than all the others, and in general the structural sheaves in schemes contains much more nilpotents than what would appear to be appropriate. If one succeeds in finding a good functorial definition of Newton polygon strata in $\mathcal{A}_{g,d}$, *it is conceivable that the resulting Newton polygon strata in $\mathcal{A}_{g,d}$ are reduced when d not divisible by p,* but non-reduced when d is divisible by a high power of p.

(iii) The Newton polygon strata $\mathcal{W}_\xi(S)$ and W_ξ above are sometimes called "*open Newton polygon strata*", while $\mathcal{W}_{\preceq\xi}(S)$ and $W_{\preceq\xi}$ are called sometimes called "*closed Newton polygon strata*". In the literature often the notation $\mathcal{W}^0_\xi(-)$ and W^0_ξ are used to denote open Newton polygon strata, while W_ξ and sometimes $\overline{W}_\xi$ are used for closed Newton polygon strata. In this chapter we have used $\mathcal{W}_\xi(-)$ and W_ξ to denote the *open* Newton strata, to be logically consistent with the notations for central leaves, and we introduced the notation $\mathcal{W}_{\preceq\xi}(-)$ and $W_{\preceq\xi}$ for closed Newton polygon strata.

3.2. The incidence relation between irreducible components of the Newton polygon stratification of $\mathcal{A}_{g,d} \otimes \mathbb{F}_p$ satisfies the following important property.

If a point x_0 of a Newton polygon stratum $W_{\xi',d}$ is in the Zariski closure of an irreducible component W of another Newton polygon stratum $W_{\xi,d}$ in $\mathcal{A}_{g,d}\otimes\mathbb{F}_p$, then $\xi'\preceq\xi$ and there exists an irreducible component W' of $W_{\xi',d}$ which contains x_0 and W' is contained in the Zariski closure of W.

In other words,

the Zariski closure in $\mathcal{A}_{g,d}\otimes\mathbb{F}_p$ of an irreducible component W of a Newton polygon stratum $W_{\xi,d}$ is the union of a subset of the following collection

$$\left\{W' \mid W' \text{ is an irreducible component of } W_{\xi'},\ \xi'\preceq\xi\right\}$$

of irreducible subvarieties of $\mathcal{A}_{g,d}\otimes\mathbb{F}_p$.

3.3. Clearly we have

$$W_{\xi,d}\subseteq (W_{\xi,d})^{\mathrm{Zar}}\subseteq W_{\preceq\xi,d},$$

where $(W_{\xi,d})^{\mathrm{Zar}}$ denotes the Zariski closure of $W_{\xi,d}$ in $\mathcal{A}_{g,d}$.

It is a conjecture by Grothendieck (1970), proved later that for d not divisible by p we have equality

$$(W_{\xi,d})^{\mathrm{Zar}} = W_{\preceq\xi,d},\quad p\nmid d.$$

Similarly for the Newton polygon strata in the equi-characteristic deformation space of a p-divisible group. For an explanation of a proof, and for references see Section 6 and see 9.4, 9.7.2.

For d divisible by p, in general the last inclusion in

$$W_{\xi,d}\subseteq (W_{\xi,d})^{\mathrm{Zar}}\subseteq W_{\preceq\xi,d}$$

is not an equality. Here is an easy example. The supersingular locus $\mathcal{W}_{3,\sigma_3}$ in $\mathcal{A}_{3,p^3}$ has an irreducible component V whose dimension is equal to 3. The subvariety $V\subset\mathcal{A}_{3,p^3}$ is an irreducible component of the locus of p-rank zero abelian varieties in $\mathcal{A}_{3,p^3}$, and it is not contained in the Zariski closure of $\mathcal{W}_{(2,1)+(1,2)}(\mathcal{A}_{3,p^3})$. This is a special case of [41, Cor. 3.4]. Also see 4.16.3.

Below we discuss many other cases, see Section 4; for a way to find counter examples see [82].

3.4. Supersingular abelian varieties. We say an elliptic curve E over a field $K\supset\mathbb{F}_p$ is *supersingular* if $E[p](k)=\{0\}$ for some (or equivalently, every) algebraically closed extension field k of K. An elliptic curve is supersingular if and only if $\mathcal{N}(E)=\sigma_1=\mathrm{NP}(1,1)$, as there are only two symmetric Newton polygons of height 2. It is a fact that for every prime number p there is at least one supersingular elliptic curve in characteristic p.

For any commutative group scheme G over a perfect field $\kappa\supset\mathbb{F}_p$, define

$$a(G):=\dim_\kappa \mathrm{Hom}(\alpha_p,G).$$

3.4.1. Theorem. *Let $k\supset K\supset\mathbb{F}_p$ be an algebraically closed field.*
(1) *For an abelian variety A of dimension g over K the following statements are equivalent*

- $\mathcal{N}(A)=\sigma_g:=\mathrm{NP}(g{*}(1,1))$;

- *$A[p^\infty]$ is k-isogenous to $(G_{1,1})^g$;*
- *for any supersingular elliptic curve E over k there exists an isogeny from E^g to A_k.*

Definition. The abelian variety is said to be supersingular if one of the above equivalent conditions hold.

(2) *If $a(A) = g > 1$, then for every supersingular elliptic curve E there is an isomorphism $E^g \xrightarrow{\sim} A_k$;* in this case the abelian variety is called *superspecial.*

For a survey of properties see [83, Section 6]

It is a fact that every supersingular abelian variety of dimension at least two is *not geometrically simple.* In contrast for every non-supersingular symmetric Newton polygon ξ of dimension two or above, there exist a geometrically simple abelian variety whose Newton polygon is ξ; see [47].

3.5. Description of irreducible components of the supersingular locus. We know that the supersingular Newton polygon stratum W_{σ_g} of $\mathcal{A}_{g,1}$ has a large number of geometrically irreducible components for $p \gg 0$. A description of the set of irreducible components of $W_{\sigma_g} \otimes \overline{\mathbb{F}_p}$ can be found in 6.6.

4 Foliations

All schemes in this section are in characteristic p. The construction of the foliation structures were suggested by the Hecke orbit problem: try to find a "small" locally closed algebraic subvariety of $\mathcal{A}_{g,d}$ containing the Hecke orbit $\mathcal{H}(x)$ of a moduli point x in $\mathcal{A}_{g,d}$ which can be described in terms of a suitable invariant of the principally polarized abelian variety A_x. Recall that a quasi-isogeny of abelian varieties (respectively p-divisible groups) is a composition of the form $\beta_2 \circ \beta_1^{-1}$, where β_1 and β_2 are both isogenies of abelian varieties (respectively p-divisible groups). Two quasi-isogenies $\beta_2 \circ \beta_1^{-1}$ and $\beta_4 \circ \beta_3^{-1}$ are equal if and only if there exist isogenies γ, δ such that $\beta_1 \circ \gamma = \beta_3 \circ \delta$ and $\beta_2 \circ \gamma = \beta_4 \circ \delta$; see 4.4.2.

Hecke correspondences on $\mathcal{A}_{g,d}$ arise from quasi-isogenies between polarized abelian varieties which respect the polarizations. Clearly points on $\mathcal{A}_{g,d}$ which correspond under a Hecke correspondence have the same Newton polygon. We are interested in the Zariski closure of the full Hecke orbit, i.e. the orbits under all polarization-preserving quasi-isogenies.

There are two subclasses of Hecke correspondences:
those which are "prime to p" and
those which are "of type α".

A prime-to-p Hecke correspondence on $\mathcal{A}_{g,d}$ comes from a polarization-preserving quasi-isogeny which can be written as $\beta_2 \circ \beta_1^{-1}$ such that the kernels of the isogenies β_1, β_2 are finite locally free group schemes whose order are prime to p, while a type-α Hecke correspondence comes from a polarization-preserving quasi-isogeny which can be written as $\beta_2 \circ \beta_1^{-1}$ such that the kernels of β_1 and β_2 are type-α finite locally free commutative group scheme, in the sense that every geometric fiber of $\mathrm{Ker}(\beta_i)$ admits a filtration with all successive quotients isomorphic to α_p), for $i = 1, 2$.

The central foliation and the isogeny foliation "separate" these two classes of Hecke correspondences:

(c1) Every central leaf in $\mathcal{A}_{g,d}$ is stable under all prime-to-p Hecke correspondences.

(c2) The image of a central leaf in $\mathcal{A}_{g,d}$ under a type-α Hecke correspondence is a central leaf (usually different from the original one).

(a1) The image of an isogeny leaf in $\mathcal{A}_{g,d}$ under a prime-to-p Hecke correspondence is the union of a finite number of isogeny leaves in $\mathcal{A}_{g,d}$.

(a2) The image of an isogeny leaf in $\mathcal{A}_{g,d}$ under a prime-to-p Hecke correspondence is a finite union of isogeny leaves in $\mathcal{A}_{g,d}$.

(a3) Suppose that $k \supset \mathbb{F}_p$ is an algebraically closed field, and $\mathcal{I}$ is an isogeny leaf over k in $\mathcal{A}_{g,d}$, and x_0 is a k-point of $\mathcal{I}$. There exists a family T of type-α Hecke correspondences parametrized by an algebraic variety such that one irreducible component of the image of x_0 under the type-α Hecke correspondences T is equal to $\mathcal{I}$.

4.1. Geometrically fiberwise constant families. We explain a notion used in considerations about central leaves.

Definition. Let K be a field, let S be a scheme over K, and let $0 \in S(K)$ be a K-point of S. Let T be an S-scheme. We say that T/S is *geometrically fiberwise constant* (abbreviated as *gfc*) if for every $s \in S$ there exists an algebraically closed field Ω and ring homomorphisms $\tau_1 : K \to \Omega$, $\tau_2 : \kappa(s) \to \Omega$ and an isomorphism

$$T_s \otimes_{\kappa(s)} \Omega \quad \cong \quad T_0 \otimes_K \Omega,$$

where $\kappa(s)$ is the residue field of s. We will see that this definition is useful is some circumstances, superfluous in some other situations, and in scheme theory it might give the wrong approach.

Discussion. Example 1. Note that any family over $\mathrm{Spec}(K[\varepsilon]/(\varepsilon^2))$ is gfc. Hence the notion gfc is inadequate for (infinitesimal) deformation theory.

Restricting $T \to S$ to $S_{\mathrm{red}} \subset S$ does not change properties of fibers over geometric points. It seems better to consider here only reduced base schemes. But then we might run into problems after inseparable extensions of K.

Specialization and gfc. Suppose $T' \to S'$ is given over K, and $S \hookrightarrow S'$ is dense open, and suppose the restriction $T \to S$ is gfc. Does the same hold for the extension T'/S' ?

Example 2. Let k be an algebraically closed field and let S' be a k-scheme of finite type. Suppose T'/S' is a smooth proper morphism whose fibers are geometrically connected algebraic curves of genus $g \geq 2$, S' is reduced, and the restriction $T' \times_{S'} S \to S$ of $T' \to S'$ to a dense open subscheme $S \hookrightarrow S'$ is gfc over S. Then T'/S' is gfc: the assumption implies that the moduli map $S' \to \mathcal{M}_g$ to the coarse moduli space $\mathcal{M}_g$ is constant when restricted to S. Since S' is reduced and $S \hookrightarrow S'$ dense, so the moduli map on $S' \to \mathcal{M}_g$ is constant.

The same holds for any other situation where there are good coarse moduli spaces, e.g. for polarized abelian varieties.

Example 3. Finite flat group schemes. Here is a gfc family where specializations gives a non-isomorphic fiber. We will use for an arbitrary prime number p the description in [100] of finite locally free group schemes of order p. Note that for the special case $p = 2$ there is an easier description that works very well. We choose $S' = \mathrm{Spec}(K[t]) \supset S = \mathrm{Spec}(K[t, 1/t])$ with $K \supset \mathbb{F}_p$ and consider $a = 0$, $b = t$, hence $ab = -p = 0 \in K[t]$. Then there is a finite, flat group scheme

$$T_{a,b} = N' \longrightarrow S' = \mathrm{Spec}(K[t])$$

of rank p. The scheme structure is given by $\mathrm{Spec}(K[t, \tau])/(\tau^p - a\tau)$. The comultiplication on the bialgebra (the group structure) is determined by $b = t$, see [100]. For $p = 2$ it is given by

$$s(\tau) = \tau \otimes 1 + 1 \otimes \tau + b{\cdot}\tau \otimes \tau.$$

We see for any value $t \mapsto t_0 \in K$ we have that $t_0 \neq 0$ implies $T_{0,t_0} \otimes \Omega \cong \mu_p$, and $T_{0,0} \cong \alpha_p$. We see "geometrically μ_p moves in a family" and it specializes to α_p. Already for such cases the notion gfc is interesting. Many more examples can be given (e.g. in the theory of EO strata: maximal gfc families of BT_1 group schemes).

In general it is not easy to decide which degenerations can happen at boundary points of gfc families of finite group schemes.

Example 4. p-divisible groups. We will define "central leaves" as "maximal gfc families" over a reduced base. We will see interesting structures are described in this way.

Here is a first instructive case: take a family $\mathcal{E} \to S$ of ordinary elliptic curves in characteristic p with non-constant j-invariant. The associated p-divisible group $X = \mathcal{E}[p^\infty] \to S$ is gfc; we see already here an interesting mix of aspects of Galois theory and of other properties.

Suppose E_0 is an ordinary elliptic curve over K, and let $\mathcal{E} \to S = \mathrm{Spf}(K[[t]])$ be the universal deformation space in characteristic p. The p-divisible group $X = \mathcal{E}[p^\infty] \to S$ is gfc; over the perfect closure $K \subset K^{(i)}$ it becomes constant, but for no finite extension of $K \subset K'$ the family $(X \to S) \otimes K'$ is constant.

We will see that the same phenomena appear for abelian varieties in positive characteristic.

In order to make a functorial concept extending the notion gfc, the concept of "sustained p-divisible groups" has been developed, see [5], not discussed in this chapter.

4.2. Central leaves

First we consider central leaves in two kinds of deformation spaces, for p-divisible groups without polarization and polarized p-divisible groups respectively. Their definitions are given in (I), (II) below.

(I) Central leaves in a deformation space in the unpolarized case. Suppose that X is a p-divisible group over a perfect field κ. Denote by $\mathrm{Def}(X)$ the equi-characteristic p deformation space of X. We write $\mathcal{C}_X = \mathcal{C}_X(\mathrm{Def}(X))$ for *the largest reduced subscheme of* $\mathrm{Def}(X)$ *where all fibers are geometrically isomorphic to* X.

Remark. We have abused the language in the phrase "the largest subscheme of $\mathrm{Def}(X)$ where all fibers are geometrically isomorphic to X". Strictly speaking that phrase is incorrect, but it has a redeeming property of being relatively short. Below is the correct and longer definition of the central leaf $\mathcal{C}_X(\mathrm{Def}(X))$ in $\mathrm{Def}(X)$.

(1) The deformation space $\mathrm{Def}(X)$ is an affine smooth formal scheme over κ, isomorphic to the formal spectrum $\mathrm{Spf}(\kappa[[t_1, \ldots, t_N]])$ of a formal power series ring in N variable, where $N = \mathrm{ht}(X) \cdot \dim(X)$. Let $R := \Gamma(\mathcal{O}_{\mathrm{Def}(X)})$ be the ring of all formal functions on $\mathrm{Def}(X)$, so that its formal spectrum $\mathrm{Spf}(R)$ is naturally identified with $\mathrm{Def}(X)$. The universal p-divisible group $\mathfrak{X} \to \mathrm{Def}(X)$ over the formal scheme $\mathrm{Def}(X)$ extends uniquely to a p-divisible group $\mathbf{X} \to \mathrm{Spec}(R)$, as follows.

The formal scheme $\mathrm{Def}(X)$ is the inductive limit of the family $\mathfrak{J}$ of closed formal subschemes, consisting of all κ-subschemes Z of the form $Z = \mathrm{Spec}(R/I)$ such that I contains a power of the maximal ideal of R. For each $Z \in \mathfrak{J}$ we have a p-divisible group $\mathfrak{X}_Z \to Z$, and for every pair $Z_1, Z_2 \in \mathfrak{J}$ with $Z_1 \hookrightarrow Z_2$ there exists a natural isomorphism $\iota_{Z_1 \hookrightarrow Z_2} : \mathfrak{X}_{Z_1} \xrightarrow{\sim} \mathfrak{X}_{Z_2} \times_{Z_2} Z_1$. Moreover for any triple $Z_1, Z_2, Z_3 \in \mathfrak{F}$ with $Z_1 \hookrightarrow Z_2 \hookrightarrow Z_3$, the transitivity relation

$$(\iota_{Z_2 \hookrightarrow Z_3} \times_{Z_2} Z_1) \circ \iota_{Z_1 \hookrightarrow Z_2} = \iota_{Z_1 \hookrightarrow Z_3}$$

is satisfied.

For each positive integer n, the inductive family of finite locally free commutative group schemes $\mathfrak{X}_Z[p^n]$ indexed by $Z \in \mathfrak{J}$ gives rise to a finite locally free commutative group scheme $\mathbf{X}_n \to \mathrm{Spec}(R)$ as follows. For each $Z \in \mathfrak{J}$, we have a commutative finite locally free commutative group scheme $\mathfrak{X}_Z[p^n]$ over Z; we write it as $\mathrm{Spec}(S_{Z,n})$. The inverse limit $\varprojlim_Z S_{Z,n}$ of these rings is a finite locally free R-algebra S_n. The spectrum $\mathbf{X}_n$ of S_n has a natural structure as a commutative finite locally free group scheme over R. The natural monomorphisms $\mathbf{X}_n \hookrightarrow \mathbf{X}_{n+1}$ defines a p-divisible group $\mathbf{X} \to \mathrm{Spec} R$. Moreover we have compatible isomorphisms

$$\mathbf{X} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/I) \xrightarrow{\sim} \mathfrak{X} \times_{\mathrm{Spf}(R)} \mathrm{Spec}(R/I)$$

(2) Implicit in the definition of $\mathcal{C}_X(\mathrm{Def}(X))$ is the following statement. There exists an ideal $J \subset R$ uniquely characterized by the properties (a)–(c) below.

(a) The quotient ring R/J is reduced.

(b) For every algebraically closed field Ω containing κ and every κ-linear ring homomorphism $h : R \to \Omega$ whose kernel contains J, the p-divisible group $\mathbf{X} \otimes_{R,h} \Omega$ is isomorphic to $X \otimes_K \Omega$ over Ω.

(c) If Ω is an algebraically closed field containing K and $h : R \to \Omega$ is a κ-linear ring homomorphism such that $\mathbf{X} \otimes_{R,h} \Omega$ is isomorphic to $X \otimes_K \Omega$ over Ω, then $\mathrm{Ker}(h) \supset J$.

This statement is a consequence of a basic finiteness result in [50, Ch. III]: For any algebraically closed field $k \supset \mathbb{F}_p$ and any p-divisible group X over k, there exists a positive integer n_0 depending only on the Newton polygon $\mathcal{N}(X)$ of X, such that for any p-divisible group Y over k, the existence of a k-isomorphism between the finite group schemes $X[p^{n_0}]$ and $Y[p^{n_0}]$ implies the existence of a k-isomorphism between the p-divisible groups X and Y.

(3) The central leaf $\mathcal{C}_X(\mathrm{Def}(X)$ is by definition the reduced closed subscheme $\mathrm{Spec}(R/J) \subset \mathrm{Spec}(R)$ of the affine scheme $\mathrm{Spec}(R)$, where J is the ideal in (2). One can regard it as the closed formal subscheme $\mathrm{Spf}(R/J)$ of the formal scheme $\mathrm{Spf}(R) = \mathrm{Def}(X)$ if one prefers. However $\mathrm{Spf}(R/J)$ has fewer points than $\mathrm{Spec}(R/J)$.

(II) Central leaves in the deformation space of a polarized p-divisible group. Suppose that (X, μ) is a polarized p-divisible group over a perfect field $\kappa \supset \mathbb{F}_p$. Let $\mathrm{Def}(X, \mu)$ be the equi-characteristic-p deformation space of (X, μ). Write $\mathcal{C}_{(X,\mu)}$ for the *largest reduced subscheme of* $\mathrm{Def}(X, \mu)$ *where all fibers are geometrically isomorphic to* (X, μ). The scheme $\mathcal{C}_{(X,\mu)}$ is called the central leaf in the deformation space $\mathrm{Def}(X, \mu)$.

Remark. As in the case for the central leaf $\mathcal{C}_X$ in the deformation space $\mathrm{Def}(X)$, the precise definition of $\mathcal{C}_{(X,\mu)}$ can be elaborated as follows.

(1) Let $R_{X,\mu}$ be the ring of all formal functions on the formal scheme $\mathrm{Def}(X,\mu)$; it is a Noetherian complete local κ-algebra which is topologically finitely generated over κ. There exists a polarized p-divisible group $(\mathbf{X},\tilde{\mu})$ over $\mathrm{Spec}(R_{X,\mu})$, unique up to unique isomorphism, whose formal completion is naturally isomorphic to the universal polarized p-divisible group $(\mathfrak{X},\tilde{\mu})$ over the formal scheme $\mathrm{Def}(X,\mu) = \mathrm{Spf}(R_{X,\mu})$.

(2) There exists an ideal $J_{X,\mu}$ in $R_{X,\mu}$, uniquely determined by the following properties.

- For every algebraically closed field $\Omega \supset \kappa$ and every κ-algebra homomorphism $h : R_{X,\mu} \to \Omega$ with $\mathrm{Ker}(h) \supset J_{X,\mu}$, the polarized p-divisible groups $(\mathfrak{X},\tilde{\mu}) \times_{\mathrm{Spec}(R_{X,\mu})} \mathrm{Spec}(\Omega)$ and $(X,\mu) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\Omega)$ are isomorphic over Ω.
- If Ω is an algebraically closed field containing κ and $h : R_{X,\mu} \to \Omega$ is a κ-linear ring homomorphism such that $(\mathfrak{X},\tilde{\mu}) \times_{\mathrm{Spec}(R_{X,\mu})} \mathrm{Spec}(\Omega)$ is isomorphic to $(X,\mu) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\Omega)$ over Ω, then $\mathrm{Ker}(h) \supseteq J_{X,\mu}$.
- The radical of the ideal $J_{X,\mu}$ is $J_{X,\mu}$ itself.

By definition the central leaf $\mathcal{C}_{X,\mu}$ in $\mathrm{Def}(X,\mu)$ is the closed subscheme $\mathrm{Spec}(R_{X,\mu}/J_{X,\mu})$ of $\mathrm{Spec}(R_{X,\mu})$.

(III) Central leaves for a family of polarized abelian varieties. Suppose that $[(B,\mu)]$ is a polarized abelian variety over a perfect field $\kappa \supset \mathbb{F}_p$, and let (A,ν) be a polarized abelian scheme over a κ-scheme S. We write $\mathcal{C}_{(B,\mu)}(S) = \mathcal{C}_{(B,\mu)}((A,\nu) \to S)$ for the largest reduced subscheme of S such that the p-divisible attached to each of its geometric fibers is κ-linearly isomorphic to (a base extensions of) $(B,\mu)[p^\infty]$.

In the special case when (A,ν) is the universal abelian scheme over $\mathcal{A}_{g,d}$ and $[(B,\mu)] =: x$ is a κ-point of $\mathcal{A}_{g,d}$, we write $\mathcal{C}(x)_d$ for the locally closed subvariety of $\mathcal{A}_{g,d} \otimes \kappa$ whose Ω-points for every algebraically closed field $\Omega \supset \kappa$ is the set

$$\mathcal{C}(x)_d(\Omega) = \{y \in \mathcal{A}_{g,d}(\Omega) \mid (B,\mu)[p^\infty] \otimes \Omega \text{ is isomorphic to } (A_y,\nu_y)[p^\infty]\}.$$

The subvariety $\mathcal{C}(x)_d$ is called the *central leaf* in $\mathcal{A}_{g,d}$ passing through x. When $d = 1$ we will abbreviate $\mathcal{C}(x)_1$ to $\mathcal{C}(x)$ if no confusion is likely. The locally closed subset $\mathcal{C}(x)_d \subset \mathcal{A}_{g,d} \otimes \kappa$ is in fact a closed subset of the Newton polygon stratum $W_{\xi,d} \otimes \kappa$ containing $\mathcal{C}(x)_d$. More information about central leaves can be found in [79].

(IV) Central leaves for a family of p-divisible groups. Suppose that $\kappa \supset \mathbb{F}_p$ is a perfect field, X_0 is a p-divisible group over κ and $X \to S$ is a p-divisible group over a κ-scheme S. The central leaf $\mathcal{C}_{X_0}(X \to S)$ is the reduced locally closed subscheme such that for every algebraically closed field $\Omega \supset \kappa$, the set of all Ω-points of $\mathcal{C}_{X_0}(X \to S)$ is

$$\mathcal{C}_{X_0}(X \to S)(\Omega) = \{s \in S(\Omega) \mid X_s \text{ is isomorphic to } X_0 \otimes_\kappa \Omega\}.$$

Discussion on a better definition. The above definition of central leaves is a "point-wise" one: a central leaf $\mathcal{C}$ over a base field κ as a constructible subset of a κ-scheme S is defined by specifying a collection $\mathfrak{C}$ of geometric points of S to be the set of all geometric points of $\mathcal{C}$. This definition relies on a finiteness result on p-divisible groups, which ensures the existence of a constructible subset $\mathcal{C}$ of S which underlies the subset $\mathfrak{C}$ of geometric points of S, and one shows that the constructible subset $\mathcal{C}$ is locally closed in the Newton polygon stratum containing $\mathcal{C}$, by checking suitable specialization properties. This definition is pleasing from a geometric point of view, but there are disadvantages. For a reduced κ-algebra R one can describe the set of all R-points of $\mathcal{C}$ in a way similar to the above point-wise definition, but one gets stuck when R is not reduced. For instance it is not easy to understand the set of all R-points of $\mathcal{C}$ for an Artinian local κ-algebra, so performing differential analysis on leaves is a challenge.

At the time of when [79] was written we did not have a better definition of central leaves. Recently a functorial approach has been see developed; see [5]. It will turn out that there is a canonical scheme structure on each of the central leaves considered above, including central leaves $\mathcal{C}(x)_d \subset \mathcal{A}_{g,d}$, $\mathcal{C}_X \subset \mathrm{Def}(X)$ and $\mathcal{C}_{X,\mu} \subset \mathrm{Def}(X,\mu)$. In case of the moduli space of polarized abelian varieties $\mathcal{A}_{g,d}$, central leaves in $\mathcal{A}_{g,d}$ according to the new definition are smooth locally closed subschemes, therefore coincide with the previously defined reduced subschemes $\mathcal{C}(x) \subset \mathcal{A}_{g,d}$.

