MEAN FIELD EQUATIONS, HYPERELLIPTIC CURVES AND MODULAR FORMS: I

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ABSTRACT. We develop a theory connecting the following three areas: (a) the mean field equation (MFE)

$$\triangle u + e^u = \rho\, \delta_0, \qquad \rho \in \mathbb{R}_{>0}$$

on flat tori $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, (b) the classical Lamé equations and (c) modular forms. A major theme in part I is a classification of developing maps f attached to solutions u of the mean field equation according to the type of transformation laws (or monodromy) with respect to Λ satisfied by f.

We are especially interested in the case when the parameter ρ in the mean field equation is an integer multiple of 4π. In the case when $\rho = 4\pi(2n+1)$ for a non-negative integer n, we prove that the number of solutions is $n+1$ except for a finite number of conformal isomorphism classes of flat tori, and we give a family of polynomials which characterizes the developing maps for solutions of mean field equations through the configuration of their zeros and poles. Modular forms appear naturally already in the simplest situation when $\rho = 4\pi$

In the case when $\rho = 8\pi n$ for a positive integer n, the solvability of the MFE depends on the *moduli* of the flat tori E_τ and leads naturally to a hyperelliptic curve $\bar{X}_n = \bar{X}_n(\tau)$ arising from the Hermite-Halphen ansatz solutions of Lamé's differential equation

$$\frac{d^2 w}{dz^2} - (n(n+1)\wp(z;\Lambda_\tau) + B)w = 0.$$

We analyse the curve $\bar{X}_n$ from both the analytic and the algebraic perspective, including its local coordinate near the point at infinity, which turns out to be a smooth point of $\bar{X}_n$. We also specify the role of the branch points of the hyperelliptic projection $\bar{X}_n \to \mathbb{P}^1$ when the parameter ρ varies in a neighborhood of $\rho = 8\pi n$. In part II, we study a "premodular form" $Z_n(\sigma;\tau)$, a real-analytic function in two variables associated to $\bar{X}_n(\tau)$, which has many symmetries and also the property that the τ-coordinates of zeros of $Z_n(\sigma;\tau)$ correspond exactly to those flat tori where the MFE with parameter $\rho = 8\pi n$ has a solution.

CONTENTS

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0. Introduction

0.1. How to study the *geometry* of a flat torus $E = E_\Lambda := \mathbb{C}/\Lambda$? There are at least two seemingly different approaches to this problem. In the first approach one studies the *Green's function* $G = G_E$ of E, characterized by

$$\begin{cases} -\triangle G = \delta_0 - \dfrac{1}{|E|} \quad \text{on } E, \\ \int_E G = 0, \end{cases} \tag{0.1.1}$$

where the Laplacian $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\partial_z \partial_{\bar z}$ on E is induced by the Laplacian of the covering space $\mathbb{C}$ of E, $|E| = \int_E dx\,dy = \frac{\sqrt{-1}}{2}\int_{\mathbb{C}/\Lambda} dz \wedge d\bar z$ is the area of E, δ_0 is the Dirac delta measure at the zero point $[0] = 0 \bmod \Lambda \in E$, and we have identified functions on E with measures on E using the Haar measure $dx\,dy = \frac{1}{2}\,|dz d\bar z|$ on E.[1] In the second approach one studies the classical *Lamé equation*

$$L_{\eta,B}\, w := w'' - (\eta(\eta+1)\wp(z) + B)w = 0 \tag{0.1.2}$$

with parameters[2] $\eta \in \mathbb{R}_{>0}$ and $B \in \mathbb{C}$, where $\wp(z) = \wp(z;\Lambda)$ is the Weierstrass elliptic function

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{w \in \Lambda \smallsetminus \{0\}} \Big(\frac{1}{(z-w)^2} - \frac{1}{w^2} \Big), \quad z \in \mathbb{C}.$$

Throughout this paper, we denote by ω_1, ω_2 a $\mathbb{Z}$-basis of the lattice Λ, $\tau = \omega_2/\omega_1$ with $\mathrm{Im}(\tau) > 0$, and $\omega_3 = -\omega_1 - \omega_2$.

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[1] So the first equation in (0.1.1) means that

$$-\int_E G \cdot \triangle(f)\, dx\,dy = f(0) - |E|^{-1} \cdot \int_E f\, dx\,dy$$

for all smooth functions $f \in C^\infty(E)$.

[2] We imposed the condition $\eta > 0$ on the real parameter η of the Lamé equation $L_{\eta,B}$ so that these Lamé equations are related to the mean field equations. The parameter η are positive integers in classical literature such as [27, 67].

0.1.1. The Green's function is closely related to the so-called *concentration phenomenon* of some non-linear elliptic partial differential equations in two dimensions. For example, consider the following *singular Liouville equation* with parameter $\rho \in \mathbb{R}_{>0}$[3]

$$\triangle u + e^u = \rho \cdot \delta_0 \quad \text{on } E. \tag{0.1.3}$$

It is proved in [12] that for a sequence of blow-up solutions u_k of (0.1.3) corresponding to $\rho = \rho_k$ with $\rho_k \to 8\pi n$, $n \in \mathbb{N}$, the set $\{p_i, \ldots, p_n\}$ of blow-up points satisfies the following equations:

$$n\frac{\partial G}{\partial z}(p_i) = \sum_{1 \le j \le n,\, j \ne i} \frac{\partial G}{\partial z}(p_i - p_j) \qquad \forall i = 1, \ldots, n. \tag{0.1.4}$$

For $n = 1$, the blow-up set consists of only one point p which by (0.1.4) is a critical point of G:

$$\frac{\partial G}{\partial z}(p) = 0. \tag{0.1.5}$$

Such a connection of (0.1.4) with the Green's function also appears in many gauge field theories in physics. The well-known examples are the Chern–Simons–Higgs equation for the abelian case, and the $\mathrm{SU}(m)$ Toda system for the non-abelian case. See [45, 51, 52] and references therein.

This connection leads to the following question: How many solutions might the system (0.1.4) have? Or an even more basic question:

How many critical points might the Green function G have?

Surprisingly, this problem has never been answered until [43], where the second and the third authors proved the following result.

Theorem A. *For any flat torus E, the Green function G has either three or five critical points.*

The statement of Theorem A looks deceptively simple at first sight. However its proof uses the non-linear PDE (0.1.3) and is not elementary.

0.1.2. In view of Theorem A it is natural to study the system (0.1.4) and the degeneracy question related to each solution of it. We will see in a moment that such an investigation leads naturally to a fundamental hyperelliptic curve $\bar{X}_n$ (which varies with n and E), and requires us to study its geometry—especially the branch points for the hyperelliptic structural map $\bar{X}_n \to \mathbb{P}^1(\mathbb{C})$. The precise definition of $\bar{X}_n$ will later be given in (0.6.6).

In the literature there are at least two situations in which one encounters this hyperelliptic curve $\bar{X}_n$. Both of them are related to the Lamé equation (0.1.2) with $\eta \in \mathbb{N}$. One of them is well-known, namely the spectral curve of the Lamé equation in the KdV theory. In §7 we will prove that this spectral curve is identical to the hyperelliptic curve $\bar{X}_n$; see Remark 7.4.3.

[3] Equation (0.1.3) with parameter $\rho < 0$ is not very interesting—it becomes too easy.

The second situation has a more algebraic flavor and is perhaps less known to the analysis community. Let

$$p(x) = 4x^3 - g_2(\Lambda)x - g_3(\Lambda).$$

The differential equation

$$p(x)\frac{d^2y}{dx^2} + \frac{1}{2}p'(x)\frac{dy}{dx} - \big(\eta(\eta+1)x + B\big)y = 0 \tag{0.1.6}$$

on $\mathbb{P}^1(\mathbb{C})$ is related to (0.1.2) by the change of variable $x = \wp(z;\Lambda)$, where $p'(x) = \frac{d}{dx}p(x)$. People have been interested in describing those parameters B so that the Lamé equation (0.1.6) has algebraic solutions only, or equivalently the global monodromy group of (0.1.6) is *finite*. This question seems not to have been fully solved, though significant progress had been achieved and there are algorithms to generate all cases; see [4, 65, 20] and references therein.

0.1.3. A related and even more classical question is:

> *When is the global monodromy group of the Lamé equation* (0.1.6) *reducible?*

That is, there is a one dimensional subspace $\mathbb{C}\cdot y_1(x)$ of the space of local solutions which is stable under the action of the fundamental group

$$\pi_1\big(\mathbb{P}^1(\mathbb{C}) \smallsetminus \{\wp(z_1;\Lambda) \mid z_1 \in \tfrac{1}{2}\Lambda/\Lambda\}\big)$$

of the complement of the 4 singular points of (0.1.6) on $\mathbb{P}^1(\mathbb{C})$. This question has a fairly complete answer:

(i) It is known that if the global monodromy group is reducible then the parameter η of the Lamé equation is an integer and the monodromy group is infinite.
(ii) If $\mathbb{C}\cdot y_1(x)$ is a one-dimensional space of solutions of (0.1.6), then $y_1(x)$ is an algebraic function in x. Moreover solutions which are not multiples of $y_1(x)$ are not algebraic in x, and the action of the monodromy group of (0.1.6) is not completely reducible (as a two-dimensional linear representation of the fundamental group).

See [65, §4.4] and [4, §3]. A solution of (0.1.2) of the form $y_1\big(\wp(z;\Lambda)\big)$ with $\eta = n$ is traditionally known as a *Lamé function*.[4]

[4]When n is even, $1 + n/2$ of the $2n+1$ Lamé functions are polynomials of degree $n/2$ in $\wp(z)$, and $3n/2$ of the form $\sqrt{(\wp(z)-e_i)(\wp(z)-e_j)}\cdot Q(\wp(z))$ for some polynomial $Q(x)$ of degree $(n/2)-1$ and some $i \neq j$ with $i, j = 1, 2$ or 3, where $e_i := \wp(\omega_i/2)$ for $i = 1, 2$ or 3.

When n is odd, $3(n+1)/2$ of the $2n+1$ Lamé functions are of the form $\sqrt{\wp(z)-e_i}\cdot Q(\wp(z))$ for some polynomial $Q(x)$ of degree $(n-1)/2$ and $i = 1, 2$ or 3. The rest $(n-1)/2$ Lamé functions are of the form $\wp'(z)\cdot Q(\wp(z)) = \sqrt{(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)}\cdot Q(\wp(z))$ for some polynomial $Q(x)$ of degree $(n-3)/2$.

0.1.4. For each fixed torus E and each $n \in \mathbb{Z}_{>0}$, those parameters B such that the monodromy group of the Lamé equation $L_{n,B}$ is reducible are characterized by the classical Theorem B below.

Theorem B. [27, 67, 53] *Suppose that $\eta = n \in \mathbb{N}$. Then there is a polynomial $\ell_n(B)$ of degree $2n+1$ in B such that equation* (0.1.2) *has a Lamé function as its solution if and only if $\ell_n(B) = 0$.*

It turns out that the algebraic curve

$$\{(B, C) \mid C^2 = \ell_n(B)\}$$

is identical to "the affine part" Y_n of a complete hyperelliptic curve $\bar{X}_n$, to be defined later in (0.5.2). Such an identification was only implicitly stated in Halphen's classic [27]. In this paper we will give a detailed and rigorous proof of the statement; see Theorem 0.7. See also [27, Ch. 12], [67, Ch. 23] and [53, Ch. 9] for traditional treatments of the Lamé equation.

0.1.5. The main theme of this paper is to explore the connection between the Liouville equation (0.1.3) and the Lamé equation (0.1.2) when the parameter ρ in (0.1.3) and the parameter η in (0.1.2) satisfies the linear relation $\eta = \rho/8\pi$.[5] Equation (0.1.3) has its origin in the prescribed curvature problem in conformal geometry. In general, for any compact Riemann surface (M, g) we may consider the following equation

$$\triangle u + e^u - 2K = 4\pi \sum_{j=1}^{n} \alpha_j \, \delta_{Q_j} \quad \text{on } M, \tag{0.1.7}$$

where $K = K(x)$ is the Gaussian curvature of the given metric g at $x \in M$, $Q_j \in M$ are distinct points, and $\alpha_j > -1$ are constants.[6] For any solution $u(x)$ to (0.1.7), the new metric

$$\tilde{g} := \frac{1}{2} e^u \cdot g$$

has constant Gaussian curvature $\tilde{K} = 1$ outside those Q_j's. Since (0.1.7) has singular sources at the Q_j's, the metric $\frac{1}{2} e^u g$ may degenerate at Q_j for each j and is called a metric on M with *conic singularities* at the points Q_j's.

Digression. There is also an application of (0.1.3) to the complex Monge-Ampère equation:

$$\det\Big(\frac{\partial^2 w}{\partial z_i \partial \bar{z}_j}\Big)_{1 \le i,j \le d} = e^{-w} \quad \text{on } (E \smallsetminus \{0\})^d, \tag{0.1.8}$$

where $(E \smallsetminus \{0\})^d$ is the d-th Cartesian product of $E \smallsetminus \{0\}$. Obviously, for any solution u to (0.1.3), the function

$$w(z_1, \ldots, z_d) = -\textstyle\sum_{i=1}^{d} u(z_i) + d \cdot \log 4$$

[5] In this article $2\eta = \rho/4\pi \in \mathbb{N}$ most of the time.

[6] The points Q_j and the constants α_j are regarded as parameters of (0.1.7)).

satisfies (0.1.8) with a logarithmic singularity along the normal crossing divisor $D = E^d \smallsetminus (E \smallsetminus \{0\})^d$. In particular, bubbling solutions to (0.1.3) will give examples of bubbling solutions to the complex Monge-Ampère equation (0.1.8), whose bubbling behavior could be understood from our theory developed in this paper. Those examples might be useful for studying the geometry related to the degenerate complex Monge–Ampère equations.

0.1.6. Equation (0.1.7) is a special case[7] of a general class of equations, called *mean field equations*:

$$\triangle u + \rho \left(\frac{he^u}{\int he^u} - \frac{1}{|M|} \right) = 4\pi \sum_{j=1}^{n} \alpha_j \left(\delta_{Q_j} - \frac{1}{|M|} \right) \quad \text{on } M, \tag{0.1.9}$$

where $h(x)$ is a *positive* C^1-function on M and ρ is a *positive* real number. Equation (0.1.9) arises not only from geometry, but also from many applications in physics. For example it appears in statistical physics as the equation for the *mean field limit* of the Euler flow in Onsager's vortex model, hence its name. Recently the equation (0.1.9) was shown to be related to the self-dual condensation of the Chern–Simons–Higgs model. We refer the readers to [9, 17, 45, 46, 51, 52] and references therein for recent developments on this subject.

Equation (0.1.9) has been studied extensively for over three decades. It can be proved that outside a countable set of critical parameters ρ, solutions u of (0.1.3) have *uniform a priori* bounds in $C^2_{loc}(M \smallsetminus \{Q_1, \ldots, Q_n\})$:

> For any closed interval I not containing any of the critical parameters and any compact subset $\Phi \subset M \smallsetminus \{Q_1, \ldots, Q_n\}$, there exists a constant $C_{I,\Phi}$ such that $|u(z)| \leq C_{I,\Phi}$ for all $z \in \Phi$ and every solution $u(z)$ of (0.1.3) with parameter $\rho \in I$;

see [5, 12, 14, 41].

The existence of uniform a priori bounds for solutions of (0.1.3) implies that the topological Leray–Schauder degree d_ρ is well-defined when ρ is a *non-critical* parameter. Recently, an explicit degree counting formula has been proved in [13, 15], which has the following consequence:

> Suppose that $\rho \in (\mathbb{R}_{>0} \smallsetminus 8\pi\mathbb{N})$, $\alpha_j \in \mathbb{N}$ for all j and the genus $g(M)$ of M is at least 1. Then $d_\rho > 0$, hence the mean field equation (0.1.9) has a solution. [8]

However when $\rho \in 8\pi\mathbb{N}_{>0}$, *a priori* bounds for solutions of (0.1.9) might not exist, and the existence of solutions becomes an intricate question. The singular Liouville equation (0.1.3) on flat tori with $\rho \in 8\pi\mathbb{N}$ is the simplest class of mean field equations where the parameter ρ is *critical*, and the existence problem for equation (0.1.3) is already a delicate one in the case when

[7] namely the case when h is the constant function 1, and $\rho = 4\pi \sum_{j=}^{n} \alpha_j$.

[8] For any natural number $m \in \mathbb{N}$, the Leray-Schauder degree d_ρ is constant in the open interval $(8\pi, 8\pi(m+1))$ by homotopy invariance of topological degree, and $d_\rho = m + 1$ in this open interval according to [15, Thm. 1.3].

$\rho = 8\pi$. In [43] the second and third authors proved that equation (0.1.3) has a solution if and only if the Green's function on the torus has *five* critical points; c.f. Theorem A in 0.1.1.

0.1.7. In this paper we will consider the case when the parameter ρ is of the form $\rho = 4\pi l$ for some positive integer $l \in \mathbb{N}$. We note that if l is odd, $\rho = 4\pi l$ is *not* a critical parameter. In this case the degree counting formula in [15, 16] gives the following result:

Theorem C. *Suppose that $l = 2n+1$ is a positive odd integer. Then the Leray-Schauder degree $d_{4\pi l}$ of* of equation (0.1.3) *is* $\frac{1}{2}(l+1) = n+1$.

Theorem C will be sharpened in corollaries 0.4.2 and 3.5.1 to:

> *Let $\rho = 4\pi(2n+1)$ for some $n \in \mathbb{Z}_{\geq 0}$. Except for a finite number of tori up to isomorphism, equation* (0.1.3) *has exactly $n+1$ solutions.*

0.2. The above sharpening of Theorem C will be established via the connection between the Liouville equation (0.1.3) and the Lamé equation (0.1.2). Indeed the equation (0.1.3) is *locally completely integrable* according to the following theorem of Liouville:

> *Any solution u to* (0.1.3) *can be expressed locally on $E \smallsetminus \{0\}$ as*

$$u(z) = \log \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2} \qquad \forall z \in E \smallsetminus \{0\}, \tag{0.2.1}$$

> *where $f(z)$ is a multi-valued meromorphic function on $\mathbb{C} \smallsetminus \Lambda$ (i.e. a meromorphic function on an unramified covering of $\mathbb{C} \smallsetminus \Lambda$) such that the right hand side of the above displayed expression is a well defined doubly periodic function on $\mathbb{C} \smallsetminus \Lambda$.*

Such a function f is called a *developing map* of the solution u of (0.1.3).

The parameter ρ does not show up explicitly in (0.2.1). It enters the picture when we consider the behavior of f near a lattice point $z \in \Lambda$.

0.2.1. When $\rho = 4\pi l$, $l \in \mathbb{N}$, it is a fact that every developing map $f(z)$ of a solution to (0.1.3) extends to a *single valued* meromorphic function on $\mathbb{C}$. Such a developing map f is not doubly periodic in general; rather it is SU(2)-automorphic for the period lattice Λ:

> *For every $\omega \in \Lambda$ there exists an element $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2)$ such that $f(z+\omega) = Tf(z) = \frac{af(z)+b}{cf(z)+d}$ for all $z \in \mathbb{C}$.*

Moreover $f(z)$ *has multiplicity $l+1$ at points of the lattice $\Lambda \subset \mathbb{C}$, and no critical point elsewhere on* $\mathbb{C}$. Conversely every meromorphic function $f(z)$ on $\mathbb{C}$ satisfying the above two properties is the developing map of a solution to (0.1.3); see Lemma 1.2.4.

0.2.2. After replacing f by Tf for a suitable element $T \in \mathrm{SU}(2)$ we get a *normalized* developing map f satisfying one of the following conditions:

(i) Type I (the monodromy of f is a Klein four):

$$\begin{aligned} f(z+\omega_1) &= -f(z) \\ f(z+\omega_2) &= \frac{1}{f(z)} \qquad \forall\, z \in \mathbb{C}. \end{aligned} \tag{0.2.2}$$

(ii) Type II (the monodromy of f is contained in a maximal torus): There exist real numbers θ_1, θ_2 such that

$$f(z+\omega_i) = e^{2\theta_i} f(z) \qquad \forall\, z \in \mathbb{C},\ \forall\, i = 1, 2. \tag{0.2.3}$$

Here ω_1, ω_2 is a $\mathbb{Z}$-basis of Λ with $\mathrm{Im}(\omega_2/\omega_1) > 0$. See §1 for more details.

0.2.3. We have seen that when the parameter ρ of the Liouville equation (0.1.3) is $4\pi l$ with $l \in \mathbb{N}$, solving (0.1.3) is equivalent to finding normalized developing maps, i.e. meromorphic functions on $\mathbb{C}$ with multiplicity $l+1$ at points of the lattice Λ and no critical points on $\mathbb{C} \smallsetminus \Lambda$, whose monodromy with respect to Λ is specified as one of the two types above. It turns out that Liouville equation (0.1.3) with $\rho/4\pi \in \mathbb{N}$ is *integrable* in the sense that the configuration of the zeros and poles of such a normalized developing map can be described by either system of polynomial equations, or as the zero locus of an explicitly defined $\mathbb{C}$-valued real analytic function on an algebraic variety. In other words the Liouville equation (0.1.3) with $\rho/4\pi \in \mathbb{N}$ is *integrable* in the sense that the problem of solving this partial differential equation is reduced to finding the zero locus of some explicit system of equations on a finite dimensional space; see Theorem 0.4 and Theorem 0.6 below.

0.2.4. Let $f(z)$ be a developing map of a solution $u(z)$ of the Liouville equation (0.1.3). Of course the formula (0.2.1) expresses u in terms of f. There is a simple way to "recover" the developing map f from u for a general parameter $\rho \in \mathbb{R}_{>0}$. Notice first that $\frac{\partial}{\partial z}$ of (0.1.3) gives

$$\frac{\partial}{\partial \bar{z}} \left(\frac{\partial^2 u}{\partial z^2} - \frac{1}{2} \left(\frac{\partial u}{\partial z} \right)^2 \right) = 4 \left(\frac{\partial e^u}{\partial z} - e^u \frac{\partial u}{\partial z} \right) = 0 \quad \text{in } E \smallsetminus \{0\},$$

which implies that $u_{zz} - \frac{1}{2} u_z^2$ is a meromorphic function on E: A simple computation using (0.2.1) gives the following formula of this meromorphic function in terms of the developing map f of u:

$$u_{zz} - \tfrac{1}{2} u_z^2 = \frac{f'''}{f'} - \frac{3}{2} \Big(\frac{f''}{f'} \Big)^2. \tag{0.2.4}$$

The right hand side of (0.2.4) is the *Schwarzian derivative* $S(f)$ of the meromorphic function f, while the left hand side is Λ-periodic, with only one singularity at 0 which is a pole of order at most 2, hence must be equal to a $\mathbb{C}$-linear combination of the Weierstrass function $\wp(z; \Lambda)$ and the constant function 1. It is not difficult to determine the coefficient of $\wp(z;\Lambda)$ in this linear combination: We know from equation (0.1.3) that

$$u(z) = 2l \cdot \log|z| + (\text{a } C^\infty\text{-function})$$

for all z in a neighborhood of $0 \in E$. A straightforward calculation shows that either $f(z)$ has a pole of order $l+1$ at $z=0$, or $f(z)$ is holomorphic at $z=0$ and its derivative $f'(z)$ has a zero of order l at $z=0$. In either case the Schwarzian derivative $S(f)$ has a double pole at 0 and

$$\lim_{z\to 0} z^2 S(f) = (l+2)(l+3) - \frac{3}{2}(l+2)^2 = -(l^2+2l)/2.$$

In other words there exists a constant $B \in \mathbb{C}$ such that

$$S(f) = -2(\eta(\eta+1)\wp(z) + B), \tag{0.2.5}$$

where $\eta := \rho/8\pi = l/2$.

On the other hand it is well-known that the "potential" of a second order linear ODE can be recovered from the Schwarzian derivative of the ratio of two linear independent (local) solutions of the ODE. In the case of the Lamé equation (0.1.2) this general fact specializes as follows.

> *If w_1, w_2 are two linearly independent local solutions to the Lamé equation $L_{\eta,B}\, w = 0$ in (0.1.2) with a general parameters η and B and $h(z) := w_1(z)/w_2(z)$, then*
>
> $$S(h) = -2(\eta(\eta+1)\wp(z) + B).$$

Combining the above discussions, we conclude:

> *If $\eta = \rho/8\pi$ then any developing map $f(z)$ of a solution $u(z)$ to* (0.1.3) *can be expressed as a ratio of two $\mathbb{C}$-linearly independent solutions of the Lamé equation* (0.1.2) *for some $B \in \mathbb{C}$.*[9]

We say that the Lamé equation $L_{\eta,B}\, w = 0$ on $\mathbb{C}/\Lambda$ *corresponds to* a solution u of (0.1.3) on $\mathbb{C}/\Lambda$ with $\rho = 8\pi\eta \in 4\pi\mathbb{N}$ if there exist two linearly independent meromorphic solutions w_1, w_2 of $L_{\eta,B}\, w = 0$ on $\mathbb{C}$ such that w_1/w_2 is a developing map of u. This is a property of the parameter B of the Lamé equation.

0.2.5. We make a simple observation about a *normalized* developing map of a solution to (0.1.3) with $\rho/4\pi \in \mathbb{N}$.

> *If $f(z)$ is normalized of type II, then $e^\lambda f(z)$ also satisfies the type II condition for all $\lambda \in \mathbb{R}$. Thus* (0.2.1) *gives rise to a* scaling *family of solutions $u_\lambda(z)$, where*

$$u_\lambda(z) = \log \frac{8e^{2\lambda}|f'(z)|^2}{(1+e^{2\lambda}|f(z)|^2)^2}. \tag{0.2.6}$$

[9] The "constant" $B \in \mathbb{C}$ depends on both the (isomorphism class of the) flat torus E and the solution $u(z)$ of (0.1.3).

Consequently if (0.1.3) *has a type II solution, then the same equation has infinitely many solutions.*

From (0.2.6), it is easy to see that $u_\lambda(z)$ blows up as $\lambda \to \pm\infty$. As we have discussed earlier, if $\rho = 4\pi l$ with $l = 2n+1$ an odd positive integer, then solutions of (0.1.3) have *a priori* bound in $C^2_{loc}(E \setminus \{0\})$. Thus we conclude that when $\rho = 4\pi l$ with l a positive odd integer, (0.1.3) has a solution because the topological degree is positive. Moreover such a solution must be of type I by the existence of uniform a priori bound on compact subsets of $E \setminus \{0\}$.

Our first main theorem in this paper says that the converse to the statement in the previous paragraph also holds. At the same time we provide a self-contained proof of the above implication without using the uniform a priori bound:

Theorem 0.3 (c.f. Proposition 1.5.1 and Theorem 2.2)**.** *Let* $\rho = 4\pi l$ *with* $l \in \mathbb{N}$*. Then equation* (0.1.3) *admits a type I solution if and only if* l *is odd.*

We will "classify" type I solutions for an odd positive integer $l = 2n+1$ in the next theorem 0.4. Let $f(z)$ be a developing map of a solution of (0.1.3) satisfying the normalized transformation formula of type I in (0.2.2). Consider the logarithmic derivative $g = (\log f)' = f'/f$, which is an elliptic function on the double cover

$$E' := \mathbb{C}/\Lambda' \to E,$$

of E, where

$$\Lambda' := \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2', \quad \omega_1' := \omega_1 \text{ and } \omega_2' := 2\omega_2.$$

Our next goal is to find all possible type I developing maps f for some solution u of the Liouville equation (0.1.3) whose parameter ρ is an odd integral multiple of 4π. To do so, we have to locate the position of poles of g, or equivalently the position of zeros and poles of f.

Theorem 0.4 (Type I evenness and algebraic integrability)**.** *Let* u *be a solution to* (0.1.3) *with* $\rho = 4\pi(2n+1)$, $n \in \mathbb{Z}_{\geq 0}$. *Let* f *be a normalized type I developing map of* u, $\Lambda' = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2 = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$, *and* $e_i := \wp(\frac{1}{2}\omega_i'; \Lambda')$ *for* $i = 1, 2$.

(1) *The solution* $u(z)$ *is even and the developing map* $f(z)$ *of* $u(z)$ *is also even; i.e.* $u(-z) = u(z)$ *for all* $z \in E$ *and* $f(z) = f(-z)$ *for all* $z \in \mathbb{C}$.
(2) *There exist* $p_1, \cdots, p_n \in \mathbb{C}$ *satisfying the following properties.*
   - $2p_i \notin \Lambda'$ *for* $i = 1, \ldots, n$,
   - $p_i \pm p_j \notin \Lambda'$ *for all* $i \neq j$, $1 \leq i, j \leq n$,
   - f *has simple zeros at* $\frac{1}{2}\omega_1$ *and* $\pm p_i$ *for* $i = 1, \ldots, n$,
   - f *has simple poles at* $\frac{1}{2}\omega_1 + \omega_2$ *and* $\pm p_i + \omega_2$ *for* $i = 1, \ldots, n$.
   - *Every zero or pole of* f *is congruent modulo* Λ' *to one of the zeros or poles listed above.*

Note that the unordered *set* $\{p_i \bmod \Lambda'\} \subset E'$ *is uniquely determined by the normalized developing map* f.

(3) *Let $q_i := p_i + \omega_2$, $i = 1, \dots, n$ for $i = 1, \dots, n$ and let*

$$z_i := \wp(p_i; \Lambda') - e_2, \qquad \tilde{z}_i = \wp(q_i; \Lambda') - e_2.$$

There exist constants μ and $C_1, \dots, C_n$ which depend only on the modular constants e_1, e_3, $g_2(\Lambda')$ and $g_3(\Lambda')$ such that the following polynomial equations hold.

$$\sum_{i=1}^{n} z_i^j - \sum_{i=1}^{n} \tilde{z}_i^j = C_j \quad \text{and} \quad z_j \tilde{z}_j = \mu \quad \forall\, j = 1, \dots, n. \tag{0.4.1}$$

(4) *Conversely let $\mu, C_1, \dots, C_n$ be the constants in* (3)*, and suppose that the $2n$-tuple $(z_1, \dots, z_n; \tilde{z}_1, \dots, \tilde{z}_n) \in \mathbb{C}^{2n}$ is a solution of the system of polynomial equations* (0.4.1)*. There is an even type I developing map f and $p_1, \dots, p_n \in \mathbb{C}$ with the following properties:*
- *f has simple zeros at $\frac{1}{2}\omega_1$ and $\pm p_i$ for $i = 1, \dots, n$.*
- *f has simple poles at $\frac{1}{2}\omega_1 + \omega_2$ and $\pm p_i + \omega_2$ for $i = 1, \dots, n$.*
- *$z_i = \wp(p_i; \Lambda') - e_2$ and $\tilde{z}_i = \wp(p_i + \omega_2; \Lambda') - e_2$ for $i = 1, \dots, n$.*

We will prove Theorem 0.4 in §2. In view of this result, it is interesting to know how many solutions the system (0.4.1) has. Since the topological degree for (0.1.3) with $\rho = 4\pi(2n+1)$ is known to be $n+1$ (by Theorem C), it is reasonable to conjecture that (0.1.3) has $n+1$ solutions, and then (0.4.1) has $(n+1)!$ solutions due to the permutation symmetry on $\{1, 2, \dots, n\}$ (c.f. [44, Conjecture 6.1] where a related version of this counting conjecture was first formulated). This conjecture had been verified previously up to $n \le 5$; see Remark 2.7. However for higher n it seems to be a non-trivial task to work on the affine polynomial system (0.4.1) directly.

We will affirm this conjecture using the connection between the Liouville equation (0.1.3) and the Lamé equation (0.1.2) discussed earlier.

Theorem 0.4.1. *For any $n \in \mathbb{N}$, the projective monodromy group of a Lamé equation $L_{n+(1/2),B}\, w = 0$ on E is isomorphic to the Klein-four group $(\mathbb{Z}/2\mathbb{Z})^2$ if and only if it* corresponds to *a type I solution of* (0.1.3)*, in the sense that there exist two meromorphic functions w_1, w_2 on $\mathbb{C}$ such that $L_{n+(1/2),B}\, w_1 = 0 = L_{n+(1/2),B}\, w_2$ and the quotient w_1/w_2 is a developing map of a solution of a Liouville equation* (0.1.3) *with $\rho = 4\pi(2n+1)$ and the monodromy group for w_1/w_2 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Moreover, each parameter B with the above property corresponds to exactly one type I solution of* (0.1.3)*.*

Theorem 0.4.1 will be proved in § 3, Theorem 3.5. Its proof shows that the number of solutions to (0.1.3) with $\rho = 4\pi(2n+1)$ is equal to the number of B's in $\mathbb{C}$ such that all solutions to $L_{n+(1/2),B}\, w = 0$ are without logarithmic singularity. A classical theorem of Brioschi, Halphen and Crawford says that there exists a polynomial $p_n(B)$ of degree $n+1$ in B whose roots are exactly the parameter values having the above property. Hence we have the following corollary which sharpens Theorem C:

Corollary 0.4.2. *Let $\rho = 4\pi(2n+1)$, $n \in \mathbb{Z}_{\geq 0}$. There exists a finite set $\mathcal{S}_n$ of tori such that for every torus not isomorphic to anyone in the exceptional set $\mathcal{S}_n$ the Liouville equation* (0.1.3) *possesses exactly $n+1$ distinct solutions.*

0.4.3. In §3 we will also give a new proof of the Brioschi–Halphen–Crawford theorem by exploring the fact that the ratio $f = w_1/w_2$ of two linearly independent solutions of the Lamé equation $L_{n+(1/2),B}\, w = 0$ satisfies the equation (0.2.5) for the Schwarzian derivative with $\eta = n + \frac{1}{2}$; see Theorem 3.2. The new proof has several advantages. It provides a convenient way to compute the polynomial $p_n(B)$ for each n. Moreover it is local in nature. Thus it can be used to treat the mean field equation with multiple singular sources of the form

$$\triangle u + e^u = 4\pi \sum_{j=1}^{l} \alpha_j\, \delta_{Q_j} \quad \text{on } E, \tag{0.4.2}$$

where $Q_1, \ldots, Q_l$ are distinct points in E and $\alpha_1, \ldots, \alpha_l$ are positive integers. In a forthcoming paper [11] we will prove that for generic $Q_1, \ldots, Q_n \in E$, equation (0.4.2) has exactly

$$\tfrac{1}{2} \textstyle\prod_{j=1}^{l} (\alpha_j + 1)$$

distinct solutions provided that $\sum_{j=1}^{l} \alpha_j$ is an odd positive integer.

An immediate consequence of Corollary 0.4.2 is a solution of the counting conjecture stated in the paragraph after 0.4:

> *There exists a finite set $\mathcal{S}_n$ of tori such that for every torus not isomorphic to anyone in the exceptional set $\mathcal{S}_n$ the polynomial system* (0.4.1) *has $(n+1)!$ solutions.*

This might be helpful when we come to study the *excess intersection* at ∞ for the projectivized version of the system of equations (0.4.1) for general n.

0.4.4. Another consequence of Theorem 0.4 is the *holomorphic dependency* of $f(z;\tau)$ on the moduli variable $\tau = \omega_2/\omega_1$ in the upper half plane $\mathbb{H}$ for normalized developing maps $f(z;\tau)$ of solution to (0.1.3) with $\rho = 4\pi(2n+1)$ as in (0.4); we have not been able to prove this statement directly from Liouville's equation (0.1.3). The modular dependency of the constants μ, C_j's in Theorem 0.4 indicates that the *normalized* developing map might be invariant under modular transformations of τ for some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. To illustrate this connection between (0.1.3) and modular forms, we will consider in §4 the simplest case $\rho = 4\pi$, where (0.1.3) has exactly one solution for any torus. In this situation we can *specify* a unique developing map $f(z;\tau)$ on $E_\tau = \mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ and $\tau \in \mathbb{H}$; see Proposition 4.2 for the definition of this function $f(z;\tau)$ and explicit formulas for it. When $f(z;\tau)$ is written as a power series

$$f(z;\tau) = a_0(\tau) + a_2(\tau) z^2 + a_4(\tau) z^4 + \cdots,$$

for each k the coefficient $a_k(\tau)$ of z^k is a *modular form* of weight k for the principal congruence subgroup $\Gamma(4)$ which is holomorphic on the upper-half plane $\mathbb{H}$ but may have poles at the cusps of the modular curve $X(4)$; see Corollary 4.5. In addition we will show that the constant term $a_0(\tau)$ of $f(z;\tau)$ is a $\mathbb{Q}(\sqrt{-1})$-rational *Hauptmodul* which is also a *modular unit*; i.e. (a) $a_0(\tau)$ is holomorphic and everywhere non-zero on $\mathbb{H}$, (b) $a_0(\tau)$ defines a meromorphic function on $X(4)$ with $\mathbb{Q}(\sqrt{-1})$-rational q-expansion at all cusps of $X(4)$, and (c) every meromorphic function on $X(4)$ is a rational function of $a_0(\tau)$. See Corollary 4.4 and Remark 4.4.1 (b).

Underlying the above statements is the fact that the function $f(z;\tau)$ satisfies a transformation law for the full modular group $\mathrm{SL}_2(\mathbb{Z})$; see Proposition 4.3 for the precise statement. Modulo a question 4.6.6 (a) on the irreducibility of certain branched covering of the upper-half plane $\mathbb{H}$, this transformation law generalizes to the case when $\rho = 4\pi(2n+1)$ for any natural number $n \in \mathbb{Z}_{\geq 0}$; see Corollary 4.6.5.

In a forthcoming paper we will consider equation (0.4.2) with multiple singular sources and show that for each k the space of modular forms of weight k arising from (0.4.2) is invariant under (suitably defined) Hecke operators.

0.5. Next we want to classify solutions of the Liouville equation (0.1.3) with $\rho = 8\pi n$ for some positive integer n. By Theorem 0.3 any solution of (0.1.3), if exists, must be of type II. Hence any solution of (0.1.3) begets infinitely many solutions. We remark that not every torus admits a solution to (0.1.3). For instance when $\rho = 8\pi$, there are no solutions to (0.1.3) for rectangular tori, while there do exist solutions for τ close to $e^{\pi i/3}$; see [43, Example 2.5, 2.6]). Indeed [43, Theorem 1.1] asserts that in the case when $\rho = 8\pi$, the Liouville equation (0.1.3) has a solution if and only the Green's function G_E for the torus has a critical point which is not a 2-torsion. Hence by Theorem A the Liouville equation (0.1.3) with $\rho = 8\pi$ has a solution for if and only if the Green function has five critical points. Let $\mathcal{M}_1 = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and let

$$\Omega_5 := \{\, \bar{\tau} \in \mathcal{M}_1 \mid G(z;\bar{\tau}) \text{ has five critical points} \,\}.$$

By the uniqueness theorem in [43, Theorem 4.1], we know that Ω_5 is open; while it is easy to see (using the holomorphic $(\mathbb{Z}/3\mathbb{Z})$-action on the torus) that the image of $e^{\pi i/3}$ in $\mathcal{M}_1$ lies in Ω_5. It is important to further investigate the geometry of this moduli subset Ω_5. In Part II of this series of papers, we shall use methods for non-linear PDE's to the Liouville equation (0.1.3) and the theory of modular forms to prove that Ω_5 is a simply connected domain and the boundary $\partial\Omega_5$ of Ω is real-analytically isomorphic to a circle, thereby settling the conjecture on the shape of Ω_5 raised in [43, §1 p. 915].[10]

[10]This phenomenon was observed in computer simulations.

0.5.1. In this paper (Part I of the series), we classify all type II solutions for general $n \in \mathbb{N}$ and study their connection with the geometry of a family of hyperelliptic curves. This will form the foundation of an investigation on certain modular forms to be developed in Part II of this series [10].

We will also consider the logarithmic derivative $g = f'/f$ of a *normalized* type II developing map f. The type II condition (0.2.3) implies that g is an elliptic function on $E = E_\Lambda$.