However there are cases with d divisible by p, where the new definition gives a non-reduced subscheme of $\mathcal{A}_{g,d}$ over a perfect field and having $\mathcal{C}(x)_d$ the underlying topological space. Moreover, over non-perfect fields subtleties appear. Under the new definition the non-perfect base fields are allowed: for a polarized g-dimensional abelian variety (B,μ) with $d := \deg(\mu)$ over a non-perfect base field $\kappa \supset \mathbb{F}_p$ the central leaf in $\mathcal{A}_{g,d}$ passing through $[(B,\mu)]$ according to the new definition is a locally closed subscheme $\mathcal{C}^{\mathrm{new}}_{[(B,\mu)]}$ of $\mathcal{A}_{g,d} \otimes \kappa$ over κ, such that for any perfect field $K \supset \kappa$ the topological space underlying $\mathcal{C}^{\mathrm{new}}_{[(B,\mu)]} \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(K)$ is equal to the central leaf $\mathcal{C}_{[(B,\mu)\otimes_\kappa K],d}$ according to the "old definition". The scheme $\mathcal{C}^{\mathrm{new}}_{[(B,\mu)]}$ may not be reduced, and it may be reduced but not geometrically reduced (the case of "hidden nilpotents"). Hence the "old" definition and construction of $\mathcal{C}(x)_d$ produces many difficulties if one tries to use it with non-perfect base fields. These phenomena, with many examples will be discussed in [6] and in [7].

4.3. Theorem [79, Theorem 2.2]. *Let $\kappa \supset \mathbb{F}_p$ be a perfect field and let S be a scheme over k. For any p-divisible group X_0 over κ and any p-divisible group $X \to S$, the subset*

$$\mathcal{C}_{X_0}(X \to S) \subset \mathcal{W}_{\mathcal{N}(X)}(X \to S)$$

is a closed subset of open the Newton polygon stratum $\mathcal{W}_{\mathcal{N}(X)}(X \to S)$ for $X \to S$.

4.4. Theorem (Isogeny correspondences for central leaves in deformation spaces without polarization). *Let $\psi : X \to Y$ be an isogeny between p-divisible groups over a perfect field $\kappa \supset \mathbb{F}_p$. Let $\mathbf{X} \to \mathrm{Def}(X)$ and $\mathbf{Y} \to \mathrm{Def}(Y)$ be the universal p-divisible groups over the respective equi-characteristic deformation spaces. There exist*

(1) *an integral κ-scheme T,*

(2) *two finite faithfully flat κ-morphisms*

$$\mathcal{C}_X(\mathrm{Def}(X)) \quad \overset{u}{\leftarrow} \quad T \quad \overset{v}{\rightarrow} \quad \mathcal{C}_Y(\mathrm{Def}(Y)),$$

(3) *an isogeny $\tilde{\psi} : u^*\mathbf{X} \to v^*\mathbf{Y}$ over T,*

(4) *an algebraic closure k of κ, a point $t \in T(k)$ above the closed point of $\mathcal{C}_X(\mathrm{Def}(X))$, and*

(5) *a k-isomorphism between the isogeny $\tilde{\psi}_t$ and the isogeny $\tilde{\psi} \otimes_\kappa k : X \otimes_\kappa k \to Y \otimes_\kappa k$.*

4.4.1. Remarks. (i) In the above we have considered both the deformation space $\mathrm{Def}(X)$ and the central leaf $\mathcal{C}_X(\mathrm{Def}(X))$ as affine schemes instead of affine formal schemes.

(ii) The central leaf $\mathcal{C}_X(\mathrm{Def}(X))$ in $\mathrm{Def}(X)$ is formally smooth over κ, in the sense that its coordinate ring is isomorphic to a formal power series over κ. Similarly $\mathcal{C}_Y(\mathrm{Def}(Y))$ is formally smooth over κ.

(iii) We explain a natural way to produce a finite locally free cover $u : T \to \mathcal{C}_X(\mathrm{Def}(Y))$ in (2). For each n we have a κ-group scheme $\mathrm{Aut}(X[p^n])$ of finite type over κ such that for every commutative κ-algebra, $\mathbf{Aut}(X[p^n])(S)$ is the group of S-automorphisms of $X[p^n] \times S$. There exists a natural number c depending only on the p-divisible group X such that for any $n, m \in \mathbb{N}$ with $m \geq c$, the image of the restriction morphism

$$\mathrm{r}_{n,m+n} : \mathbf{Aut}(X[p^{m+n}]) \to \mathbf{Aut}(X[p^n])$$

is a finite subgroup scheme $\mathbf{Aut}^{\mathrm{st}}(X[p^n])$ of $\mathbf{Aut}(X[p^n])$ over κ and is independent of $m \geq c$.

Consider the Isom-scheme $\mathbf{Isom}_{\mathcal{C}_X(\mathrm{Def}(X))}(X[p^n], \mathbf{X}[P^n])$, so that for every scheme S over $\mathrm{Def}(X)$, the set of S-points of $\mathbf{Isom}_{\mathcal{C}_X(\mathrm{Def}(X))}(X[p^n], \boldsymbol{X}[P^n])$ is the set of all S-isomorphisms from $X[p^n] \times S$ to $\boldsymbol{X}[p^n] \times_{\mathrm{Def}(X)} S$. It is a fact that this Isom-scheme is a torsor for the Aut-scheme $\mathbf{Aut}(X[p^n])$ over the central leaf $\mathcal{C}_X(\mathrm{Def}(X))$, i.e. it is faithfully flat over $\mathcal{C}_X(\mathrm{Def}(X))$. For $n, m \in \mathbb{N}$ with $m \geq c$, the image of the restriction morphism of fppf sheaves

$$\mathrm{r}_{n,m+n} : \mathbf{Isom}_{\mathcal{C}_X(\mathrm{Def}(X))}(X[p^{m+n}], \boldsymbol{X}[p^{m+n}]) \to \mathbf{Isom}_{\mathcal{C}_X(\mathrm{Def}(X))}(X[p^n], \boldsymbol{X}[p^n])$$

is independent of m, and is represented by a closed subscheme $\mathbf{Isom}^{\mathrm{st}}_{\mathcal{C}_X(\mathrm{Def}(X))}(X[p^n], \boldsymbol{X}[p^n])$ of $\mathbf{Isom}_{\mathcal{C}_X(\mathrm{Def}(X))}(X[p^n], \boldsymbol{X}[p^n])$, which is a torsor for the stabilized Aut-group scheme $\mathbf{Aut}^{\mathrm{st}}(X[p^n])$.

Choose n to be a sufficiently large natural number so that $\mathrm{Ker}(\psi) \subset X[p^n]$. Let

$$T := \mathbf{Isom}^{\mathrm{st}}_{\mathcal{C}_X(\mathrm{Def}(X))}(X[p^n], \boldsymbol{X}[p^n]),$$

and let $u : T \to \mathcal{C}_X(\mathrm{Def}(X))$ is the structural morphism of T. We have obtained a finite faithfully flat morphism $u : T \to \mathcal{C}_X(\mathrm{Def}(X))$ for the first half of the statement (2).

(iv) Let $\zeta : X[p^n] \times T \xrightarrow{\sim} \boldsymbol{X}[p^n] \times_{\mathrm{Def}(X)} T$ be the universal isomorphism over the stabilized Isom-scheme T. The p-divisible group

$$\left(\boldsymbol{X} \times_{\mathrm{Def}(X)} T\right) / \zeta(\mathrm{Ker}(\psi) \times T)$$

over T is a deformation of Y. The morphism $T \to \mathrm{Def}(Y)$ corresponding to this deformation factors through the closed subscheme $\mathcal{C}(\mathrm{Def}(Y)) \subset \mathrm{Def}(Y)$, and give us a morphism $v : T \to \mathcal{C}(\mathrm{Def}(Y))$. It is easy to see from the above construction that properties (4), (5) are satisfied

(v) The morphism $u : T \to \mathcal{C}(\mathrm{Def}(X))$ explained in (iii) is finite and faithfully flat by construction. It is not difficult to check that v is a finite. The flatness of v is a consequence of these properties of u and v and the formal smoothness of the central leaves in (ii): The fact that u is finite flat implies that T is the spectrum of a Noetherian complete Cohen-Macaulay semi-local κ-algebra R_1. The morphism v corresponds to a κ-linear ring homomorphism from a formal power series ring R_2 of the same dimension as T. The fact that v is a finite morphism implies that every regular system of parameters of R_2 is a regular sequence in the semi-local ring R_1. It follows from the local criterion of flatness that v is flat.

Theorem 4.4 was formulated in terms of isogenies. In many situations it is more natural to use quasi-isogenies. We recall the definition of quasi-isogenies below, and invite the readers to extend the statement 4.4 when we are given a quasi-isogeny $\psi : X \dashrightarrow Y$ instead of an isogeny.

4.4.2. Quasi-isogenies. A *quasi-isogeny* $\psi : A \dashrightarrow B$ from an abelian A to an abelian variety B is an equivalence class of diagrams $A \xleftarrow{u} C \xrightarrow{v} B$, where u, v are isogenies; two diagrams $A \xleftarrow{u_1} C_1 \xrightarrow{v_1} B$ and $A \xleftarrow{u_2} C_2 \xrightarrow{v_2} B$ are declared to be equivalent if there exist diagrams of isogenies $A \xleftarrow{u_3} C_3 \xrightarrow{v_3} B$ and $C_1 \xleftarrow{w_1} C_3 \xrightarrow{w_2} C_2$ such that

$$u_1 \circ w_1 = u_3 = u_2 \circ w_2 \quad \text{and} \quad v_1 \circ w_1 = v_3 = v_2 \circ w_2.$$

The quasi-isogeny ψ represented by a diagram $A \xleftarrow{u} C \xrightarrow{v} B$ will be written as $\psi = v \circ u^{-1}$. Thus $v_1 \circ u_1^{-1}$ is declared to be equal too $v_2 \circ u_2^{-1}$ if they can both be formally written as $v_3 \circ u_3^{-1}$; for instance $u_1 \circ w_1 = u_3$ and $v_1 \circ w_1 = v_3$ implies that $v_3 \circ u_3^{-1} = (u_1 \circ w_1) \circ (v_1 \circ w_1)^{-1}$, which is "equal to " $(u_1 \circ w_1) \circ (w_1^{-1} \circ v_1^{-1})$. Note that we have implicitly used the fact that isogenies admits a calculus of right fractions in the above definition. Of course isogenies also admits a calculus of left fractions. Formally inverting isogenies on the left, on the right, or bilaterally all lead to the same notion of quasi-isogenies. Note also that the composition $\zeta \circ \psi$ of two quasi-isogenies $\psi : A \dashrightarrow B$ and $\zeta : B \dashrightarrow C$ is again a quasi-isogeny.

Given a quasi-isogeny $\psi : A \dashrightarrow B$ represented by a diagram $A \xleftarrow{u} C \xrightarrow{v} B$ of isogenies, there exists an isogeny $A \xrightarrow{w} C$ and a non-zero number n such that $u \circ w = [n]_A$. Therefore the quasi-isogeny ψ is represented by the diagram $A \xleftarrow{[n]_A} A \xrightarrow{v \circ w} B$. This means that we can identify the set of all quasi-isogenies from A to B as the subset of all elements of $\psi \in \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that there exists a non-zero integer n such that $n \cdot \psi$ is an isogeny from A to B.

Suppose that (A, μ) and (B, ν) are polarized abelian varieties over the same base field. A quasi-isogeny $\psi : A \dashrightarrow B$ is said to respects the polarizations if $\mu = \psi^* \nu := \psi^t \circ \nu \circ \psi$ as quasi-isogenies from A to A^t.

The notion of quasi-isogenies between p-divisible groups is defined similarly; same for the notion of polarization-preserving quasi-isogenies between p-divisible groups.

4.5. Isogeny correspondences for central leaves in $\mathcal{A}_g \otimes \mathbb{F}_p$.

Let $\psi : A \dashrightarrow B$ be a quasi-isogeny between abelian varieties over a perfect field $\kappa \supset \mathbb{F}_p$, and let λ, μ be polarizations on A and B, of degrees d_1 and d_2 respectively, and suppose that $\psi^*(\mu) = \lambda$. Then there exist finite locally free morphisms

$$\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_{g,d_1} \otimes \kappa) \ \xleftarrow{u}\ T \ \xrightarrow{v}\ \mathcal{C}_{(B,\mu)[p^\infty]}(\mathcal{A}_{g,d_2} \otimes \kappa).$$

In addition there exists a polarization-preserving quasi-isogeny $\boldsymbol{\psi} : u^*(\boldsymbol{A}, \boldsymbol{\lambda}) \dashrightarrow (\boldsymbol{B}, \boldsymbol{\mu})$ and a geometric point $\bar{t}$ of T such that $\boldsymbol{\psi}_{\bar{t}}$ coincides with (the pull-back to $\bar{t}$ of) ψ. Here $(\boldsymbol{A}, \boldsymbol{\lambda})$ and $(\boldsymbol{B}, \boldsymbol{\mu})$ are the universal polarized abelian schemes over $\mathcal{A}_{g,d_1}$ and $\mathcal{A}_{g,d_2}$ respectively. See [79, 3.16].

(i) The morphisms u, v can be constructed in a similar way as in (iii)–(iv) of the remark after theorem 4.4. Choose positive integers a, m, n with $p \nmid a, b$, $m \le n$, $b \le a$, such that $bp^m \psi$ is an isogeny and $\mathrm{Ker}(bp^m\psi) \subseteq A[ap^n]$. Choose a finite etale cover

$$u_1 : T_1 \to \mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_{g,d_1} \otimes \kappa)$$

over which there exists an isomorphism $\zeta_1 : (A, \lambda)[a]_{T_1} \xrightarrow{\sim} (\boldsymbol{A}, \boldsymbol{\lambda})[a]_{T_1}$ which can be lifted etale locally over T_1 to isomorphisms $(A, \lambda)[a^i]_{T_1} \xrightarrow{\sim} (\boldsymbol{A}, \boldsymbol{\lambda})[a^i]_{T_1}$ for all $i \ge 1$. Let

$$u_2 : T \to T_1$$

be the stabilized Isom-scheme $\mathbf{Isom}^{\mathrm{st}}_{T_1}((A, \lambda)[p^n], (\boldsymbol{A}, \boldsymbol{\lambda})[p^n])$ over T_1, which is a torsor over T_1 for the stabilized Aut-group scheme $\mathbf{Aut}^{\mathrm{st}}((A, \lambda)[p^n])$. Here $(\boldsymbol{A}, \boldsymbol{\lambda})$ denotes the universal polarized abelian scheme over $\mathcal{A}_{g,d_1}$. The composition $u := u_2 \circ u_1 : T \to \mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_{g,d})$ is a finite locally free cover, and over T we have a isomorphism from to $(A,\lambda)[ap^n]_T$ to $(\boldsymbol{A},\boldsymbol{\lambda})[ap^n]_T$. The morphism $v : T \to \mathcal{C}_{(B,\mu)[p^\infty]}(\mathcal{A}_{g,d_2} \otimes \kappa)$ is defined by an isogeny $\boldsymbol{\psi}_1 : (\boldsymbol{A},\boldsymbol{\lambda})_T \to (\boldsymbol{B},\boldsymbol{\mu})$ of polarized abelian schemes over T whose kernel is contained in $\boldsymbol{A}[ap^n]$, such that the quasi-isogeny $\boldsymbol{\psi} = (bp^m)^{-1}\boldsymbol{\psi}_1$ respects the polarizations, and there exists a geometric point $\bar{t}$ of T over $[(A,\lambda)]$ such that $\boldsymbol{\psi}_{\bar{t}}$ coincides with ψ.

(ii) **The invariant** $c(\xi) = \mathrm{cdp}(\xi)$. The existence of isogeny correspondences between central leaves in $\mathcal{A}_g \otimes \mathbb{F}_p$ implies that the irreducible components of central leaves in $\mathcal{A}_g \otimes \mathbb{F}_p$ with the same Newton polygon (but with possibly different polarization degrees) all have the same dimension:

> *Let $\kappa \supset \mathbb{F}_p$ be a perfect field. For any two points $x \in \mathcal{A}_{g,d_1}(\kappa)$ and $y \in \mathcal{A}_{g,d_2}(\kappa)$ with the same Newton polygon ξ, we have*
>
> $$\dim(\mathcal{C}(x)_{d_1}) = \dim(\mathcal{C}(y)_{d_2}) =: c(\xi)$$

The above equality holds in the strong sense that all irreducible components of $\mathcal{C}(x)_{d_1}$ and $\mathcal{C}(y)_{d_2}$ have the same dimension. In other words for any symmetric Newton polygon ξ all geometric irreducible components of central leaves in the open Newton polygon stratum $W_{\xi,d}$ have the same dimension $c(\xi)$, which depends only on the symmetric Newton polygon ξ and is independent of the polarization degree d.

4.6. The central stream. Suppose $[(A,\lambda)] = x \in \mathcal{A}_{g,1}$ such that $A[p^\infty]$ is the minimal p-divisible group associated with $\xi = \mathcal{N}(A)$, i.e. $A[p^\infty] \otimes k \cong H_\xi \otimes k$. In this case the corresponding central leaf $\mathcal{C}(x)$ in $\mathcal{A}_{g,1}$ will be denoted by $\mathcal{Z}_\xi$:

$$\mathcal{Z}_\xi := \mathcal{C}(x) \subset W_\xi \subset \mathcal{A}_{g,1},$$

and will be called the *central stream* in the open Newton polygon stratum W_ξ in $\mathcal{A}_{g,1}$. We will see this particular central leaf plays an important role in many considerations. Note that implicit in the definition of the central stream is the fact that up to non-unique isomorphisms there is only one principal polarization on the minimal p-divisible group attached to a symmetric Newton polygon.

4.7. Isogeny leaves.

4.7.1. Type-α quasi-isogenies. An isogeny $u : A \to B$ of abelian varieties over a field $\kappa \supset \mathbb{F}_p$ is a *type-α* isogeny, or an *α-isogeny* for short, if there exists an extension field $k \supset \kappa$ and a finite filtration of $\mathrm{Ker}(u) \otimes_\kappa k$ by subgroup schemes over k such that all successive quotients are isomorphic to α_p.

A quasi-isogeny $\psi : A \dashrightarrow B$ of abelian varieties which can be represented by a diagram $A \xleftarrow{u} C \xrightarrow{v} B$ with α-isogenies u, v is said to be an *type-α quasi-isogeny*, or an α-quasi-isogeny. Clearly the inverse of an α-quasi-isogeny is an α-quasi-isogeny, and the dual of an α-quasi-isogeny is again an α-quasi-isogeny. It is not difficult to check that the composition of two type-α quasi-isogenies is a type-α quasi-isogeny.

The notion of type-α quasi-isogenies between p-divisible groups is defined similarly.

4.7.2. Definition. Let $\kappa \supset \mathbb{F}_p$ be a perfect field, and let k be an algebraic closure of κ. Let $x_0 = [(A_{x_0}, \mu_{x_0})]$ be a κ-point of $\mathcal{A}_{g,d} \otimes \kappa$, where A_{x_0} is an abelian variety over κ and μ_{x_0} is a degree d polarization on A_{x_0}. Let $\mathfrak{I}_k$ (respectively $\mathfrak{I}_\kappa$ be the family of integral closed subschemes $Z \subset \mathcal{A}_{g,d} \otimes k$ (respectively $Z \subset \mathcal{A}_{g,d} \otimes \kappa$) such that for every algebraically closed field Ω containing k (respectively κ) and every geometric point $y = [(A_y, \mu_y)] \in \mathcal{A}_{g,d}(\Omega)$, there exists a polarization preserving type-α quasi-isogeny

$$\psi : (A_{x_0}, \mu_{x_0}) \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\Omega) \dashrightarrow (A_y, \mu_y).$$

(i) An *isogeny leaf in $\mathcal{A}_{g,d}$ over k* is a maximal element of $\mathfrak{I}_k$.

(ii) An *isogeny leaf in $\mathcal{A}_{g,d}$ over κ* is a maximal element of $\mathfrak{I}_\kappa$, or equivalently is a reduced closed subscheme of $\mathcal{A}_{g,d} \otimes \kappa$ whose k/κ-base change is a $\mathrm{Gal}(k/\kappa)$-orbit of an isogeny leaf in $\mathcal{A}_{g,d}$ over k.

(iii) Define $\mathcal{I}(x_0)_{d,\kappa}$ to be the union of the finitely many isogeny leaves over κ in $\mathcal{A}_{g,d} \otimes \kappa$ containing x_0. Equivalently $\mathcal{I}(x_0)_{d,\kappa}$ is the reduced closed κ-subscheme of $\mathcal{A}_{g,d} \otimes \kappa$ whose k/κ-base change is the union of all isogeny leaves in $\mathcal{A}_{g,d} \otimes k$ which contain x_0.

Remarks. (a) The definition of $\mathcal{I}(x_0)_d$ above, following [79], is a "point-wise" definition. For more information and a "better" definition, see [106], [102], [103].

(b) The definition of isogeny leaves conforms with the genearl idea of foliation structure expounded § 2, but the isogeny foliation does not result from any common "invariant" of polarized abelian varieties as in 2.1. Had we strictly adhered to the paradigm in 2.1, we would have defined

> "the isogeny leaf passing through a κ-point $x = [(B, \mu)]$ of $\mathcal{A}_{g,d}$"

for an algebraically closed field $\kappa \supseteq \mathbb{F}_p$ as

> "the algebraic subvariety of $\mathcal{A}_{g,d} \otimes k$ over k whose k-points is the subset of all $y = [(C, \nu)] \in \mathcal{A}_{g,d}(k)$ such that there exists an α-quasi-isogeny $\psi : B \dashrightarrow C$ with the property that $\psi^*(\nu) = \mu$."

This sounds fine at first sight, however there is a problem. As we will see in 4.7.3, generally there does not exist a constructible subset of $\mathcal{A}_{g,d} \otimes k$ which has the desired property, and this happens unless B is a supersingular abelian variety. Instead there exist *countably many* closed algebraic subvarieties $\{Z_j \mid j \in \mathbb{N}\}$ of $\mathcal{A}_{g,d}$ over k such that the union $\cup_j Z_j(k)$ of k-points of these subvarieties is the subset of $\mathcal{A}_{g,d}(k)$ in the quoted subparagraph above.

Thus when we adopted the definition of isogeny leaves in 4.7, we have effectively made a modification of the paradigm 2.1: instead of requiring that the set of points of $\mathcal{A}_g$ with a fixed value of an "invariant" is an algebraic subvariety of $\mathcal{A}_g$, we require only that it is the union of a countable number of algebraic subvarieties of $\mathcal{A}_g$. Of course "forming a countable union of algebraic subvariety" is a rather unpleasant operation in algebraic geometry. In 4.7.4 isogeny leaves "connected to a point x_0" in $\mathcal{A}_{g,d}$ are explained from a different perspective, via a morphism from a scheme $\alpha\text{-QIsog}_\infty^{\mathrm{red}}(x_0) \to \mathcal{A}_{g,d}$. Since the definition of $\alpha\text{-QIsog}_\infty^{\mathrm{red}}(x_0)$ is long, the readers are advised to skip 4.7.4 in the first reading.

4.7.3. An example. Here is an example which exhibits the problem with the excessive expectation that the orbit of every k-point x_0 of $\mathcal{A}_{g,d} \otimes k$ under all possible α-quasi-isogeny correspondences over algebraically closed fields containing k is the set of all geometric points of a constructible subset of $\mathcal{A}_{g,d} \otimes k$ over k.

Let $\mathcal{C}(x_0)_d$ be a non-supersingular central leaf in $\mathcal{A}_{g,d}$ passing through an $\mathbb{F}_q$-point $x_0 \in \mathcal{A}_{g,d}(\mathbb{F}_q)$, and $q = p^r$ is a power of p.

Assume for simplicity that the p-rank of A_{x_0} is 0, i.e. neither 0 nor 1 is a slope of $A_{x_0}[p^\infty]$.

The central leaf $\mathcal{C}(x_0)_d$ is a smooth locally closed subvariety of $\mathcal{A}_{g,d} \otimes \mathbb{F}_q$. Let $x = [(A_x, \mu_x)] \in \mathcal{A}_{g,d}(\Omega)$ be a geometric point lying over a maximal point η of the $\mathbb{F}_q$-scheme $\mathcal{C}(x_0)_d$, where Ω is an algebraically closed field which contains the function field $\mathbb{F}_q(\eta)$. Clearly for every $m \in \mathbb{N}$, the base change of $[(A_x, \mu_x)]$ under the absolute Frobenius $t \mapsto t^{q^m}$ of Ω is also a geometric point $x^{(q^m)} := [(A_x^{(q^m)}, \mu_x^{(q^m)})]$ of $\mathcal{C}(x_0)_d$. The set of all these points $x^{(q^m)}$ is an infinite subset of the algebraic variety $\mathcal{C}(x_0)_d$.

If the "excessive expectation" were true, there would be a positive-dimensional integral locally closed algebraic subvariety $\mathcal{I} \subset \mathcal{C}(x_0)_d \otimes_{\mathbb{F}_q} \Omega$ such that every geometric point y of $\mathcal{I}$ represents an isomorphism class $[(A_y, \mu_y)]$ of a polarized abelian variety which is in correspondence with (A_{x_0}, μ_{x_0}) via an α-quasi-isogeny which respects the polarizations. However the above property of $\mathcal{I}$ implies the existence of a polarization preserving quasi-isogeny from the base change $(A_\mathcal{I}, \mu_\mathcal{I})$ to $\mathcal{I}^{\rm norm}$ of the universal polarized abelian scheme over $\mathcal{A}_{g,d}$ to the constant polarized abelian scheme $(A_{x_0}, \mu_{x_0}) \times \mathcal{I}^{\rm norm}$, where $\mathcal{I}^{\rm norm}$ is the normalization of $\mathcal{I}$. The last property leads to a contradiction in a number of ways, for instance using the fact that homomorphisms between abelian varieties do not deform.

We refer the reader to 8.4 and 8.6 for further discussions of a specific case, where $g = 3$, $d = 1$, the Newton polygon $\mathcal{N}(x_0)$ of A_{x_0} is $\mathrm{NP}\big((2,1)+(1,2)\big)$, $\dim(W_{\mathcal{N}(x_0)}) = 3$, $\dim \mathcal{C}(x_0) = 2$.