0.5.2. As was explained in 0.2.4, the Liouville equation (0.1.3) is related to the Lamé equation (0.1.2) whose parameter η is equal to $\rho/8\pi$. In the case when η is a positive integer n, there are explicit formulas for solutions, of the Lamé equation (0.1.2), called the *Hermite–Halphen ansatz*; c.f. [29, I–VII] and [27, pp. 495-498]:

> *For any $a_1, \ldots, a_n \in \mathbb{C} \smallsetminus \Lambda$ such that the images $[a_i] \in E$ of a_i under the projection $\mathbb{C} \to E = \mathbb{C}/\Lambda$, $i = 1, \ldots, n$, represent n* mutually distinct *points in $E \smallsetminus \{[0]\}$, the function*

$$w_a(z) = e^{z \cdot \sum_{i=1}^n \zeta(a_i;\Lambda)} \prod_{i=1}^n \frac{\sigma(z - a_i; \Lambda)}{\sigma(z; \Lambda)} \tag{0.5.1}$$

> *is a solution to* (0.1.2) *for some $B \in \mathbb{C}$ if and only if $\{[a_i]\} \in Y_n \subset \mathrm{Sym}^n(E \smallsetminus \{0\})$, where*

(0.5.2)

$$Y_n := \left\{ \{[a_1], \ldots, [a_n]\} \;\middle|\; \begin{array}{l} [a_i] \in E\smallsetminus\{0\}\ \forall i,\ \ [a_i] \neq [a_j] \text{ for all } i \neq j, \\ \sum_{1 \le j \le n,\, j \ne i} \left(\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)\right) = 0 \\ \text{for } i = 1, \ldots, n. \end{array} \right\}.$$

> *Moreover if $\{[a_i]\}_{i=1}^n$ is a point of Y_n and $w_a(z)$ is a solution of a Lamé equation* (0.1.2) *with $\eta = n$, then $B = (2n-1)\sum_{i=1}^n \wp(a_i)$.*

Note that $w_b(z) \in \mathbb{C}^\times \cdot w_a(z)$ if $b = (b_1, \ldots, b_n)$ and $b_i \equiv a_i \pmod{\Lambda}$ $i = 1, \ldots, n$.

0.5.3. The following properties are known from classical literature.

(i) Each ansatz solution $w_a(z)$ of $L_{n,B}\, w = 0$ satisfies

$$w_a(z + \omega) = e^{\sum_{i=1}^n \zeta(a_i;\Lambda)\omega - \sum_{i=1}^n a_i \eta(\omega;\Lambda)} \cdot w_a(z) \quad \forall\, \omega \in \Lambda.$$

In other words $w_a(z)$ is a common eigenvector for the global monodromy representation of $\Lambda = \pi_1(E)$ on the 2-dimensional space of local solutions of the Lamé equation $L_{n,B}\, w = 0$.

(ii) Every one-dimensional eigenspace of the monodromy representation of a Lamé equation $L_{n,B}\, w = 0$ is of the form $\mathbb{C} \cdot w_a(z)$ for some a such that $B = (2n-1)\sum_{i=1}^n \wp(a_i)$. In other words the map $\pi : Y_n \to \mathbb{C}$ given by $\{[a_i]\}_{i=1}^n \mapsto (2n-1)\sum_{i=1}^n \wp(a_i)$ is surjective. Note that $\pi \circ \iota = \pi$ for the involution $\iota : \{[a_i]\}_{i=1}^n \mapsto \{[-a_i]\}_{i=1}^n$ on Y_n.

(iii) For every $B \in \mathbb{C}$, the set $\pi^{-1}(B)$ is an orbit of the involution ι, and $\pi^{-1}(B)$ is a singleton if and only if $L_{n,B}\, w = 0$ has a Lamé function as a solution.

The above properties tell us that Y_n can be regarded as the parameter space of all one-dimensional eigenspaces of the monodromy representations on the solutions of the Lamé equation $L_{n,B}\, w = 0$ on E when the parameter B varies over $\mathbb{C}$. This and the fact that $\pi : Y_n \to \mathbb{C}$ is a double cover drives home the compelling picture that Y_n can be regarded as a "spectral curve" for the global monodromy representation.[11] The algebraic structure on Y_n is explained in 0.5.4 below.

0.5.4. The analytic set of solutions of the system of equations

$$\sum_{1\le j\le n,\, j\neq i} \big(\zeta(a_i - a_j;\Lambda) + \zeta(a_j;\Lambda) - \zeta(a_i;\Lambda)\big) = 0 \quad \forall\, i = 1,\ldots,n \tag{0.5.3}$$

in variables $(a_1,\ldots,a_n)$ under the constraint that

$$a_i \notin \Lambda \quad \forall\, i = 1,\ldots,n \ \text{ and } \ a_i - a_j \notin \Lambda \ \ \forall\, i \neq j$$

descends to a locally closed algebraic subvariety of

$$\mathrm{Sym}^n(E \smallsetminus \{0\}) = (E \smallsetminus \{0\})^n / S_n$$

because Y_n is stable under the symmetric group S_n, and the classical addition formula (c.f. [67, 20·53 Example 2])

$$\frac{1}{2}\,\frac{\wp'(z) + \wp'(w)}{\wp(z) - \wp(w)} = \zeta(z - w) - \zeta(z) + \zeta(w)$$

for elliptic functions allows us to express the definition of Y_n algebraically: Let $\tilde{\Delta}$ be the divisor of E^n consisting of all points of E^n where at least two components are equal, and let Δ be the image of $\tilde{\Delta}$ in $\mathrm{Sym}^n E$. Denote by $\tilde{U}$ the algebraic variety $(E \smallsetminus \{[0]\})^n \smallsetminus \Delta$.

> *Y_n is the closed subvariety of $\mathrm{Sym}^n(E \smallsetminus \{[0]\}) \smallsetminus \Delta$ whose inverse image $\tilde{Y}_n$ in the affine algebraic variety $\tilde{U}$ is defined by the system of equations*

$$\sum_{1\le j\le n,\, j\neq i} \frac{y_i + y_j}{x_i - x_j} = 0, \qquad \forall\, i = 1,\ldots,n. \tag{0.5.4}$$

Here x_i, y_i are the pull-back via the i-th projection of coordinates of the Weierstrass form $y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$ of $E = \mathbb{C}/\Lambda$. For each pair (i,j) with $i \neq j$, the regular function $\frac{y_i+y_j}{x_i-x_j}$ on the affine open subset $\tilde{U}_{(x_i-x_j)}$ of $\tilde{U}$ where $x_i \neq x_j$ extends to a regular function on $\tilde{U}$, therefore the above description defines an affine closed subvariety $\tilde{Y}_n$ of $\tilde{U}$.

[11] This is more than an analogy: Y_n is indeed a spectral curve in KdV theory. It parametrizes one-dimensional common eigenspaces for the commutator subring of the differential operator $\frac{d^2}{dz^2} - n(n+1)\wp(z)$ in the ring of linear differential operators in one variable.

In view of the algebraic structure of Y_n, the classically known facts recalled in 0.5.3 means that Y_n is "the affine part" of a hyperelliptic curve and the $\pi : Y_n \to \mathbb{C}$ is the restriction to Y_n of the hyperelliptic projection. On the other hand, solutions to the Liouville equation (0.1.3) with $\rho = 8\pi n$ admit the following description.

Theorem 0.6 (Type II evenness and Green/algebraic system)**.** *Let u be a solution to* (0.1.3) *with $\rho = 8\pi n$ on $E = \mathbb{C}/\Lambda$ and let f be a* normalized *developing map of u of type II.*

(1) *The developing map f is non-zero at points of Λ.*
(2) *There are $2n$ elements $p_1, \dots, p_n, q_1, \dots, q_n \in \mathbb{C}$ with the following properties.*
  - *$[p_1], \dots, [p_n], [q_1], \dots, [q_n]$ are $2n$* distinct *points in E, where $[p_i] := p_i \bmod \Lambda$ for all i and similarly for the $[q_i]$'s.*
  - *f has simple zeros at points above $[p_1], \dots, [p_n]$ and simple poles at points above $[q_1], \dots, [q_n]$.*
  - *f is holomorphic and non-zero at every point of $\mathbb{C}$ which is not congruent modulo Λ to one of $\{p_1, \dots, p_n, q_1, \dots, q_n\}$.*

(3) *The zeros and poles of the developing map f are related by*
$$\{[q_1], \dots, [q_n]\} = \{[-p_1], \dots, [-p_n]\}.$$
(4) *There is a unique even solution in the one-parameter scaling family of solutions $u_\lambda(z) = \log \frac{8\,e^{2\lambda}\,|f'(z)|^2}{(1+e^{2\lambda}|f(z)|^2)^2}$ of* (0.1.3) *with parameter $\lambda \in \mathbb{R}$.*
(5) *The "zero points" $p_1, \dots, p_n \in E$ of f satisfy the following n equations:*

$$\sum_{i=1}^{n} \wp'(p_i; \Lambda) \cdot \wp^r(p_i; \Lambda) = 0, \quad r = 0, \dots, n-2, \tag{0.6.1}$$

$$\sum_{i=1}^{n} \frac{\partial G}{\partial z}(p_i) = 0, \tag{0.6.2}$$

where $G(z)$ is the Green's function of E.

(6) *The meromorphic function $g := \frac{d}{dz} \log f = f'/f$ on E is* even *and is determined by the points $[p_1], \dots, [p_n] \in E$, while the normalized developing map f is determined up to $\mathbb{C}^\times$ by $[p_1], \dots, [p_n]$, via the following formulas:*

$$g(z) = \sum_{i=1}^{n} \frac{\wp'(p_i; \Lambda)}{\wp(z; \Lambda) - \wp(p_i; \Lambda)}, \qquad f(z) = f(0) \cdot \exp \int_0^z g(\xi)\, d\xi. \tag{0.6.3}$$

Conversely, if $\{[p_1], \dots, [p_n]\}$ is a set of n distinct points of $E \smallsetminus \{0\}$ which satisfies equations (0.6.1) *and* (0.6.2)*, and*

$$\{[p_1], \dots, [p_n]\} \cap \{[-p_1], \dots, [-p_n]\} = \emptyset, \tag{0.6.4}$$

then the function f defined by (0.6.3) *is a type II normalized developing map of a solution of* the Liouville equation (0.1.3) *with $\rho = 8\pi n$.*

Theorem 0.6 will be proved in §5. While the type I system is purely algebraic, the type II system is somewhat transcendental as it involves the Green function of E. It is natural to isolate the Green equation $\sum \nabla G(p_i) = 0$ and consider the remaining $n-1$ algebraic equations (0.6.1) first.

0.6.1. Let $a = \{a_1, \ldots, a_n\}$ be an unordered set of complex numbers with *distinct* images in $E \smallsetminus \{0\} = (\mathbb{C} \smallsetminus \Lambda)/\Lambda$ such that the equation (0.6.1) is satisfied with $p_i = a_i$ for $i = 1, \ldots, n$. Let $x = \wp(z; \Lambda)$, $y = \wp'(z; \Lambda)$, so that the torus E is given by the Weierstrass equation

$$y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda).$$

Let $(x_i, y_i) = (\wp(a_i; \Lambda), \wp'(a_i; \Lambda))$ for $i = 1, \ldots, n$. Then the system of $n-1$ equations (0.6.1) takes the algebraic form

$$\sum_{i=1}^{n} x_i^r \cdot y_i = 0, \qquad i = 0, 1, \ldots, n-2. \tag{0.6.5}$$

Recalled that the algebraic variety Y_n is defined by the system of equations (0.5.4).

Theorem 0.6.2 (= Theorem 5.8.3)**.** *For any set of* distinct *complex numbers* $x_1, \ldots, x_n$*, the two systems of* linear *equations in variables* $y_1, \ldots, y_n$ (0.6.5) *and* (0.5.4) *are equivalent.*

Define $X_n \subset \mathrm{Sym}^n E$ by

$$X_n = \left\{ \{(x_i, y_i)\}_{i=1}^n \;\middle|\; \begin{array}{l} (x_i, y_i) \in E \smallsetminus E[2] \;\; \forall\, i,\; x_i \neq x_j \;\; \forall\, i \neq j \\ \sum_{i=1}^n x_i^r \cdot y_i = 0 \;\text{ for } r = 0, 1, \ldots, n-2 \end{array} \right\}, \tag{0.6.6}$$

where $E[2] = \frac{1}{2}\Lambda/\Lambda$ is the subset of 2-torsion points of E. This variety X_n is an affine algebraic curve, which will be called the (n-th) *Liouville curve*. Theorem 0.6.2 implies that

$$X_n = \left\{ \{[a_i]\}_{i=1}^n \in Y_n \;\middle|\; \wp(a_i) \neq \wp(a_j) \text{ whenever } i \neq j,\; \wp'(a_i) \neq 0 \;\forall\, i \right\}.$$

The following theorem says that the Liouville curve X_n is the unramified locus of the Lamé curve $\bar{X}_n$ for the hyperelliptic projection.

Theorem 0.7 (Hyperelliptic structure on $X_n \subset Y_n \subset \bar{X}_n$)**.**

(1) *Let* $a = \{[a_i]\}_{i=1}^n$ *be a point of* Y_n*. The corresponding* B *in the Lamé equation, in the sense of the Hermite–Halphen ansatz recalled in* 0.5.1*, is given by*

$$B_a = (2n-1) \sum_{i=1}^{n} \wp(a_i).$$

(2) *The map* $\pi : Y_n \to \mathbb{C}$ *defined by* $a \mapsto B_a$ *is a proper surjective branched double cover.*

(3) *The map* $\pi : Y_n \to \mathbb{C}$ *has a natural extension to a proper morphism*

$$\bar{\pi} : \bar{X}_n \to \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\},$$

where $\bar{X}_n$ is the closure of X_n in $\mathrm{Sym}^n E$*, for both the Zariski and the complex topologies.*

(4) *The restriction*

$$\pi|_{X_n} : X_n \to \pi(X_n) =: U_n$$

of π to the Zariski open subset $X_n \subset Y_n$ is a finite étale double cover of the Zariski open subset $U_n \subset \mathbb{C}$. Points of the finite set $\bar{X}_n \smallsetminus X_n$ are precisely the ramification points of $\bar{\pi}$

(5) *The curve $\bar{X}_n$ is a (possibly singular) hyperelliptic curve of arithmetic genus n and $\bar{\pi}$ is the hyperelliptic structural morphism. Moreover $\bar{X}_n = Y_n \cup \{[0]^n\}$ is the union of Y_n with a* single point $[0]^n := \{[0], \dots, [0]\} \in \mathrm{Sym}^n E$

(6) *The curve $\bar{X}_n$ is stable under the involution $\tilde{\iota}$ of* $\mathrm{Sym}^n E$*, defined by*

$$\iota \colon \{P_1, \dots, P_n\} \mapsto \{-P_1, \dots, -P_n\} \qquad \forall\, P_1, \dots, P_n \in E.$$

The restriction $\bar{\iota}$ of $\tilde{\iota}$ to $\bar{X}_n$ is the hyperelliptic involution *on $\bar{X}_n$. The set of ramification points $\bar{X}_n \smallsetminus X_n$ of $\bar{\pi}$ coincides with the fixed point set of the hyperelliptic involution $\bar{\iota}$ on $\bar{X}_n$.*

(7) *The map π induces a bijection from the finite set $Y_n \smallsetminus X_n$ to the finite set $\mathbb{C} \smallsetminus U_n$. A point $\{[a_i]\}_{i=1}^n$ of Y_n lies in $Y_n \smallsetminus X_n$ if and only if the function w_a as defined in* 0.5.1 *is a Lamé function. Hence $\#(Y_n \smallsetminus X_n) = 2n+1$ when $Y_n \smallsetminus X_n$ is counted with multiplicities inherited from $\mathbb{C} \smallsetminus U_n$ when $\mathbb{C} \smallsetminus U_n$ is identified with the set of roots of the polynomial $\ell_n(B) = 0$ of degree $2n+1$ in Theorem B.*

(8) *The inverse image of the point $\infty = \mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{C}$ under $\bar{\pi}$ consists of the single point 0^n. This point 0^n "at infinity" is a smooth point of $\bar{X}_n$ for every torus E.*

0.7.1. A complete proof of Theorem 0.7 is given in §7, after some preparation in §6 on characterizations of Y_n and X_n related to the Lamé equations; see Theorems 7.3, 7.4, Corollary 7.5.2 and Proposition 7.7. In particular, the affine hyperelliptic curve Y_n is defined by the explicit equation

$$C^2 = \ell_n(B; g_2, g_3) \quad \text{with} \quad \deg \ell_n(B) = 2n+1.$$

This curve, called *the n-th Lamé curve*, is smooth for generic tori. It is an irreducible algebraic curve since the degree of ℓ_n is odd.

Due to its fundamental importance, we offer several proofs, from both the analytic and the algebraic perspectives, for (part of) the theorem. We must mention that the polynomial $\ell_n(B)$ in Theorem B has been treated in the literature in several different contexts, including the investigation of Lamé equations with algebraic solutions [4, 20] and the mathematical physics related to Lamé equations. Thus a substantial portion of Theorem 0.7 overlaps with existing literature. However there are a number of issues for which we were unable to locate satisfactory treatments in the literature.

For instance, why the closure $\bar{X}_n$ of Y_n in $\mathrm{Sym}^n E$ coincides with the *projective hyperelliptic model*[12] of the affine curve $C^2 = \ell_n(B)$ at the infinity point, instead of the closure in $\mathbb{P}^2$ of the latter curve.

In this paper we attempt to provide a self-contained account of the hyperelliptic structure of $\bar{X}_n$, from both the analytic and the algebraic point of view, for the convenience of the readers. The readers will find in our treatment the precise behavior of the local structure near every $a \in \bar{X}_n \smallsetminus X_n$, *including the infinity point*, and also "the meaning" of the coordinate C of the Lamé curve in various contexts (c.f. Theorem 7.4 and Remark 7.4.2, as well as formulas (5.6.4), (7.1.2), (7.1.4) and (7.4.1)). It is also worth mentioning that the coefficients of the polynomial $\ell_n(B)$ are effectively computable by simple recursive relations (c.f. Theorem 7.4 and (7.3.6)).

0.7.2. Theorem 0.6 tells us that algebraic geometric structure

$$\bar{\pi}_n : (\bar{X}_n, Y_n, X_n) \to (\mathbb{P}^1(\mathbb{C}), \mathbb{C}, U_n)$$

provides a scaffold for analyzing the mean field equation $\triangle u + e^u = 8\pi n \delta_0$ on a flat torus: a necessary and sufficient condition for a point to be attached to a type II solution of the mean field equation (0.1.3) with parameter $\rho = 8\pi n$ is that $\{p_1, \ldots, p_n\}$ satisfies the Green equation (0.6.2). Of course one wish to pursue the above thread to bring about a complete analysis of the set of all solutions of (0.1.3). The case $n = 1$, where $\bar{X}_1 = E$, has been successfully treated in [43] with a combination of two techniques. Naturally one would like to extend these methods to higher values of n.

0.7.3. The first technique is to use the double cover $E \to \mathbb{P}^1(\mathbb{C}) \cong S^2$ and the evenness of u to transform the equation to another one on S^2 (with more singular sources). To extend this step to a general positive integer n, we believe that the hyperelliptic structure $\bar{\pi} : \bar{X}_n \to \mathbb{P}^1(\mathbb{C})$ is the right replacement of E.

It will be shown in Part II of this series of articles that the map

$$\sigma : \bar{X}_n \to E, \qquad \{p_1, \ldots, p_n\} \mapsto \sigma(\{p_1, \ldots, p_n\}) = \sum p_i$$

is a branched covering of degree $\frac{1}{2}n(n+1)$, and the rational function

$$\mathbf{z}([a]) := \zeta(\textstyle\sum a_i) - \sum \zeta(a_i) \qquad [a] = \{[a_1], \ldots, [a_n]\} \in \bar{X}_n$$

on $\bar{X}_n$ is a primitive generator of the extension field $K(\bar{X}_n)$ over $K(E)$. Using this, a "pre-modular form" $Z_n(\sigma; \tau)$ for $\tau \in \mathbb{H}$ and $\sigma \in E_\tau$ will be constructed, which has the property that non-trivial solutions to the Green equation (0.6.2) on $\bar{X}_n$ correspond exactly to the zeros of the single function $Z_n(\sigma; \tau)$.