4.7.4. Another look at isogeny leaves through a moduli space of quasi-isogenies. Let (A_{x_0}, λ_{x_0}) be a polarized abelian variety over a perfect field $\kappa \supset \mathbb{F}_p$ corresponding to a point $x_0 \in \mathcal{A}_{g,d}(\kappa)$. Assume for simplicity that neither 0 nor 1 appear in the slopes of A_{x_0}; in other words the p-rank of A_{x_0} is 0. Later we will indicate how to modify the construction without this simplifying assumption.

Construction of a moduli space of α-quasi-isogenies. We will define a moduli space of α-quasi-isogenies $\alpha\text{-QIsog}_\infty(x_0)$ as a direct limit of κ-schemes $\mathrm{Isog}_m(x_0)$ of finite type, and a reduced version $\alpha\text{-QIsog}_\infty^{\rm red}(x_0)$, which is a reduced κ-scheme locally of finite type. There is a natural κ-morphism $\pi_\infty^{\rm red} : \alpha\text{-QIsog}_\infty^{\rm red}(x_0) \to \mathcal{A}_{g,d}$ such that the restriction of $\pi_\infty^{\rm red}$ to any irreducible component Z of $\alpha\text{-QIsog}_\infty^{\rm red}(x_0)$ defines a finite surjective morphism from Z to an isogeny leaf in $\mathcal{A}_{g,d}$ when the perfect base field is algebraically closed.

(i) Definitions of $\mathrm{Isog}_m(x_0)$ and $\alpha\text{-QIsog}_\infty^{\rm red}(x_0)$.
For each $m \in \mathbb{N}$, let $\mathrm{Isog}_m(x_0)$ be the κ-scheme such that for every κ-scheme S, there is a functorial bijection between $\mathrm{Isog}_m(x_0)(S)$ and the set of all isomorphism classes of triples $(B \to S, \mu, \rho)$, where $(B \to S, \mu)$ is a polarized abelian scheme over S of degree d, and $\rho : A_{x_0} \times_{\mathrm{Spec}(\kappa)} S \to B$ is an isogeny over S such that $\rho^*\mu = [p^m]^*_{A_{x_0}} \lambda_{x_0}$. Note that an isogeny ρ as above defines a polarization preserving quasi-isogeny $p^{-m}\rho$, and the assumption on the p-rank of A_{x_0} implies that ρ is an α-isogeny and $p^{-m}\rho$ is an α-quasi-isogeny. Let

$$(B_{\mathrm{Isog}_m} \to \mathrm{Isog}_m(x_0),\ \mu_{\mathrm{Isog}_m},\ \rho_{\mathrm{Isog}_m} : A_{x_0} \times \mathrm{Isog}_m \to B_{\mathrm{Isog}_m})$$

be the universal triple over $\mathrm{Isog}_m(x_0)$, so that μ_{Isog_m} is a polarization of the abelian scheme B_{Isog_m} of degree d, and let $\psi_m := p^{-m} \cdot \rho_{\mathrm{Isog}_m}$ be the universal α-quasi-isogeny over $\mathrm{Isog}_m(x_0)$. Let

$$\pi_m : \mathrm{Isog}_m(x_0) \to \mathcal{A}_{g,d}$$

be the proper morphism defined by the polarized abelian scheme $(B_{\mathrm{Isog}_m}, \mu_{\mathrm{Isog}_m})$. For each $m \in \mathbb{N}$, the triple $(B_{\mathrm{Isog}_m} \to \mathrm{Isog}_m(x_0),\ \mu_{\mathrm{Isog}_m},\ p \cdot \rho_{\mathrm{Isog}_m})$ defines a morphism

$$j_{m+1,m} : \mathrm{Isog}_m(x_0) \to \mathrm{Isog}_{m+1}(x_0).$$

Moreover we have $\pi_{m+1} \circ j_{m+1,m} = \pi_m$ for every $m \in \mathbb{N}$. The schemes $\mathrm{Isog}_m(x_0)$ form an inductive system with the $j_{m+1,m}$'s as the transition maps. Let

$$\alpha\text{-QIsog}_\infty(x_0) := \varinjlim_m \mathrm{Isog}_m(x_0)$$

be its inductive limit in the category of ringed spaces. The α-quasi-isogenies ψ_m glue to a universal polarization-preserving α-quasi-isogeny

$$\boldsymbol{\psi}_\infty : (A_{x_0}, \lambda_{x_0}) \times \alpha\text{-QIsog}_\infty(x_0) \dashrightarrow (\boldsymbol{B}, \boldsymbol{\mu})$$

over α-$\mathrm{QIsog}_\infty(x_0)$.

(ii) The definition of $\bar{\pi}_\infty : \alpha\text{-QIsog}_\infty(x_0)/\Gamma_{x_0} \to \mathcal{A}_{g,d} \otimes \kappa$.
Assigning a quasi-isogeny to its target gives us a morphism

$$\pi_\infty : \alpha\text{-QIsog}_\infty(x_0) \to \mathcal{A}_{g,d} \otimes \kappa\,.$$

Intuitively speaking, α-$\mathrm{QIsog}_\infty(x_0)$ is the moduli space of polarization-preserving α-quasi-isogenies from (A_{x_0}, μ_{x_0}) to polarized abelian schemes in characteristic p, and $\mathrm{Isog}_m(x_0)$ is the locus in α-$\mathrm{QIsog}_\infty(x_0)$ where the p^m times the universal quasi-isogeny is an isogeny.

Let Γ_{x_0} be the group consisting of all polarization-preserving quasi-isogenies γ from (A_{x_0}, μ_{x_0}) to itself such that $p^m \cdot \gamma$ is an isogeny for some $m \in \mathbb{N}$. The description of α-$\mathrm{QIsog}_\infty(x_0)$ as the moduli space of polarization-preserving α-quasi-isogenies from (A_{x_0}, μ_{x_0}) to polarized abelian schemes gives a right action of Γ_{x_0} on α-$\mathrm{QIsog}_\infty(x_0)$, so that an element $\gamma \in \Gamma_{x_0}$ sends the universal quasi-isogeny ψ over α-$\mathrm{QIsog}_\infty(x_0)$ to $\psi \circ \gamma$. It is easy to see directly from the definition that the morphism $\pi_\infty : \alpha\text{-QIsog}_\infty(x_0) \to \mathcal{A}_{g,d} \otimes \kappa$ is fixed under the action of Γ_{x_0}, i.e. $\pi_\infty \circ \gamma = \pi_\infty \circ \gamma$ for every $\gamma \in \Gamma_{x_0}$.

(iii) Definitions of α-$\mathrm{QIsog}^{\mathrm{red}}_\infty(x_0)$ **and** $\bar{\pi}^{\mathrm{red}}_\infty : \alpha\text{-QIsog}^{\mathrm{red}}_\infty(x_0)/\Gamma_{x_0} \to \mathcal{A}_{g,d} \otimes \kappa$.
The ringed space α-$\mathrm{QIsog}_\infty(x_0)$ is not a scheme; instead it is an adic formal scheme. Pursuing the "point-wise" approach we have adopted so far, we will focus on the ringed space α-$\mathrm{QIsog}^{\mathrm{red}}_\infty(x_0)$ whose structure sheaf is the quotient of the structure sheaf of α-$\mathrm{QIsog}_\infty(x_0)$ by its own radical. This ringed space $\mathrm{Isog}^{\mathrm{red}}_\infty(x_0)$ is a reduced κ-scheme locally of finite type over κ and is naturally isomorphic to the inductive limit of the reduced schemes $\mathrm{Isog}^{\mathrm{red}}_m(x_0)$:

$$\alpha\text{-QIsog}^{\mathrm{red}}_\infty(x_0) \xleftarrow{\sim} \varinjlim_m \mathrm{Isog}^{\mathrm{red}}_m(x_0)\,,$$

where $\mathrm{Isog}^{\mathrm{red}}_m(x_0)$ is the reduced scheme attached to $\mathrm{Isog}_m(x_0)$.

The right action of Γ_{x_0} on α-$\mathrm{QIsog}_\infty(x_0)$ induces a right action of Γ_{x_0} on the reduced scheme α-$\mathrm{QIsog}^{\mathrm{red}}_\infty(x_0)$. The morphism $\pi_\infty : \alpha\text{-QIsog}_\infty(x_0) \to \mathcal{A}_{g,d} \otimes \kappa$ induces a morphism

$$\pi^{\mathrm{red}:}_\infty \alpha\text{-QIsog}^{\mathrm{red}}_\infty(x_0) \to \mathcal{A}_{g,d} \otimes \kappa\,,$$

which is fixed by the action of Γ_{x_0} on the source scheme α-$\mathrm{QIsog}^{\mathrm{red}}_\infty(x_0)$. Passing to the quotient, we get a morphism

$$\bar{\pi}^{\mathrm{red}}_\infty : \alpha\text{-QIsog}^{\mathrm{red}}_\infty(x_0)/\Gamma_{x_0} \to \mathcal{A}_{g,d} \otimes \kappa\,.$$

Properties of α-$\mathrm{QIsog}^{\mathrm{red}}_\infty(x_0)$ **and** $\bar{\pi}^{\mathrm{red}}_\infty : \alpha\text{-QIsog}^{\mathrm{red}}_\infty(x_0)/\Gamma_{x_0} \to \mathcal{A}_{g,d} \otimes \kappa$.

(1) The reduced scheme α-$\mathrm{QIsog}^{\mathrm{red}}_\infty(x_0)$ is locally of finite type over κ, and is a locally finite union of its irreducible components, each irreducible component being of finite type over κ. The same statement holds for α-$\mathrm{QIsog}^{\mathrm{red}}_\infty(x_0)/\Gamma_{x_0}$.

(2) The set of all geometric points of $\mathcal{A}_{g,d}$ connected to the κ-point $x_0 \in \mathcal{A}_{g,d}(\kappa)$ via α-quasi-isogenies is the image under $\pi_\infty^{\mathrm{red}}$ of all geometric points of the reduced κ-scheme $\alpha\text{-QIsog}_\infty^{\mathrm{red}}(x_0)/\Gamma_{x_0}$.

(3) The morphism $\pi_\infty^{\mathrm{red}} : \alpha\text{-QIsog}_\infty^{\mathrm{red}}(x_0)/\Gamma_{x_0}\mathcal{A}_{g,d} \otimes \kappa$ induces an *injection* on geometric points.

(4) The restriction of $\pi_\infty^{\mathrm{red}}$ to any irreducible component Z of $\alpha\text{-QIsog}_\infty^{\mathrm{red}}(x_0)/\Gamma_{x_0}$ is a finite morphism. In other words $\pi_\infty^{\mathrm{red}}$ induces a radicial morphism from Z to its schematic image in $\mathcal{A}_{g,d}\otimes\kappa$. When κ is algebraically closed, the image under $\pi_\infty^{\mathrm{red}}$ of any irreducible component of $\alpha\text{-QIsog}_\infty^{\mathrm{red}}(x_0)/\Gamma_{x_0}$ is an isogeny leaf in $\mathcal{A}_{g,d}$.

(5) The κ-scheme $\alpha\text{-QIsog}_\infty^{\mathrm{red}}(x_0)/\Gamma_{x_0}$ is *not* of finite type over κ unless A_{x_0} is supersingular. Here we need to remember that we have made the simplifying assumption that the p-rank of A_{x_0} is 0.

Construction of the moduli space $\alpha\text{-QIsog}_\infty(x_0)$ when $p\text{-rk}(A_{x_0}) > 0$.
Here we indicate how to modify the above construction when we no longer assumes that the p-rank of A_{x_0} is 0. The p-divisible group $A_{x_0}[p^\infty]$ admits a natural filtration

$$0 \subset A_{x_0}[p^\infty]_{[1,1]} \subset A_{x_0}[p^\infty]_{(0,1]} \subset A_{x_0}[p^\infty],$$

where $A_{x_0}[p^\infty]_{[1,1]}$ is the maximal p-divisible subgroup of $A_{x_0}[p^\infty]$ of multiplicative type, and $A_{x_0}[p^\infty]/A_{x_0}[p^\infty]_{(0,1]}$ is the maximal quotient etale p-divisible group of $A_{x_0}[p^\infty]$. We have assumed that the base field $\kappa \supset \mathbb{F}_p$ for (A_{x_0}, μ_{x_0}) is perfect. This implies that there exist abelian varieties C_m over κ, $m \in \mathbb{N}$, with $C_0 = A_{x_0}$, and isogenies

$$v_{m,m+1} : C_{m+1} \to C_m$$

such that $\mathrm{Ker}(v_{m,m+1}) = C_m[p^\infty]_{(0,1]}[p]$ for all $m \in \mathbb{N}$. Here the p-divisible group $C_m[p^\infty]_{(0,1]}$ is defined in the same way as $A_{x_0}[p^\infty]_{(0,1]}$. Let $v_m : C_m \to A_{x_0}$ be the composition

$$v_m := v_{0,1} \circ v_{1,2} \circ \cdots \circ v_{m-1,m}.$$

Clearly $\mathrm{Ker}(v_m) = B_m[p^\infty]_{(0,1]}[p^m]$, the truncation at level m of the maximal connected p-divisible subgroup of $B_m[p^\infty]$.

For each $m \in \mathbb{N}$, let $\alpha\text{-QIsog}_m(x_0)$ be the moduli space classifying triples $(B \to S, \nu, \rho)$, where $(B \to S, \nu)$ is a polarized abelian scheme over S of degree d, and $\rho : C_m \times_{\mathrm{Spec}(\kappa)} S \to B$ is an isogeny over S such that $\rho^*\nu = v_m^*\mu$ and

$$C_m[p^\infty]_{[1,1]}[p^m] \times_{\mathrm{Spec}(\kappa)} S \subseteq \mathrm{Ker}(\rho) \subseteq C_m[p^\infty]_{(0,1][p^m]} \times_{\mathrm{Spec}(\kappa)} S\,.$$

A triple $(B \to S, \nu, \rho)$ as above defines a polarization preserving α-quasi-isogeny $\rho \circ v_m^{-1}$, because the kernels of v_m and ρ both contain $C_m[p^\infty]_{[1,1]}[p^m]$. Over the moduli space $\alpha\text{-QIsog}_m(x_0)$ we have a universal polarization-preserving α-quasi-isogeny ψ_m such that $p^m\cdot\psi_m$ is an isogeny. These moduli spaces form an inductive system, where the transition maps $j_{m+1,m} : \alpha\text{-QIsog}_m(x_0) \to \alpha\text{-QIsog}_{m+1}(x_0)$ sends an S-point $(B \to S, \nu, \rho)$ of $\alpha\text{-QIsog}_m(x_0)$ to the S-point $B \to S, \nu, \rho \circ v_{m,m+1})$ of $\alpha\text{-QIsog}_{m+1}(x_0)$.

Just as before, the inductive limit of the moduli spaces $\alpha\text{-QIsog}_m(x_0)$ gives us moduli space of polarization-preserving α-quasi-isogenies $\mathrm{Isog}_\infty(x_0)$ with right action by Γ_{x_0}, together with a Γ_{x_0}-invariant morphism $\pi_\infty : \alpha\text{-QIsog}_\infty(x_0) \to \mathcal{A}_{g,d} \otimes \kappa$. Killing all nilpotent elements in the structure sheaf of $\mathrm{Isog}_\infty(x_0)$, we get a locally finite reduced κ-scheme $\alpha\text{-Isog}_\infty^{\mathrm{red}}(x_0)$, together with a morphism $\pi_\infty^{\mathrm{red}} : \alpha\text{-Isog}_\infty^{\mathrm{red}}(x_0) \to \mathcal{A}_{g,d} \otimes \kappa$ which descends to a morphism $\pi_\infty^{\mathrm{red}} : \alpha\text{-Isog}_\infty^{\mathrm{red}}(x_0)/\Gamma_{x_0} \to \mathcal{A}_{g,d} \otimes \kappa$.

Remark and Discussion. (a) The construction in 4.7.4 of the moduli space α-$\mathrm{QIsog}_\infty(x_0)$ of α-quasi-isogenies is a naive characteristic-p version of the Rapoport-Zink spaces [91].

(b) The action of Γ_{x_0} on α-$\mathrm{QIsog}_\infty(x_0)$ actually extends to a larger group $\tilde{\Gamma}_{x_0}$, consisting of all polarization-preserving p-power quasi-isogenies $\tilde{\gamma}$ from $(A_{x_0}[p^\infty], \mu_{x_0}[p^\infty])$ to itself: one only needs to observe that α-$\mathrm{QIsog}_\infty(x_0)$ can also be interpreted as the moduli space of polarization-preserving α-quasi-isogenies from $(A_{x_0}[p^\infty], \mu_{x_0}[p^\infty])$ to polarized p-divisible groups in characteristic p.

(c) Similar to the question of finding a natural/good scheme structure for Newton polygon strata, one can ask for a natural/good scheme structure for isogeny leaves in $\mathcal{A}_{g,d}$. In some sense the natural object is the ringed space α-$\mathrm{QIsog}_\infty(x_0)/\Gamma_{x_0}$, which is not a scheme but an adic formal scheme, and the scheme α-$\mathrm{QIsog}_\infty^{\mathrm{red}}(x_0)/\Gamma_{x_0}$ defined in 4.7.4 (iii) is the reduced scheme attached to the largest ideal of definition of this formal scheme. Every ideal of definition of the formal scheme α-$\mathrm{QIsog}_\infty(x_0)/\Gamma_{x_0}$ defines a scheme structure for every isogeny leaf in $\mathcal{A}_{g,d}$, but there does not seem to be a "truly natural" one.

4.8. The almost product structure on $W_{\xi,d}$. The construction of this "almost product structure" hinges on an important property of central leaves in $\mathcal{A}_{g,d}$, which was used in the construction 4.5 (i) of isogeny correspondence between central leaves in $\mathcal{A}_{g,d}$: for every $n \in \mathbb{N}$ and every $a \in \mathbb{N}_{>0}$ not divisible by p, there exists a finite locally free cover $T \xrightarrow{u} \mathcal{C}(x_0)$ of the central leaf $\mathcal{C}(x_0) \subset \mathcal{A}_{g,d} \otimes \kappa$ and an isomorphism

$$\delta : (A, \lambda)[ap^n] \times_{\mathrm{Spec}(\kappa)} T \xrightarrow{\sim} (\boldsymbol{A}, \boldsymbol{\lambda})[ap^n] \times_{\mathcal{C}(x_0)} T,$$

where $(\boldsymbol{A}, \boldsymbol{\lambda})$ is the universal polarized abelian scheme over $\mathcal{A}_{g,d} \otimes \kappa$, $a, n \in \mathbb{N}_{>0}$, $a \nmid p$.

The way we constructed isogeny correspondences between central leaves goes like this: give a quasi-isogeny $\psi : (A, \lambda) \dashrightarrow (B, \mu)$, choose a sufficiently large ap^n and trivialize the ap^n-torsion subgroup scheme of the universal polarized abelian scheme over a finite locally free cover T of the given central leaf, then produce a suitable quasi-isogeny $\boldsymbol{\psi}$, from the pull-back to T of the universal polarized abelian scheme over the given central leaf in $\mathcal{A}_{g,d_1}$ to another polarized abelian scheme over T, such that the given quasi-isogeny ψ appears as a geometric fiber of $\boldsymbol{\psi}$. The latter polarized abelian scheme is geometrically fiberwise constant, so we obtain a morphism from T to a central leaf in $\mathcal{A}_{g,d_2}$; compare with 4.1.

In the above construction we used "only" one quasi-isogeny ψ. Since every Newton polygon stratum $W_{\xi,d}$ is a disjoint union of central leaves, a natural idea is to "produce" the whole Newton polygon stratum $W_{\xi,d}$ through a family of quasi-isogeny correspondences with a fixed central leaf in a Newton polygon stratum W_{ξ,d_1}. Among plausible candidates of families of isogenies, the moduli spaces α-$\mathrm{QIsog}_\infty(x_0)$ of α-quasi-isogenies in 4.7.4 and its relatives seems particular attractive for the above purpose. This idea gives rise to what will be called an *almost product structure* of a Newton polygon stratum $W_{\xi,d}$, in the sense that there is a finite surjective morphism, from the product of a finite flat cover $\mathcal{C}'$ of a central leaf with a finite cover $\mathcal{I}'$ of an isogeny leaf, to the given Newton polygon stratum $W_{\xi,d}$, so that the images of $\mathcal{C}' \times \{\mathrm{pt}\}$ are central leaves and images of $\{pt\} \times \mathcal{I}'$ are isogeny leaves. A precise statement is in the following paragraph, with the assumption that the Newton polygon ξ is neither ordinary nor supersingular, to avoid the uninteresting cases: isogeny leaves in $W_{\xi,d}$ are 0-dimensional if ξ is ordinary, while central leaves in $W_{\xi,d}$ are 0-dimensional if ξ is supersingular.

Let $k \supset \mathbb{F}_p$ be an algebraically closed field. Let $W_{\xi,d}$ be a Newton polygon stratum over k, where the symmetric Newton polygon ξ is neither ordinary nor the supersingular. Let W_1 be an irreducible component of $W_{\xi,d}$. There exist

- *an irreducible component $\mathcal{I}_1$ of $\mathcal{I}(x)$,*

- *a finite flat surjective morphism* $C' \to \mathcal{C}(x)$,
- *a finite surjective morphism* $\mathcal{I}' \to \mathcal{I}_1$, *and*
- *a finite surjective morphism* $\Phi : C' \times \mathcal{I}' \to W'$

such that

(1) *For any* $y' \in C'$ *lying above a point* $y \in \mathcal{C}(x)$, *the image*

$$\Phi(\{y'\} \times \mathcal{I}' \subset W_{\xi,d})$$

of $\{y'\} \times \mathcal{I}' \subset W_{\xi,d}$ *under* Φ *is an isogeny leaf and is an irreducible component of* $\mathcal{I}(y)$.

(2) *For any* $z' \in \mathcal{I}'$ *lying above a point* $z \in \mathcal{I}_1$, *the image*

$$\Phi(C' \times \{x''\}) \subset W' \subseteq W_{\xi,d}$$

of $C' \times \{x''\}$ *is the central leaf* $\mathcal{C}(y)$.

Note that irreducible components of a Newton polygon stratum $W_{\xi,d}$ may have different dimensions, so the dimension of the isogeny leaf $\mathcal{I}_1$ in the above statement may depend on the choice of the irreducible component W' of $W_{\xi,d}$.

The invariant $i(\xi)$. For every symmetric Newton polygon ξ we define

$$i(\xi) := \dim(W_{\xi,1}) - c(\xi)$$

for symmetric Newton polygons ξ. We also use the notation $c(\xi) = \mathrm{cdp}(\xi)$ and $i(\xi) = \mathrm{idpp}(\xi)$. We see from the almost product structure (and the fact that every open Newton polygon stratum W_ξ in $\mathcal{A}_{g,1}$ is equi-dimensional) that all isogeny leaves in W_ξ have the same dimension, and $i(\xi) = \dim(\mathcal{I}(x))$ for every $x \in W_{\xi,1}$. We will see that the inequality

$$i(\xi) \leq \dim(\mathcal{I}(x))_d) \quad \forall\, x \in W_{\xi,d}$$

holds for all d, and that the strict inequality $i(\xi) < \dim(\mathcal{I}(x)_d)$ holds in many cases. In other words isogeny leaves in a single α-quasi-isogeny class in $\mathcal{A}_{g,d}$ may have different dimensions.

For $i(-)$ als see [102], [111].

Some examples. (i) For the *ordinary* symmetric Newton polygon $\rho_{g,2g} = \mathrm{NP}\big((g,0)+(0,g)\big)$ of dimension g, isogeny leaves in $W_{\rho_{g,2g},d}$ are points and the Newton polygon $W_{\rho_{g,2g},d}$ itself is a central leaf.

(ii) For the *almost ordinary* symmetric Newton polygon $\xi = (g-1,0)+(1,1)+(0,g-1)$, isogeny leaves in $W_{\xi,d}$ are points, and $W_{\xi,d}$ itself is a central leaf.

(iii) For the *supersingular* symmetric Newton polygon $\sigma_g = \mathrm{NP}\big(g{\cdot}(1,1)\big)$, central leaves in $W_{\sigma_g,d}$ are finite, and every irreducible component of $W_{\sigma_g,d}$ is an isogeny leaf.

(iv) For $\xi = \mathrm{NP}\big((g-1,1)+(1,g-1)\big)$, isogeny leaves in $W_{\xi,d}$ are rational curves,

$$c(\xi) = \tfrac{(g+1)(g-2)}{2} = \tfrac{g(g+1)}{2} - g - 1,$$

and $W_{\xi,d}$ is equi-dimensional of dimensional of dimension $g(g-1)/2 = (g(g+1)/2) - g$. This example can be used to illustrate interesting phenomena.

4.9. Some finite sets with combinatorial structures For every Newton polygon ζ the dimension of the Newton polygon stratum $\mathcal{W}_\zeta(\mathrm{Def}(X)$ in the deformation space of a p-divisible group X with Newton polygon ζ, as well as the dimension of the central leaf $\mathcal{C}(\mathrm{Def}(X))$ and the isogeny leaf $\mathcal{I}(\mathrm{Def}(X))$ can be expressed directly in terms of ζ. Similarly for every symmetric Newton polygon ξ, the dimensions of the Newton polygon stratum $W_\xi = W_{\xi,1}$ in $\mathcal{A}_{g,1}$, the dimension of central leaves in $W_{g,d}$, possible dimensions of $W_{\xi,d}$ and isogeny leaves in $\mathcal{A}_{g,d}$ can be expressed directly in terms of ξ; see [82].

We have found it convenient to express these dimensions as the cardinalities of finite sets which come with specific combinatorial structures related to Newton polygons. Part of the attraction of these finite subsets is that for they are tied to results on the formal completions of strata or leaves at suitably chosen points whose Newton polygon coincides with the given Newton polygon ζ (respectively symmetric Newton polygon ξ). These subsets will be defined below.

4.9.1. Notation. Let ζ be a Newton polygon.

(i) Notation for the position of a point $(x, y) \in \mathbb{Q} \times \mathbb{Q}$ relative to ζ:

$(x, y) \preceq \zeta$ if (x, y) is on or above ζ,

$(x, y) \precneqq \zeta$ if (x, y) is strictly above ζ,

$(x, y) \succeq \zeta$ if (x, y) is on or below ζ,

$(x, y) \succneqq \zeta$ if (x, y) is strictly below ζ.

(ii) Definition of the polygon ζ^* *opposite to* ζ.