[12]See 7.6.1.e for the definition of the projective hyperelliptic model defined by $\ell_n(B)$.

0.7.4. The second technique employed for the case $\rho = 8\pi$ is to use the *method of continuity* to connect the equation for $\rho = 8\pi$ to the known case when $\rho = 4\pi$ by establishing the non-degeneracy of the linearized equations of (0.1.3). For general ρ, such a non-degeneracy statement is out of reach at this moment. However, since equation (0.1.3) has a solutions u_η for every $\rho = 8\pi\eta \notin 8\pi\mathbb{N}$, it is natural to study the limiting behavior of u_η as $\eta \to n$. If the limit does not blow up, it will converge to a solution u for $\rho = 8\pi n$. For the blow-up case, we will establish a connection between the location of the blow-up set and the hyperelliptic geometry of $Y_n \to \mathbb{P}^1(\mathbb{C})$:

Theorem 0.7.5. *Let $S = \{p_1, \ldots, p_n\}$ be an element of* $\mathrm{Sym}^n E$ *such that $p_i \neq p_j$ whenever $i \neq j$. Suppose that S is the blow-up set of a sequence of solutions u_k of the Louiville equation* (0.1.3) *with parameter ρ_k such that $\rho_k \to 8\pi n$ as $k \to \infty$. Then $S \in Y_n$. Moreover,*

(1) *If $\rho_k \neq 8\pi n$ for every k then S is a branch (or ramification) point of Y_n.*
(2) *If $\rho_k = 8\pi n$ for all k then S is* not *a branch point of Y_n.*

The proof of Theorem 0.7.5 will be given in §8. Theorems 0.7.5 and 0.7 provide rather precise information on the blow-up set of sequences of solutions of (0.1.3), which we believe will play a fundamental role in future research on the mean field equations.

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1. Liouville equations with singular source

1.1. A Theorem of Liouville.

1.1.1. We begin with a quick review of a classical theorem of Liouville.

Proposition 1.1.2. *Every $\mathbb{R}$-valued C^2 solution u of the differential equation*

$$\triangle u + e^u = 0 \tag{1.1.1}$$

in a simply connected domain $D \subset \mathbb{C}$ can be expressed in the form

$$u = \log \frac{8|f'|^2}{(1+|f|^2)^2} \tag{1.1.2}$$

where f is a holomorphic function on D whose derivative f' does not vanish on D. Conversely for every meromorphic function f on an open subset $V \subset \mathbb{C}$ with at most simple poles whose derivative does not vanish on V, the function $\log \frac{8|f'|^2}{(1+|f|^2)^2}$ is a smooth function which satisfies equation (1.1.2).

A proof of Liouville's theorem 1.1.2 is given in 1.1.8 for the convenience of the readers.

Definition 1.1.3. Let u be a real-valued C^2-function on a domain $D \subset \mathbb{C}$ and satisfies equation (1.1.1) on D.

A *developing map* f of u is a meromorphic function f defined on a (not necessarily connected) covering space $\pi : \tilde{D} \to D$ of D such that

$$u(z) = \log \frac{8|f'(\tilde{z})|^2}{(1+|f(\tilde{z})|^2)^2} \quad \text{for every } \tilde{z} \in \tilde{D} \text{ and } z = \pi(\tilde{z}). \tag{1.1.3}$$

For any pole $\tilde{z}_0 \in \tilde{D}$ of f, the equality (1.1.3) for $z = z_0$ means that the right hand side of (1.1.3) has a finite limit as $z \to \tilde{z}_0$, and this limit is equal to $u(\pi(\tilde{z}_0))$.

Remark 1.1.4. (a) It is easy to see that every developing map $f : \tilde{D} \to \mathbb{P}^1(\mathbb{C})$ of a C^2 solution u of (1.1.1) on D has no critical point on $\tilde{D}$. In other words the holomorphic map $f : \tilde{D} \to \mathbb{P}^1(\mathbb{C})$ is étale. More explicitly this means that the derivative f' of the meromorphic function f does not vanish at every point where f is holomorphic, and f has at most simple poles.

(b) The proof of Liouville's theorem 1.1.2 in 1.1.8 provides another interpretation of developing maps: a developing map f for a solution u of (1.1.1) is an orientation-preserving local isometry, from a covering space of D with the Riemannian metric $\frac{1}{2}\, e^u\,(dx^2 + dy^2)$, to $\mathbb{P}^1(\mathbb{C})$ with the Fubini–Study metric (or equivalently the unit sphere S^2 with the standard metric) which has constant Gaussian curvature 1.

Developing maps are not unique. In Lemma 1.1.5 below we show that different developing maps of a solution u are related by special *unitary* Möbius transformations.

Lemma 1.1.5. *Let u be a C^2 solution of equation* (1.1.1) *on a domain $D \subset \mathbb{C}$. Let f be a developing map for u on a covering space $\tilde{D}$ of D as in as in Definition* 1.1.3.

(1) *The solution u of* (1.1.1) *and its developing map f are related by*[13]

$$u_{zz} - \frac{1}{2}u_z^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{1.1.4}$$

(2) *Let U be an element of* $\mathrm{PSU}(2)$ *represented by a 2×2 special unitary matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{C}$. The function $Uf := \frac{af+b}{cf+d}$ is also a developing map of u on $\tilde{D}$.*

(3) *Assume that the covering space $\tilde{D}$ of D is connected. Suppose that $\tilde{f}$ is another developing of u on $\tilde{D}$. There exists an element $T \in$* $\mathrm{PSU}(2)$ *such that $\tilde{f} = Tf$.*

[13]The right hand side of equation (1.1.4) is the *Schwarzian derivative* $S(f)$ of f; the equality here means that the Schwarzian derivative of f descends to the function $u_{zz} - \frac{1}{2}u_z^2$ on D.

Proof. The statements (1) and (2) are easily verified by direct calculations. It remains to prove (3). Notice first that the Schwarzian derivatives of f and $\tilde{f}$ are equal by (1). Hence there exists a Möbious transformation T, say represented by an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, such that $\tilde{f} = Tf = \frac{af+b}{cf+d}$.[14] From

$$\log \frac{8|f'|^2}{(1+|f|^2)^2} = \log \frac{8|(Tf)'|^2}{(1+|Tf|^2)^2}, \qquad (Tf)' = \frac{f'}{(cf+d)^2},$$

we deduce that $|af+b|^2 + |cf+d|^2 = 1 + |f|^2$ on $\tilde{D}$. Hence the quality

$$|az+b|^2 + |cz+d|^2 = 1 + |z|^2$$

holds on $\mathbb{C}$ because meromorphic maps are open. Applying $\partial\bar{\partial}\log$ to both sides of the last displayed equality, we see that the Möbius transformation T preserves the Fubini–Study metric on $\mathbb{P}^1(\mathbb{C})$, or equivalently the spherical metric on the 2-sphere S^2. So T is an element of $\mathrm{PSU}(2)$, because $\mathrm{PSU}(2)$ is the group of all orientation preserving isometries of $\mathbb{P}^1(\mathbb{C})$. $\square$

Remark 1.1.6. In the notation of Lemma 1.1.5, let V be an element of $\mathrm{SU}(2)$ such that $VUV^{-1} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for some $\theta \in \mathbb{R}$. Then the two developing maps $V\tilde{f}$ and Vf of u are related by

$$V\tilde{f} = e^{2i\theta} Vf.$$

1.1.7. Lemmas 1.1.5 can be reformulated as follows. See also 1.1.8.

Let u be a C^2 solution of equation (1.1.1) on a domain $D \subset \mathbb{C}$. There exists a (not necessarily connected) covering space $\pi_u : \mathcal{D}_{u,\mathrm{univ}} \to D$, a left action of $\mathrm{PSU}(2)$ *on $\mathcal{D}_u$, and a meromorphic function $f_{u,\mathrm{univ}} : \mathcal{D}_{u,\mathrm{univ}} \to \mathbb{P}^1(\mathbb{C})$ on $\mathcal{D}_{u,\mathrm{univ}}$ satisfying the following properties.*

(a) *$\pi : \mathcal{D}_{u,\mathrm{univ}} \to D$ is a left principle homogeneous space for* $\mathrm{PSU}(2)$.
(b) *$f_{u,\mathrm{univ}}$ is a developing map for u.*
(c) *For any open subset $U \subset D$ and any developing map f for $u|_U$ on a covering space $\pi : \tilde{U} \to U$ of U, there exists a unique holomorphic map $g : \tilde{U} \to \mathcal{D}_{u,\mathrm{univ}}$ such that $f(\tilde{z}) = f_{u,\mathrm{univ}}(g(\tilde{z}))$ for all $\tilde{z} \in \tilde{U}$.*

1.1.8. A PROOF OF PROPOSITION 1.1.2. From the perspective of differential geometry, equation (1.1.1) is simply the prescribed Gaussian curvature equation for the metric

$$g = \frac{1}{2}\, e^u \,(dx^2 + dy^2)$$

[14]Here we have used the assumption that $\tilde{D}$ is connected and a basic property of Schwarzian derivatives: if $S(g_1) = S(g_2)$ for two locally non-constant meromorphic functions g_1 and g_2, then g_1 and g_2 differ by a Möbius transformation. This is consequence of the special case that $S(g) = 0$ if and only if g is a linear fractional transformation and the cocycle property of Schwarzian derivatives: $S(g \circ h)(z) = S(g)(h(z)) \cdot h'(z)^2 + S(h)(z)$.

on the domain D to have Gaussian curvature $K_g = 1$. Given that u is a solution of the equation (1.1.1), a simple calculation shows that

$$K_g = -e^{-u} \triangle u = 1. \tag{1.1.5}$$

So at any given point $z_1 \in D$, there exists a meromorphic function $f_1(z)$ on a neighborhood of $V(z_1)$ of z_1 such that $u(z) = \log \frac{8|f_1'(z)|^2}{(1+|f_1(z)|^2)^2}$ for all $z \in V(z_1)$, because the Fubini–Study metric on $\mathbb{C} = \mathbb{P}^1(\mathbb{C}) \smallsetminus \{\infty\}$ is

$$\frac{4\, dz\, d\bar{z}}{(1+z\bar{z})^2}.$$

The collection of germs of all such local developing maps f_1 form a locally constant sheaf $\mathcal{D}ev(u)$ over D with monodromy group $\mathrm{PSU}(2)$ according to Lemma 1.1.5. Note that $\mathcal{D}ev(u)$ is the locally constant sheaf attached to the covering space $\mathcal{D}_{u,\mathrm{univ}}$ in 1.1.7. The locally constant sheaf $\mathcal{D}ev(u)$ has a global section g because the domain D is simply connected. This global section g is a developing map of u on D. The holomorphic map $\tilde{g}$ from D to $\mathbb{P}^1(\mathbb{C})$ defined by the meromorphic function g is étale as we have remarked in 1.1.4 (b), hence the image $\tilde{g}$ misses some point Q of $\mathbb{P}^1(\mathbb{C})$. Pick an element $U \in \mathrm{PSU}(2)$ which sends Q to ∞. Then $f := Ug$ is a holomorphic function on U which is also a developing map of u. □

1.2. Liouville theory on tori with isolated singular data. It is a challenge to extend the Liouville theory recalled in the previous section to oriented Riemann surfaces and with singular sources. In this paper we will consider the genus one case, so the Riemann surface will be a flat torus E. Moreover we put just one singular source on E, and we will make this singular source the additive unity 0 for a holomorphic group law on E.

1.2.1. We choose and fix a non-zero global holomorphic one-form β on E, so that integrating β along paths starting from 0 gives an isomorphism $\int_? \beta : E \xrightarrow{\sim} \mathbb{C}/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}$, so that β is the pull-back of the one-form $dz = dx + \sqrt{-1}dy$ descended to $\mathbb{C}/\Lambda$. The flat torus E will be identified with $\mathbb{C}/\Lambda$ in the rest of this paper. Let ω_1, ω_2 be a $\mathbb{Z}$-basis of Λ such that $\tau := \omega_2/\omega_1$ satisfies $\mathrm{Im}(\tau) > 0$. Let $\omega_3 := -\omega_1 - \omega_2$, so that $\omega_1 + \omega_2 + \omega_3 = 0$.

We will consider the *mean field equation*

$$\triangle u + e^u = \rho \cdot \delta_0, \quad \rho \in \mathbb{R}_{>0} \tag{1.2.1}$$

on $E = \mathbb{C}/\Lambda$, where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, δ_0 is the Dirac measure at 0 and we have identified L^1-functions with (signed) measures using the Lebesgue measure $dx\, dy$ on $\mathbb{C}/\Lambda$, so the equation (1.2.1) means that

$$\int_E (u \cdot \triangle h + e^u \cdot h)\, dx\, dy = \rho \cdot h(0)$$

for every smooth function h on E. The corresponding geometric problem is the equation

$$K_g = -e^{-u}\triangle u = 1 - \rho e^{-u}\delta_0$$

for the Gaussian curvature K_g of the metric $g = \frac{1}{2}e^u(dx^2 + dy^2)$ on E, which has a highly non-classical character.

We will be mostly interested in the case when the parameter ρ of the equation (1.2.1) is an integer multiple of 4π. This integrality condition on ρ implies that every developing map of a solution of (1.2.1) is meromorphic locally on E (and not just on $E \smallsetminus \{0\}$).

Lemma 1.2.2. *Let u be a solution of* (1.2.1) *on E, where the parameter $\rho = 4\pi l$ for a positive integer l. Let f_1 be a developing map of the restriction to $E \smallsetminus \{0\}$ of u, so that f_1 is a holomorphic function on a universal covering U of $E \smallsetminus \{0\}$ whose derivative does not vanish on U. Then f_1 extends to a meromorphic function on a covering space of E in the following sense: There exists a covering space $\gamma : \tilde{E} \to E$ of E such that the following statements hold.*

(a) *The holomorphic function f_1 on U descends to a function f_2 on the covering space $\gamma^{-1}(E \smallsetminus \{0\})$ of $E \smallsetminus \{0\}$*
(b) *The holomorphic function f_2 on the open subset $\gamma^{-1}(E \smallsetminus \{0\})$ of $\tilde{E}$ extends to a meromorphic function on $\tilde{E}$.*

Equivalently, f_2 defines a holomorphic map from $\tilde{E}$ to $\mathbb{P}^1(\mathbb{C})$.

Proof. This statement is local at $0 \in E$. A proof can be found in [18, 44, 54], based on the following inequality: For a punctured disk $\Delta_\epsilon^\times$ with a small radius ϵ we have

$$\infty > \int_{\Delta_\epsilon^\times} e^u\, dA = \int_{\Delta_\epsilon^\times} \frac{8|f'|^2}{(1+|f|^2)^2}\, dA,$$

where the right hand side is the spherical area under the inverse stereographic projections covered by $f(\Delta_\epsilon^\times)$.

Alternatively, from the well-known formula

$$\triangle \log \sqrt{x^2 + y^2} = 2\pi \cdot \delta_{(0,0)}$$

on $\mathbb{R}^2$, one sees that every holomorphic map from a neighborhood V of $0 \in E$ to $\mathbb{P}^1(\mathbb{C})$ with multiplicity $l + 1$ at 0 is a developing map of a solution of (1.2.1) in V. Since any two local developing maps of any local solution of (1.2.1) differ by an element of $\mathrm{PSU}(2)$, we conclude that every developing map of every solution of (1.2.1) in a neighborhood of $0 \in E$ "is" a meromorphic function in a neighborhood of $0 \in E$. □

Remark 1.2.3. As an immediate consequence of the fact that $\triangle \log |z| = 2\pi \cdot \delta_0$, one sees that if a meromorphic function f on an open neighborhood U of $0 \in \mathbb{C}$ such that the locally L^1 function $u = \log \frac{8|f'|^2}{(1+|f|^2)^2}$ satisfies $\triangle u + e^u = \rho \cdot \delta_0$ on U for some real number ρ, then $\rho = 4\pi \cdot l$, where $l + 1 \in \mathbb{N}$ is the multiplicity of f at 0. So the parameter ρ in the equation (1.2.1) must be in $4\pi \cdot \mathbb{Z}_{\geq 0}$ if a developing map of a solution u is a meromorphic function on $\mathbb{C}$. Note also that the equation (1.2.1) has no solution when $\rho = 0$, for otherwise the elliptic curve has a metric with constant Gaussian curvature 1, contradicting the Gauss–Bonnet theorem.

Lemma 1.2.4. *Let u be a solution of the of equation* (1.2.1) *on E. Assume that the parameter ρ is of the form $\rho = 4\pi l$ where l is a positive integer.*

(1) *There exists a meromorphic function f on the universal covering $\mathbb{C}$ of E which is a developing map of u. Let $\tilde{f} : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ be the holomorphic map corresponding to the meromorphic function f on $\mathbb{C}$.*
(2) *For every $T \in \mathrm{PSU}(2)$, the meromorphic function Tf is also a developing map of u. Moreover every developing map of u is equal to Tf for some element $T \in \mathrm{PSU}(2)$.*
(3) *Suppose that z_0 is an element of the lattice Λ. The holomorphic map $\tilde{f} : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ has multiplicity $l+1$ at z_0. In other words either f is holomorphic at z_0 and f' has a zero of order l at z_0, or f has a pole of order $l+1$ at z_0.*
(4) *The holomorphic map $\tilde{f} : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ has no critical point outside Λ. In other words if $z_1 \in \mathbb{C} \smallsetminus \Lambda$, then either f is holomorphic at z_1 and $f'(z_1) \neq 0$, or f has a simple pole at z_1.*

Proof. The statement (1) is a corollary of Lemma 1.2.2. The statement (2) is a consequence of the interpretation of developing maps as local isometries from the conformal metric $\frac{1}{2}\, e^u \, dz\, d\bar{z}$ to the Fubini–Study metric $\frac{4\, dz\, d\bar{z}}{(1+z\bar{z})^2}$ and the fact that $\mathrm{PSU}(2)$ is the group of all orientation preserving isometries of $\mathbb{P}^2(\mathbb{C})$ with the Fubini–Study metric; c.f. 1.1.8. The statement (3) is a consequence of the last paragraph of the proof of Lemma 1.2.2. The statement (4) follows from 1.1.4 (a). ☐

Remark 1.2.5. We discuss how to relate solutions of (1.2.1) on $\mathbb{C}/\Lambda$ and $\mathbb{C}/(t\cdot\Lambda)$ for $t \in \mathbb{C}^\times$. Suppose that $u(z;\Lambda)$ is a solution of the singular Liouville equation (1.2.1) on $\mathbb{C}/\Lambda$ and $f(z;\Lambda)$ is a developing map on $\mathbb{C}$ for $u(z;\Lambda)$. It is easy to check that $u(w;t\Lambda) := u(t^{-1}w;\Lambda) - \log(t\bar{t})$ is a solution of (1.2.1) on the elliptic curve $\mathbb{C}/t\Lambda$ whose universal covering is the complex plane $\mathbb{C}_w$ with coordinate $w = tz$. Moreover $f(t^{-1}w;\Lambda)$ is a developing map for the solution $u(w;t\Lambda)$ on $\mathbb{C}_w$.

Of course the above "gauge transformation rules" reflects the fact that the three terms of equation (1.2.1) scale differently when the coordinate of $\mathbb{C}$ changes from z to $w = tz$ for a non-zero constant t: the equation (1.2.1) is better written as

$$2\sqrt{-1}\,\partial\bar{\partial}u + \tfrac{\sqrt{-1}}{2}e^u\, dz\wedge d\bar{z} = \rho\cdot\delta_0,$$

where the last term δ_0 is the *δ-measure* at $[0] \in \mathbb{C}/\Lambda$. The second term $\frac{\sqrt{-1}}{2}e^u\, dz\wedge d\bar{z}$ in the above equation depends on the choice of a global holomorphic 1-form on the elliptic curve, while the other two terms do not.