Suppose that the Newton polygon ζ has height h, with slopes $1 \geq s_1 \geq \cdots \geq s_h \geq 0$ with $s_j \in \mathbb{Q}$ as in 1.1, so that ζ is the graph of a continuous piecewise linear function on $[0, h]$ whose derivative is equal to s_{h-i+1} on the interval $(i-1, i)$, a convex function. The polygon ζ^* is the graph of the continuous piecewise linear function whose derivative is equal to s_i on the interval $(i-1, i)$, a concave function. Note that ζ^* is not a Newton polygon.

4.9.2. $\diamond(\zeta)$ and $\mathrm{udim}(\zeta)$ (Newton polygon strata in deformation space w/o polarization)

We define a function $\zeta \mapsto \diamond(\zeta)$ on the set of all Newton polygons with values in the set of all finite subsets of $\mathbb{N} \times \mathbb{N}$ as follows. Suppose that ζ is a Newton polygon of height h and dimension, so that (h, d) is the end point of ζ. Define $\diamond(\zeta)$ by

$$\diamond(\zeta) := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < d, \quad y < x, \quad (x, y) \preceq \zeta\}.$$

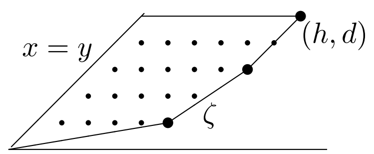
Associated to the function $\zeta \mapsto \diamond(\zeta)$ is the $\mathbb{N}$-valued function $\zeta \mapsto \mathrm{udim}(\zeta)$ defined by

$$\boxed{\mathrm{udim}(\zeta) := \#(\diamond(\zeta)).}$$

The significance of $\mathrm{udim}(\zeta)$ is that it is the dimension of the Newton polygon stratum $\mathcal{W}_\zeta(\mathrm{Def}(X))$ in the deformation space of a p-divisible group with Newton polygon ξ; see 4.12 (i).

The picture below is to be read in two ways. Schematically it is a rendering of the definition of $\diamond(\zeta)$ for general Newton polygons, therefore the end point is labeled as (h, d). It also illustration a special case, where the Newton polygon ζ shown has height $h = 11$ and dimension $d = 5$, the set $\diamond(\zeta)$ consists of the lattice points in solid black circles, with $\#(\diamond(\zeta)) = 22$. The corresponding geometric statement is that the dimension of the Newton polygon stratum $(\mathcal{W}_\zeta(\mathrm{Def}(X))$ in the deformation space $\mathrm{Def}(X)$ of a p-divisible group X with Newton polygon ζ is 22.

Illustration.



$\zeta = \mathrm{NP}(2\times(1,0)+(2,1)+(1,5))$
$= \mathrm{NP}(6*\frac{1}{6}+3*\frac{2}{3}+2*\frac{1}{1});\quad h=11,\ d=5$
$\mathrm{udim}(\zeta)=\#(\Diamond(\zeta))=22.$

4.9.3. $\Diamond(\zeta;\zeta^*)$ and $\mathrm{cdu}(\zeta)$ (for central leaves in deformation space without polarization)

For every Newton polygon ζ, define a finite subset $\Diamond(\zeta,\zeta^*)\subset\mathbb{N}\times\mathbb{N}$ and a natural number $\mathrm{cdu}(\zeta)$ by

$$\Diamond(\zeta;\zeta^*):=\{(x,y)\in\mathbb{Z}\times\mathbb{Z}\mid(x,y)\preceq\zeta,\quad(x,y)\not\succeq\zeta^*\},$$

and

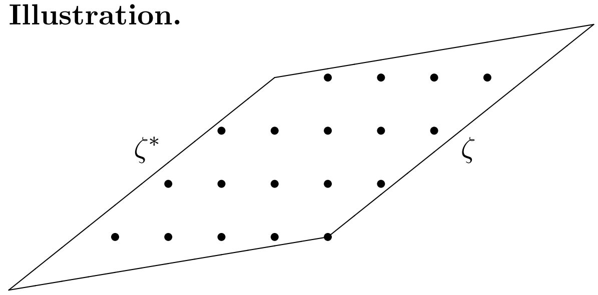
$$\boxed{\mathrm{cdu}(\zeta):=\#\left(\Diamond(\zeta;\zeta^*)\right)}$$

where ζ^* is the polygon opposite to ζ as defined in 4.9.1 (ii).

Note: The acronym "cdu" comes from "<u>d</u>imension of <u>c</u>entral leaf, <u>u</u>npolarized case". It turns out that the dimension $\mathrm{udim}(\mathcal{C}_X(\mathrm{Def}(X)))$ of the central leaf in the deformation space $\mathrm{Def}(X)$ of a p-divisible group X with Newton polygon ζ is $\#(\Diamond(\zeta;\zeta^*))$; see 4.12 (ii).

The picture below depicts a Newton polygon $\zeta=\mathrm{NP}((4,1)+(1,5))$ in the notation of 1.1 (ii) and its opposite polygon ζ^*. The Newton polygon ζ is the concatenation of two Newton polygons $\zeta_1=\mathrm{NP}((4,1))$ and $\zeta_2=\mathrm{NP}((1,5))$, whose heights and dimensions are $h_1=5$, $d_1=4$, $h_2=6$, $d_2=1$ respectively; the height h and dimension d of ζ are given by $h=11=h_1+h_2$ and $d=5=d_1+d_2$. The set $\Diamond(\zeta;\zeta^*)$ consists of the lattice points in black dots, and $\#(\Diamond(\zeta;\zeta^*))=19$. The last equality means that the dimension of the central leaf $\dim(\mathcal{C}(\mathrm{Def}(X)))$ in the deformation space of a p-divisible group X with Newton polygon ξ is 19 according to 4.13. Three ways of expressing the dimension 19 are shown, illustrating the formulas for $\#(\Diamond(\zeta;\zeta^*))$ in 4.10.

Illustration.



$\#\left(\Diamond(\zeta;\zeta^*)\right)=19;\quad(\frac{4}{5}-\frac{1}{6})\cdot5\cdot6=19,$

$d_1h_2-d_2h_1=4\cdot6-1\cdot5=19,\quad d_1c_2-d_2c_1=4\cdot5-1\cdot1=19.$

4.9.4. $\triangle(\xi)$ and $\mathrm{sdim}(\xi)$ (for the Newton polygon stratum $W_\xi\subset\mathcal{A}_{g,1}$.)
For every *symmetric* Newton polygon ξ, let $g=\dim(\xi)=\frac{1}{2}\mathrm{ht}(\xi)$, define a finite subset $\triangle(\xi)\subset\mathbb{N}\times\mathbb{N}$ and a natural number $\mathrm{sdim}(\xi)\in\mathbb{N}$ by

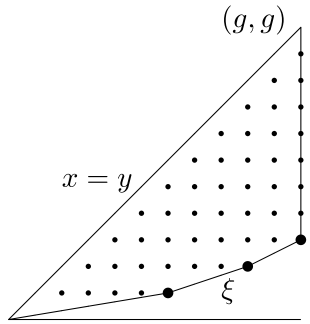
$$\triangle(\xi):=\{(x,y)\in\mathbb{Z}\times\mathbb{Z}\mid y<x\le g,\quad(x,y)\ \text{ on or above }\ \xi\}$$

and

$$\boxed{\mathrm{sdim}(\xi):=\#(\triangle(\xi))}\,.$$

Theorem 4.12 (ii) says that $\mathrm{sdim}(\xi)$ is the dimension of the open Newton polygon stratum W_ξ in $\mathcal{A}_{g,1} \otimes \overline{\mathbb{F}_p}$.

Illustration:



The picture above depicts the definition of $\triangle(\xi)$ in a schematic way and shows only the first half of the Newton polygon ξ, with x-coordinate in the interval $[0, g]$. When understood "as is", the symmetric Newton polygon ξ with its first half shown is

$$\xi = \mathrm{NP}((5,1) + (2,1) + 2{\cdot}(1,1) + (1,2) + (1,5))$$

with height $h = 22$, dimension $g = 11$, and slope sequence

$$\left(6 * \tfrac{5}{6}, 3 * \tfrac{2}{3}, 4 * \tfrac{1}{2}, 3 * \tfrac{1}{3}, 6 * \tfrac{1}{6}\right).$$

Elements of $\triangle(\xi)$ corresponds to solid circles; among them those lying on ξ are darkened. We have

$$\mathrm{sdim}(\xi) = \#(\triangle(\xi)) = 9,$$

which indicates that $\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)) = 48$; see 4.12 (ii).

4.9.5. $\triangle(\xi;\xi^*)$ and $\mathrm{cdp}(\xi)$ (for central leaves in $W_{\xi,d} \subset \mathcal{A}_{g,d}$.)
For any symmetric Newton ξ, define a finite subset $\triangle(\xi;\xi^*) \subset \mathbb{N} \times \mathbb{N}$ and a natural number $\mathrm{cdp}(\xi)$ by

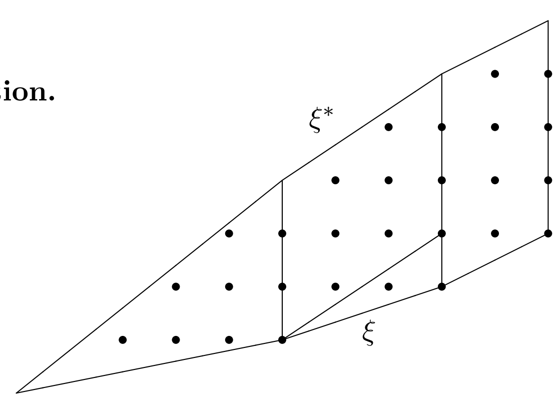
$$\triangle(\xi;\xi^*) := \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid (x,y) \preceq \xi, \quad (x,y) \gneqq \xi^*, \quad x \leq g\}$$

and

$$\boxed{\mathrm{cdp}(\xi) := \#\left(\triangle(\xi;\xi^*)\right)}$$

Note: The acronym "cdp" comes from "dimension of central leaf, polarized case".

Illustration.



In the above picture ξ is the symmetric Newton polygon

$$\xi = \mathrm{NP}\big((4,1)+(2,1)+2*(1,1)+(1,2)+(1,4)\big)$$

In the notation in the combinatorial formula 4.11 for $\mathrm{cdp}(\xi)$, we have $r=2$, $s=2$, $\mu_1=\mu_2=1$, $m_1=4$, $n_1=1$, $m_2=2$, $n_2=1$. The first half of the polygons ξ and ξ^* are shown. The subset $\triangle(\zeta;\zeta^*)\subset\mathbb{N}\times\mathbb{N}$ consists of 28 solid circles shown. So $\mathrm{cdp}(\xi)=\#\triangle(\zeta;\zeta^*)=28$, which indicates that $c(\xi)=\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_{g,c}\otimes\mathbb{F}_p))=28$; see 4.14.

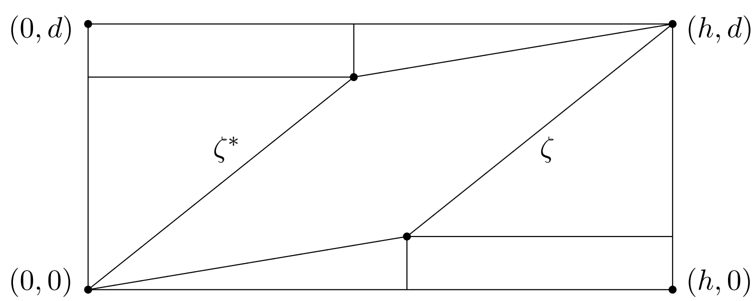
4.10. Combinatorial Lemma for $\mathrm{cdu}(\zeta)=\#(\diamondsuit(\zeta,\zeta^*))$, **the unpolarized case.**
Let $\zeta=\mathrm{NP}\big(\sum_{1\le i\le u}\mu_i*(m_i,n_i)\big)$ be a Newton polygon of height h and dimension d. where $\gcd(m_i,n_i)=1$ for all i and

$$\tfrac{m_1}{m_1+m_2}>\cdots>\tfrac{m_u}{m_u+n_u}.$$

Let $d_i:=\mu_i{\cdot}m_i$, $c_i:=\mu_i{\cdot}n_i$, and $h_i:=\mu_i{\cdot}(m_i+n_i)$ for $i=1,\ldots,u$. Let $\nu_i:=m_i/(m_i+n_i)=d_i/(d_i+c_i)$, which appears h_i times in the slope sequence of ζ. for each i.
The following equalities hold.

$$\mathrm{cdu}(\zeta):=\#\left(\diamondsuit(\zeta;\zeta^*)\right)=\sum_{i=1}^{i=h}\big(\underline{\zeta}^*(i)-\underline{\zeta}(i)\big)=\sum_{1\le i<j\le u}(d_ic_j-d_jc_i)=$$
$$=\sum_{1\le i<j\le u}(d_ih_j-d_jh_i)=\sum_{1\le i<j\le u}h_j{\cdot}h_i{\cdot}(\nu_i-\nu_j).$$

The last two equalities are obvious, because $h_i=c_i+d_i$ and $d_i=\nu_i\cdot h_i$ for each i, so the real content of this lemma lies in the second and third equalities. This following picture may help as a "visual aid" for readers who like to find a proof themselves.



$h=h_1+\cdots+h_u$

$d=d_1+\cdots+d_u$

4.11. Combinatorial Lemma for $\mathrm{cdp}(\xi)$, **the polarized case.** Let ξ be a symmetric Newton polygon of height $2g$ and dimension d, written in the form

$$\xi=\mathrm{NP}\big(\mu_1*(m_1,n_1)+\cdots+\mu_s*(m_s,n_s)+r*(1,1)+\mu_s*(n_s,m_s)+\cdots+\mu_1{\cdot}(n_1,m_1)\big)$$

with $r,s\ge 0$, $m_i>n_i$ *and* $\gcd(m_i,n_i)=1$ *for* $i=1,\ldots,s$, *and*

$$\tfrac{m_1}{m_1+n_1}>\tfrac{m_2}{m_2+n_2}>\cdots>\tfrac{m_s}{m_s+n_s}.$$

- Let $d_i=\mu_i{\cdot}m_i$, and $c_i=\mu_i{\cdot}n_i$, $h_i=d_i+c_i$ for $i=1,\ldots,s$.
- Let $\nu_i:=m_i/(m_i+n_i)$ for $i=1,\ldots,s$, $\nu_{s+1}:=1/2$,
  and $\nu_j:=n_{2s+2-j}/(m_{2s+2-j}+n_{2s+2-j})=1-\nu_{2s+2-j}$ for $j=s+2,s+3,\ldots,2s+1$.

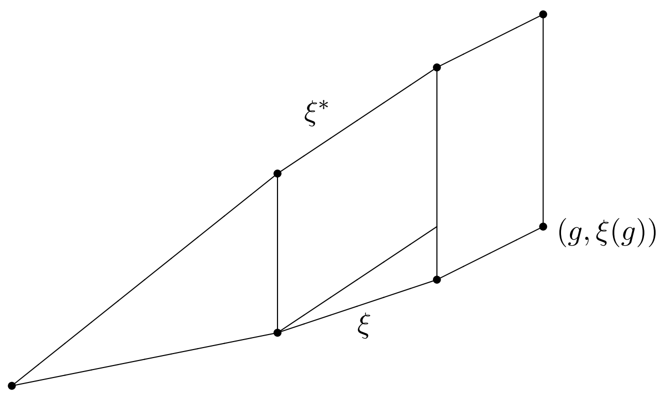
Let ξ^* be the polygon opposite to ξ as defined in 4.9.1 (ii). Let $\underline{\xi}$ and $\underline{\xi}^*$ be the piecewise linear functions on $[0,2g]$ whose graphs are ξ and ξ^* respectively.

The following equalities hold.

$$\#\left(\triangle(\xi;\xi^*)\right) =: \mathrm{cdp}(\xi) = \frac{1}{2}\mathrm{cdu}(\xi) + \frac{1}{2}(\underline{\xi}^*(g) - \underline{\xi}(g)) = \sum_{1\leq j\leq g} (\underline{\xi}^*(j) - \underline{\xi}(j)) =$$

$$= \sum_{1\leq i\leq s}\left(\frac{1}{2}\cdot d_i(d_i+1) - \frac{1}{2}\cdot c_i(c_i+1)\right) + \sum_{1\leq i<j\leq s}(d_i - c_i)h_j + \left(\sum_{i=1}^{i=s}(d_i - c_i)\right)\cdot r =$$

$$= \frac{1}{2}\sum_{1\leq i\leq s}(2\nu_i - 1)h_i(h_i+1) + \frac{1}{2}\sum_{1\leq i<j\neq s+2-i}(\nu_i - \nu_j)h_ih_j.$$



4.12. Theorem (Dimension formula for Newton polygon strata) (i) *Let X be a p-divisible group over a perfect field $\kappa \supset \mathbb{F}_p$ with Newton polygon ζ. The dimension of the Newton polygon stratum in the equi-characteristic deformation space* $\mathrm{Def}(X)$ *is*

$$\dim(\mathcal{W}_\zeta(\mathrm{Def}(X)) = \mathrm{udim}(\zeta).$$

(ii) *Let ξ be a symmetric Newton polygon of dimension g. All irreducible components of the open Newton polygon stratum $W_{\xi,1}$ in the moduli space of g-dimensional principally polarized abelian varieties $\mathcal{A}_{g,1}\otimes\mathbb{F}_p$ have the same dimension, which is*

$$\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1}\otimes\mathbb{F}_p)) = \mathrm{sdim}(\xi).$$

(In fact $W_{\xi,1}$ is irreducible unless ξ is the super-singular Newton polygon σ_g.)

4.13. Theorem. (Dimension formula for central leaves, the unpolarized case.) *Let X be a p-divisible group over a perfect field $\kappa \supset \mathbb{F}_p$ with Newton polygon ζ. The dimension of the central leaf $\mathcal{C}_X(\mathrm{Def}(X))$ in the characteristic-p deformation space* $\mathrm{Def}(X)$ *of X is*

$$\dim(\mathcal{C}_X(\mathrm{Def}(X)) \;=\; \mathrm{cdu}(\zeta).$$

4.14. Theorem (Dimension formula for central leaves, the polarized case). *Let (A,λ) be a polarized abelian variety over a perfect field $\kappa\supset\mathbb{F}_p$. Let $(X,\lambda) = (A,\lambda)[p^\infty]$ and let $\xi = \mathcal{N}(A)$. Then*

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right) \;=\; \mathrm{cdp}(\xi).$$

4.15. Definition. For a Newton polygon ζ we define $\mathrm{idu}(\zeta)$ as the $\underline{\mathrm{d}}$imension of an $\underline{\mathrm{i}}$sogeny leaf in the $\underline{\mathrm{u}}$npolarized case as in [79].

4.15.1. Theorem (Dimension formula for isogeny leaves in the unpolarized case). *Suppose*

$$\zeta = \mathrm{NP}((m_1, n_1), \cdots, (m_a, n_a))$$

(repetitions in the slopes allowed). *We have:*

$$\mathrm{idu}(\zeta) = \#\left(\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid \rho^* \precneqq (x,y) \prec \zeta^*\}\right) =$$

$$= \sum_i \; (m_i - 1)(n_i - 1)/2 \;+\; \sum_{i>j} \; m_i n_j,$$

where ρ is the ordinary Newton polygon with $\zeta \preceq \rho$. See [82], 4.7.

4.16. Theorem 4.12 tells us the dimension of open Newton polygon strata $W_{\xi,d} \subset \mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}$ when the polarization degree d is prime to p. We like to know what the dimension of irreducible components of $\mathcal{W}_\xi(\mathcal{A}_{g,d})$ can be when d is divisible by p. Note that $W_{\xi,d}$ need not be equidimensional.

4.16.1. Definitions. Define natural numbers $\mathrm{minsd}(\xi)$ and $\mathrm{maxsd}(\xi)$ by

- $\mathrm{minsd}(\xi) := \mathrm{Min}\,\{\dim(T) \mid T \text{ is an irreducible component of } W_{g,d},\ d \geq 1\}$,
- $\mathrm{maxsd}(\xi) := \mathrm{Max}\,\{\dim(T) \mid T \text{ is an irreducible component of } W_{g,d},\ d \geq 1\}$.

4.16.2. Theorem (dimensions of open Newton polygon strata). *For any symmetric Newton polygon ξ we have*

$$\mathrm{minsd}(\xi) = \mathrm{sdim}\xi = \#\left(\triangle(\xi, \xi^*)\right) = c(\xi) + \mathrm{idpp}(\xi),$$

and

$$\mathrm{maxsd}(\xi) = c(\xi) + \mathrm{idu}(\xi).$$

See [82], 6.3. Recall from 4.8 that $c(\xi) = \mathrm{cdp}(\xi)$ and $i(\xi) = \mathrm{idpp}(\xi)$. Theorems 4.12, 4.14 and the first displayed formula above give a combinatorial formula for $\mathrm{idpp}(\xi)$. Note that this implies there is an irreducible component T of $\mathcal{W}_{\preceq\xi}(\mathcal{A}_{g,d})$ for some d where the maximum $\dim(T) = \mathrm{cdp}(\xi) + \mathrm{idu}(\xi)$ is achieved.

4.16.3. Example. For every g and the supersingular Newton polygon $\sigma = \sigma_g$ we have $\mathrm{idu}(\sigma) = g(g-1)/2$; this equals the dimension of the p-rank zero locus $\mathcal{V}_0(\mathcal{A}_{g,d})$ for every d. Hence for appropriate d there is an irreducible component $T \subset \mathcal{W}_\sigma(\mathcal{A}_{g,d})$ that is also an irreducible component of $\mathcal{V}_0(\mathcal{A}_{g,d})$; in this case T is not contained in $\mathcal{W}_\xi(\mathcal{A}_{g,d})^{\mathrm{Zar}}$ for any $\xi \neq \sigma$ with $f(\xi) = 0$; this produces examples for any $g > 2$ where the analogue of the Grothendieck conjecture in the polarized case does not hold.

4.16.4. Discussion. Earlier we have defined *open and closed Newton polygon strata.* For a symmetric ξ we have

$$W_{\xi,d} \subseteq (W_{\xi,d})^{\mathrm{Zar}} \subseteq \mathcal{W}_{\preceq\xi}(\mathcal{A}_{g,d}) \subseteq \mathcal{A}_{g,d}.$$

By the Grothendieck conjecture (and the fact that it has been proved) we know

$$(W_{\xi,1})^{\mathrm{Zar}} = \mathcal{W}_{\preceq\xi}(\mathcal{A}_{g,1});$$

see 9.4, see 9.7.2. However for the non-principally polarized case the analogous fact does not hold. That means there are symmetric Newton polygons $\xi_1 \prec \xi_2$ and an irreducible component $T_1 \subset W_{\xi_1,d}$ such that

$$T_1 \subseteq \mathcal{W}_{\preceq\xi_2}(\mathcal{A}_{g,d}) \text{ (by definition) but } T_1 \not\subseteq (W_{\xi_2,d})^{\mathrm{Zar}}.$$

For proofs and more information, see [79], [82], [83].

5 The EO stratification

We have seen a stratification and foliations originating from two p-adic invariants of p-divisible groups, the Newton polygon and the isomorphism class of a (polarized) p-divisible group respectively. In this section we consider two other p-adic invariant, the (isomorphism class of the) p-kernel $X[p$ (respectively $(X,\mu)[p]$) of a p-divisible group X (respectively a polarized p-divisible group (X,μ).) Two basic ideas breathed life into this attempt. Kraft showed how to classify group schemes of a given rank annihilated by p over an algebraically closed field of characteristic p; there are only *finitely many* isomorphism classes. An idea by Raynaud (and by Moret-Bailly) was explained to us and extended by Ekedahl: the "circular" (F, V^{-1})-structure on a polarized BT_1 group scheme can be considered in families of fiber-wise constant group schemes. These lines of thought were brought together for $\mathcal{A}_g = \mathcal{A}_{g,1}\otimes\mathbb{F}_p$ in [76], constructing what is now called Ekedahl–Oort strata in $\mathcal{A}_{g,1}\otimes\mathbb{F}_p$; later these ideas were extended to other moduli spaces and to Shimura varieties, e.g. see [106], [105].

5.1. Kraft cycles. The problem of classifying commutative finite group schemes N over an algebraically closed field $k \supset \mathbb{F}_p$ which are killed by $[p]_N$ was considered and solved by Kraft. The result is that the set of isomorphism classes of such group schemes N with order not exceeding a given constant is *finite*. For a description see [45]; [54], Section 2; [76]; [81], Section 1.

5.2. Suppose that (A,λ) is a *principally polarized* abelian variety over $k = \overline{k} \supset \mathbb{F}_p$. In this case $(A,\lambda)[p]$ is a finite group scheme annihilated by p, where the polarization on A induces a non-degenerate alternating pairing $[-,-] : A[p] \times A[p] \to \mu_p$ on $N = A[p]$; here we have a complete classification (in case $k = \overline{k} \supset \mathbb{F}_p$): any such finite group scheme annihilated by p with such a pairing is the same as the data of an *elementary sequence*, see [76], Section 9, in particular Theorem 9.4:

Definition. An *elementary sequence* of length g is a sequence φ of natural numbers

$$\varphi = (0 = \varphi(0), \varphi(1), \cdots, \varphi(g))$$

such that $\varphi(j) \le \varphi(j+1) \le \varphi(j)+1$ for $j = 0,\dots,g-1$. There exists a partial ordering on the set $\Phi(g)$ of all elementary sequences of length g defined by

$$\varphi \preceq \varphi' \quad\Longleftrightarrow\quad \varphi(i) \le \varphi'(i) \;\; \forall\, i = 1,\dots,g.$$

Instead of this ordering considered one can also use the ordering $\varphi_1 \subseteq \varphi_2$ defined by $S_{\varphi_1} \subset (S_{\varphi_2})$. However these notions are not the same, see 5.3.(v).

Theorem. *Let $k \supset \mathbb{F}_p$ be an algebraically closed field. There is a canonical bijection between the following two sets*

- *the set of isomorphism classes of pairs $(N, [-,-])$, where N is a commutative group schemes of rank $2g$ over k annihilated by p and $[-,-] : N \times N \to \mu_p$ is a perfect alternating pairing, and*
- *the set of all elementary sequences of length g.*

In the above theorem, the map $(N, [-,-]) \mapsto \mathrm{ES}(N) = \varphi$ is given by the restriction to the subgroup scheme $V[N] = N[F]$ a *canonical filtration* on N; see [76, 2.2] for the construction of this map. For the reverse direction, given an elementary sequence φ, one first construct the *final sequence* associated to φ, and use it to construct a commutative group scheme over k killed by $[p]$, together with a perfect alternating paring $[-,-]$ and a filtration on N which is stable under V and F^{-1}. For full details see [76].