Lemma 1.2.6. *Let f be a meromorphic function on the universal covering $\mathbb{C}$ of $E = \mathbb{C}/\Lambda$. This function f is a developing map of a solution u of* (1.2.1) *with parameter $\rho = 4\pi l \in 4\pi\mathbb{N}$ if and only if the following conditions hold.*

(1) *The holomorphic map $\tilde{f} : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ corresponding to f has multiplicity $l + 1$ at every point above Λ, and it has no critical point outside Λ.*
(2) *For every $\omega \in \Lambda$, there exists a unique element $T \in \mathrm{PSU}(2)$ such that*

$$f(z + \omega) = (Tf)(z) \quad \forall z \in \mathbb{C}.$$

Proof. The condition (1) means that the equality $\triangle u + e^u = 4\pi\, l \cdot \sum_{\omega\in\Lambda} \delta_\omega$ holds for the function $u := \frac{8|f'|}{(1+|f|^2)2}$ on $\mathbb{C}$. The condition (2) means that u descends to a function on $\mathbb{C}/\Lambda$. □

Definition 1.2.7. Let u be a solution u of the equation (1.2.1) on E, where the parameter $\rho \in 4\pi \cdot \mathbb{Z}_{\geq 0}$. Let f be a meromorphic function on $\mathbb{C}$ which is a developing map of u.

(1) The *monodromy representation* ρ_f of the fundamental group $\pi_1(E) \cong \Lambda$ of E attached to the developing map f is the group homomorphism $\rho_f : \Lambda \to \mathrm{PSU}(2)$ such that

$$f(z + \omega) = (\rho(\omega)f)(z) \quad \forall \omega \in \Lambda,\ \forall z \in \mathbb{C}.$$

(2) The *monodromy* of the solution u of equation (1.2.1) is the PSU(2)-conjugacy class of the homomorphism $\rho_f : \Lambda \to \mathrm{PSU}(2)$, which depends only on u and not on the choice of developing map f.

1.3. Monodromy constraints. Next we review the monodromy constraints on a developing map f of a solution of (1.2.1) on E, resulting from the fact that the fundamental group Λ of E is a free abelian group of rank two. By Lemma 1.2.6, there exist $T_1 = \rho_f(\omega_1), T_2 = \rho_f(\omega_2) \in \mathrm{PSU}(2)$ with the following properties:

$$\begin{aligned} f(z + \omega_1) &= T_1 f, \\ f(z + \omega_2) &= T_2 f. \end{aligned} \tag{1.3.1}$$

In addition $T_1T_2 = T_2T_1$ in PSU(2) because Λ is commutative.

Lemma 1.3.1. *Let Γ be a commutative subgroup of* PSU(2)*.*

(1) *Suppose that Γ is isomorphic to the Klein-four group $K_4 \cong (\mathbb{Z}/2\mathbb{Z})^2$.*
  (1a) *Γ is conjugate to Γ_0, where Γ_0 is the image in* PSU(2) *of*

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

  (1b) *The centralizer subgroup of Γ in* PSU(2) *is equal to Γ.*
  (1c) *The centralizer subgroup of Γ in $\mathrm{PSL}_2(\mathbb{C})$ is also equal to Γ.*
  (1d) *The normalizer subgroup $\mathrm{N}_{\mathrm{PSU}(2)}(\Gamma)$ is isomorphic to the symmetric group S_4. In other words, $\mathrm{N}_{\mathrm{PSU}(2)}(\Gamma)$ is a semi-direct product of Γ*

and $\mathrm{PSU}(2)(\Gamma)/\Gamma \cong S_3$*, and the conjugation action of* $\mathrm{N}_{\mathrm{PSU}(2)}(\Gamma)$ *on* Γ *induces an isomorphism*

$$\mathrm{N}_{\mathrm{PSU}(2)}(\Gamma)/\Gamma \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{grp}}(\Gamma) \cong \mathrm{Perm}(\Gamma \smallsetminus \{\mathrm{Id}\}) \cong S_3,$$

where $\mathrm{Perm}(\Gamma \smallsetminus \{0\})$ *is the permutations group of the set* $\Gamma \smallsetminus \{\mathrm{Id}\}$.

(1e) *The subset* $\{x \in \mathrm{PSL}(2,\mathbb{C}) \mid x \cdot \Gamma \cdot x^{-1} \subset \mathrm{PSU}(2)\}$ *of* $\mathrm{PSL}_2(\mathbb{C})$ *is equal to* $\mathrm{PSU}(2)$.

(2) *If* Γ *is not isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$*, then* Γ *is contained in a maximal torus of* $\mathrm{PSU}(2)$*; i.e. there exists an element* $T_0 \in \mathrm{PSU}(2)$ *such that* $T_0 \cdot \Gamma \cdot T_0^{-1}$ *is contained in the image in* $\mathrm{PSU}(2)$ *the diagonal maximal torus*

$$\left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix} : \theta \in \mathbb{R}/\pi\mathbb{Z} \right\} \subset \mathrm{SU}(2)$$

Proof. The spectral theorem tells us that every element of $\mathrm{U}(2)$ is conjugate in $\mathrm{U}(2)$ to a diagonal matrix. Using this it is easy to verify the following assertion, whose proof is omitted here.

Suppose that u *is a non-trivial element of* $\mathrm{PSU}(2)$.

- *If* $u^2 \neq 1$ *in* $\mathrm{PSU}(2)$*, then the centralizer* $\mathrm{Z}_{\mathrm{PSU}(2)}(u)$ *of* u *is a maximal torus in* $\mathrm{PSU}(2)$*, i.e. a conjugate of the image of the diagonal maximal torus*

$$\left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix} : \theta \in \mathbb{R}/\pi\mathbb{Z} \right\}$$

- *If* u *is an element of order two in* $\mathrm{PSU}(2)$*, then the centralizer subgroup* $\mathrm{Z}_{\mathrm{PSU}(2)}(u)$ *of* u *in* $\mathrm{PSU}(2)$ *is a semi-direct product of a maximal torus of* $\mathrm{PSU}(2)$ *with a group of order two, equal to the normalizer of a maximal torus. Moreover* $\mathrm{Z}_{\mathrm{PSU}(2)}(u)$ *contains a unique subgroup which is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$.

The statement (2) follows, so do (1a), (1b) and (1c).

To prove (1d), by (1a) and (1b) it suffices to show that the normalizer subgroup $\mathrm{N}_{\mathrm{PSU}(2)}(\Gamma_0)$ on Γ_0 contains a subgroup S of order 6 which intersect Γ trivially. Let δ be the image of $\begin{pmatrix} 0 & -e^{-\pi\sqrt{-1}/4} \\ e^{\pi\sqrt{-1}/4} & 0 \end{pmatrix}$ in $\mathrm{PSU}(2)$ and let γ be the image of $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ \sqrt{-1} & \sqrt{-1} \end{pmatrix}$ in $\mathrm{PSU}(2)$. It is straightforward to check that δ has order 2 and induces a transposition on $\Gamma \smallsetminus \{\mathrm{Id}\}$, γ has order 3 and $\delta \cdot \gamma \cdot \delta^{-1} = \gamma^{-1}$. It follows that $\mathrm{N}_{\mathrm{PSU}(2)}(\Gamma_0)$ is a semi-direct product $\Gamma_0 \rtimes S_3$, so $\mathrm{N}_{\mathrm{PSU}(2)}(\Gamma_0)$ is isomorphic to S_4. We have proved (1d). (Alternatively, it is well known that $\mathrm{PSU}(2)$ contains a finite subgroup isomorphic to S_4. The statement (1d) also follows from this fact, (1a) and (1b).)

Finally the statement (1e) follows from (1a), (1d) and (1c): Suppose that $x \in \mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{Ad}(x)(\Gamma_0) = x \cdot \Gamma_0 \cdot x^{-1} \subset \mathrm{PSU}(2)$. By (1a) and (1d), there exists $y \in \mathrm{PSU}(2)$ such that $y \cdot x$ commutes with every element of Γ_0. By (1c) $y \cdot x \in \Gamma_0$, hence $x \in y^{-1} \cdot \Gamma_0 \subset \mathrm{PSU}(2)$. □

Remark. The group $\mathrm{N}_{\mathrm{PSU}(2)}(\Gamma)$ is also isomorphic to $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})/\{\pm \mathrm{I}_2\}$, the quotient of $\mathrm{SL}_2(\mathbb{Z})$ by the subgroup generated by the principal congruence subgroup of level 4 and $\{\pm \mathrm{I}_2\}$.

Corollary 1.3.2. *Let $\rho : \Lambda \to \mathrm{PSU}(2)$ be a group homomorphism.*

(i) *If the image of ρ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, then ρ is conjugate to the homomorphism which sends ω_1 to the image of $\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$ and ω_2 to the image of $\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$.*

(ii) *If the image of ρ is not isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, then there exists real numbers θ_1, θ_2 such that ρ is conjugate to the homomorphism which sends ω_i to $\begin{pmatrix} e^{\sqrt{-1}\theta_1} & 0 \\ 0 & e^{\sqrt{-1}\theta_2} \end{pmatrix}$ for $i = 1, 2$.*

Lemma 1.3.3 below follows from Corollary 1.3.2 and Lemma 1.2.4 (1), (2).

Lemma 1.3.3. *Let u be solution of equation* (1.2.1) *where the parameter $\rho > 0$ is an integer multiple of 4π.*

Type I. *If the image in* $\mathrm{PSU}(2)$ *of the monodromy of u is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, then there exists a developing map f of u such that*

$$\begin{aligned} f(z+\omega_1) &= -f(z) \quad \forall z, \\ f(z+\omega_2) &= \frac{1}{f(z)} \quad \forall z. \end{aligned} \tag{1.3.2}$$

The choice of f is unique up to sign and inverse. More precisely, the set $\{f, -f, f^{-1}, -f^{-1}\}$ is uniquely determined by the solution u.

Type II. *Suppose that the image in* $\mathrm{PSU}(2)$ *of the monodromy of u is not isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. There exists a developing map f of u and two real numbers θ_1, θ_2 such that*

$$\begin{aligned} f(z+\omega_1) &= e^{2i\theta_1} f(z) \quad \forall z, \\ f(z+\omega_2) &= e^{2i\theta_2} f(z) \quad \forall z. \end{aligned} \tag{1.3.3}$$

Moreover, if $\{\theta_1, \theta_2\} \not\subseteq \frac{1}{2}\mathbb{Z}$ then the set $\mathbb{C}_1^\times \cdot f \cup \mathbb{C}_1^\times \cdot f^{-1}$ is uniquely determined by u, where $\mathbb{C}_1^\times := \{w \in \mathbb{C} : |w| = 1\}$.

Definition 1.3.4. (a) Let f be a solution of equation (1.2.1) where the parameter $\rho > 0$ is an integer multiple of 4π. If the image of the monodromy representation ρ_f of f is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, then say that f is *of type I*; otherwise we say that f is *of type II*.

(b) A developing map f which satisfies equation (1.3.2) (respectively (1.3.3)) will be said to be *normalized* of type I (respectively type II).

Lemma 1.3.5. *Let f be a developing map of a solution u of equation* (1.2.1) *where the parameter $\rho > 0$ is an integer multiple of 4π.*

(1) *If f is of type I and $T \in \mathrm{PGL}_2(\mathbb{C})$ is a linear fractional transformation such that $T \cdot f$ is again a developing map of a solution of equation* (1.2.1) *with the same parameter ρ, then $T \in \mathrm{PSU}(2)$ and Tf is a developing map of the same solution u of (1.2.1).*

(2) *Suppose that f is of type II. There exists a closed subgroup A of $\mathrm{PGL}_2(\mathbb{C})$, conjugate to the image in $\mathrm{PGL}_2(\mathbb{C})$ of the diagonal non-compact real torus $A_0 := \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^{\times}_{>0} \right\}$, such that the following statements hold.*

(2a) *$T \cdot f$ is a developing map of a solution u_{Tf} of equation* (1.2.1) *with the same parameter ρ.*

(2b) *$u_{T_1 f} \neq u_{T_2 f}$ for any two distinct elements T_1, T_2 in A.*

Proof. The statement (1) follows from Lemma 1.2.6 and Lemma 1.3.1 (1e). To show (2), we may assume that f is normalized of type II and take A to be the image in $\mathrm{PGL}(2, \mathbb{C})$ of A_0. Then the statement (2a) follows from Lemma 1.2.6 and Lemma 1.3.3. The statement (2b) follows from Lemma 1.1.5 (3) because the only element of A which is conjugate in $\mathrm{PGL}_2(\mathbb{C})$ to an element in $\mathrm{PSU}(2)$ is the unity element of A. □

1.3.6. *Logarithmic derivatives of normalized developing maps.* In this article we approach the mean field equations (1.2.1) with $\rho = 4\pi l$, $l \in \mathbb{N}_{>0}$ through the logarithmic derivative

$$g := (\log f)' = \frac{f'}{f}.$$

of a normalized developing map f of a solution of (1.2.1). Recall that such developing maps are meromorphic functions f on $\mathbb{C}$ satisfying 1.2.6 (1) and either of equations (1.3.2), (1.3.3).

Lemma 1.3.7. *Suppose that f is a normalized developing map of a solution of* (1.2.1)*, and $l := \rho/4\pi$ is a positive integer. Let $g := f'/f$.*

(1) *The developing map f on $\mathbb{C}$ is holomorphic and non-zero at every point of Λ; i.e. $f(\Lambda) \subset \mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, \infty\}$.*

(2) *The meromorphic function g on $\mathbb{C}$ has a zero of order l at every point of Λ, no zeros and at most simple poles on $\mathbb{C} \smallsetminus \Lambda$.*

(3) *If f is of type II, then g descends to a meromorphic function on $E = \mathbb{C}/\Lambda$.*

(4) *If f is of type I, then g descends to a meromorphic function $\bar{g}$ on the double cover $E' = \mathbb{C}/\Lambda'$, where $\Lambda' = \mathbb{Z} \cdot \omega_1 + \mathbb{Z} \cdot 2\omega_2$.*

Proof. The statements (3) and (4) are immediate from the equations (1.3.2) and (1.3.3) for normalized developing maps.

Clearly g has at most simple poles on $\mathbb{C}$. Lemma 1.2.4 (4) implies that g has no zeros on $\mathbb{C} \setminus \Lambda$. For any point $z_0 \in \Lambda$, if f has either a zero or a pole at z_0, then g will have a simple pole at every point of Λ, and the meromorphic function $\bar{g}$ on $E' = \mathbb{C}/\Lambda'$ defined by g will have no zero but at least one pole, a contradiction. Therefore f has values in $\mathbb{C}^\times$ in a neighborhood of z_0; we have proved the statement (1). Lemma 1.2.4 (3) then implies that g has a zero of order l at every point of Λ. We have proved statement (2). $\square$

1.4. Type I solutions. In this subsection we will show that the existence of solution of (1.2.1) such that the image of the monodromy representation is $(\mathbb{Z}/2\mathbb{Z})^2$ implies that the parameter $l = \rho/4\pi$ is an *odd* positive integer.

1.4.1. *Notation for type I.*

- Let $\omega_1' = \omega_1$, $\omega_2' = 2\omega_2$ and let $\Lambda' := \mathbb{Z} \cdot \omega_1' + \mathbb{Z} \cdot \omega_2'$.
- Let $\wp(z) = \wp(z; \Lambda')$ be the Weierstrass $\wp$-function for the lattice $\Lambda' \subset \mathbb{C}$.
- Let $\zeta(z) = \zeta(z; \Lambda') = -\int^z \wp(u)du = z^{-1} + \cdots$ be the Weierstrass ζ-function and let $\sigma(z) = \sigma(z; \Lambda') = \exp \int^z \zeta(u)du = z + \cdots$ be the Weierstrass σ-functions for $\Lambda' \subset \mathbb{C}$.
- Let g be the logarithmic derivative of the normalized developing map f of a type I solution u of (1.2.1). Let $\bar{g}$ be the function on E' defined by g.

The standard references for elliptic functions are [67, Ch. 20], [1, Ch. 7] and [39, Ch. 18 §1]; we have followed the notation in [1, Ch. 7]:[15]

- ω_1', ω_2' form a $\mathbb{Z}$-basis of the lattice Λ' with $\mathrm{Im}(\omega_2'/\omega_1') > 0$. Note that the latter condition means that (ω_1', ω_2') is an oriented basis for the standard orientation of the complex plane.
- $\eta_i = \eta(\omega_i'; \Lambda')$ for $i = 1, 2$, where $\omega \mapsto \eta(\omega; \Lambda')$ is the $\mathbb{Z}$-linear function from Λ' to $\mathbb{C}$ such that

  $$\zeta(z + \omega; \Lambda') = \zeta(z) + \eta(\omega; \Lambda') \qquad \forall\, z \in \mathbb{C},\ \forall\, \omega \in \Lambda'.$$

- The classical Legendre relation

  $$\eta_1 \cdot \omega_2' - \eta_2 \cdot \omega_1' = 2\pi\sqrt{-1}$$

  means that

  $$\eta(\alpha)\beta - \eta(\beta)\alpha = 2\pi\sqrt{-1}\,\psi(\alpha, \beta) \quad \forall\, \alpha, \beta \in \Lambda',$$

  where $\psi : \Lambda' \times \Lambda' \to \mathbb{Z}$ is the alternating pairing on Λ' which sends an oriented $\mathbb{Z}$-basis (ω_1', ω_2') of Λ' to 1.

[15] The notation in [39] is the same as in [1] except that $\mathrm{Im}(\omega_1/\omega_2) > 0$. The notation in [67] is: $2\omega_1', 2\omega_2'$ form a $\mathbb{Z}$-basis of Λ' with $\mathrm{Im}(\omega_2'/\omega_1') > 0$, and $\zeta(z + 2\omega_i'; \Lambda') = \zeta(z) + 2\eta_i(\Lambda')$ for $i = 1, 2$.

1.4.2. Recall that the type I condition implies that

$$g(z+\omega_1) = g(z) \quad \forall z, \qquad (1.4.1)$$
$$g(z+\omega_2) = -g(z) \quad \forall z.$$

According to Lemma 1.3.7, the meromorphic function $\bar{g}$ on E' has zeros of order l at the two points of Λ/Λ', no zeros and at most simple poles elsewhere on E'.

From 1.3.7 (1), the principal divisor $(\bar{g})$ of the meromorphic function $\bar{g}$ on E' has the form

$$(\bar{g}) = \ell \cdot 0_{E'} + \ell \cdot [\omega_2]_{E'} - \sum_{P \in \bar{g}^{-1}(\infty)} P,$$

where $[\omega_2]_{E'} = \omega_2 \bmod \Lambda'$ is the image of ω_2 in E', and $\sum_{P\in\bar{g}^{-1}(\infty)} P$ is the polar divisor $(\bar{g})_\infty$ of $\bar{g}$, an effective divisor of degree $2l$ which is a sum of 2ℓ distinct points of E'. Clearly the sum of the polar divisor under the group law of E' is equal to ℓ times the 2-torsion point $[\omega_2]_{E'}$. We know from the condition $g(z+\omega_2) = -g(z)$ that the polar divisor $(\bar{g})$ of $\bar{g}$ is stable under the translation by the 2-torsion point $[\omega_2]_{E'}$. Let $P_1, \ldots, P_l$ be a set of representatives of the quotient of $\bar{g}^{-1}(\infty)$ under the translation action by $[\omega_2]_{E'}$. The sum $\mu_{E'}(P_1, \ldots, P_l) = P_1 +_{E'} \cdots +_{E'} P_l$ of this set of representatives under the group law of E' is a 2-torsion point because the sum of the polar divisor $(\bar{g})_\infty$ is $[2]_{E'}([\omega_2]_{E'})$. Moreover it is clear that the image of $\mu_{E'}(P_1, \ldots, P_l)$ in the quotient group $E'[2]/\{0_{E'}, [\omega_2]_{E'}\}$ is independent of the choice of representatives $P_1, \ldots, P_l$. The following lemma says that this image is equal to the *non-trivial* element of $E'[2]/\{0_{E'}, [\omega_2]_{E'}\}$.