Remark. Instead of elementary sequences one can use the language of Weyl cosets, see [54], [22, § 7] and [58]. Also see [104].

Definition. For every elementary sequence φ, define $S_\varphi \subset \mathcal{A}_g = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ to be the locally closed subset of $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ whose geometric points $[(A, \lambda)]$ satisfies the property that the elementary sequence $\mathrm{ES}(A[p], [-,-]_\lambda)$ associated to $(A, \lambda)[p]$ is φ.

Remark. In [22, § 7], one finds a description of S_φ as a degeneracy locus, giving a *scheme-theoretic definition* of these strata.

Remark. Note that the EO stratification is constructed on $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$. One can define EO strata in $\mathcal{A}_{g,d} \otimes \mathbb{F}_p$ by considering the p-adic invariant which associate to any polarized g-dimensional abelian variety (B, μ) over an algebraically closed field k the isomorphism class of $(B[p], \mu[p])$. For d divisible by p, the classification problem for such pairs is not as easy as the case when d is not divisible by p. The EO strata for irreducible components of $\mathcal{A}_{g,d} \otimes \mathbb{F}_p$ seems difficult to describe explicitly; determining their properties remain a challenge. Although the Newton polygon stratification and the foliations are also considered on $\mathcal{A}_{g,d} \otimes \mathbb{F}_p$, but for EO this has not been done (we think for good reasons).

5.3. Properties of EO strata.

(i) Each EO stratum is locally closed and *quasi-affine*, i.e. isomorphic to an open subset of an affine scheme. See (vi) and (vii) below for a strengthened version.

(ii) The dimension of S_φ is equal to $\varphi(1) + \cdots + \varphi(g)$; see also [56], where this dimension is computed as the length of the corresponding Weyl group element.

(iii) The elementary sequence $\sigma_{\min} = (0, \ldots, 0)$ is the minimal element of the poset Φ_g, also called the *superspecial elementary sequence*. The EO stratum $S_{\sigma_{\min}}$ in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is a finite set consisting of moduli points of superspecial principally polarized g-dimensional abelian varieties. For every elementary sequence $\varphi \neq \sigma_{\min}$, the dimension of the EO stratum S_φ is strictly positive.

(iv) The boundary inside $\mathcal{A}_g \otimes \mathbb{F}_p$ of an EO stratum is a union of EO strata.

(v) For elements of Φ_g we write $\varphi_1 \preceq \varphi_2$ if $\varphi_1(i) \leq \varphi_2(i)$ for every $1 \leq i \leq 2g$. We write $\varphi_1 \subseteq \varphi_2$ if $S_{\varphi_1} \subset (S_{\varphi_2})^{\mathrm{Zar}}$. We know that

$$\varphi_1 \preceq \varphi_2 \quad \Rightarrow \quad \varphi_1 \subseteq \varphi_2.$$

However these notions are not the same, is some case the opposite implication does not hold. See [76], 11.1 and 14.3; see [107]

(vi) The EO stratification on $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ extends to a stratification, also indexed by $\Phi(g)$, on the *minimal compactification* (also called the Satake compactification) $\mathcal{A}^*_{g,1} \otimes \mathbb{F}_p$ of $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$; for details see [76, 6.3]. For each $\varphi \in \Phi(g)$, the stratum T_φ is a locally closed subset of $\mathcal{A}^*_{g,1} \otimes \mathbb{F}_p$, such that $T_\varphi \cap \mathcal{A}_{g,1} \otimes \mathbb{F}_p = S_\varphi$.

The idea is to add to S_φ points in the boundary $\mathcal{A}^*_g \setminus \mathcal{A}_g$ that are in the closure of S_φ corresponding to semiabelian varieties that have the same φ but truncated to the abelian variety associated with the boundary point.

(vii) Each EO stratum T_φ of $\mathcal{A}^*_{g,1} \otimes \mathbb{F}_p$ is *quasi-affine* for each φ; see [76, 6.5]. It follows that for every elementary sequence $\varphi \neq \sigma_{\min}$, the Zariski closure of S_φ contains a point of $S_{\sigma_{\min}}$.

(viii) For $\varphi = (0, \cdots, 0, 1)$ the EO stratum S_φ is *connected*; this was proved by Ekedahl, published in [76]. This leads to a new proof of the fact (proved earlier by Faltings and Chai) that

$$\mathcal{A}_{g,1} \otimes \mathbb{F}_p \text{ is absolutely irreducible, see [76, 1.5].}$$

5.4. Irreducibility of EO strata. For a given elementary sequence φ we like to determine the number of components of $S_\varphi \otimes \overline{\mathbb{F}_p}$, and whether it is irreducible in particular. As is many other cases of stratifications or foliations, here it also turns out that strata not contained in the supersingular locus are irreducible. For strata contained in the supersingular locus the number of irreducible components can be computed as a class number, which turns out to be greater than one for large p in many cases.

Proposition. *Let $r = \lceil \frac{g}{2} \rceil$, so that either $g = 2r - 1$ or $g = 2r$. Then*

$$S_\varphi \subseteq W_{\sigma_g} \quad \Leftrightarrow \quad \varphi(r) = 0,$$

where W_{σ_g} is the supersingular locus in $\mathcal{A}_{g,1} \otimes \overline{\mathbb{F}_p}$.

This was indicated in a correspondence with Harashita. A proof can be found in [29, Prop. 5.2].

Theorem (T. Ekedahl and G. van der Geer). *Let $S_\varphi \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ be an* EO *stratum. Suppose that $\varphi(r) \neq 0$, or equivalently that $S_\varphi \not\subseteq W_{\sigma_g}$. Then the* EO *stratum S_φ is geometrically irreducible.*
See [15, Th. 11.5].

Theorem (S. Harashita, M. Hoeve). *Let φ be an elementary sequence with $\varphi(r) = 0$. The number of irreducible components of $S_\varphi \otimes \overline{\mathbb{F}_p}$ can be computed as a class number, and for φ fixed and $p \to \infty$ this class number goes to infinity.*
See the theorem in [29, §1], [32, Cor. 1.3], and [23].

6 Irreducibility

In this section we sketch a proof of Theorem 6.1 on the irreducibility of non-supersingular Newton polygon strata, which is [74, Conjecture 3]. For irreducibility of EO strata, see 5.4. Several concepts and results in this section are also discussed from a historical perspective, see Section 9.

6.1. Theorem. *Every non-supersingular Newton polygon stratum $W_\xi \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is geometrically irreducible.*

Here $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is the moduli space of g-dimensional principally polarized abelian varieties in characteristic p, and where ξ is a symmetric Newton polygon not equal to the supersingular Newton polygon σ and $W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)$ is the open Newton polygon stratum in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ consisting of all g-dimensional principally polarized abelian varieties in characteristic p with Newton polygon ξ. Note that the Zariski closure of W_ξ is the closed subset of $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ consisting of all principally polarized g-dimensional abelian varieties in characteristic p with Newton polygon $\preceq \xi$. The last statement is part of Grothendieck's conjecture on stratification by Newton polygons.

Remark. The special case $W_\rho \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$, the fact that the moduli space of principally polarized abelian varieties in characteristic p is geometrically irreducible we find in [16] and in [76, 1.4].

6.2. Theorem. Let $k \supset \mathbb{F}_p$ be an algebraically closed field. *Let $x \in \mathcal{A}_{g,d}(k)$ be a k-point of the moduli space of polarized g-dimensional abelian varieties with polarization degree d, where d is a positive integer, and we assume that x is* not *in the supersingular locus of $\mathcal{A}_{g,d} \otimes k$. The central leaf $\mathcal{C}(x)$ in $\mathcal{A}_{g,d}$ containing x is geometrically irreducible.*

6.3. Theorem. *Let $k \supset \mathbb{F}_p$ be an algebraically closed field. Let $d \in \mathbb{N}_{>0}$ be a positive integer. Let $x \in \mathcal{A}_{g,d}(k)$ be a point of $\mathcal{A}_{g,d}$ which is* not *in the supersingular locus. The p-adic monodromy representation $\rho_{\mathcal{C}(x)} : \pi_1(\mathcal{C}(x), x) \to \mathrm{Aut}\big((A_x, \lambda_x)[p^\infty]\big)$ for the central leaf $\mathcal{C}(x) \subset \mathcal{A}_{g,d} \otimes k$ is surjective.*

For further information and precise formulation of theorems 6.1–6.3, see 3.1, 4.1, 5.6 of [4]. A proof of theorem 6.1 is sketched in 6.4–6.6 below.

6.4. Step 1. *For every g-dimensional symmetric Newton polygon $\xi \neq \sigma_g$, every algebraically closed field $k \supset \mathbb{F}_p$ and for every irreducible component W of the Newton polygon stratum $W_\xi \subset \mathcal{A}_{g,1} \otimes k$, the Zariski closure of W in $\mathcal{A}_{g,1} \otimes k$ contains an irreducible component of a non-empty Newton polygon stratum W_ζ of $\mathcal{A}_{g,1} \otimes k$ for some symmetric Newton polygon $\zeta \precneqq \xi$.*

In other words for every g-dimensional non-supersingular symmetric Newton polygon ξ, no irreducible component of W_ξ is closed inside $\mathcal{A}_{g,1}$.

6.4.1 Tool 1: EO stratification. The basic idea about the EO stratification has been discussed in §5. For the proof of theorems 6.1–6.3, we need basic properties of the EO stratification in 5.3 (i)–(vii). We will also use the property 3.2 of the Newton polygon stratification

6.4.2 Tool 2: Finite Hecke orbits.
Proposition. *Let ℓ be a prime number not equal to p, and let k be an algebraically closed field. Let $g, d > 0$ be positive integers, and let $x = [(A_x, \lambda_x)] \in \mathcal{A}_{g,d}(k)$ be a k-point of $\mathcal{A}_{g,d}$. The ℓ-adic Hecke orbit $\mathcal{H}_\ell{\cdot}x$ is finite if and only if A_x is supersingular.*

See [1, Prop. 1, p. 448]. Here the ℓ-adic Hecke orbit $\mathcal{H}_\ell \cdot x$ is the countable subset of $\mathcal{A}_{g,d}(k)$ consisting of all points $[(A_y, \lambda_y)]$ such that there exists a polarization preserving quasi-isogeny $\psi : (A_x, \lambda_x) \dashrightarrow (A_y, \lambda_y)$ represented by a diagram of the from $(A_x, \lambda_x) \xleftarrow{u} (B, \mu) \xrightarrow{v} (A_y, \lambda_y)$, where u, v are isogenies whose kernels are killed by suitable powers of ℓ.

Proof of Step 1. Consider the set of all EO types appearing on W, with the partial ordering $\subseteq$ for elementary sequences, see 5.3.(v). Let φ be a minimal element for this ordering in this finite poset.

Remark. Finally we will see that this minimal EO type in W_ξ corresponds to the central stream $\mathcal{Z}_\xi$, but a priori we do not know whether any point of $\mathcal{Z}_\xi$ is contained in the irreducible component W.

Let $x \in (W \cap S_\varphi)(k) \subset W_\xi(k)$. By 6.4.1 Tool 2 (the proposition above), we know that the ℓ-adic Hecke orbit of x is not finite. Hence there exists an irreducible component $S \subset S_\varphi \cap W$ of positive dimension. Thus S is a positive dimensional locally closed subvariety of the quasi-affine subvariety $T_\varphi \subset \mathcal{A}^*_{g,1} \otimes k$. It follows that the Zariski closure $(S_\varphi \cap W)^{\mathrm{Zar}}$ of $S_\varphi \cap W$ in $\mathcal{A}^*_{g,1} \otimes k$ contains a point y which is not in the EO stratum T_φ of $\mathcal{A}^*_{g,1} \otimes k$. However this point y may not be in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$. What we really need is the existence of a point in the Zariski closure of $S_\varphi \cap W$ in $\mathcal{A}_{g,1}$ which is not in S_φ. Clearly such a point y' cannot be in W_ξ because of the minimality of the EO type φ, so y' belongs to a Newton polygon stratum $W_{\xi'}$ with $\xi' \precneqq \xi$. By 3.2, there is an irreducible component W' of $W_{\xi'}$ which is contained in the Zariski closure of W, and step 1 follows by induction.

Claim. *The closure S^{Zar} of S in $\mathcal{A}_{g,1}$ is not equal to S.*

We prove the above claim using degeneration theory of abelian varieties in [17, Ch. 2]. The idea is to produce from a degeneration from S to y another degeneration from S to a point y' in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ which is not a point of S.

There exists a principally polarized semi-abelian scheme (G, μ) over the power series ring $k[[t]]$ such that the abelian part B_0 of the closed fiber together with the principal polarization ν_0 induced by μ corresponds to the point y, and the generic fiber $(G, \mu)_{k((t))}$ corresponds to a morphism from $\mathrm{Spec}\big(k((t))\big)$ to the dense open subset of W where W is smooth over k. The degeneration theory says that (G, μ) is a "quotient" $\tilde{G}/\Gamma$, where $\tilde{G}$ is an extension of a principally polarized abelian scheme (B, ν) over $k[[t]]$ by a torus T over $k[[t]]$, and Γ is a principally polarized "period subgroup" of $\tilde{G}(k((t)))$ of rank r, where $r = \dim(T)$. The elementary sequence of $(B_0, \nu_0)[p]$ is equal to $(\varphi'(r+1) - r, \varphi'(r+2) - r, \ldots, \varphi'(g) - r)$, and $\varphi'(1) = 1,\ \varphi'(2) = 2, \ldots,\ \varphi'(r) = r$. Similarly the elementary sequence of $(B, \nu)[p]_{k((t))}$ is $\varphi(r+1) - r, \varphi(r+2) - r, \ldots, \varphi(g) - r$. The slope sequence of the Newton polygon of the $B_{k((t))}$ is obtained from ξ by deleting r 0's and r 1's, where $r = \dim(T)$.

Next one takes $R = k[[t, u]]$, let $\tilde{G}' = \tilde{G} \times_{\mathrm{Spec}(k[[t]])} \mathrm{Spec}(R)$, and construct a principally polarized period subgroup $\Gamma' \in \tilde{G}'(\mathrm{frac}(R))$ extending the principally polarized period subgroup Γ, such that the quotient principally polarized semi-abelian scheme (G', μ') over R satisfies the following properties.

(i) $(G', \mu') \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k[[t]]) \cong (G, \mu)$

(ii) $(G', \mu') \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k((u)))$ is an abelian variety.

It follows that the elementary sequence of $(G', \mu') \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k((u)))[p]$ is φ', and the slope sequence of $(G', \mu') \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k((u)))$ is obtained from the slope sequence of (B_0, ν_0) by adding r 0's and r 1's. Mover the Newton polygon of the generic fiber of (G', μ') is ξ. Let U be the largest open subscheme of $\mathrm{Spec}(R)$ over which G' is an abelian scheme. The principally polarized abelian scheme $(G', \mu') \times_{\mathrm{Spec}(R)} U$ over U defines a morphism $U \to W_\xi$ which factors through $W \subset W_\xi$, and the claim follows. $\square$ Claim

We finish the proof of Step 1. As $y \notin S_\varphi \cap W_\xi$ and as φ is minimal on W we see that $y \notin W$. As W is a closed subset of W_ξ, it follows that $y \notin W_\xi$. As Newton polygons go up under specialization (by Grothendieck) we see that $y \in W_\zeta$ for some $\zeta \precneqq \xi$. By 3.2, the Zariski closure of W in $\mathcal{A}_{g,1} \otimes k$ contains an irreducible component of W_ζ. $\square$ Step 1

6.5. Step 2. (i) *For every symmetric Newton polygon ξ and every irreducible component W of the open Newton polygon stratum W_ξ, there is an irreducible component T of the supersingular Newton polygon stratum W_σ such that T is contained in the Zariski closure of W.*

(ii) *For every pair of symmetric Newton polygons $\zeta \preceq \xi$, the incidence relation for Zariski closure of irreducible components gives a well-defined* surjective *map*

$$\mathrm{cosp} : \pi_0(W_\zeta) \to \pi_0(W_\xi).$$

In other words, every irreducible component of W_ζ is contained in the Zariski closure of a unique irreducible component of W_ξ, and the Zariski closure of every irreducible component of W_ξ contains at least one irreducible component of W_ζ.

6.5.1. Tool 3: Purity. *In a family of p-divisible groups, if the Newton polygon jumps, it already jumps in codimension one. More precisely, let S be a locally Noetherian scheme in characteristic p, and let $X \to S$ be family of p-divisible groups. Suppose that there exists a Newton polygon ζ such that the Newton polygon stratum $\mathcal{W}_\zeta(X \to S)$ is a dense open subset of S. Then the complement $Z := S \smallsetminus \mathcal{W}_\zeta(X \to S)$ of $\mathcal{W}_\zeta(X \to S)$ has codimension one at every maximal point of Z.* See [39, 4.1], [78], [116], [101], [108].

6.5.2. Tool 4: Deformations with constant Newton polygon. *For any* principally *polarized abelian variety (A, λ) there exists a deformation with generic fiber (A', λ') with $\mathcal{N}(A) = \mathcal{N}(A')$ and $a(A') \leq 1$.*
See [39], (5.12), and [77], (3.11) and (4.1).

6.5.3 Notation. For every symmetric ξ of height $2g$ we define a non-negative integer $\mathrm{sdim}(\xi)$. For the definition of $\mathrm{sdim}(\xi)$, see [75], 3.3, or [77], 1.9; also see 4.9.3. Here is the definition. We write $(b, c) \in \mathbb{Z} \times \mathbb{Z}$, and $\xi \precneqq (b, c)$ if the point (b, c) is strictly below the polygon ξ and $0 < b \leq 2g$, $0 \leq c$. We write

$$\mathrm{sdim}(\xi) = \frac{1}{2} g(g+1) - \#(\{(b, c) \in \mathbb{Z} \times \mathbb{Z} \mid \xi \precneqq (b, c), \quad 0 < b \leq g, \quad 0 \leq c\}).$$

Note that for $\zeta \precneqq \xi$ we have

$$\mathrm{sdim}(\sigma) = [\frac{g^2}{4}] \leq \mathrm{sdim}(\zeta) < \mathrm{sdim}(\xi) \leq \mathrm{sdim}(\rho) = \frac{1}{2} g(g+1).$$

We will see that $W_\xi \subset \mathcal{A}_{g,1}$ is pure of dimension $\mathrm{sdim}(\xi)$. Note that different components of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ may have different dimensions: our uniform formula using $\mathrm{sdim}(\xi)$ works for *principally polarized* abelian varieties.

6.5.4. Tool 5: Cayley–Hamilton. We refer to [75, 3.5] for the this method in deformation theory; see also 9.12. The statement below is a consequence of this method combined with 6.5.2.

Let $k \supset \mathbb{F}_p$ be an algebraically closed field.

(a) *Let W_ξ be a Newton polygon stratum of $\mathcal{A}_{g,1} \otimes k$. The a-number of the generic point of each irreducible component of W_ξ is at most 1, and the dimension of every irreducible component of W_ξ is equal to* $\mathrm{sdim}(\xi)$.

(b) *Let $x \in \mathcal{A}_{g,1}(k)$ be a k-point of $\mathcal{A}_{g,1} \otimes k$ and let $\zeta := \mathcal{N}(A_x)$. Assume that $a(A_x) = 1$. Then for each symmetric Newton polygon $\xi \succeq \zeta$, the Zariski closure W_ξ^{Zar} in $\mathcal{A}_{g,1}$ of the open Newton polygon stratum W_ξ contains x and is smooth at x.*

(c) *Let x and ζ be as in* (ii). *For any two symmetric Newton polygons $\xi_1, \xi_2 \succeq \zeta$, we have*

$$\left(W_{\xi_1}^{\mathrm{Zar}}\right)^{/x} \supseteq \left(W_{\xi_2}^{\mathrm{Zar}}\right)^{/x} \quad \Longleftrightarrow \quad \xi_1 \succeq \xi_2\,,$$

where $\left(W_{\xi_i}^{\mathrm{Zar}}\right)^{/x}$ is the formal completion of $W_{\xi_i}^{\mathrm{Zar}}$ at x.

6.5.5. Corollary. (i) *For any irreducible component W of an open Newton polygon stratum W_ξ in $\mathcal{A}_{g,1} \otimes k$, its boundary $(W^{\mathrm{Zar}} \smallsetminus W) \subset \mathcal{A}_{g,1}$ in $\mathcal{A}_{g,1} \otimes k$ is a union of irreducible components of smaller Newton polygon strata. More precisely, $(W^{\mathrm{Zar}} \smallsetminus W) \subset \mathcal{A}_{g,1}$ is the union of a subset of the finite collection*

$$\{\, W' \subseteq \mathcal{A}_{g,1} \otimes k \mid W' \ \text{ is an irreducible component of } W_{\xi'} \text{ for some } \ \xi' \precneqq \xi \,\}$$

of algebraic subvarieties in $\mathcal{A}_{g,1} \otimes k$.

(ii) *Every irreducible component W of $W_\xi \subset \mathcal{A}_{g,1}$ contains in its closure a point in the supersingular locus, in fact W^{Zar} contains at least one irreducible component of W_{σ_g}.*

Proof. For any point $y \in W^{\mathrm{Zar}} \smallsetminus W$ we know by the dimension formula the dimension of the related NP stratum, and by Purity we know the dimension of every irreducible component of $W^{\mathrm{Zar}} \smallsetminus W$ have pure codimension one in W. Newton polygons $\precneqq \xi$ have dimension at most $\mathrm{sdim}(\xi) - 1$. From this we see that irreducible components of $W^{\mathrm{Zar}} \smallsetminus W$ generically are irreducible components of lower strata; by induction the proof of (i) is finished.

Consider in the boundary $W^{\mathrm{Zar}} \smallsetminus W$ an irreducible component W' of a Newton polygon stratum W_ζ of smallest dimension in W^{Zar}. Then $W' \subset \mathcal{A}_{g,1}$ is closed. By 6.4 we conclude $\zeta = \sigma$. □

Remark. We can deduce 6.5.5.(ii) directly from 6.4 by considering ℓ-Hecke correspondences on boundary points of W, and applying descending induction on Newton polygons. Also see [72].

6.5.6. Remark. Using 6.5.2 and 6.5.3 a conjecture by Grothendieck, see [75] and [77] follows. See 9.4.

6.5.7. Explanation. Eventually we will see: every $W = W_\xi$ contains the supersingular locus W_σ, and every chain of Newton polygons $\xi_1 \precneqq \cdots \precneqq \xi_e$ corresponds with a chain of inclusions

$$(W_{\xi_1})^{\mathrm{Zar}} \subsetneqq \cdots \subsetneqq (W_{\xi_e})^{\mathrm{Zar}}$$

inside $\mathcal{A}_{g,1}$.

6.5.8. Proof of Step 2. We have seen in **6.5.5.(ii)** that W contains a supersingular point. By Purity 6.5.2, we know that Newton polygons jump in codimension one. By 6.5.3 (iii) we know the dimensions of all Newton polygon strata in $\mathcal{A}_{g,1} \otimes k$. It is known that the poset of all symmetric Newton polygons of height $2g$ is ranked in the sense that all maximal chains between two elements of the posets have the same length, and for any symmetric Newton polygon ξ of height $2g$ the distance from ξ to the ordinary Newton polygon $\rho_{g,2g}$ is $\mathrm{sdim}(\xi)$. So the difference between the codimensions of W_σ and W_ξ in $\mathcal{A}$ is precisely the length of the longest chain of symmetric Newton polygons between ξ and σ. The above facts imply that the Zariski closure of W *contains an irreducible component of* W_σ. We have proved part (i) of Step 2.

Conversely given an irreducible component T of the supersingular Newton polygon W_σ in $\mathcal{A}_{g,1} \otimes k$, we choose a point $x \in T$ with $a(A_x) = 1$, which exists by 6.5.3,. An application of 6.5.3 shows the surjectivity of the map $\mathrm{cosp} : \pi_0(W_\xi) \to \pi_0(W_\sigma)$. The argument actually establishes more, that we have a well-defined surjection $\mathrm{cosp} : \pi_0(W_{\xi_1}) \to \pi_0(W_{\xi_2})$ for every pair (ξ_1, ξ_2) of symmetric Newton polygons of height $2g$ with $\xi_1 \prec \xi_2$. This ends the proof of Step 2. □

6.6. Irreducibility and prime-to-p monodromy.

6.6.1. Tool 6: Characterization of irreducible components W_σ.

Let $k \supset \mathbb{F}_p$ be an algebraically closed field, and let E be a supersingular elliptic curve over k. There is a canonical bijective map

$$\pi_0(W_\sigma) \quad \xrightarrow{\sim} \quad \Lambda_{g,g-1}$$

from the set $\pi_0(W_\sigma)$ of irreducible components of the supersingular locus $W_\sigma \subset \mathcal{A}_{g,1} \otimes k$ to the set $\Lambda_{g,g-1}$ of all isomorphism classes of a polarization μ on E^g such that

$$\mathrm{Ker}(\mu) = \mathrm{Ker}\left(F^{g-1} : E^g \to (E^g)^{(p^{g-1})}\right).$$

See [48], 3.6 and 4.2; this uses [67], 2.2 and 3.1.

6.6.2. Tool 7: Transitivity of Hecke correspondences on $\pi_0(W_\sigma)$

Let ℓ be a prime number different from p. The ℓ-adic Hecke correspondences operate transitively on $\pi_0(W_\sigma)$.

By 6.6.1 the statement is translated into the statement of that ℓ-adic Hecke correspondences operate transitively on the set of isomorphism classes of certain polarizations on a superspecial abelian variety. Use [14, pp. 158–159] to describe the set of isomorphism classes of such polarizations, and apply the strong approximation theorem. See [89, Thm. 7.12, p. 42] for the strong approximation theorem for simply connected reductive algebraic groups over global fields.

6.6.3. Tool 8: Irreducibility of subvarieties stable under Hecke correspondences

Let $k \supset \mathbb{F}_p$ be an algebraically closed field and let ℓ be a prime number different from p. Let $Z \subset \mathcal{A}_{g,1} \otimes k$ be a locally closed subscheme smooth over $\mathrm{Spec}(k)$*, such that Z is stable under all ℓ-adic Hecke correspondences. Assume that the ℓ-adic Hecke correspondences operate transitively on the set $\pi_0(Z)$ of irreducible components of Z. Assume moreover that Z is not contained in the supersingular locus $W_{\sigma_g} \subset \mathcal{A}_{g,1} \otimes k$. Then Z is irreducible.* See [1], 4.4.