Lemma 1.4.3. *Notation as above. The sum $P_1 +_{E'} \cdots +_{E'} P_l$ in E' of any set of representatives of the quotient $\bar{g}^{-1}(\infty)/\{0_{E'}, [\omega_2]_{E'}\}$ is congruent to the non-zero 2-torsion point $\frac{\omega_1}{2} \bmod \Lambda'$ modulo the subgroup $\{0_{E'}, [\omega_2]_{E'}\}$ of the group of all 2-torsion points of E'.*

1.4.4. Lemma 1.4.3 is a consequence of a more precise statement Lemma 1.4.5; the latter uses Weierstrass σ-function. The σ-function is essentially the odd theta function $\theta_{11}(z)$ with half-integer characteristics, up to rescaling of the z-variable, a harmless factor $-\pi \cdot e^{-\eta_2 z^2/2}$ and the product

$$\theta_{00}(0) \cdot \theta_{01}(0) \cdot \theta_{10}(0)$$

of three even theta constants. (We adopt the notations of theta functions from [49].) We recall some of the basic properties of the σ-function below.

(i) The function

$$\sigma(z) = \sigma(z;\Lambda') = z \cdot \prod_{\omega \in \Lambda' \smallsetminus \{0\}} \left[\left(1 - \frac{z}{\omega}\right) \cdot \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)\right]$$

is an entire odd function on $\mathbb{C}$, with simples zeros on points of Λ' and non-zero elsewhere. In addition $\zeta(z)$ satisfies the following transformation law for translation by elements of Λ'.

$$\sigma(z+\alpha) = \epsilon(\alpha)\cdot e^{\eta(\alpha)(z+\frac{\alpha}{2})}\cdot\sigma(z) \qquad \forall\, z\in\mathbb{C},\ \forall\,\alpha\in\Lambda', \tag{1.4.2}$$

where $\epsilon:\Lambda'\to\{\pm 1\}$ is the quadratic character on Λ' given by

$$\epsilon(\alpha) = \begin{cases} 1 & \text{if } \alpha\in 2\Lambda' \\ -1 & \text{if } \alpha\notin 2\Lambda' \end{cases}$$

(ii) Suppose that m is a positive integer and $a_1,\dots,a_m;b_1,\dots,b_m$ are elements of $\mathbb{C}$. The meromorphic function

$$h(z;a_1,\dots,a_m;b_1,\dots,b_m) := \frac{\prod_{i=1}^m \sigma(z-a_i)}{\prod_{i=1}^m \sigma(z-b_i)}$$

on $\mathbb{C}$ is Λ'-periodic if and only if $\sum_{i=1}^m a_i = \sum_{i=1}^m b_i$. Moreover if $\sum_{i=1}^m a_i = \sum_{i=1}^m b_i$, then the principle divisor of the meromorphic function on $E' = \mathbb{C}/\Lambda'$ defined by $\prod_{i=1}^m \sigma(z-a_i)/\prod_{i=1}^m \sigma(z-b_i)$ is

$$\sum_{i=1}^m [a_i]_{E'} - \sum_{i=1}^m [b_i]_{E'},$$

where $[a_i]_{E'}$ (respectively $[b_i]_{E'}$) is the image of a_i (respectively b_i) in E' for $i=1,\dots,m$.

(iii) Suppose that $(a_1,\dots,a_m;b_1,\dots,b_m)$ and $(a'_1,\dots,a'_m;b'_1,\dots,b'_m)$ are two $2m$-tuples of complex numbers such that $\sum_{i=1}^m a_i = \sum_{i=1}^m b_k$, $\sum_{i=1}^m a'_i = \sum_{i=1}^m b'_i$, $a'_i \equiv a_i \pmod{\Lambda'}$ and $b'_i \equiv b_i \pmod{\Lambda'}$ for $i = 1,\dots,m$. Then

$$\frac{\prod_{i=1}^m \sigma(z-a_i)}{\prod_{i=1}^m \sigma(z-b_i)} = \frac{\prod_{i=1}^m \sigma(z-a'_i)}{\prod_{i=1}^m \sigma(z-b'_i)}$$

(iv) Let h be a non-constant meromorphic function on E'. Let

$$a_1,\dots,a_m,b_1,\dots,b_m$$

be elements of $\mathbb{C}$ such that the principle divisor of h is equal to $\sum_{i=1}^m [a_i]_{E'} - \sum_{i=1}^m [b_i]_{E'}$. Then there exists a constant $A\in\mathbb{C}^\times$ such that

$$h([z]_{E'}) = A\cdot\frac{\prod_{i=1}^m \sigma(z-a_i)}{\prod_{i=1}^m \sigma(z-b_i)} \qquad \forall\, z\in\mathbb{C}.$$

Note that (ii) and (iii) are consequences of the transformation law (1.4.2) in (i), and (iv) follows from (ii).

Lemma 1.4.5. [44] *Let l be a positive integer. Let $p_1,\dots,p_l;q_1,\dots,q_l$ be elements in $\mathbb{C}\smallsetminus\Lambda$ satisfying*

$$\sum p_i + \sum q_i = l\omega_2 \tag{1.4.3}$$

and $p_i+\omega_2 \equiv q_i \pmod{\Lambda'}$ for $i=1,\dots,l$. Let

$$h(z) = \frac{\sigma^l(z)\sigma^l(z-\omega_2)}{\prod_{i=1}^l \sigma(z-p_i)\prod_{i=1}^l \sigma(z-q_i)} \tag{1.4.4}$$

be the meromorphic function on $\mathbb{C}$ *attached to the* $4l$*-tuple*

$$(0, \ldots, 0, \omega_2, \ldots, \omega_2; p_1, \ldots, p_l, q_1, \ldots, q_l)$$

as in 1.4.4 (ii). *Note that* $h(z)$ *descends to a meromorphic function on* E' *whose principle divisor is*

$$l \cdot 0_{E'} + l \cdot [\omega_2]'_E - \sum_{i=1}^{l} [p_i]_{E'} - \sum_{i=1}^{l} [q_i]_{E'}$$

according to 1.4.4 (ii).

(a) *The function* $h(z)$ *satisfies* $h(z + \omega_2) = -h(z)$ *if and only if*

$$\sum_{i=1}^{l} p_i \equiv \tfrac{1}{2}\omega_1 \pmod{\Lambda}.$$

Namely, $\sum_{i=1}^{l} p_i$ *is congruent to either* $\frac{1}{2}\omega_1$ *or* $\frac{1}{2}\omega_1 + \omega_2$ *modulo* Λ'.

(b) *Suppose that* $p_1 + \cdots + p_l \equiv \frac{1}{2}\omega_1 \pmod{\Lambda'}$. *There exist elements* $p'_1, \ldots, p'_l; q'_1, \ldots, q'_l$ *in* $\mathbb{C}$ *satisfying the following conditions.*
(b1) $p'_i \equiv p_i \pmod{\Lambda'}$ *and* $q'_i \equiv q_i \pmod{\Lambda'}$ *for* $i = 1, \ldots, l$,
(b2) $\sum_{i=1}^{l} p'_i = \frac{1}{2}\omega_1$,
(b3) $q'_i = p'_i + \omega_2$ *for* $i = 1, \ldots, l-1$ *and* $q'_l = p'_l + \omega_2 - \omega_1$,
(b4) $h(z) = \dfrac{\sigma^l(z)\sigma^l(z-\omega_2)}{\prod_{i=1}^{l} \sigma(z-p'_i) \prod_{i=1}^{l} \sigma(z-q'_i)}$.

(c) *Suppose that* $p_1 + \cdots + p_l \equiv \frac{1}{2}\omega_1 + \omega_2 \pmod{\Lambda'}$. *There exist elements* $p'_1, \ldots, p'_l; q'_1, \ldots, q'_l$ *in* $\mathbb{C}$ *satisfying the following conditions.*
(c1) $p'_i \equiv p_i \pmod{\Lambda'}$ *and* $q'_i \equiv q_i \pmod{\Lambda'}$ *for* $i = 1, \ldots, l$,
(c2) $\sum_{i=1}^{l} p'_i = \frac{1}{2}\omega_1 + \omega_2$
(c3) $q'_i = p'_i + \omega_2$ *for* $i = 1, \ldots, l-1$, *and* $q'_l = p'_l - \omega_2 - \omega_1$,
(c4) $h(z) = \dfrac{\sigma^l(z)\sigma^l(z-\omega_2)}{\prod_{i=1}^{l} \sigma(z-p'_i) \prod_{i=1}^{l} \sigma(z-q'_i)}$.

Proof. Clearly (b2) and (b3) implies that

$$\sum_{i=1}^{m} p'_i + \sum_{i=1}^{m} q'_i = l \cdot \omega_2,$$

therefore (b4) follows from (b1)–(b3) by 1.4.4 (iii). Similarly (c1)–(c3) implies (c4).

We will prove (a), (b1)–(b3), and (c1)–(c3) simultaneously from the transformation law (1.4.2) and the condition (1.4.3).

Because $p_i + \omega_2 \equiv q_i \pmod{\Lambda'}$ for each i, (1.4.3) implies that $\sum_{i=1}^{l} p_i \equiv m \cdot \frac{\omega_1}{2} + n \cdot \omega_2 \pmod{\Lambda'}$ for integers $m, n \in \{0, 1\}$. By adjusting p_l we may assume that $\sum_{i=1}^{l} p_i = m \cdot \frac{\omega_1}{2} + n \cdot \omega_2$. Let

- $p'_i := p_i$ for $i = 1, \ldots, l$,
- $q'_i := p_i + \omega_2$ for $i = 1, \ldots, l-1$, and
- $q'_l := p_l + \omega_2 - (m\omega_1 + 2n\omega_2)$.

By 1.4.4 (iii) $h(z) = \dfrac{\sigma^l(z)\sigma^l(z-\omega_2)}{\prod_{i=1}^l \sigma(z-p_i') \prod_{i=1}^l \sigma(z-q_i')}$ holds. It remains to use the transformation law (1.4.2) to see whether $h(z+\omega_2) = -h(z)$.

There are only four possibilities for the pair (m,n), namely

$$(m,n) = \quad \text{(i) } (0,0), \quad \text{(ii) } (1,0), \quad \text{(iii) } (0,1), \quad \text{or} \quad \text{(iv) } (1,1).$$

One verifies by direction calculations that $h(z+\omega_2) = h(z)$ if $m = 0$, while $h(z+\omega_2) = -h(z)$ if $m = 1$.

For instance when $(m,n) = (0,0)$, $h(z+\omega_2)$ and $h(z)$ differ by the factor of automorphy

$$\frac{(-1)^l \exp(l\eta_2 z)}{(-1)^l \exp\left[\eta_2 \sum_{i=1}^l (z-p_i)\right]} = 1,$$

meaning that $h(z+\omega_2) = h(z)$.

When $(m,n) = (1,0)$, $h(z+\omega_2)$ and $h(z)$ differ by the factor

$$\frac{(-1)^{l-(l+1)} \exp(l\eta_2 z)}{\exp\left[\eta_2 \sum_{i=1}^{l-1}(z-p_i) + (\eta_2-\eta_1)(z-p_l-\frac{1}{2}\omega_1) + \eta_1(z-p_l-\frac{1}{2}\omega_1)\right]}$$

which is -1 since $\sum p_i = \frac{1}{2}\omega_1$. The other two cases are checked similarly. We have proved lemmas 1.4.5 and 1.4.3. □

1.4.6. In order to construct type I solutions from the elliptic function g, we need to find all the other constraints imposed on its poles.

Let $p_1, \ldots, p_l$ be points of $\mathbb{C}$ such that $\bigcup_{i=1}^l p_i + \Lambda'$ are the simple zeroes of the developing map f and $\bigcup_{i=1}^l p_i + \omega_2 + \Lambda'$ are the simple poles of f. Let $P_i := p_i \bmod \Lambda'$ and let $Q_i := p_i + \omega_2 \bmod \Lambda'$ for $i = 1, \ldots, l$. We know that

$$P_1, \ldots, P_l, Q_1, \ldots, Q_l$$

are $2l$ distinct points of $(\mathbb{C} \smallsetminus \Lambda)/\Lambda' = E' \smallsetminus \{0_{E'}, [\omega_2]_{E'}\}$; equivalently

$$p_i - p_j \notin \Lambda \quad \forall\, i \neq j,\ 1 \leq i,j \leq l.$$

We also know that

$$\sum_{i=1}^{\ell} p_i \equiv \frac{\omega_1}{2} \pmod{\Lambda}$$

according to Lemma 1.4.3. By 1.4.4 (iv) we know that there exists a constant $A \in \mathbb{C}^\times$ such that

$$g(z) = A \cdot \frac{\sigma^l(z) \cdot \sigma^l(z-\omega_2)}{\prod_{i=1}^l \sigma(z-p_i) \cdot \prod_{i=1}^l \sigma(z-q_i)}, \tag{1.4.5}$$

where $q_1, \ldots, q_l$ are elements of $\mathbb{C}$ such that

$$q_i \equiv p_i + \omega_2 \pmod{\Lambda'}\ \forall\, i, \quad \text{and} \quad \sum_{i=1}^l p_i + \sum_{i=1}^l q_i = l\omega_2. \tag{1.4.6}$$

Notice that the residue of $g(z)$ at $z = p_j$ is given by Ar_j for $j = 1, \ldots, l$, where

$$r_j = \frac{\sigma^l(p_j) \cdot \sigma^l(p_j - \omega_2)}{\prod_{i=1,\neq j}^{l} \sigma(p_j - p_i) \cdot \prod_{i=1}^{l} \sigma(p_j - q_i)} \qquad \text{for } j = 1, \ldots, l. \tag{1.4.7}$$

It is immediate from 1.4.4 (ii) that the formula (1.4.7) for r_j is independent of the choice of $q_1, \ldots, q_l$ satisfying (1.4.6), with $p_1, \ldots, p_l$ fixed, and also independent of the choice of $p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_l$ in their respective congruence classes modulo Λ' when the q_i's and $p_1 + \cdots + p_{j-1} + p_{j+1} + \cdots + p_l$ are fixed. One checks by a routine calculation that the right hand side of the formula (1.4.7) remains the same when p_j is replaced by $p_j + \alpha$ and q_j is replaced by α for any element $\alpha \in \Lambda'$. So the right hand side of the formula (1.4.6) is a meromorphic function of $(P_1, \ldots, P_l) \in E' \times \cdots \times E'$.

Lemma 1.4.7. *Let $p_1, \ldots, p_l$ be elements of $\mathbb{C}$ such that $\bigcup_{i=1}^{l} p_i + \Lambda'$ are the zeroes of the developing map f and $\bigcup_{i=1}^{l} p_i + \omega_2 + \Lambda'$ are the poles of f. Let $q_1, \ldots, q_l$ be elements of $\mathbb{C}$ satisfying the conditions in* (1.4.6). *Then*

$$r_1 = r_2 = \cdots = r_l. \tag{1.4.8}$$

where the non-zero complex numbers $r_1, \ldots, r_l$ are defined by (1.4.7) *and the elements $q_1, \ldots, q_l \in \mathbb{C}$ appearing in the formula* (1.4.7) *satisfies the conditions in* (1.4.6).

Proof. Write $g = f'/f$ as in (1.4.5) for a suitable constant $A \in \mathbb{C}^{\times}$. Then $Ar_j = 1$ for $j = 1, \ldots, l$, hence $r_1 = r_2 = \cdots = r_l$. $\square$

Proposition 1.4.8. *Let $p_1, \ldots, p_l$ be elements of $\mathbb{C}$ with the following properties.*

(i) $\sum_{i=1}^{l} p_i \equiv \omega_1/2 \pmod{\Lambda}$,
(ii) $p_i - p_j \not\equiv 0 \pmod{\Lambda}$ *whenever $i \neq j$, and*
(iii) *the residue equalities* (1.4.8) *hold, where $r_1, \ldots, r_l$ are defined by* (1.4.7) *and the elements $q_1, \ldots, q_l \in \mathbb{C}$ satisfy the conditions in* (1.4.6).

Let h be an elliptic function on E' defined by (1.4.4). *Let $A := r_1^{-1}$ and let $g_1 := A \cdot h$.*

(a) $s_1 = \cdots = s_l = -r_1 = \cdots = -r_l$, *where*

$$s_j = \frac{\sigma^l(q_j) \cdot \sigma^l(q_j - \omega_2)}{\prod_{i=1}^{l} \sigma(q_j - p_i) \cdot \prod_{i=1,\neq j}^{l} \sigma(q_j - q_i)} \quad \text{for } j = 1, \ldots, l. \tag{1.4.9}$$

Consequently the residue of the simple pole P_i (respectively Q_i) of the meromorphic function $A \cdot h$ on E' is equal to 1 *(respectively* -1*). Here $P_i := p_i \bmod \Lambda' \in E'$ and $Q_i := q_i \bmod \Lambda' \in E'$ for $i = 1, \ldots, l$.*

(b) *If h is an odd function, then the following statements hold.*
  (b1) *The subset $\{P_1, \ldots, P_l\} \subset E' \smallsetminus \{0_{E'}, [\omega_2]'_E\}$ is stable under the involution of E' induced by "multiplication by -1".*
  (b2) *Exactly one of $P_1, \ldots, P_l$ is a two-torsion point of E'; this point is either $\frac{\omega_1}{2} \bmod \Lambda'$ or $\frac{\omega_1}{2} + \omega_2 \bmod \Lambda'$.*

(b3) *l is an odd integer.*

(c) *Conversely suppose that the condition* (b1) *is satisfied, or equivalently conditions all* (b1)–(b3) *hold. Then h is an odd function; i.e. $h(-z) = -h(z)$ for all $z \in \mathbb{C}$.*

(d) *Assume h is an odd function, or equivalently that conditions* (b1)–(b3) *hold.*

(d1) *There exists a normalized type I developing map f_1 of a solution of* (1.2.1) *with parameter $\rho = 4\pi l$ such that $f_1'/f_1 = g_1$.*

(d2) *f_1 and $-f_1$ are the only normalized type I developing map whose logarithmic derivative is g_1.*

(d3) *f_1 is an even function, i.e. $f_1(-z) = f_1(z)$ for all $z \in \mathbb{C}$.*

Proof. We know that $h(z + \omega_2) = -h(z)$ for all $z \in \mathbb{C}$ by Lemma 1.4.5 (a). So the statement (a) follows from the assumption that $r_1 = \cdots = r_l$.

The set $\{P_1, \ldots, P_l\}$ is the set of all (simple) poles with residue r_1 of the meromorphic differential hdz on E'. The assumption that h is odd means that hdz is invariant under "multiplication by -1", so the statement (b1) follows. The statement (b2) follows because of assumption (i). The statement (b3) follows from (a) and (b2).

Suppose that (b1)–(b3) hold. Let $n = (2l-1)/2$. After renumbering the p_i's we may assume that $P_{n+1} = -P_1, P_{n+2} = -P_2, \ldots, P_{2n} = -P_n$ and $P_l = [\omega_2]_{E'}$. According to 1.4.4 (iii), we have

(1.4.10)
$$h(z) = \frac{\sigma^{2n+1}(z) \cdot \sigma^{n+1}(z-\omega_2) \cdot \sigma^n(z+\omega_2)}{[\prod_{i=1}^n \sigma(z-p_i) \cdot \sigma(z+p_i)] \cdot [\prod_{i=1}^n \sigma(z-p_i-\omega_2) \cdot \sigma(z+p_i+\omega_2)] \cdot \sigma(z-\frac{\omega_1}{2}) \cdot \sigma(z+\frac{\omega_1}{2}-\omega_2)}$$

Using the fact that $\sigma(z)$ is an odd function, we get

$$\begin{aligned} \frac{h(-z)}{h(z)} &= \frac{\sigma(z+\omega_2) \cdot \sigma(z-\frac{\omega_1}{2}) \cdot \sigma(z+\frac{\omega_1}{2}-\omega_2)}{\sigma(z-\omega_2) \cdot \sigma(z+\frac{\omega_1}{2}) \cdot \sigma(z-\frac{\omega_1}{2}+\omega_2)} \\ &= (-1) \cdot e^{\eta_2 \cdot z} \cdot e^{\eta_1 \cdot z} \cdot e^{(\eta_1-\eta_2)\cdot z} = -1 \end{aligned}$$

by the transformation law for the σ-function. We have proved (c).