The end of the proof of 6.1. We know from 6.6.2 the ℓ-adic Hecke correspondences operate transitively on the set of irreducible components of the non-supersingular open Newton polygon stratum W_ξ. Apply 6.6.2 to the smooth locus of W_ξ, we conclude that W_ξ is geometrically irreducible. □

6.7. Remark. Suppose φ appears on $W = W_\xi \subset \mathcal{A}_{g,1}$, i.e. $S_\varphi \cap W \neq \emptyset$. In Step 1 we have seen that the Zariski closure of $S_\varphi \cap W$ contains a point with or a smaller EO type or a smaller Newton polygon (or both). Indeed it is true that $(S_\varphi \cap W)^{\mathrm{Zar}}$ contains the central stream $\mathcal{Z}_\xi$, see [30, Cor. 1.1] and [105, Thm 1.7]. If we would know this a priori. the proof of Step 1 would be easier (as the central stream is an EO stratum); however the logic in our proof works the other way around.

6.8. Results for non-principally polarized abelian varieties? In 6.1–6.7 above we have worked on $\mathcal{A}_{g,1}$. We note that analogous statements for non-principally polarized abelian varieties fail in many cases. The analogy of the Grothendieck conjecture has many counter examples, see [82]; formulas for dimensions of Newton polygon strata and for EO strata can be different, inclusion relations as in 6.5 may fail, and the irreducibility of the various strata may fail. However the dimension formula for central leaves works uniformly on $\mathcal{A}_g$. EO strata seem difficult to handle on $\mathcal{A}_g$. Still Purity holds, but the Cayley–Hamilton method breaks down in general. Indeed, in many places in this section it is essential that we are working with principally polarized abelian varieties.

6.9. Remarks on supersingular moduli. The story began with studying (ir)reducibility of strata defined by supersingular (polarized) abelian varieties. Eichler [13] and Igusa [35] showed that the number of supersingular j-invariants is roughly $p/12$, hence bigger than one for $p \to \infty$; in fact, for $p \in \{2, 3, 5, 7, 13\}$ there is precisely one supersingular j-invariant, for all other primes there are at least two supersingular j-invariants. Eichler showed this number can be computed as a class number.

What can be said about the number of irreducible components of the supersingular locus? This question is not easily answered for moduli of non-principally polarized supersingular abelian varieties (dimension of strata and number of components depend not only on p but may also depend on invariants given by the polarization). However for the supersingular Newton polygon stratum $W_\sigma \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ we now know that the dimension of every component equals $[p^2/4]$ (conjectured by Tadao Oda and FO in [67] and proved by K-Z.Li and FO in [48]) The number of components of $W_\sigma \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is a class number (Serre, Katsura, Ibukiyama, FO); this number is computed by Hashimoto and Ibukiyama in many cases, see [40], [34], [41] and [48].

6.10. We cannot resist the temptation to give, as an illustration of the preceding sentence, an elementary proof of the following fact.

> *For $g = 2$ the supersingular locus $W_{\sigma_g} \subset \mathcal{A}_{2,1} \otimes \mathbb{F}_p$ is pure of dimension one and for $p > 1500$ this supersingular Newton polygon stratum is* reducible.

For an abelian variety A over an algebraically closed field k of characteristic p we know that $a(A) = g = \dim(A)$ implies that A is superspecial, see 3.4. For $g = 2$ supersingular abelian surfaces with $a = 1$ move in a one dimensional family, e.g. see [70, §4]. In [59] we find a further analysis, of a one-dimensional family of principally polarized supersingular abelian surfaces. Such a one-dimensional family of abelian surfaces arises as the Jacobian of a family $\mathcal{C} \to \mathbb{P}^1$ of curves of genus two parametrized by $\mathbb{P}^1$. In [59, II.2.5] we find that such a one-dimensional family of supersingular curves of genus 2 contains exactly $5p - 5$ singular fibers, and every singular fiber is a union of two supersingular elliptic curves joined transversally at one point. We will abuse notation and denote such a stable curve of genus two by $E \cup E'$, where E, E' are the two irreducible components.

We know the number h_p of isomorphism classes of supersingular elliptic curves over an algebraically field k of characteristic p satisfies

$$\frac{p}{12} < h_p < \frac{p}{12} + 2.$$

The number of isomorphism classes of curves $E' \cup E''$ is $(h_p^2 + h_p)/2$. Let's fix p and suppose we know that the supersingular Newton polygon stratum $W_\sigma \subset \mathcal{A}_{2,1} \otimes \mathbb{F}_p$ is geometrically irreducible. Then W_σ is the image of one Moret-Baily family; this has at most $5p-5$ singular fibers and all curves of the form $E' \cup E''$ appear on this family. It follows that

$$5p > 5p - 5 \geq (h_p^2 + h_p)/2,$$

which implies that

$$12^2{\cdot}2{\cdot}5 \cdot p > 12^2(h_p^2 + h_p) > p(p+12).$$

We conclude that the supersingular locus W_σ in the moduli space $\mathcal{A}_{g,2} \otimes \overline{\mathbb{F}_p}$ of principally polarized abelian surfaces is geometrically reducible for $p > 1500$.

Later we proved by studying ramification behavior, or by the fact that the number of irreducible components is given by a class number, and a computation by Hashimoto and Ibukiyama shows that

$$W_\sigma \subset \mathcal{A}_{2,1} \otimes \overline{\mathbb{F}_p} \text{ is irreducible} \quad \Longleftrightarrow \quad p \leq 11,$$

see [40], [34].

7 CM **lifting**

We recall questions and results as stated in [2]. We start with an abelian variety A_0 (respectively a p-divisible group X_0) over a finite field κ, and we ask whether there exists a CM lift for A_0 (respectively X_0) over a local integral domain $(R, \mathfrak{m})$ of generic characteristic 0 with residue field $R/\mathfrak{m} = \kappa$.

We know from results of Honda and Tate that after an extension $\kappa \hookrightarrow \kappa'$ of finite base fields and an isogeny $A_0 \otimes_\kappa \kappa' \to B_0$ or $X_0 \otimes_\kappa \kappa' \to Y_0$, a CM lift exists.

7.1. An isogeny is necessary. In [73] we see *examples of abelian varieties and p-divisible groups p-divisible groups over $\overline{\mathbb{F}_p}$ which do not admit a* CM *lift* to characteristic 0. In fact, in this paper it is shown that for any symmetric Newton polygon ξ of height $2g$ with $g \geq 3$ and p-rank $f(\xi) \leq g-3$, there exists such an example with Newton polygon ξ. Hence the analogous statement holds for abelian varieties over $\overline{\mathbb{F}_p}$. In [2], Chapter 3 the method has been generalized and the result holds for Newton polygons which are not of the form

$$\mathrm{NP}\big(\mu_1 * (0,1) + \mu_2 * (1,0) + \mu_3 * (1,m) + \mu_4 * (m,1)\big)$$

with $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{N}$ and $\mu_3 + \mu_4 \leq 1$.

Here is the basic idea of the proof. Suppose we are given a symmetric Newton polygon ξ of height $2g$ with p-rank $f(\xi) \leq g-3$. For such Newton polygons there exists an abelian variety A over $\overline{\mathbb{F}_p}$ with $a(A) = 2$ and Newton polygon ξ such that there exists $\alpha_p \cong N \subset A$ with $a(A/N) = 1$. The $\mathbb{P}^1$-family of (α_p)-quotients of A is studied, and one sees from the theory of complex multiplication of abelian varieties that at most finitely many of these quotients admit a CM lift; see [73]. A proof using CM theory for p-divisible groups is in [2, Chap. 3].

7.2. Theorem. *For any finite field κ and any abelian variety A_0 over κ there exists a κ-isogeny $A_0 \to B_0$ such that B_0 admits a* CM *lifting* to an abelian scheme B over a Noetherian local domain $(R, \mathfrak{m})$ of generic characteristic 0 with $R/\mathfrak{m} = \kappa$. See [2], Theorem 4.1.1.

This is the central result of the book: "*a field extension is not necessary for the existence of a CM lift*". The proofs in [2] is a bit complicated. They all follow the same general strategy, but there are different routes to get to the end. One of the proofs reduces the general case to the case of two-dimensional supersingular p-divisible group, which can be handled directly. We will explain the proof for the case of an abelian surface with CM by $\mathbb{Q}(\zeta_5)$ in 7.5; the argument applies to p-divisible groups verbatim.

7.3. CM lifting to normal domains: the residual reflex condition. In the statement of the preceding Theorem 7.2 one can ask whether the CM lift of B_0 can be achieved over a *normal* domain in mixed characteristics $(0, p)$. The answer is negative in general. On can test this by the *residual reflex condition* in [2, 2.1.5]. Below is an easy example from [2, 2.3]. We will also use it to illustrate 7.2.

7.4. Example. *Let p be a prime number p that is either $\equiv 2$ or $\equiv 3 \pmod 5$. let $\pi = \zeta_5{\cdot}p$, a p^2-Weil number. Hence* (by Honda and Tate) *there exists an abelian variety A_0 over $\kappa = \mathbb{F}_{p^2}$ with endomorphisms by $\mathbb{Z}[\zeta_5]$ having π as κ-Frobenius. Then no abelian variety B_0 over $\mathbb{F}_{p^2}$ isogenous to A_0 admits a* CM *lift to a mixed characteristics* $(0, p)$ normal local *domain.*

PROOF. The condition on p means that p is inert in $\mathbb{Q}(\zeta_5)/\mathbb{Q}$. The endomorphism algebra $\mathrm{End}^0_{\mathbb{F}_{p^2}}(A_0)$ of A_0 is equal to $\mathbb{Q}(\zeta_5)$. The CM field $\mathbb{Q}(\zeta_5)$ has four CM types, each consisting of a pair of non-conjugate embeddings for $\eta_1, \eta_2 : \mathbb{Q}(\zeta_5) \hookrightarrow \mathbb{C}$. The reflex field of each CM type of $\mathbb{Q}(\zeta_5)$ is $\mathbb{Q}(\zeta)$ itself.

Suppose that there exists an $\mathbb{F}_{p^2}$-isogeny $A_0 \to B_0$ and a CM lift $(B, \iota : \mathbb{Q}(\zeta_5) \to \mathrm{End}^0_R(B))$ of $(B_0, \iota_0 : \mathbb{Q}(\zeta_5) \to \mathrm{End}^0_R(B_0))$ to a normal local domain $(R, \mathfrak{m})$ of mixed characteristics $(0, p)$ with residue field $\mathbb{F}_{p^2}$. We will see that this leads to a contradiction.

The fraction field L of R contains the reflex field of the CM type Φ of (B, ι), i.e $L \supset \mathbb{Q}(\zeta_5)$. Therefore the normal domain R contains the ring of integers $\mathbb{Z}[\zeta_5]$ in $\mathbb{Q}(\zeta_5)$. Because $\mathbb{Z}[\zeta_5]/p\zeta_5 \cong \mathbb{F}_{p^4}$, the inclusion homomorphism $\mathbb{Z}[\zeta_5] \hookrightarrow R$ induces a ring homomorphism

$$\mathbb{F}_{p^4} \cong \mathbb{Z}[\zeta_5]/p\mathbb{Z}[\zeta_5] \longrightarrow R/\mathfrak{m} \cong \mathbb{F}_{p^2}.$$

This is a contradiction. □

7.5. Proof of existence of CM lifting of the abelian surface A_0 in 7.4. Let A_0 an abelian surface over $\mathbb{F}_{p^2}$ with $\mathrm{End}_{\mathbb{F}_{p^2}}(A_0) = \mathbb{Z}[\zeta_5]$. We claim that the pair $(A_0, \iota_0 : \mathbb{Z}+p\mathbb{Z}[\zeta_5] \hookrightarrow \mathrm{End}(A_0))$ admits a lift to a local domain of characteristics $(0, p)$ with residue field $\mathbb{F}_{p^2}$, where the CM structure ι_0 on the abelian surface A_0 is the restriction to the subring $\mathbb{Z} + p\mathbb{Z}[\zeta_5]$ of the action of $\mathbb{Z}[\zeta_5]$ on A_0.

PROOF. It follows easily from an analysis of the $(\mathbb{Z}[\zeta_5] \otimes_{\mathbb{Z}} \mathbb{Z}_p)$-linear Dieudonné module of A_0 that there exists an abelian surface B_0 over $\mathbb{F}_{p^4}$ with an action $\iota_1 : \mathbb{Z}[\zeta_5] \to \mathrm{End}(B_0)$ by $\mathbb{Z}[\zeta_5]$ and an $\mathbb{F}_{p^4}$-isogeny $\alpha_0 : B_0 \to A_0 \otimes_{\mathbb{F}_{p^2}} \mathbb{F}_{p^4}$ with the following properties:

(i) The a-number $a(B_0)$ of the supersingular abelian variety B_0 is 1. In other words there is a subgroup scheme G_0 of $B_0[p]$ such that $G_0 \otimes_{\mathbb{F}_{p^4}} F$ is the unique subgroup scheme of $B_0 \otimes_{\mathbb{F}_{p^4}} F$ of order p, for every extension field F of $\mathbb{F}_{p^4}$.

(ii) $\mathrm{Ker}(\alpha_0)$ is isomorphic to α_p, hence it is the unique subgroup scheme of B_0 of order p.

(iii) The action of $\mathbb{Z}[\zeta_5]$ on $\mathrm{Lie}(B_0)$ corresponds to a pair of isomorphism $\mathbb{Z}[\zeta_5]/p\mathbb{Z}[\zeta_5] \longrightarrow \mathbb{F}_{p^4}$ which are *not* stable under the action of the element of order 2 in

$$\mathrm{Aut}_{\mathrm{ring}}(\mathbb{Z}[\zeta_5]/p\mathbb{Z}[\zeta_5]) \cong \mathbb{Z}/4\mathbb{Z}.$$

In other words The action of $\mathbb{Z}[\zeta_5]$ on $\mathrm{Lie}(B_0)$ corresponds to a CM type for $\mathbb{Q}(\zeta_5)$.

The property (iii) above implies that the lifting problem for the CM abelian variety $(B_0, \mathbb{Z}[\zeta_5] \hookrightarrow \mathrm{End}_{\mathbb{F}_{p^4}}(B_0))$ is unobstructed and the deformation space is $\mathrm{Spf}(\Lambda(\mathbb{F}_{p^4}))$, where $\Lambda(\mathbb{F}_{p^4})$ is the ring of p-adic Witt vectors with entries in $\mathbb{F}_{p^4}$, and the universal $\mathbb{Z}[\zeta_5]$-linear formal abelian scheme $(\boldsymbol{B}, \mathbb{Z}[\zeta_5] \to \mathrm{End}(\boldsymbol{B}))$ over the deformation space

$$\mathrm{Def}(B_0, \mathbb{Z}[\zeta_5] \to \mathrm{End}(B_0))$$

is algebraizable; see 7.6 below. Take a suitable finite extension field E of $\mathrm{frac}(\Lambda(\mathbb{F}_{p^4}))$ and an non-zero $\mathcal{O}_E$-point Q of $\boldsymbol{B}[p]$, and let P be the finite flat subgroup of $\boldsymbol{B}[p] \times_{\mathrm{Spf}(\Lambda(\mathbb{F}_{p^2}))} \mathrm{Spf}(\mathcal{O}_E)$ of order p generated by Q in $\boldsymbol{B}[p] \times_{\mathrm{Spf}(\Lambda(\mathbb{F}_{p^4}))} \mathrm{Spf}(\mathcal{O}_E)$. The closed fiber P_0 of P is a subgroup scheme of $B_0 \otimes_{\mathbb{F}_{p^4}} \kappa_E$, where κ_E is the residue field of E. Therefore $P_0 = G_0 \otimes_{\mathbb{F}_{p^4}} \kappa_E$. Let

$$\boldsymbol{A} = \Big(\boldsymbol{B} \times_{\mathrm{Spf}(\Lambda(\mathbb{F}_{p^4}))} \mathrm{Spf}(\mathcal{O}_E)\Big)/P\,.$$

Clearly the action of $(\mathbb{Z} + p\mathbb{Z}[\zeta_5]$ on $\boldsymbol{B}$ induces an action of $\mathbb{Z} + p\mathbb{Z}[\zeta_5]$ on $\boldsymbol{A}$, and the closed fiber of $\boldsymbol{A}$ together with the action of $\mathbb{Z} + p\mathbb{Z}[\zeta_5]$ coincides with the base field extension via $\mathbb{F}_{p^2} \hookrightarrow \kappa_E$ of the CM structure $(A_0, \mathbb{Z} + p\mathbb{Z}[\zeta_5] \longrightarrow \mathrm{End}(A_0))$.

Let $\mathrm{Def}(A_0, \mathbb{Z} + p\mathbb{Z}[\zeta_5] \to \mathrm{End}(A_0))$ be the equi-characteristic-p deformation space of the $(\mathbb{Z} + p\mathbb{Z}[\zeta_5])$-linear abelian variety A_0. Its base extension to $\mathrm{Spf}(\mathcal{O}_E)$ has an irreducible component of mixed characteristics $(0,p)$, over which the universal abelian scheme together with the action of $\mathbb{Z} + p\mathbb{Z}[\zeta_5]$ is algebraizable. Therefore the same holds for $\mathrm{Def}(A_0, \mathbb{Z} + p\mathbb{Z}[\zeta_5] \to \mathrm{End}(A_0))$ before base change. In other words there exists a complete Noetherian local domain $(R, \mathfrak{m})$ and an $(\mathbb{Z}+p\mathbb{Z}[\zeta_5])$-linear abelian scheme over R whose closed fiber is the $(\mathbb{Z} + p\mathbb{Z}[\zeta_5])$-linear abelian variety A_0. □

More details can be found in [2, 4.1.2–4.1.3]. CM p-divisible groups similar to the CM p-divisible group $A_0[p^\infty]$ are called "toy models" in [2].

7.6. Criterion of algebraicity for CM formal abelian schemes In general for lifting problems, a formal lifting over a complete Noetherian local ring R need not to give an actual lifting, as the lifted proper formal scheme over R might not be the formal completion of a proper scheme over R. In the problem of lifting abelian varieties with complex multiplications, there is a simple criterion for whether a formal CM abelian scheme is algebraizable:

> *A formal abelian scheme $(\boldsymbol{X}, \iota)$ with CM by K over a complete Noetherian local domain $(R, \mathfrak{m})$ of mixed characteristics $(0,p)$ with $[K:\mathbb{Q}] = 2\dim_R(\boldsymbol{X})$ is algebraizable if and only if the action of K on the tangent space of the generic fiber of $\boldsymbol{X}$ corresponds to a CM type for K.*

see [2, 2.2.3].

In 7.5, the deformation theory of $(A_0, \mathbb{Z}[\zeta_5] \hookrightarrow \mathrm{End}(A_0))$ is unobstructed and the deformation space $\mathrm{Def}(A_0, \mathbb{Z}[\zeta_5] \hookrightarrow \mathrm{End}(A_0))$ is isomorphic to $\mathrm{Spf}(\Lambda(\mathbb{F}_{p^2})$. However the action of $\mathbb{Z}[\zeta_5]$ on the Lie algebra of the universal formal abelian scheme over $\mathrm{Def}(A_0, \mathbb{Z}[\zeta_5] \hookrightarrow \mathrm{End}(A_0))$ corresponds to a pair of complex conjugate embeddings of $\mathbb{Z}[\zeta_5]$ into $\Lambda(\overline{\mathbb{F}_p})$. Therefore the universal formal abelian scheme over $\mathrm{Def}(A_0, \mathbb{Z}[\zeta_5] \hookrightarrow \mathrm{End}(A_0))$ is not algebraic.

8 Generalized Serre–Tate coordinates

8.1. Let $x \in W_{\xi,d}(\overline{\mathbb{F}_p})$ be an $\overline{\mathbb{F}_p}$-point of a Newton polygon stratum $W_{\xi,d}$ in $\mathcal{A}_{g,d} \otimes \overline{\mathbb{F}_p}$. If ξ is not the ordinary Newton polygon $\rho_{g,2g} = \mathrm{NP}\big(g*(1,0) + g*(0,1)\big)$ nor the "almost ordinary"

$\mathrm{NP}\big((g-1)*(1,0)+(1,1)+(g-1)*(0,1)\big)$, we do not know of a simple structural description for either the formal completion $(\mathcal{A}_{g,d}\otimes\overline{\mathbb{F}_p})^{/x}$ of $\mathcal{A}_{g,d}\otimes\overline{\mathbb{F}_p}$ at x nor the formal completion $W_{\xi,d}^{/x}$ at a point x of $W_{\xi,d}$ at x, which is analogous to the structure of the formal completion of $\mathcal{A}_{g,d}\otimes\overline{\mathbb{F}_p}$ at ordinary points given by the Serre–Tate theory.[4] Admittedly we have not been able to formulate a convincing heuristic against the existence of good generalized Serre–Tate canonical coordinates, nevertheless we are convinced that is the phenomenon.

Next one might attempt to find a generalization of Serre–Tate coordinates in the following sense: for an algebraically closed point a point $x\in\mathcal{A}_{g,d}(k)$ with $\xi=\mathcal{N}(A_x)$, find a good structural description of the formal completion $W_{\xi,d}^{/x}$ at x of the Newton polygon stratum $W_{\xi,d}\subset\mathcal{A}_{g,d}\otimes k$. A variant version in the quest of a good theory of generalized Serre–Tate coordinates is to replace Newton polygon strata by central leaves in $\mathcal{A}_{g,d}\otimes k$:

> *Find a good structural description of the formal completion $\mathcal{C}(x)^{/x}$ of the central leaf $\mathcal{C}(x)$ in $\mathcal{A}_{g,d}\otimes k$, where $x\in\mathcal{A}_{g,d}(k)$ and $k\supset\mathbb{F}_p$ is an algebraically closed field.*

8.2. It turns out that the latter question has a good answer. The general phenomenon is that the formal completions $\mathcal{C}(x)^{/x}$ of central leaves in $\mathcal{A}_{g,d}\otimes k$ as above can be "built up" from p-divisible formal groups over k, in the following sense. There is a collections $\mathcal{J}$ of morphisms of smooth formal schemes over k, with the following properties.

(i) The source and target of each morphism $\mathcal{J}$ is isomorphic to the formal spectrum of a formal power series ring over k in a finite number of variables.

(ii) Each morphism in $\mathcal{J}$ has a natural structure as a torsor for a connected commutative formal group over k which is an extension of a p-divisible formal group over k by a connected commutative finite group scheme over k.

(iii) The diagram of the morphisms in $\mathcal{J}$ is isomorphic to the diagram of a finite partially ordered set with a unique maximal element and a unique minimal element. The maximal element corresponds to $\mathcal{C}(x)^{/x}$, while the minimal element corresponds to a point $\mathrm{Spec}(k)$.

8.3. While exploring the properties of the "generalized Serre–Tate coordinates" for formal completions of central leaves, we encountered the difficulty that the definition of a central leaf $\mathcal{C}(x)$ is a "pointwise" one which does not lend to good functorial considerations. This difficulty was overcome by the introduction of a notion of *sustained p-divisible groups*, which replaces and generalizes the notion of "geometrically fiberwise constant p-divisible groups", see 4.1. A survey of this theory can be found in [5], and a full account will appear in a planned monograph [7] on the Hecke orbit problem. We describe an example below to provide an idea about the generalized Serre–Tate coordinates for central leaves.

8.4. Example. For more information see 8.6 below. We study the following special case:

$$g=3,\ \text{and a Newton polygon } \xi=\mathrm{NP}((2,1)+(1,2)).$$

[4] In this connection, we note that one cannot expect the existence of a good analog to the non-ordinary situation of the Serre–Tate canonical lifting of ordinary abelian varieties. For instance one can show the non-existence of liftings of non-ordinary points in $\mathcal{A}_{g,1}(\overline{\mathbb{F}_p})$ to characteristic 0 which are equivariant with respect to all prime-to-p correspondences.

i.e. slopes 2/3 and 1/3 both with multiplicity 3. Let $x = [(A_x, \lambda_x)]$ be a k-point of $\mathcal{A}_{3,1}$, where $k \supset \mathbb{F}_p$ is an algebraically closed point, such that

$$A_x[p^\infty] \cong G_{2,1} \times G_{1,2}.$$

The central leaf $\mathcal{C}(x)$ is what we called the central stream in $W_\xi \subset \mathcal{A}_{g,1} \otimes k$, and denoted by $\mathcal{Z}(\xi)$. We know that $\dim(\mathcal{C}(x)) = 2$ and $\dim(W_\xi) = 3$. In this case:

Result (generalized Serre–Tate coordinates, in this special case). *The formal completion $\mathcal{Z}_\xi^{/x}$ at x of the central stream $\mathcal{Z}_\xi$ has a natural structure as an isoclinic p-divisible group over k with slope* $(2/3) - (1/3) = 1/3$ *and height* 6.

We note that on W_ξ we have two foliations, *the central leaves and the isogeny leaves.* For $x \in \mathcal{Z}_\xi$ the isogeny leaf $\mathcal{I}(x)$ has two nonsingular branches: one is obtained by deforming $G_{2,1} \to X_0 \to (G_{1,2}/\alpha_p)$, the other by deforming $(G_{2,1}/\alpha_p) \to X_0 \to G_{1,2}$. This geometric structure (a 3-fold W_ξ with a singularity along a non-singular surface $\mathcal{Z}_\xi$ where two branches meet) is reflected in the group structures we are going to describe.

General remarks.

- One has "generalized Serre–Tate coordinates" for central leaves in the equi-characteristic-p deformation spaces of (unpolarized) p-divisible groups and polarized p-divisible groups, in the sense of 8.2.
- In the special case when the p-divisible group in question is a product of two isoclinic p-divisible group, the central leave in the deformation space has a natural structures as an isoclinic p-divisible formal groups. The example above falls into this case.
- In the slightly more general situation of p-divisible groups with two slopes, the central leaves in the equi-characteristic-p deformation space has a natural structure as torsors for an isoclinic p-divisible formal group.

8.5. The generalized Serre–Tate coordinates in 8.4. Let $k \supset \mathbb{F}_p$ be an algebraically closed field. let X_0 be the p-divisible group $A_x[p^\infty]$ and let μ_0 be the polarization on X_0 induced by the polarization μ_x on A_x. We have assumed that X_0 is isomorphic to $G_{1,2} \times G_{2,1}$; let E_0 be the split extension of $G_{1,2}$ by $G_{2,1}$ underlying X_0.