Assume again that (b1)–(b3) hold, so that $l = 2n+1$ is odd and $h(z)$ is an odd function. Then $g_1(z)$ is an odd meromorphic function on E' which has simple poles with residue 1 at $P_1, \ldots, P_{2n+1}$ and has simple poles with residue -1 at $Q_1, \ldots, Q_{2n+1}$. From the proof of 1.4.8 (d) may and so assume that $p_1 = p_{n+i}$ for $i = 1, \ldots, n$, and $p_n = \frac{\omega_1}{2}$, $q_n = \frac{\omega_1}{2} + \omega_2$, so that $h(z)$ is given by equation (1.4.10). For each of the $4n+2$ poles of $g_1(z)\,dz$, the integral along a sufficiently small circle around the pole is $\pm 2\pi$. Hence the line integral $\displaystyle\int_0^z g_1(w)\,dw$ is well-defined as an element of $\mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$ and the function

$$f_2(z) = \exp \int_0^z g_1(w)\,dw$$

is a well-defined meromorphic function on $\mathbb{C}$ with simple poles at points in the union $\bigcup_{i=1}^{2n+1} q_i + \Lambda'$ of Λ'-cosets, simple zeros at points in $\bigcup_{i=1}^{2n+1} p_i +$

Λ', neither zero nor pole elsewhere on $\mathbb{C}$. In particular $f_2(z)$ is holomorphic and non-zero at points of Λ. Notice that $f_2(z) = f_2(-z)$ for all $z \in \mathbb{C}$ because g_1 is odd.

The fact that $g_1(w)dw$ is invariant under translation by ω_1 implies that

$$f_2(z+\omega_1) = \int_0^{\omega_1} g_1(w)\,dw \cdot f_2(z) \qquad \forall\, z \in \mathbb{C}\,.$$

Similarly the fact that $g_1(w+\omega_2) = -g_1(w)$ implies that

$$f_2(z+\omega_2)\cdot f_2(z) = \int_0^{\omega_2} g_1(w)\,dw.$$

To prove (d) it suffice to show that

$$\int_0^{\omega_1} g_1(w)\,dw \equiv \pi\sqrt{-1} \pmod{2\pi\sqrt{-1}\mathbb{Z}}, \tag{1.4.11}$$

for then $f_1(z) = \sqrt{f_2(\omega_2)^{-1}}\cdot f_2(z)$ will be a normalized developing map of type I (for a solution of equation (1.2.1) with $\rho = 2n+1$), for either of the two square roots of $f_2(\omega_2)^{-1}$. Clearly these are the only two normalized developing maps of type I whose logarithmic derivatives are equal to g_1.

To compute the integral $\int_0^{\omega_1} g_1(w)\,dw$ modulo $2\pi\sqrt{-1}\mathbb{Z}$, let C_ϵ be the path from 0 to ω_2, obtained from the oriented line segment $\overrightarrow{0\omega_2}$ from 0 to ω_2 near by replacing the ϵ-neighborhood of each pole of $g_1(w)dw$ by the half circle of radius ϵ to the right of $\overrightarrow{0\omega_2}$, for all sufficiently small $\epsilon > 0$. Clearly the integral $\int_{C_\epsilon} g_1(w)\,dw$ is independent of ϵ. Write C_ϵ the union of the small half circles and the "straight part" C'_ϵ of C_ϵ. Let m_1 (respectively m_2) be the number of poles of g_1 with residue 1 (respectively -1) on the line segment $\overrightarrow{0\omega_2}$.

The fact that $g_1(w)\,dw$ is invariant under multiplication by -1 implies that the integral of $g_1(w)$ over C'_ϵ is 0, so $\int_0^{\omega_1} g_1(w)\,dw$ converges to $(m_1 - m_2)\cdot\pi\sqrt{-1}$ as $\epsilon \to 0^+$. In other words

$$\int_{C_\epsilon} g_1(w)\,dw = (m_1 - m_2)\cdot \pi\sqrt{-1}$$

for all (sufficiently small) $\epsilon > 0$. On the other hand the assumptions (b1) and (b2) tells us that $m_1 - m_2$ is an odd integer. We have proved the statements (d1)–(d3). □

1.5. Type II scaling families and blow-up points. In type II, it follows from (1.3.3) that $g = f'/f$ is an elliptic function on E. From §1.2, g has zero only at $z = 0$. Thus by Lemma 1.2.4,

$$g(z) = A\,\frac{\sigma^l(z)}{\prod_{i=1}^l \sigma(z-p_i)} \tag{1.5.1}$$

for p_i's being simple zeros/poles of f with $\sum p_i = 0$. Now the Weierstrass function σ is with respect to E. Also the points p_i's are unique up to elements in Λ as long as the constraint $\sum p_i = 0$ is satisfied.

Proposition 1.5.1. *For $\rho = 4\pi l$ with l being odd, there are no type II, i.e. blow-up, solutions to the mean field equation*

$$\triangle u + e^u = \rho\delta_0 \quad \text{on } E.$$

Proof. If there is a solution u with developing map f, then $g = f'/f$ is elliptic on E with residues at p_i, $i = 1, \ldots, l$, being ± 1. Since l is odd, the sum of residues of g is non-zero, which contradicts to the classical fact that the sum of residues of an elliptic function must be zero. □

Therefore we may set $l = 2n$. Let $p_1, \ldots, p_n$ be zeros and $p_{n+1}, \ldots, p_{2n}$ be poles of f. The residue of g at $z = p_j$ is given by Ar_j with

$$r_j = \frac{\sigma^l(p_j)}{\prod_{i=1,\neq j}^{l} \sigma(p_j - p_i)}. \tag{1.5.2}$$

Then we have equations

$$r_1 = \cdots = r_n = -r_{n+1} = \cdots = -r_{2n}. \tag{1.5.3}$$

Recall that

$$f(z) = f(0) \exp \int_0^z g(w)\, dw.$$

Lemma 1.5.2. *In order for f to verify (1.3.3), it is equivalent to require that the periods integrals are purely imaginary:*

$$\int_{L_i} g(z)\, dz \in i\,\mathbb{R}, \quad i = 1, 2.$$

Another characteristic feature for type II is that any solution must exist in an one parameter scaling family of solutions. To see this, notice that if f is a developing map of solution u then $e^\lambda f$ also satisfies (1.3.3) for any $\lambda \in \mathbb{R}$. In fact $e^\lambda f$ is a developing map of u_λ defined by (0.2.6):

$$u_\lambda(z) = \log \frac{8e^{2\lambda}|f'(z)|^2}{(1 + e^{2\lambda}|f(z)|^2)^2}$$

and it is clear that u_λ is a scaling family of solutions of (1.2.1).

Let z_0 be a zero of f. We know that $z_0 \not\equiv 0$ and $f'(z_0) \neq 0$. Thus

$$u_\lambda(z_0) \sim 2\lambda \to +\infty \quad \text{as} \quad \lambda \to +\infty$$

while if $f(z) \neq 0$ then

$$u_\lambda(z) \sim -2\lambda \to -\infty \quad \text{as} \quad \lambda \to +\infty.$$

Points like z_0 are referred as *blow-up* points.

Thus as $\lambda \to +\infty$, the blow-up set of u_λ consists of the zeros of f. Similarly, as $\lambda \to -\infty$, the blow-up sets of u_λ consists of the poles of f.

Remark 1.5.3. In general it is very hard to solve the residue equations (1.4.8) (for type I) and (1.5.3) (for type II) directly, though some simple cases had been treated in [43, 44] for $\rho = 4\pi, 8\pi$ and 12π.

2. Type I solutions: Evenness and algebraic integrability

Let $\rho = 4\pi l$, $l \in \mathbb{N}$. Let u be a type I solution and f be a developing map of u. In this section we will prove Theorem 0.4 stated in the introduction. Proposition 1.5.1 proves that if l is odd then the solution is of type I. We will start by proving the converse in Theorem 2.2, i.e., if the solution is of type I then l must be odd. At the same time the evenness of u will follow.

2.1. The evenness of solutions. Recall that the logarithmic derivative

$$g = (\log f)' = \frac{f'}{f}$$

of the developing map f is an elliptic function on $E' = \mathbb{C}/\Lambda'$ with $\Lambda' = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$. For the ease of notations we will use $\omega_1' = \omega_1$ and $\omega_2' = 2\omega_2$. In the following all the elliptic functions are with respect to the torus E'.

Since g has zero at $z = 0$ of order l, it also has zero of order l at $z = \omega_2$. There are no other zeros hence it has simple poles at $p_1, \ldots, p_l$ and $q_1, \ldots, q_l$ where p_i's are simple zeros of f and q_i's are simple poles of f modulo Λ'. We may assume that

$$q_i = p_i + \omega_2, \quad i = 1, \ldots, l.$$

From

$$f(z) = f(0) \exp \int_0^z g(w)\, dw,$$

the residues of g are 1 at p_i's and -1 at q_i's. Thus we may write g as

$$g(z) = \sum_{i=1}^{l} (\zeta(z - p_i) - \zeta(z - p_i - \omega_2)) + c \tag{2.1.1}$$

By (1.4.1), it is easily seen that $c = l\eta_2/2$.

There are also other useful equivalent forms of g:

$$\begin{aligned} g(z) &= \frac{1}{2}\sum_{i=1}^{l} (2\zeta(z - p_i) - \zeta(z - p_i - \omega_2) - \zeta(z - p_i + \omega_2)) \\ &= -\frac{1}{2}\sum_{i=1}^{l} \frac{\wp'(z - p_i)}{\wp(z - p_i) - e_2} \\ &= -\frac{1}{2}\sum_{i=1}^{l} \frac{d}{dz}\log(\wp(z - p_i) - e_2) \end{aligned}$$

by the addition formula.

Remark 2.1.1. The middle formula says that up to a constant $g(z)$ is the sum of slopes of the l lines from the point $(\wp(\omega_2), \wp'(\omega_2)) = (e_2, 0)$ to the points $(\wp(z - p_i), \wp'(z - p_i))$ of the torus E' under the standard cubic embedding into $\mathbb{C}^2 \cup \{\infty\}$, for $i = 1, \ldots, l$.

The only constraint that remains is the order of zero of g at $z = 0$. Namely

$$0 = g(0) = g'(0) = \cdots = g^{(l-1)}(0).$$

Notice first that

$$2g(0) = \sum \frac{\wp'(p_i)}{\wp(p_i) - e_2} =: \sum s(p_i)$$

is the first (degree one) symmetric polynomial of the slops $s(p_i)$. It is reasonable to expect that some of the higher derivatives $g^{(m)}(0)$ are also higher degree symmetric polynomials of slops. The expectation turns out to be true only for m even and for odd degree polynomials:

Proposition 2.1.2. *The even order derivatives $g^{(2j)}(0)$, $j = 0, \ldots, [\frac{l-1}{2}]$, of g from a basis of the odd degree symmetric polynomials in s_i's up to degree l for l being odd and up to degree $l-1$ for l being even.*

Proof. Consider the slope function

$$\begin{aligned} s(z) &= \frac{d}{dz}\log(\wp(z) - e_2) = \frac{\wp'(z)}{\wp(z) - e_2} \\ &= -2\zeta(z) + \zeta(z+\omega_2) + \zeta(z-\omega_2) \\ &= -2(\zeta(z) - \zeta(z-\omega_2) - \eta_2/2). \end{aligned} \tag{2.1.2}$$

By differentiating the last equation, we get

$$\begin{aligned} \frac{1}{2}s'(z) &= \wp(z) - \wp(z-\omega_2) \\ &= \wp(z) - e_2 - \frac{\mu}{\wp(z) - e_2} \end{aligned} \tag{2.1.3}$$

where we have used the half period formula with

$$\mu = (e_1 - e_2)(e_3 - e_2) = e_1e_3 - (e_1 + e_3)e_2 + e_2^2 = 2e_2^2 + e_1e_3.$$

Also

$$\begin{aligned} \frac{1}{2}s'' &= \wp' + \frac{\mu\wp'}{(\wp - e_2)^2} \\ &= s\Big(\wp - e_2 + \frac{\mu}{\wp - e_2}\Big). \end{aligned}$$

(Notice the variations on signs with (2.1.3).) Then we have

Lemma 2.1.3. *The slope function satisfies the following ODE:*

$$s'' = \frac{1}{2}s^3 - 6e_2 s. \tag{2.1.4}$$

Proof. We will compute s'' in a different way, namely

$$s' = \frac{\wp''}{\wp - e_2} - \frac{\wp'\wp'}{(\wp - e_2)^2} = \frac{6\wp^2 - \frac{1}{2}g_2}{\wp - e_2} - s^2. \tag{2.1.5}$$

It is elementary to see

$$6\wp^2 - \frac{g_2}{2} = 6(\wp - e_2)^2 + 12e_2(\wp - e_2) + 6e_2^2 - \frac{g_2}{2}$$

and

$$6e_2^2 - \frac{g_2}{2} = 6e_2^2 + 2(e_1e_2 + e_3e_2 + e_1e_3) = 2(2e_2^2 + e_1e_3) = 2\mu.$$

Thus (2.1.5) becomes

$$s' = 12e_2 - s^2 + 6(\wp - e_2) + \frac{2\mu}{\wp - e_2}.$$

Then

$$\begin{aligned} s'' &= -2ss' + 6\wp' - \frac{2\mu s}{\wp - e_2} \\ &= -24e_2s + 2s^3 - 12s(\wp - e_2) - \frac{4\mu s}{\wp - e_2} + 6s(\wp - e_2) - \frac{2\mu s}{\wp - e_2} \\ &= -24e_2s + 2s^3 - 6s\Big(\wp - e_2 + \frac{\mu}{\wp - e_2}\Big) \\ &= -24e_2s + 2s^3 - 3s'', \end{aligned}$$

where the last equality is by (2.1.4). The lemma follows. □

To proceed to higher even derivatives, we notice that

$$(s^k)'' = (ks^{k-1}s')' = k(k-1)s^{k-2}(s')^2 + ks^{k-1}s''. \tag{2.1.6}$$

By (2.1.3) and (2.1.4),

$$\begin{aligned} (s')^2 &= 4\Big(\wp - e - \frac{\mu}{\wp - e}\Big)^2 \\ &= 4\Big(\wp - e + \frac{\mu}{\wp - e}\Big)^2 - 16\mu = \Big(\frac{s''}{s}\Big)^2 - 16\mu \end{aligned}$$

which is an even degree polynomial in s of degree 4 by Lemma 2.1.3. Thus $(s^k)''$ is odd in s of degree $k+2$ if k is odd. By induction we then have that $s^{(2j)}$ is a degree $2j+1$ odd polynomial in s.

The proposition now follows easily from

$$2g^{(2j)}(0) = \sum_{i=1}^{l} s^{(2j)}(p_i) \tag{2.1.7}$$

and general facts on symmetric polynomials. □

Now we are ready to prove

Theorem 2.2. *Let $\rho = 4\pi l$. If the developing map f satisfies the type I relation* (1.3.2)*, then l is odd. Furthermore $g(-z) = -g(z)$ and $u(-z) = u(z)$.*

Proof. Consider the polynomial

$$S(x) = \prod_{i=1}^{l} (x - s(p_i)).$$

By Proposition 2.1.2, the relations

$$0 = g(0) = g''(0) = \cdots = g^{(2[\frac{l-1}{2}])}(0)$$

lead to the vanishing of all odd symmetric polynomials of $s(p_i)$'s in the expansion of $S(x)$.

If $l = 2n$, then $S(x)$ consists of only even degrees and its roots $s(p_i)$ must appears in pairs. Without loss of generality we may assume that

$$s(p_1) = -s(p_{n+1}), s(p_2) = -s(p_{n+2}), \ldots, s(p_n) = -s(p_{2n}). \tag{2.2.1}$$

Notice that the slope equation

$$\frac{\wp'(a)}{\wp(a) - e_2} = s(a) = -s(b) = -\frac{\wp'(b)}{\wp(b) - e_2}$$

leads to $b = -a$ or $b = a + \omega_2$. To see this, notice that under the cubic embedding $z \mapsto (\wp(z), \wp'(z))$, $s(a)$ is slope of the line ℓ_a connecting the images of $z = \omega_2$ and $z = a$, with the unique third intersection point being $z = -a - \omega_2$ and $s(-a - \omega_2) = s(a)$. Thus the slope function defines a branched double cover

$$s : E' \to \mathbb{P}^1(\mathbb{C}).$$

(From (2.1.3), it has 4 branch points given by $\wp(z) = e_2 \pm \sqrt{\mu}$.)

In particular the line with slope $-s(a) = s(-a)$ and passing through $(e_2, 0)$ must be $\ell_{-a} \equiv \ell_{a+\omega_2}$. That is, $b = -a$ or $b = a + \omega_2$ as claimed.

In our case (2.2.1), we must conclude $p_{n+1} = -p_1$ since $p_1 + \omega_2 = q_1$ can not appear in p_i's. In the same way we conclude that

$$p_i = -p_{i+n}, \quad i = 1, \ldots, n. \tag{2.2.2}$$

In particular $\sum p_i = 0$. But this violates $\sum p_i \equiv \frac{1}{2}\omega_1$ modulo Λ' (which follows from $g(z + \omega_2) = -g(z)$ in Lemma 1.4.5), hence l is odd.

For $l = 2n + 1$, $S(x)$ is a polynomial with odd degree terms only. In particular there is a root $x = 0$ of $S(x)$ and we may assume that $s(p_{2n+1}) = 0$ (namely $p_{2n+1} = \frac{1}{2}\omega_1$ or $\frac{1}{2}(\omega_1' + \omega_2') = \frac{1}{2}\omega_1 + \omega_2$).

Consider the polynomial $S(x)/x$ with only even degree terms. In exactly the same manner as above we conclude that (2.2.2) still holds and

$$S(x) = x \prod_{i=1}^{n} (x - s(p_i))(x + s(p_i)).$$

It is clear that now $g(-z) = -g(z)$. Then $f(-z) = f(z)$, which implies that u is an even function. $\square$

2.3. The polynomial system. The remaining statements in Theorem 0.4 which have not been proved yet are that these points $[p_1], \ldots, [p_n]$ on E are determined by the polynomial equations (0.4.1) in $\wp(p_i)$'s.

Philosophically this follows easily from (2.1.1) and (2.1.3). Indeed it is clear that the odd order derivatives of g at $z = 0$ will involve only rational expressions with denominator being powers of $\wp(p_i) - e_2$ and with at most even derivatives $\wp(z)^{(2j)}(p_i)$ in the numerator (all expressions in $-p_i$ are transformed into expressions in p_i). The latter can be written as polynomials in $\wp(p_i)$ and thus the polynomial system is obtained.

Proof of Theorem 0.4. To write down the complete set of polynomial equations explicitly, recall that we have

$$\begin{aligned} g(z) &= \sum_{i=1}^{l} (\zeta(z - p_i) - \zeta(z - p_i - \omega_2) - \eta_2/2), \\ -g'(z) &= \sum_{i=1}^{l} (\wp(z - p_i) - \wp(z - p_i - \omega_2)), \\ -g^{(m+1)}(z) &= \sum_{i=1}^{l} (\wp^{(m)}(z - p_i) - \wp^{(m)}(z - p_i - \omega_2)) \quad \forall m \in \mathbb{Z}_{\geq 0}, \end{aligned} \tag{2.3.1}$$

and the half period formula (let $\tilde{\wp}(p) = \wp(p + \omega_2)$)

$$\tilde{\wp} = e_2 + \frac{\mu}{\wp - e_2}$$

where $\mu = (e_1 - e_2)(e_3 - e_2)$. Equivalently $(\wp - e_2)(\tilde{\wp} - e_2) = \mu$.

In the proof of Theorem 2.2, the even order derivatives $g^{(2j)}(0) = 0$, $j = 0, \ldots, n$, leads to the evenness of solutions. We will show that the remaining odd order differentiations $g^{(2j+1)}(0) = 0$, $j = 0, \ldots, n-1$, leads to the desired polynomial system.