(1) Define deformations spaces $\mathcal{D}(X_0, \mu_0)$ and $\mathcal{D}(X_0)$ as follows. Let

$$\mathcal{D}(E_0, \mu_0) = \mathrm{Def}(G_{2,1} \to X_0 \to G_{1,2}, \lambda)$$

be the equi-characteristic-p deformation spaces such that for every Artinian commutative local k-algebra $(R, \mathfrak{m})$ endowed with an isomorphism $\epsilon : R/\mathfrak{m} \xrightarrow{\sim} k$, the set of all (R, ϵ)-points of $\mathcal{D}(E_0, \mu_0)$ is the set of all isomorphism classes of triples

$$\big(E = (0 \to (G_{2,1})_{/R} \to X \to (G_{1,2})_{/R} \to 0), \quad \boldsymbol{\mu}, \quad \delta : \epsilon^* E \xrightarrow{\sim} E_0\big),$$

where

- X is a p-divisible group over R, and E is a short exact sequence making X an extension of $G_{1,2} \otimes R$ by $G_{2,1} \otimes R$,

– $\boldsymbol{\mu}$ is a polarization on X, and

– δ is an isomorphism from the closed fiber of $(X, \boldsymbol{\mu})$ to (X_0, μ_0), which induces the identity maps on $G_{1,2}$ and $G_{2,1}$.

Similarly let $\mathcal{D}(E_0)$ be the equi-characteristic-p deformation spaces such that for every Artinian commutative local k-algebra $(R, \mathfrak{m})$ endowed with an isomorphism $\epsilon : R/\mathfrak{m} \xrightarrow{\sim} k$, the set of all isomorphism classes of pairs

$$\big(E = (0 \to (G_{2,1})_{/R} \to X \to (G_{1,2})_{/R} \to 0), \quad \delta : \epsilon^* E \xrightarrow{\sim} E_0\big),$$

where

– X is a p-divisible group over R, and E is a short exact sequence making X an extension of $G_{1,2} \otimes R$ by $G_{2,1} \otimes R$, and

– δ is an isomorphism from the closed fiber of X to X_0, which induces the identity maps on $G_{1,2}$ and $G_{2,1}$.

(2) Using the Grothendieck–Messing deformation theory for p-divisible groups, one checks without difficulty that the deformation space $\mathcal{D}(E_0)$ is formally smooth over k of dimension 4. i.e. it is isomorphic to the formal spectrum of the ring of all formal power series over k in 4 variables. The Baer sum construction defines a group law on the formal scheme $\mathcal{D}(E_0)$, so that $\mathcal{D}(X_0)$ is a 4-dimensional commutative smooth formal group over k. Similarly $\mathcal{D}(X_0, \mu_0)$ is a 3-dimensional commutative smooth formal group over k.

(3) Every finite dimensional commutative formal G group over a perfect base field of characteristic p has a maximal p-divisible formal subgroup G_{divsub}, a maximal p-divisible quotient group G_{divquot}, a maximal unipotent smooth formal subgroup G_{unipsub} and a maximal unipotent quotient smooth formal group G_{unipquot}. They fit in exact sequences

$$0 \to G_{\mathrm{unipsub}} \to G \to G_{\mathrm{divquot}} \to 0 \quad \text{and} \quad 0 \to G_{\mathrm{divsub}} \to G \to G_{\mathrm{unipquot}} \to 0\,.$$

Moreover the intersection $G_{\mathrm{unipsub}} \cap G_{\mathrm{divsub}}$ is a finite subgroup scheme over k, so the natural maps

$$G_{\mathrm{divsub}} \to G_{\mathrm{divquot}} \quad \text{and} \quad G_{\mathrm{divsub}} \to G_{\mathrm{unipquot}}$$

are isogenies with $G_{\mathrm{unipsub}} \cap G_{\mathrm{divsub}}$ as their kernels.

(4) One can show that the maximal p-divisible subgroup $\mathcal{D}(E_0)_{\mathrm{divsub}}$ of $\mathcal{D}(E_0)$ is the central leaf in $\mathcal{D}(E_0)$. Similarly the maximal p-divisible subgroup $\mathcal{D}(E_0, \mu_0)_{\mathrm{divsub}}$ of $\mathcal{D}(E_0, \mu_0)$ is the central leaf in $\mathcal{D}(E_0, \mu_0)$, which is also the formal completion $\mathcal{Z}(x_0)^{/x_0}$ of the central stream $\mathcal{Z}(x_0)$ in $\mathcal{A}_{3,1} \otimes k$ passing through x_0.

(5) The maximal unipotent subgroup $\mathcal{D}(E_0)_{\mathrm{unipsub}}$ of $\mathcal{D}(E_0)$ is one-dimensional and coincides with the maximal unipotent subgroup $\mathcal{D}(E_0, \mu_0)_{\mathrm{unipsub}}$ of $\mathcal{D}(E_0, \mu_0)$. This one-dimensional smooth formal group is an irreducible component of the isogeny leaf in both $\mathcal{D}(E_0)$ and $\mathcal{D}(E_0, \mu_0)$. So $\mathcal{D}(E_0, \mu_0)_{\mathrm{unipsub}}$ is an irreducible component of the formal completion at x_0 of an isogeny leaf in $\mathcal{A}_{3,1} \otimes k$ which passes through x_0.

(6) Through x there is another branch of the isogeny leaf in the deformation spaces for X_0 and (X_0, μ_0). It can also be expressed by a similar construction as follows. Let E_1 be the split extension

$$0 \to G_{1,2} \to X_0 \to G_{2,1} \to 0$$

with $G_{1,2}$ as the sub and $G_{2,1}$ as the quotient. Define equi-characteristic-p deformation spaces $\mathcal{D}(E_1)$ and $\mathcal{D}(E_1, \mu_0)$ in the same ways as $\mathcal{D}(E_0)$ and $\mathcal{D}(E_1, \mu_0)$. One finds that $\mathcal{D}(E_1)$ is a one-dimensional unipotent commutative smooth formal group over k, and the natural map $\mathcal{D}(E_1, \mu_0) \to \mathcal{D}(E_1)$ is an isomorphism. Clearly $\mathcal{D}(E_1, \mu_0)$ is an irreducible component of the isogeny leaf in $\mathrm{Def}(X_0)$, and also an isogeny leaf in $\mathrm{Def}(X_0, \mu_0)$. Thus we get an irreducible component of an isogeny leaf in $\mathcal{A}_{g,1} \otimes k$ passing through x_0, and it is different from $\mathcal{D}(E_0, \mu)_{\mathrm{unipsub}}$; in fact is easy to see that the tangent spaces at x_0 of these two irreducible components are transversal. We explain this in the following example.

8.6. Example (isogeny leaves in a particular case). We choose for ξ the symmetric Newton polygon of height 6 with slopes 2/3 and 1/3, i.e. $\xi = \mathrm{NP}((2,1) + (1,2))$. We choose some level structure $n \geq 3$ prime to p (in order to have a universal family), defined over a finite field κ, and write $\mathcal{A} = \mathcal{A}_{3,1,n} \otimes \kappa$ and $W = W_\xi = \mathcal{W}_\xi(\mathcal{A})$. We know that $\dim(W) = 3$, and that central leaves have dimension 2 and isogeny leaves have dimension 1 in W. We will give an explicit geometric description of this W_ξ.

We write $Z' = Z_\xi \otimes \kappa$ for the central leaf in W. We write $\mathcal{B}' \to Z'$ for the restriction of the universal family over $\mathcal{A}$ to Z. As Z' is smooth over κ we have a slope filtration over Z'; see [114]. In fact over Z' there exists a slope filtration

$$0 \to X' \longrightarrow \mathcal{G}' := \mathcal{B}'[p^\infty] \longrightarrow Y' \to 0.$$

Note that every geometric fiber of $X' \to Z'$ can be descended to $G_{2,1}/\kappa$ and every geometric fiber of $Y' \to Z$ can be descended to $G_{1,2}/\kappa$.

Suppose

$$0 \to G_{2,1} \longrightarrow H \longrightarrow G_{1,2} \to 0$$

is an exact sequence over a field K such that $\alpha_p \times \alpha_p$ embeds into H over a field K. We say $\alpha_p \subset H$ is a *special direction* if either $\alpha_p \subset G_{2,1}$ or

$$(\alpha_p \subset H) \otimes k = (\{0\} \times \alpha_p) \subset (G_{2,1} \times G_{1,2}) \cong H \otimes k.$$

It is easy to see that if $\alpha_p = N \subset H$ is a special direction then $a(H/N) = 2$, and if it is not a special direction then $a(H/N) = 1$.

8.6.1. Construction. We produce an exact sequence

$$0 \to \mathcal{N}' \to Y_1' \xrightarrow{\varphi'} Y' \to 0;$$

over a purely inseparable cover $Z \to Z'$ we construct a commutative diagram with exact horizontal sequences.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{Q} \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d, "\psi"] & Y_1 \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & X \arrow[r] & \mathcal{G} \arrow[r] & Y \arrow[r] & 0;
\end{tikzcd}$$

such that

- every fiber of $\mathcal{Q}' \to Z$ and every fiber of $\mathcal{N}' \to Z$ is isomorphic with α_p (i.e. $\mathcal{N}'$ is an α-sheaf of rank one); let λ' be the quasi-polarization on $\mathcal{G}'$;
- after an appropriate inseparable base change we can lift the $\mathcal{N}'$-cover $Y_1 \to Y$ to an $\mathcal{N}$-cover $\psi : \mathcal{H} \to \mathcal{G}$. Here $\mathcal{G} = \mathcal{G}' \times_{Z'} Z$, etc.;
- write $\mu = \psi^*(\lambda)$ and $\mathcal{K} = \mathrm{Ker}(\mu) = \mathrm{Ker}(\mathcal{H} \to \mathcal{G} \to \mathcal{G}^t \to \mathcal{H}^t)$
- with $\mathcal{K} = \mathcal{Q} \times \mathcal{N}$;
- inside $\mathcal{K}$ the subgroups $\mathcal{Q}$ and $\mathcal{N}$ give the two special directions in every fiber of $\mathcal{K} \to Z$.
- the quasi-polarization $\psi^*(\lambda)$ on $\mathcal{H}$ gives the usual pairing on its kernel $\mathcal{K}$.

This will be carried out as follows. Over Z' we have the universal family $\mathcal{B}'$, and duality gives $\mathcal{B}'^t \to Z$; here we have the slope filtration

$$0 \to Y'^t \longrightarrow \mathcal{B}'^t[p^\infty] \longrightarrow X'^t \to 0.$$

There is a unique finite locally free group scheme $\mathcal{M}' \subset Y'^t$ with fibers isomorphic to α_p; this we see as follows; the generic fiber $Y'^t \otimes \kappa(Z')$ has a-number equal to one, hence contains a (unique) subgroup isomorphic with α_p. All geometric fibers of $Y'^t \to Z'$ have a-number equal to one; hence this subgroup in the generic fiber extends uniquely to such a group scheme $\mathcal{M}' \subset Y'^t$; note $\mathcal{M}' \subset Y'^t[p] \subset \mathcal{B}'^t$. From the exact sequence

$$0 \to \mathcal{M}' \longrightarrow Y'^t \longrightarrow Y'^t/\mathcal{M} \to 0$$

and the duality theorem (see [68] Theorem 19.1, and 1.6.1), we have

$$0 \to \mathcal{N}' := \mathcal{M}'^D \longrightarrow Y_1' := (Y'^t/\mathcal{M}')^t \xrightarrow{\psi} Y'^{tt} = Y' \to 0.$$

After a purely inseparable base change $Z' \to Z$ this $\mathcal{N}'$-cover lifts to an exact sequence

$$0 \to \mathcal{N} \longrightarrow \mathcal{H} \longrightarrow \mathcal{G} \to 0;$$

note there is no $\mathcal{H}'$ but we write $\mathcal{H}$ in order to keep notation coherent. The rest of the construction follows from this.

8.6.2. Isogeny leaves. Using $\mathrm{Ker}(\mu) = \mathcal{K} \to Z$, an α-sheaf of rank 2, we obtain a $\mathbb{P}^1$-fiber bundle $\mathcal{P} \to Z$. Every point of $\mathcal{P}$ corresponds with an embedding of α_p in a fiber of $\mathcal{K} \to Z$ up to an isomorphism on α_p. Note that $\mathcal{P} \to Z$ has two sections, corresponding in every fiber with the two special directions,

$$d_\infty : Z \to \mathcal{P} \quad \text{defined by} \quad \mathcal{Q} \subset \mathcal{K} = \mathcal{Q} \times \mathcal{N},$$

and

$$d_0 : Z \to \mathcal{P} \quad \text{defined by} \quad \mathcal{N} \subset \mathcal{K} = \mathcal{Q} \times \mathcal{N}.$$

Over $\mathcal{P} \to Z$ we have the tautological finite flat group scheme with fibers isomorphic with α_p

$$\mathcal{T} \quad \subset \quad \mathcal{K} \times_Z \mathcal{P} \quad \subset \quad \mathcal{D} \times_Z \mathcal{P}.$$

Note that every fiber of $\mathcal{T}$ is isotropic in that fiber of $\mathcal{K}$. Hence

the polarization $\mu \times_Z \mathcal{P}$ descends to a principal polarization Λ on $\mathcal{D}/\mathcal{T} \to Z$.

The pair $(\mathcal{D}/\mathcal{T}, \Lambda)$ defines the moduli morphism $\Psi : \mathcal{P} \longrightarrow W = W_\xi$.

8.6.3. We observe the following properties:

(1) For any field K and any $z \in Z(K)$ the image of the morphism $\Psi_z : \mathcal{P}_z \to W = W_\xi$ is an irreducible isogeny leaf. The morphism Ψ is surjective.

(2) The composite

$$Z \xrightarrow{d_0} \mathcal{P} \xrightarrow{\Psi} Z$$

is the identity on Z.

(3) The restriction of the morphism Ψ maps

$\mathcal{P} \smallsetminus (d_0(Z) \cup d_\infty(Z))$ surjectively onto $W(a = 1)$; this morphism is bijective and purely inseparable.

(4) For a perfect field K, and a point $z = [(A, \lambda)] \in Z(K)$ the image

$$\Psi(d_\infty(z)) = [(A, \lambda)^{(p^{-1})}] \in Z(K).$$

(5) For any $x \in W(a = 1)$ there is exactly one irreducible isogeny leaf passing through x.

(6) For every $z \in Z$ over $\mathcal{P}_z \cong \mathbb{P}^1$ there exists the slope filtration on $\mathcal{D}/\mathcal{T}$ when restricted to $\mathcal{P}_z \setminus \{\infty\}$; a filtration with slopes in the opposite direction exists on $\mathcal{D}/\mathcal{T}$ when restricted to $\mathcal{P}_z \setminus \{0\}$.

(7) For any $z \in Z(K)$, where K is a perfect field, there are exactly two branches of an irreducible isogeny leaf over K containing z. These two are described in (2) and (4) above and in 8.2, 8.4, 8.5.

(8) One branch $\mathcal{I}_{z,0}$ comes from $\mathcal{P}_z$ attaching via $\Psi(d_0)(z)$. The slope filtration on $\mathcal{G}'_z$ extends to $\mathcal{I}_{z,0}$.

One branch $\mathcal{I}_{z,\infty}$ comes from $\mathcal{P}_{z^{(p)}}$ via $\Psi(d_\infty)(z^{(p)})$. For z defined over a perfect field, on $\mathcal{I}_{z,\infty}$ there is a filtration in the opposite way.

(9) Every local branch is a germ of an isogeny leaf passing through $z = [(A, \lambda)]$. Irreducible components are rational curves. Two local branches come from the same irreducible component if and only if (A, λ) can be descended to $\mathbb{F}_p$.

(10) The branch $\mathcal{I}_{z,0}$ is tangent to $\mathcal{Z}_\xi$. The branch $\mathcal{I}_{z,\infty}$ is transversal to $\mathcal{Z}_\xi$ and to $\mathcal{I}_{z,0}$.

These facts follow from the construction; we explain (4). Over K we have $z = [(A, \lambda)] \in Z(K)$ and $\varphi : Y'_1 \to Y'$. There is an isomorphism over K

$$\varphi' : Y'_1 \cong Y'^{p^{-1}} \longrightarrow Y' \cong G_{1,2}$$

and ψ is up to an isomorphism the Frobenius. Then we take

$$0 \to X' \longrightarrow H' \longrightarrow Y'^{(p^{-1})} \to 0,$$

and in order to obtain $\Psi(d_\infty(z))$ we push out by

$$X' \cong G_{2,1} \longrightarrow X/\alpha_p \cong X'^{(p^{-1})},$$

isomorphic to the Verschiebung. Thus we obtain the conclusion in (4).

8.6.4 As a corollary we see that there is *no quotient morphism $W \to Q'$ where the fibers are exactly connected chains of isogeny leaves*: such a map would identify $z = [(A, \lambda)]$ and $[(A, \lambda)^{(p^{-1})}]$. Fibers of such a quotient map restricted Z would have discrete fibers, but of unbounded cardinality.

9 Some historical remarks

We discuss some aspect of the time line for these topics. They have their roots in earlier questions and results.

9.1. From Gauss to CM liftings. An expectation by Gauss preludes the Weil conjectures, and an attempt by Hasse and results of Deuring led to CM liftings.
Notation. We write RH for the classical Riemann hypothesis, and we write pRH for the analogous conjectures and theorems about zeta functions attached to varieties over finite fields in characteristic p.

9.1.1. Carl Friedrich Gauss considered solutions of equations modulo p; in his "Last Entry" (1814) Gauss discussed (as we would say now) the number of points on an elliptic curve over the prime field of characteristic p, see [20].

> *Observatio per inductionem facta gravissima theoriam residuorum biquadraticorum cum functionibus lemniscaticis elegantissime nectens. Puta, si $a+bi$ est numerus primus, $a-1+bi$ per $2+2i$ divisibilis, multitudo omnium solutionum congruentiae* $1 = xx + yy + xxyy \pmod{a + bi}$ *inclusis* $x = \infty, \quad y = \pm i, \quad x = \pm i, \quad y = \infty$ *fit* $= (a-1)^2 + bb$.

The text of the "Tagebuch" was rediscovered in 1897, and was edited and published by Felix Klein; see [44], with the Last Entry on page 33. A later publication appeared in [20]. For a brief history see [21], page 97. In translation:

> A most important observation made by induction (empirically) which connects the theory of biquadratic residues most elegantly with the lemniscatic functions. Suppose, if $a+bi$ is a prime number, $a-1+bi$ divisible by $2+2i$, then the number of all solutions of the congruence $1 = xx + yy + xxyy \pmod{a + bi}$ including $x = \infty, \quad y = \pm i; \quad x = \pm i, \quad y = \infty$, equals $(a-1)^2 + bb$.

Interesting aspect: in writing his question Gauss writes an equality sign, adding "modulo p" (actually Gauss wrote $\pmod{a+bi}$ which amounts to the same). Felix Klein in his edition [44] of the Tagebuch, "corrected" this to an equivalence sign. In many other instances we see that Gauss knew very well the difference between "=" and "≡"; no correction was necessary. One could give an interpretation of this Tagebuch notation that Gauss (in some sense) considered an object in characteristic p (as we would do no, instead of solutions in characteristic zero, reduced modulo p). We know that this point of view was only slowly accepted in the history of arithmetic algebraic geometry.

For the nonsingular curve E over $\mathbb{F}_p$ with $p \equiv 1 \pmod 4$ defined as the normalization of the completed plane curve given by

$$1 = X^2 + Y^2 + X^2Y^2, \quad \text{Gauss expected that} \quad \#(E(\mathbb{F}_p)) = (a-1)^2 + b^2,$$

where $p = a^2 + b^2$ and $a - 1 \equiv b \pmod 4$. This is the way Gauss wrote his expectation, and the way we would like to express this now. Equivalently:

$$\#\left(\{(x,y) \in (\mathbb{F}_p)^2 \mid x^2 + y^2 + x^2y^2 = 1\}\right) \quad = \quad (a-1)^2 + b^2 - 4; \quad p \equiv 1 \pmod 4.$$

It is clear from his text that Gauss knew very well that the projective plane model of this curve has two points at infinity, and that for $p \equiv 1 \pmod 4$ the nonsingular branches through these nodes are rational over $\mathbb{F}_p$.

This expectation by Gauss (later proved by Herglotz and many others) is a special case of conjectures formulated and proved by E. Artin, Hasse, Weil and many others. For the fascinating story of the Last Entry see [20], [44], [21], [84].

For any field K of $\mathrm{char}(K) \neq 2$ the normalization of a complete model of the affine curve given by $1 = X^2 + Y^2 + X^2Y^2$ is K-isomorphic to the elliptic curve E given by $U^2 = V^3 + 4V$. We see that such a curve is supersingular for all $p \equiv 3 \pmod 4$, and for such cases $E(\mathbb{F}_q) = 1 + q$. For all $p \equiv 1 \pmod 4$ this curve is ordinary, and the number of rational points is given by Gauss (over $\mathbb{F}_p$): over $\mathbb{F}_q$ this determines the number of rational points: $\alpha, \beta = a \pm b\sqrt{-1}$, and $q = p^n$:

$$\#(E(\mathbb{F}_p)) = 1 - 2a + p, \quad \pi = \mathrm{Frob}_{E/\mathbb{F}_p} = a \pm b\sqrt{-1}, \quad \#(E(\mathbb{F}_q)) = 1 - (\alpha^n + \beta^n) + q.$$

For another instance where Gauss approaches such a topic see § 358 of his 1801 "Disquisitiones Arithmeticae" [19], see [51]; for an explanation and references see [97], Theorem 2.2.

9.1.2. Emil Artin in his PhD-thesis (1921/1924) discussed the number of rational points on an elliptic curve in positive characteristic. Forty (40) special cases were computed, and Artin conjectured what the outcome should be. An analogy with the classical Riemann Hypothesis was noted. At that moment many people thought that solving this analogous pRH of the Riemann Hypothesis would be as difficult as proving the classical RH about the classical Riemann zeta function.

9.1.3. Helmut Hasse (and F.K.Schmidt and several others) tried to prove pRH for elliptic curves in positive characteristic in the period of time $1933 - 1937$ (in the language of function fields in characteristic p). In this attempt we see an interesting difference between two approaches. In the first a lift of a special kind of an elliptic curve (we would say now, a CM lift) from characteristic p to characteristic zero was studied; here the difference between an ordinary and a supersingular elliptic curve was noted by Hasse; this proof was not completed by Hasse, as the full liftability result was not proved. In a second approach Hasse basically found the characteristic p approach, proving that the Frobenius endomorphism π of an elliptic curve over a field with q elements satisfies $\overline{\pi}{\cdot}\pi = q$, a profound basis for later proofs of the pRH / the Weil conjectures. The "Frobenius operator" was already constructed by Hasse in 1930.

The duality operation $E \mapsto E^t$ defines the Rosati involution $-^\dagger$ for endomorphisms, which coincides with complex conjugation. For the Frobenius map $F_E = F : E \to E^{(p)}$ and elliptic curves E defined over $\mathbb{F}_q$ one shows that the dual map F^t satisfies $F^t{\cdot}F = p$; e.g. see [12, VII.5]. For $q = p^n$ with $\pi = \pi_{E/\mathbb{F}_q}$ and $\pi^\dagger = \overline{\pi}$ this results in

$$E \cong (E^{(q)})^t \to \cdots \to \left((E^{(p^2)})^t \to \left((E^{(p)})^t \xrightarrow{(F_E)^t} E \xrightarrow{F_E} E^{(p)}\right) \xrightarrow{F_{E^{(p)}}} E^{(p^2)}\right) \to \cdots \to E^{(q)} \cong E$$

$$\overline{\pi}\cdot\pi = \pi^{\dagger}\cdot\pi = F\circ\cdots\circ(F\circ(F\circ F^t)\circ F^t)\circ\cdots\circ F^t = p^n = q.$$

From this we see that

$$\mid\pi\mid = \sqrt{q} \quad \text{and} \quad \#(E(\mathbb{F}_q)) = \#(\mathrm{Ker}(\pi - 1 : \mathrm{E}\to\mathrm{E})) = \mathrm{Norm}(\pi-1) = 1 - (\pi+\overline{\pi}) + q.$$

Here we see a modern proof for the expectation in the Last Entry of Gauss, and a prelude to the Weil conjectures. The argument above is basically contained in the second proof by Hasse.

9.1.4. André Weil discussed this on several occasions with his German colleagues, insisting that one should consider these properties as *geometric aspects* of objects in positive characteristic, and not "only as" algebraic aspects (such as valuations in function fields). For example, see the letter of Weil to Hasse on 17 July 1936, see [86], page 619.

This resulted later (1949) in the formulation of the Weil conjectures (using a hypothetical analogue of the Lefschetz fixed point formula, which was proved much later). An interesting aspect of mathematics: the difference between aspects of *number theory*, as in the German school 1930–1940, and the *geometric approach* to this problem. See [51], [53], [86] for discussions and surveys. We do not discuss later developments in algebraic geometry around the Weil conjectures (Serre, Dwork, Grothendieck, Deligne and many others).

9.1.5. In 1941 Max Deuring took up these questions raised by Hasse, see [10].

The Frobenius $\pi : E \to E$ of an elliptic curve over a finite field can be induced on the space of differentials on E; if this map is non-zero (respectively zero) we say the Hasse invariant of E equals 1 (respectively equals 0). Hasse found that elliptic curves with Hasse invariant zero behave quite differently from those with Hasse invariant equal to one.

In classical language an elliptic curve with complex multiplication over the complex numbers was said to have a *singular j-invariant.* Deuring discussed the case, already noted by Hasse, then denoted by "the Hasse invariant is zero", that even more endomorphisms can be present in positive characteristic, and invented the word *supersingular*, for elliptic curves that have a rank 4 algebra of endomorphisms over $\overline{\mathbb{F}_p}$. The following statements are equivalent:

- The Hasse invariant of E equals zero;
- $\mathrm{rank}_{\mathbb{Z}}\left(\mathrm{End}(E)\otimes\overline{\mathbb{F}_p})\right) = 4.$
- There are no points of order p on $E(\overline{\mathbb{F}_p})$.
- **Definition.** Such an elliptic curve E is called *supersingular.*

Note that a more correct wording here would be "an elliptic curve with supersingular j-invariant". This terminology "supersingular" by Deuring is now applied to abelian varieties in general, see 3.4. Elliptic curves with non-zero Hasse invariant, i.e. non-supersingular, are called *ordinary.* In Deuring's paper we find the full lifting theorem for CM elliptic curves:

every pair (E_0, β_0), with $\beta_0 \in \mathrm{End}(E_0)$, can be lifted to characteristic zero.

For another proof and a generalization see [71], Th. 14.6. This result by Deuring finished the first proof of the pRH by Hasse, although at that time we already had a better of the pRH for elliptic curves. These considerations of Deuring obtained their natural place later in workd og Tate, [98], the Honda-Tate theory [33], [99], and the full theory of CM liftings as completed and surveyed in [2]:

a question raised by Hasse in 1933*, solved by Deuring for elliptic curves, developed into a whole theory of* CM *liftings for abelian varieties, completed* 70 *years later,*

and

theory developed for elliptic curves over a finite field by Deuring (1941)*, the Weil conjectures for abelian varieties and a suggestion by Mumford to Tate gave rise to the theorem* (Tate, 1966) *that any abelian variety over a finite field is a* CM *abelian variety, and to many theorems and conjectures about ℓ-adic representations.*

9.2. Moduli spaces. In 1857, discussing what we now call Riemann surfaces of genus p, Riemann wrote: "*... und die zu ihr behörende Klasse algebraischer Gleichungen von 3p-3 stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen.*" See [94], Section 12. Therefore, we use the word "moduli" as the number of parameters on which deformations of a given geometric object depend.

There were several attempts to build a solid foundation for the concept of a "moduli space". Grothendieck and Mumford both were working on this concept. To many it came as a surprise that such a space for curves of genus 2 over an arbitrary base actually exists, see [36]. For some time it confused mathematicians who were carrying out constructions of "universal objects", under the technical term of "representable functors". In 1961 Samuel wrote in [95]:

> Signalons aussitôt que le travail d'Igusa ne résoud pas, pour les courbes de genre 2, le "problème des modules" tel qu'il a été posé par Grothendieck à diverses reprises dans de Séminaire.

After the difference between "fine moduli schemes" (representing a functor) and "coarse moduli schemes" (where the points correspond to isomorphism classes of the objects considered) was clarified we can work with these concepts, especially thanks to the pioneering work by Mumford in [60]. This enables us to work with moduli spaces in all characteristics, and over arbitrary base rings. These methods are crucial for all considerations in this chapter.

9.3. Manin. In the influential memoir [50], 1963, we see a new field of research opened. Almost all results discussed in that memoir are in some way connected with this present article. Earlier work by Barsotti and Dieudonné find their natural place in this paper. The central consideration of the paper gives a classification of p-divisible groups over an algebraically closed field of positive characteristic, and relations with the theory of abelian varieties in characteristic p. We find here the formulation of the "Manin conjecture": *for every prime number p any symmetric Newton can be realized by an abelian variety in characteristic p* (several proofs of this conjecture were given). In [50] we also find the first step in describing strata where the Newton polygon is constant. We have described in this survey results on stratifications and foliations building on this work by Manin. The way how the Newton polygon strata can fit together under specialization is discussed in the following development.

9.4. A conjecture by Grothendieck. We work in positive characteristic p. Under a specialization of an abelian variety (or of a p-divisible group) the Newton polygon "goes up" as Grothendieck proved [25], page 149, [42], 2.3.2. This relation between Newton polygons that every point of ζ is on or above τ is a partial ordering, written as $\zeta \preceq \tau$.

In 1970 Grothendieck conjectured the converse. In [25], the appendix, we find a letter of Grothendieck to Barsotti, and on page 150 we read: "$\cdots$ *The wishful conjecture I have in mind now is the following: the necessary conditions* $\cdots$ *that G' be a specialization of G are also sufficient. In other words, starting with a BT group $G_0 = G'$, taking its formal modular* *deformation $\cdots$ we want to know if every sequence of rational numbers satisfying $\cdots$ these numbers occur as the sequence of slopes of a fiber of G as some point of S."*

Suppose given a p-divisible group X_0 and Newton polygons $\mathcal{N}(X_0) = \zeta \preceq \tau$.
Does there exist an equi-characteristic deformation $X \to S$ with $0 \in S$ and $\mathcal{N}(X_\eta) = \tau$ realizing $\zeta \preceq \tau$ as the Newton polygons of a special, respectively the generic fiber?

One can also study this question for quasi-polarized p-divisible groups. We will discuss a proof of the unpolarized, and of the principally polarized case below, see 9.7.2.

9.5. CM **abelian varieties.** An abelian variety with "sufficiently many complex multiplications" is called a CM abelian variety. For definitions and properties see [2], Chapter 1. In characteristic zero these are defined over a number field (we think this was observed by Shimura for the first time). As Hasse, Deuring and Tate proved, any abelian variety defined over a finite field is a CM abelian variety, see [98]. Conversely, there are many CM abelian varieties in characteristic p that cannot be defined over a finite field. However, as Grothendieck proved, a CM abelian variety is isogenous with an abelian variety defined over a finite extension of the prime field, see [69], [110].

9.6. Lifting problems. Suppose given an object G_0 in algebraic geometry over a field $\kappa \supset \mathbb{F}_p$; one can think of: an algebraic curve, an algebraic curve with an automorphism, a higher dimensional algebraic variety, a finite group scheme, a polarized abelian variety, a CM abelian variety, a CM Jacobian, and many more situations.

9.6.1. Unobstructed problems. In many cases (local) moduli functors (in mixed characteristic) are (pro)-representable. If one shows moreover that this functor has a characteristic zero fiber, e.g. when the functor is unobstructed or when the functor is flat over $W_\infty(\kappa)$ for a perfect κ, it follows that a lifting is possible.

This approach was successful for complete non-singular curves, see [24], and for principally polarized abelian varieties (Grothendieck), in which cases one shows that the obstructions vanish. As an application Grothendieck determined the structure of the prime-to-p part of the etale fundamental group of a curve in characteristic p, using the characteristic zero result via a lifting to characteristic zero in [24, X.3.10].

An interesting and crucial case is the Serre–Tate lifting theory of ordinary abelian varieties, see [49] (1964), giving rise also to Serre–Tate canonical coordinates in mixed characteristic around the moduli point of an ordinary polarized abelian variety; for an explanation see [43]. These have many applications.

9.6.2. Counter examples. In a second class of situations, where an obstruction to lifting does not (or cannot be shown to) vanish, the moduli problem is not (or may not be) formally smooth, different ideas have to be invented. One can try to construct a counter example to the lifting problem.

In some cases this is easy, e.g. for a curve C with an automorphism group $G = \mathrm{Aut}(G)$ that violates the Hurwitz bound the pair (C, G) cannot be lifted to characteristic zero. Many more examples of a curve with a group of automorphisms can be handled along the same lines.

Serre (1981) gave an example of an algebraic variety that cannot be lifted to characteristic zero, see [96]. Many more examples can be given, e.g. of algebraic surfaces, see the appendix of [85], reproducing a letter by Deligne.

This inspired [87] (1986) were we did find an explicit example of a Jacobian of an ordinary curve where the canonical lifting of its Jacobian is not a Jacobian (at the same time Dwork and Ogus gave a more general approach, with a general answer for $g \geq 4$, but asking for an explicit example; see below).

9.6.3. Infinitesimal mod p^2 methods. Suppose you want to prove or to disprove a certain claim in characteristic zero. Of course, a very old method consists of reducing modulo p; if then you obtain a negative answer in characteristic p, you conclude the answer is negative in characteristic zero.

There is an even more subtle version of this, comparing "infinitesimal deformations modulo p and modulo p^2". In this situation we have a kind of differential calculus, with $p = \epsilon$ and $\epsilon^2 = 0$. We know several instances where this method was successful. Also this idea can be used in lifting problems.

Raynaud's proof of the Manin–Mumford conjecture. The Manin–Mumford conjecture is a fact about curves and abelian varieties over the complex numbers: *for a curve C inside an abelian variety A with* genus$(C) \geq 2$ *the number of torsion points of A situated on C is finite.*

This does not look as a theorem in positive characteristic; for a curve C_0 inside an abelian variety A_0 over the same finite field, every point in $C_0(\overline{\mathbb{F}_p})$ is a torsion point in A_0: *the statement corresponding to the Manin–Mumford conjecture over an algebraically closed field of positive characteristic is not true.* However a beautiful (first) proof of this conjecture in characteristic zero was given by Raynaud in 1983, by the "mod p^2 / mod p" method, see [92]: first reducing to a number field, and then choosing carefully a reduction mod p^2, showing that only finitely many of the torsion points in $C_0(\overline{\mathbb{F}_p})$ have survived as torsion points mod p^2, magic!). For a second (different) proof see [93], and many other publications (Hrushovski, Pink, Roessler).

Dwork–Ogus theory for canonical liftings. The question by Katz whether the canonical lifting of an ordinary Jacobian from positive to zero characteristic is again a Jacobian was studied (for every $g \geq 4$) by Dwork and Ogus (1986). In their paper [11] we find a "mod p^2 / mod p" method showing that outside a smaller closed set of ordinary Jacobians (of genus at least 4) even a canonical lifting to $p^2 = 0$ the canonical lift is not a Jacobian. This method has important consequences and applications, e.g. see [38], [57].

Deligne–Illusie and degeneration of the Hodge spectral sequence. The degeneration of the Hodge spectral sequence for a smooth projective algebraic variety over a field of characteristic zero is a basic fact in algebraic geometry. In [8] we find a beautiful proof (1987) by Deligne and Illusie of this theorem by "mod p^2 / mod p" methods.

9.7. The Mumford method. Suppose you have a lifting problem (or a deformation problem in equi-characteristic) which is not formally smooth, how do you try to show that however a lifting actually exist? In other words: there is a general theory, but that does not provide either a negative nor a positive answer to your lifting problem; how do you proceed? We explain a method, initiated by David Mumford, in the special case it was used as suggested by Mumford.

9.7.1. Theorem (Mumford, Norman–Oort). *Any polarized abelian variety (A_0, λ_0) over $\kappa \supset \mathbb{F}_p$ can be lifted to characteristic zero.* See [63], [62], [109].

In the Mumford method you first deform the situation in equi-characteristic-p to a "better"

situation (usually, a non-canonical process), and you hope that the first step arrives at a situation where you know a lifting does exist. In the case at hand we know that an ordinary polarized abelian variety can be lifted (by the Serre–Tate theory). In order to study the situation we consider the Newton polygon of A_0. A quite non-trivial computation shows that you can deform (A_0, λ_0) to a new polarized abelian variety with a strictly lower Newton polygon, see [63], Lemma 3.3 for details. After a finite number of such deformations you arrive at an ordinary polarized abelian variety (B_0, μ_0); this finishes the first (computational, non-canonical) *first step*. Then, Serre–Tate theory tells you that (B_0, μ_0) can be lifted to characteristic, and this *second step* finishes the proof.
Remark. The computation showing the deformation as in the first step uses the *theory of displays*, invented by Mumford for this situation. It enables you to write down explicitly in one stroke a deformation (of a p-divisible group). This theory was further developed in [115].

9.7.2. A conjecture by Grothendieck: see 9.4.
Theorem. *For p-divisible groups, for principally quasi-polarized p-divisible groups and for principally polarized abelian varieties this conjecture by Grothendieck holds.* See [75], [39], [77].
Remark. There are many examples showing that this result is not correct in general for non-principal polarizations. In [41] we find an example of non-principally polarized supersingular 3-folds, that cannot be deformed, as polarized abelian varieties, to abelian varieties with Newton polygon $\mathrm{NP}((2,1),(1,2))$. A way of producing examples where the analogue of the Grothendieck conjecture does not hold for polarized abelian varieties and for quasi-polarized p-divisible groups has been described in [82]; see 4.16.3.

The proofs follow the Mumford method (as we realized much later). We define

$$a(G) := \dim_\kappa \mathrm{Hom}(\alpha_p, G),$$

where κ is a perfect field, and G a commutative group scheme over κ. The crucial case in the theorem is that of (unpolarized) p-divisible groups, and $a(X) > 0$.

Also see 9.14, especially [90], for another application of the Mumford method.

9.8. Methods: deformations to $a \leq 1$.
Theorem (the "Purity theorem"). *If in a family of p-divisible groups (say, over an irreducible scheme) the Newton polygon jumps, then it already jumps in codimension one.*
See [39] Th. 4.1; several other proofs have been given later. This very non-trivial result will be one of the main tools.

9.9. Catalogues. Let us fix a prime number p and coprime positive integers m, n. We try to "classify" all p-divisible groups isogenous with $G_{m,n}$.

In general there is no good theory of (global) moduli spaces for p-divisible groups (and there are various ways to remedy this). We use the (new) notion of a "catalogue": a family where every object studied up to geometric isomorphism appears at least once; note that a catalogue is far from unique (e.g. pulling back by any surjective morphism to the base gives again a catalogue). In our case this is a family $\mathcal{G} \to S$, i.e. a p-divisible group over some base scheme S, such that every isogenous $G \sim G_{m,n}$ defined over an algebraically closed field appears as at least one geometric fiber in $G \to S$. You can rightfully complain that this is a rather vague notion, that a catalogue is not unique (e.g. the pull back by a surjective morphism again is a catalogue), etc. However this notion has some advantages:

Theorem (catalogues). *Suppose given p, m, n as above. There exists a catalogue $\mathcal{G} \to T$ over $\mathbb{F}_p$ for the collection of p-divisible groups isogenous with $G_{m,n}$ such that T is geometrically irreducible.* See [39], Theorem 5.11.

9.10. Theorem. *Suppose G_0 is a p-divisible group; there exists a deformation to G_η such that*

$$\mathcal{N}(G_0) = \mathcal{N}(G_\eta) \quad \textit{and} \quad a(G_\eta) \leq 1.$$

E.g. see [77], 3.10. We sketch a proof of the theorem on catalogues, using the Purity Theorem, see [39]. We write $H = H_{m,n}$, and let $r := (m-1)(n-1)/2$. We see that for every p-divisible group G isogenous to $G_{m,n}$ there exists an isogeny $\varphi : H \to G$ of degree exactly $\deg(\varphi) = p^r$, see [39], 5.8. We construct $\mathcal{G} \to T$ as the representing object of isogenies $\varphi : H \times S \to G/S$ of this degree (it is easy to see that such a functor is representable).

Using this definition we see that the formal completion at $[(G_0, \varphi)] = s \in T$ embeds in $\mathrm{Def}(G_0)$, i.e. $T^{/s} \hookrightarrow \mathrm{Def}(G_0)$. Furthermore we compute the longest chain of Newton polygons between $\mathcal{N}(G_{m,n})$ and the ordinary one: this equals $mn - r$ (an easy combinatorial fact). From these two properties, using 9.8, we deduce: *every component of T has dimension at least r.*

We make a stratification of T (using combinatorial data, such a thing like "semi-modules"). We show (using explicit equations) that every stratum is geometrically irreducible, and that there is one stratum, characterized by $a(G) = 1$, of dimension r, and *that all other strata have dimension less than r.* These considerations do not contain deep arguments, but the proofs are rather lengthy and complicated.

From these two aspects the proof follows: any component of T on which generically we would have $a > 1$ would have dimension strictly less than r, which contradicts "Purity". Hence the locus where $a = 1$ is dense in T, and we see that T is geometrically irreducible.

9.11. We sketch a proof of 9.7.2, see [77] for details. By 9.10 we conclude the deformation property 9.7.2 for simple p-divisible groups. Then we study groups filtered by simple subfactors, and deformation theory of such objects. By the previous result we can achieve a deformation where all simple subfactors are deformed within the isogeny class to $a \leq 1$. Then we write down an explicit deformation ("making extensions between simple subfactors non-trivial") in order to achieve $a(G_\eta) \leq 1$, see [77], Section 2 for details. This finishes the first step in the proof of 9.7.2.

Remark. This method of catalogues for p-divisible groups works fine for simple p-divisible groups. However the use of "catalogues" for non-isoclinic groups does not seem to give what we want; it is even not clear that nice catalogues exist in general. Note that we took isogenies of the form $\varphi : H \times S \to G/S$; however over a global base scheme monodromy groups need not be trivial, and this obstructs the existence of one catalogue which works in all cases (to be considered in further publications).

9.12. Methods: Cayley–Hamilton This is taken entirely from [75]. In general it is difficult to read off from a description of a p-divisible group (e.g. by its Dieudonné module) its Newton polygon. However in the particular case that its a-number is at most one this can be done. This we describe here. The marvel is a new idea which produces for a given element in a given Dieudonné module a polynomial (in constants and in F) which annihilates this element (but, in general, is does not annihilate other elements of the Dieudonné module). This idea for constructing this polynomial comes from the elementary theorem in linear algebra: every endomorphism of a vector spaces is annihilated by its characteristic polynomial. As we work in our case with an operator which does not commute with constants, things are not that elementary. The method we propose works for $a(G_0) = 1$, but it breaks down if $a(G_0) > 1$ in an essential way.

9.13. Theorem (of Cayley–Hamilton type). *Let G_0 be a p-divisible group over an algebraically closed field $k \supset \mathbb{F}_p$ with $a(G_0) \leq 1$. In $\mathcal{D} = \mathrm{Def}(G_0)$ there exists a coordinate system $\{t_j \mid j \in \diamond(\rho)\}$ and an isomorphism $\mathcal{D} \cong \mathrm{Spf}(k[[t_j \mid j \in \diamond(\rho)]])$ such that for any $\gamma \succeq \mathcal{N}(G_0)$ we have*

$$\mathcal{W}_\gamma(\mathcal{D}) = \mathrm{Spf}(R_\gamma), \quad \textit{with} \quad R_\gamma := k[[t_j \mid j \in \diamond(\gamma)]] = k[[t_j \mid j \in \diamond(\rho)]]/(t_j \mid j \notin \diamond(\gamma)).$$

See 4.9.2 for the definition of the finite sets $\diamond(\gamma)$. Recall also that ρ is the ordinary Newton polygon with $\rho \succeq \mathrm{NP}(G_0)$.

9.13.1. Corollary. *Let G_0 be a p-divisible group over a field K with $a(G_0) \leq 1$. In* $\mathrm{Def}(G_0)$ *every Newton polygon $\gamma \succeq \mathcal{N}(G_0)$ is realized.*

These methods allow us to give a **proof for the Grothendieck conjecture**. In fact, starting with G_0 we use 9.10 in order to obtain a deformation to a p-divisible group with the same Newton polygon and with $a \leq 1$ (the first step in the Mumford method). For that group the method 9.12 of Cayley-Hamilton type can be applied, which shows that it can be deformed to a p-divisible group with a given lower Newton polygon (the second step in the Mumford method). Combination of these two specializations shows that the Grothendieck conjecture 9.7.2 is proven. $\square$

9.14. Lifting an automorphism of an algebraic curve. In [71], Section 7 and in [74], Section 4 we see the

Conjecture. *A pair (C_0, φ_0) of an algebraic curve C and $\varphi_0 \in \mathrm{Aut}(C_0)$ can be lifted to characteristic zero*

for any (C_0, φ_0) over a field of characteristic $p > 0$. In general (for automorphisms of order divisible by p) this problem is obstructed; the general moduli approach does not give an answer, but also no counter example was known. After proofs for special cases, and several attempts, the final result is a proof for this conjecture, see [64] and [90]; see the survey paper [65] for details. We just mention that the proof for the "favorable" situation, needed for the *first step*, is contained in [64] (here finding the good condition, and the proof that lifting does work in such cases is far from obvious). The deformation argument needed in the *second step* of the Mumford method is contained in [90] (also a non-trivial step).

Some notations used in this paper.
We write k for an algebraically closed field, κ for a field in characteristic p;
for a perfect κ we write $\Lambda = \Lambda_\infty(\kappa)$ for the ring of infinite Witt vectors.
We write $\mathcal{A}_g$ either for $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$, or for $\cup_d \mathcal{A}_{g,d} \otimes \mathbb{F}_p$. However in Section 7 and in the beginning of Section 8 moduli spaces in mixed characteristic are used.
For $\zeta_1 \preceq \zeta_2$ see 1.1.

For an algebraic scheme W over a field K we write $\Pi_0(W)$ for the set of irreducible components of $W \otimes \overline{K}$.

Notations $\mathcal{W}_\zeta(-)$, W_ζ, $W_{\zeta,d}$, etc. will be used in the theory of Newton polygon strata.
We use $\mathcal{C}(x)$ for the central leaf passing through x, and $\mathcal{Z}_\xi$ for the central stream in $W_\xi \subset \mathcal{A}_{g,1}$. For a given symmetric Newton polygon ξ the dimension of every central leaf in $\mathcal{A}_g$ does not depend on the leaf chosen (arbitrary degree of polarization); this dimension is denoted by $c(\xi)$. For ξ fixed the dimension of the related Newton polygon W_ξ stratum inside $\mathcal{A}_{g,1}$ is denoted by $\mathrm{sdim}(\xi)$; note that other components of $\mathcal{W}_\xi(\mathcal{A}_g)$ (non-principally polarized) can have different dimensions. We write $i(\xi) = \mathrm{sdim}(\xi) - c(\xi)$; any isogeny leaf inside $\mathcal{A}_{g,1}$ has this dimension; other isogeny leaves in $\mathcal{A}_g$ can have a different dimension. For more information on definitions and on properties, see Section 4. We will also use the notation $c(\xi) = \mathrm{cdp}(\xi)$ and $i(\xi) = \mathrm{idpp}(\xi)$.
For $\triangle(-)$, $\diamond(-)$, $\mathrm{sdim}(-)$, $c(\xi) = \mathrm{cdp}(\xi)$, $i(\xi) = \mathrm{idpp}(\xi)$, $\mathrm{minsd}(-)$, $\mathrm{maxsd}(-)$, etc., see Section 4.

In Section 5 we defined the notion of an elementary sequence φ and the related EO stratum $S_\varphi \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$. One can define EO strata for other components of $\mathcal{A}_g$, for other moduli spaces; however these are not considered here.
For a p-divisible group X we will say that $X[p^n]$ is a BT_n, an n-truncated Barsotti-Tate group.
For an abelian variety A over a field $\kappa \supset \mathbb{F}_p$ we write $f(A)$, the p-rank of A, for the integer with the property that $A[p](k) \cong (\mathbb{Z}/p)^{f(A)}$, where k is any algebraically closed field containing κ. If $f(A) = \dim(A)$ we say A is ordinary; if $f(A) = \dim(A) - 1$ we say A is almost ordinary.

Let f be an integer with $0 \le f \le g$; we write $\mathcal{V}_f(\mathcal{A}_g)$ for the (Zariski closed) locus of polarized abelian varieties of p-rank at most f.

10 Some Questions.

10.1. Is there a good functorial (scheme-theoretic) definition of Newton polygon strata? If so, which Newton polygon strata are reduced?

10.2. Suppose given a perfect field $\kappa \supset \mathbb{F}_p$, and an algebraic curve C_0 with an automorphism φ_0 over κ. Is there a lifting of (C_0, φ_0) to a *normal* mixed characteristic domain with residue class field κ? We do not know a proof, nor a counter example.

We note that $(C_0, \varphi_0) \otimes k$ admits a lifting to a mixed characteristic discrete valuation ring with residue class field k, where k is an algebraically closed field containing κ.

10.3. Suppose given a symmetric Newton polygon ξ. *How can one decide which* EO-*strata have a non-empty intersection with* W_ξ ? Partial answers are known.

If an elementary sequence φ belongs to an abelian variety with $a = 1$, then φ appears on every NP-stratum with that p-rank.

If φ belongs to an abelian variety where the p-divisible group is minimal, then φ appears on precisely one NP-stratum and it gives the central stream for that Newton polygon stratum.

In other cases we know some restrictions and partial results.

10.4. What is the boundary of a central leaf? *Suppose $\xi' \subsetneqq \xi$ and let $C \subset W_\xi$ be a central leaf. Try to describe $C^{\mathrm{Zar}} \cap W_{\xi'}$.*

See [28], [105] and see Question 13 by S. Harashita in the Appendix: Questions in Arithmetic Algebraic Geometry in this book.

10.5. Quotients with fibers central leaves. Consider the open Newton polygon stratum $W_\xi \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ for some symmetric Newton Polygon ξ of height $2g$.
Expectation. *There exists a surjective morphism*

$$g : W_\xi \longrightarrow Q$$

to some scheme Q such that the fibers are exactly the central leaves in W_ξ.

Is there a functorial definition of this quotient morphism? Do we obtain fibers with good scheme structure as given by the sustained techniques?

10.6. Quotients with isogeny leaves as fibers. Consider a symmetric Newton polygon ξ. We expect in general there is no quotient map of W_ξ contracting exactly isogeny leaves; compare with the special case 8.6.4.

10.6.1. Question. Let $I^0 \subset W_\xi(a=1)$ be the open subscheme of an irreducible isogeny leaf I where $a=1$, and let $x \in I^0$ be a point of I^0.

$$\text{What is the intersection } I^0 \cap \mathcal{C}(x) \text{ set-theoretically?}$$

10.6.2. Question. Suppose $I_1^0, I_2^0 \subset W_\xi(a=1)$ are the open part of different irreducible isogeny leaves where $a=1$. Is it true that

$$I_1^0 \cap I_2^0 = \emptyset \quad ?$$

10.6.3. Question. Does there exist a quotient morphism $W_\xi(a=1) \longrightarrow Q'$ where the fibers are exactly open parts in irreducible isogeny leaves?

10.7. Expectation. Consider a symmetric Newton polygon ξ. Suppose ξ is not the ordinary and not the almost ordinary Newton polygon. We know (working on $\mathcal{A}_{g,1,n}$ for sufficiently high level structure), considering the non-smooth locus $\mathrm{Sing}(-)$, that

$$\mathrm{Sing}(W_\xi) \subset W_\xi(a>1); \quad \text{is it true that} \quad \mathrm{Sing}(W_\xi) \stackrel{?}{=} W_\xi(a>1) \ ?$$

Is

$$W_\xi(a>1) \stackrel{?}{=} (W_\xi(a=2))^{\mathrm{Zar}}?$$

10.7.1. Expectation. Is it true that

$$W_\xi(a>1) \quad \text{is geometrically irreducible for} \quad \sigma \precneqq \varphi?$$

Remark. For $n \in \mathbb{Z}_{\geq 2}$ and $\xi = \mathrm{NP}((n,1)+(1,n))$ this is indeed the case, as in this case $\mathrm{Sing}(W_\xi) = \mathcal{Z}_\xi$ and arguments analogous as in 8.6 can be used to prove this fact.

Remark. Results by Harashita show indeed that for the supersingular locus (for $g>1$) we have:

$$(W_\sigma(a=2))^{\mathrm{Zar}} = W_\sigma(a \geq 2),$$

and $(W_\sigma(a=2))^{\mathrm{Zar}}$ is connected and of codimension one in W_σ; see [26] for this and more general results on strata in W_σ given by the a-number; also see [27], [29], [30] for more information on EO strata in Newton polygon strata.

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