To calculate $g^{(2j+1)}(0)$, we first notice that

Lemma 2.3.1. *For every $k \in \mathbb{N}$, $(\wp^k)''$ is a degree $k+1$ polynomial in $\wp$. Indeed*

$$(\wp^k)'' = 2k(2k+1)\wp^{k+1} - \frac{g_2}{2}k(2k-1)\wp^{k-1} - k(k-1)g_3\wp^{k-2}.$$

Proof. Since $(\wp^k)' = k\wp^{k-1}\wp'$, we get

$$(\wp^k)'' = k(k-1)\wp^{k-2}(\wp')^2 + k\wp^{k-1}\wp''.$$

The lemma follows from Weierstrass' cubic relations between $\wp'$ and $\wp$. □

Now we set $x_i = \wp(p_i)$, $\tilde{x}_i = \tilde{\wp}(p_i) = \wp(p_i + \omega_2)$ for $i = 1, \ldots, n$. It is clear that $(x_i - e_2)(\tilde{x}_i - e_2) = \mu$ for all $i = 1, \ldots, n$.

During the following computations, we assume that $p_{2n+1} = \frac{1}{2}\omega_1$ and $p_{n+i} = -p_i$ for $i = 1, \ldots, n$. For the other case $p_{2n+1} = \frac{1}{2}\omega_1 + \omega_2$, we could replace f by $1/f$ to reduce to the former case, since f and $1/f$ give rise to the same solution u.

For $j = 0$ we have from (2.3.1) that

$$-g'(0) = 2\sum_{i=1}^{n} x_i + e_1 - 2\sum_{i=1}^{n} \tilde{x}_i - e_3 = 0.$$

This is the degree one equation ($m = 1$) with $c_1 = -\frac{1}{2}(e_1 - e_3) \neq 0$.

For $j = 1$, since

$$-g''' = \sum_{1}^{l} \wp'' - \sum_{1}^{l} \tilde{\wp}'' = 6\sum_{1}^{l} \wp^2 - 6\sum_{1}^{l} \tilde{\wp}^2,$$

the equation $g'''(0) = 0$ becomes

$$\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \tilde{x}_i^2 = -\tfrac{1}{2}(e_1^2 - e_3^2).$$

This is the degree two equation ($m = 2$) with $c_2 = -\frac{1}{2}(e_1^2 - e_3^2)$.

The general case follows from Lemma 2.3.1. Suppose that $g^{(2j+1)}(0) = 0$ gives rise to a new polynomial relation $\sum_{i=1}^{n} x_i^j - \sum_{i=1}^{n} \tilde{x}_i^j = c_j$. A further double differentiation increases the degree of the polynomial in $\wp$ by one, hence it gives rise to a new relation $\sum_{i=1}^{n} x_i^{j+1} - \sum_{i=1}^{n} \tilde{x}_i^{j+1} = c_{j+1}$, with the universal constant c_{j+1} being determined by c_1, c_2, g_2, g_3 recursively.

Therefore we conclude that $x_i = \wp(p_i)$, $\tilde{x}_i = \wp(p_i + \omega_2)$, $i = 1, \ldots, n$, satisfy the polynomial system:

$$\sum_{i=1}^{n} x_i^j - \sum_{i=1}^{n} \tilde{x}_i^j = c_j, \quad j = 1, \ldots, n,$$
$$(x_i - e_2)(\tilde{x}_i - e_2) = \mu, \quad i = 1, \ldots, n,$$

which is easily seen to be equivalent to the system (0.4.1).

Conversely, any solution of the polynomial system gives rise to a function g which satisfies

$$g^{(j)}(0) = 0, \quad j = 0, 1, \ldots, 2n.$$

From g, the developing map f is then constructed by Proposition 1.4.8. $\square$

Remark 2.3.2. In the next section we will prove that except for a finite set of conformal equivalence classes of tori, the mean field equation (0.1.3) has exactly $n+1$ solutions for $\rho = 4\pi l$ with $l = 2n+1$. This implies that, except for those tori, the above polynomial system has exactly $n+1$ solutions up to permutation symmetry by S_n. Equivalently it has $(n+1)!$ solutions.

Since the $c_j(\tau)$'s are all holomorphic in τ, solutions $(x_i(\tau), \tilde{x}_i(\tau))$ of the polynomial system, hence the developing map $f(z;\tau)$, should then depend on τ holomorphically. It is not so obvious how to prove the holomorphic dependence of $f(z;\tau)$ in the moduli space of tori by other methods.

Example 2.4. For $\rho = 4\pi$, $l = 1$ and $n = 0$. Then $p_1 = \frac{1}{2}\omega_1$. The polynomial system is empty and the solution u is unique. This was first proved in [43].

Example 2.5. Consider the case $\rho = 12\pi$, i.e. $l = 3$ and $n = 1$. Let $p_1 = a$. $p_2 = -a$ and $p_3 = \frac{1}{2}\omega_1$. Then the equation $g'(0) = 0$ becomes

$$2\Big((\wp(a) - e_2) - \frac{\mu}{\wp(a) - e_2}\Big) + (e_1 - e_3) = 0.$$

That is, we get a degree 2 polynomial in $\wp(a)$:

$$(\wp(a) - e_2)^2 + \tfrac{1}{2}(e_1 - e_3)(\wp(a) - e_2) - \mu = 0$$

and then

$$\wp(a) = e_2 + \tfrac{1}{4}(e_3 - e_1) \pm \tfrac{1}{4}\sqrt{(e_3 - e_1)^2 + 16(e_1 - e_2)(e_3 - e_2)}.$$

These are exactly the solutions obtained in [44] via a different method. In particular there are precisely two solutions of the mean field equation on any torus E with non-zero discriminant $(e_3 - e_1)^2 + 16(e_1 - e_2)(e_3 - e_2) \neq 0$ for the double cover E', and with $\rho = 12\pi$. The case with zero discriminant will be discussed in Example 3.6.

Example 2.6. Consider the case $\rho = 20\pi$, i.e. $l = 5$ and $n = 2$. The full set of polynomial equations in x_i's and $\tilde{x}_i$'s is given by

$$\begin{aligned}
x_1 + x_2 - \tilde{x}_1 - \tilde{x}_2 &= c_1 = -\tfrac{1}{2}(e_1 - e_3),\\
x_1^2 + x_2^2 - \tilde{x}_1^2 - \tilde{x}_2^2 &= c_2 = -\tfrac{1}{2}(e_1^2 - e_3^2),\\
(x_1 - e_2)(\tilde{x}_1 - e_2) &= \mu,\\
(x_2 - e_2)(\tilde{x}_2 - e_2) &= \mu.
\end{aligned}$$

Now the number of solutions N'_n (here $n = 2$) for $x_1.x_2, \tilde{x}_1, \tilde{x}_2$ can be calculated by the Bezout theorem to be $N'_2 = 1 \times 2 \times 2 \times 2 - r_2^\infty = 8 - r_2^\infty$ where r_2^∞ is the number of solutions at ∞, counted with multiplicity, of the projectivized system of polynomial equations. The projective system is

$$\begin{aligned}
X_1 + X_2 - \tilde{X}_1 - \tilde{X}_2 &= c_1 X_0,\\
X_1^2 + X_2^2 - \tilde{X}_1^2 - \tilde{X}_2^2 &= c_2 X_0^2,\\
(X_1 - e_2 X_0)(\tilde{X}_1 - e_2 X_0) &= \mu X_0^2,\\
(X_2 - e_2 X_0)(\tilde{X}_2 - e_2 X_0) &= \mu X_0^2.
\end{aligned}$$

And the infinity solutions are given by setting $X_0 = 0$:

$$\begin{aligned}
X_1 + X_2 = \tilde{X}_1 + \tilde{X}_2, &\qquad X_1^2 + X_2^2 = \tilde{X}_1^2 + \tilde{X}_2^2,\\
X_1 \tilde{X}_1 = 0, &\qquad X_2 \tilde{X}_2 = 0.
\end{aligned}$$

This shows that $\{X_1, X_2\} = \{\tilde{X}_1, \tilde{X}_2\}$. Since these four variables are not all zero, it is easy to see that there are precisely two solutions given by

$$\begin{aligned}
P_1 : &\quad X_1 = 0 = \tilde{X}_2, \quad X_2 = \tilde{X}_1 \neq 0,\\
P_2 : &\quad X_2 = 0 = \tilde{X}_1, \quad X_1 = \tilde{X}_2 \neq 0.
\end{aligned}$$

It remains to compute the multiplicity of P_1 and P_2. Consider P_1 first. Since it is in the chart $\tilde{U}_1 := \{\tilde{X}_1 \neq 0\}$, in terms of $y_i = X_i/\tilde{X}_1$, $i = 1, 2$, $\tilde{y}_2 = \tilde{X}_2/\tilde{X}_1$ and $y_0 = X_0/\tilde{X}_1$, P_1 has coordinates $(y_0, y_1, y_2, \tilde{y}_2) = (0, 0, 1, 0)$ and the system at point P_1 reads as $f_i = 0$, $i = 1, \ldots, 4$, where

$$\begin{aligned}
&f_1 = y_1 + y_2 - 1 - \tilde{y}_2 - c_1 y_0,\\
&f_2 = y_1^2 + (y_2 - 1)^2 + 2(y_2 - 1) - \tilde{y}_2^2 - c_2 y_0^2,\\
&f_3 = (y_1 - e_2 y_0)(1 - e_2 y_0) - \mu y_0^2 = y_1 + \cdots,\\
&f_4 = (y_2 - e_2 y_0)(\tilde{y}_2 - e_2 y_0) - \mu y_0^2 = (y_2 - 1)\tilde{y}_2 + \tilde{y}_2 + \cdots.
\end{aligned}$$

From these expressions, the appearance of degree one monomial in each f_i shows that the local analytic coordinates $(y_0, y_1, y_2 - 1, \tilde{y}_2)$ at the point P_1 can be replaced by f_1, f_3, f_2, f_4 accordingly, and thus the multiplicity is one. Indeed $P_1 = (0, 0, 1, 0)$ is a simple point of $\{f_i = 0\}$ by computing the Jacobian

$$\det \frac{\partial(f_1, f_2, f_3, f_4)}{\partial(y_0, y_1, y_2, \tilde{y}_2)}(0, 0, 1, 0) = e_1 - e_3 \neq 0.$$

Similarly the multiplicity at P_2 is one. Thus $r_2^\infty = 2$ and $N_2' = 8 - 2 = 6$.

Since any reordering of p_i's leads to the same solution, also it is easy to see that for generic tori we do not have any solution with $x_1 = x_2$, so finally

$$N_2 = N_2'/2! = 3 = 2 + 1.$$

Remark 2.7. The above method can be extended to the case $n = 3$, $\rho = 28\pi$ to show that $N_3 = 4$ since in this case the infinity solutions are still zero dimensional. It fails for $n \geq 4$ since positive dimensional intersections at infinity do occur and excess intersection theory is needed. The cases $n = 4$ and $n = 5$ were recently settled in [42] where the infinity solutions are one dimensional.

3. Lamé for type I: Finite monodromies

In this section we prove Theorem 0.4.1 (c.f. Theorem 3.5).

3.1. From mean field equations to Lamé. The second order equation

$$L_{\eta,B}\, w := w''(z) - (\eta(\eta+1)\wp(z) + B)w(z) = 0 \tag{3.1.1}$$

is known as the Lamé equation with two parameters η and B; the parameter η is called the *index* and B is the called the *accessary parameter*.

3.1.1. Recall that for any two linearly independent solutions w_1 and w_2 of a general second order ODE $w'' = Iw$, the Schwarzian derivative

$$S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2$$

of $h = w_1/w_2$ satisfies $S(h) = -2I$, hence for any two linear independent local solutions w_1, w_2 of the Lamé equation (3.1.1) we have

$$S(\tfrac{w_1}{w_2}) = -2(\eta(\eta+1)\wp(z) + B).$$

Conversely if h_1 is meromorphic function with $S(h_1) = -2(\eta(\eta+1)\wp(z) + B$, then $S(h_1) = S(\frac{w_1}{w_2})$ for a chosen pair of linearly independent solutions w_1, w_2 of (3.1.1), therefore h_1 is equal to a linear fractional transformation of $\frac{w_1}{w_2}$, or equivalently there exists a pair of linearly independent solutions w_3, w_4 of (3.1.1) such that $h_1 = w_3/w_4$.

3.1.2. Suppose that u is a solution of the mean field equation

$$\triangle u + e^u = \rho\delta_0 \tag{3.1.2}$$

on a flat torus $E = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and f is a developing map of u on a covering space of the punctured torus $E \smallsetminus \{0\}$. As in §1, locally u is expressed in f via

$$u(z) = \log \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}.$$

Let $\eta := \rho/8\pi$. By (1.1.4), we have

$$\begin{aligned} S(f) &:= \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2 \\ &= u_{zz} - \frac{1}{2}u_z^2 = -2\eta(\eta+1)\frac{1}{z^2} + O(1), \end{aligned} \tag{3.1.3}$$

where the last equality follows from the asymptotic expansion

$$u \sim 4\eta \log|z|$$

at $z = 0$ and that u is smooth outside $z = 0$ in E. The expression of $S(f)$ in u shows that it is a meromorphic function on E, which is holomorphic outside $\{0\}$ and its polar part at $z = 0$ is given by the last expression in (3.1.3). Therefore there exists a constant $B = B(E, \eta, u)$ such that

$$S(f) = u_{zz} - \tfrac{1}{2}u_z^2 = -2(\eta(\eta+1)\wp(z) + B). \tag{3.1.4}$$

It follows that there exist two linearly independents solutions w_1 and w_2 of the Lamé equation (3.1.1) with accessary parameter $B(E, \eta, u)$ such that $f = w_1/w_2$.

3.1.3. The Lamé equation (3.1.1) had been studied in the classical literature in two special cases, very extensively in case when the index η is a positive integer, and somewhat less so in the case when the index η is a half-integer, i.e. $2\eta = 2n + 1$ is an odd positive integer. We have seen in the previous sections that the former case corresponds to type II solutions while the latter case is for type I solutions. The main objective of this section is to prove that for any odd positive integer $2n + 1$, on all but a finite number of isomorphism classes of elliptic curves, there are precisely $n + 1$ solutions to the mean field equation $\triangle u + e^u = 4\pi(2n+1)\delta_0$.

The following theorem is due to Brioschi [7], Halphen [27, pp. 471–473] and Crawford [19] in the late nineteenth century; see [19] for a complete proof. See also [53, pp. 162–164] for a succinct presentation of Halphen's transformation as well as [67, p. 570] for Crawford's procedure for analyzing Brioschi's solution.

Theorem 3.2. *Let n be a non-negative integer.*

(a) *There exists a monic polynomial $p_n(B;\Lambda) = p_n(B, g_2(\Lambda), g_3(\Lambda))$ of degree $n+1$ in B with coefficients in $\mathbb{Z}[\frac{g_2(\Lambda)}{4}; \frac{g_3(\Lambda)}{4}]$ such that the Lamé equation $L_{n+1/2,B}\, w = 0$ on $\mathbb{C}/\Lambda$ has all solutions free from logarithm at $z = 0$ if and only if $p_n(B;\Lambda) = 0$.*

This polynomial $p_n(B, g_2, g_3) \in \mathbb{Z}[\frac{1}{2}][B, g_2, g_3]$ is homogeneous of weight $n+1$ if B, g_2, g_3 are given weights $1, 2, 3$ respectively.

(b) *For any lattice Λ outside a finite subset S_n of homothety classes of lattices in $\mathbb{C}$, the polynomial $p_n(B;\Lambda)$ has $n+1$ distinct roots.*

Proof. The logarithm-free solutions of the Lamé equation $L_{n+1/2,B}\, w = 0$ were first discovered by Brioschi [7, p. 314], but the underlying structure are more transparently exhibited using Halphen's transformation [27, p. 471] as carried out in detail by Crawford [19]. The statement (a) is proved in [19]; see also [53, p. 164] for a presentation of Crawford's proof. A slightly different proof of (a) following the same train of ideas can be found in [3, p. 26–28].

Crawford's proof provides a recursive formula for $p_n(B;\Lambda)$. When Λ is of the form $\mathbb{Z} + \sqrt{-1}a\mathbb{Z}$ with $a \in \mathbb{R}_{>0}$, this recursive formula also produces a *Sturm sequence* starting with $p_n(B)$, therefore $p_n(B;\Lambda)$ has $n+1$ distinct real roots; see [19, p. 94].[16] This implies that the discriminant of the polynomial $p_n(B;\Lambda)$, which is a modular form for $\mathrm{SL}_2(\mathbb{Z})$, is not identically 0. The statement (b) follows. See §3.3 for remarks on Sturm's theorem used in Crawford's proof. □

Remark 3.2.1. We will give an alternative proof of part (a) of Theorem 3.2 in §3.4, which is essentially local near $z = 0$. Our proof not only provides a new construction of the polynomial $p_n(B)$, it also generalizes to the case with multiple singular sources. This generalization will be presented in a later work; c.f. [11].

3.3. Remark on Sturm's theorem. Crawford's proof in [19, p. 94] that the polynomial $p_n(B;\Lambda)$ has $n+1$ distinct real roots for rectangular tori uses a fact closely related to Sturm's theorem on real roots of polynomials over $\mathbb{R}$, not found in standard treatment of this topic, such as [64, 11.3] and [33,

[16]The statement that $p_n(B)$ has $n+1$ distinct real roots was proved in [19, p. 94] under the condition that the x-coordinates of the three non-trivial two-torsion points, $e_i = \wp(\omega_i/2;\Lambda)$ for $i = 1, 2, 3$, are real numbers. This is the case when the lattice Λ is of the form Λ_τ with $\tau \in \sqrt{-1}\mathbb{R}_{>0}$.

5.2].[17] We have been able to find only one reference of this fact, as a "starred exercise" in [63, p. 149 ex. 30]. In Proposition 3.3.3 below we provide a mild generalization of the usual form of Sturm's theorem for the convenience of the readers. Its corollary 3.3.4 is equivalent to [63, p. 149 ex. 30].

Definition 3.3.1. A sequence of non-zero polynomials

$$f_0(x), f_1(x), \ldots, f_m(x) \in \mathbb{R}[x]$$

is a *Sturm sequence* on $(a, b]$ if the following two properties hold.

(i) $f_m(x)$ is either positive definite or negative definite on $(a, b]$.

(ii) Suppose that $\xi \in (a, b]$ and $f_i(\xi) = 0$ for some i with $1 \le i \le m-1$. Then $f_{i-1}(\xi)$ and $f_{i+1}(\xi)$ have opposite signs (in the sense that either they are both non-zero with opposite signs, or are both zero.[18])

Remark. There is an extra condition in the conventional definition of a Sturm sequence: $f_1(\xi)$ *and* $f_0(\xi)$ *have the same sign for every root* ξ *of* $f_0(x)$ *in* $(a, b]$. This condition has been dropped in Definition 3.3.1 above.

3.3.2. Definition. Let $f_0(x), f_1(x), \ldots, f_m(x)$ be a Sturm sequence.

(1) For every real number ξ, define $\sigma(\xi)$ to be the total number of changes of signs in the sequence $(f_0(\xi^+), f_1(\xi^+), \ldots, f_{m-1}(\xi^+), f_m(\xi))$.[19]

(2) Define a $\{-1, 0, 1\}$-valued "local index" function $\epsilon_{f_0(x)}$ on $\mathbb{R}$ attached to a real polynomial $f_0(x) \in \mathbb{R}[x]$ as follows.

- Suppose that $f_0(\xi) = 0$[20] and $\mathrm{mult}_{x=\xi}\, f_0(x)$ is *odd*.[21] Define

$$\epsilon_{f_0(x)}(\xi) := \begin{cases} 1 & \text{if } f_0(\xi^+) \text{ and } f_1(\xi) \text{ have the same sign} \\ -1 & \text{if } f_0(\xi^+) \text{ and } f_1(\xi) \text{ have opposite signs} \end{cases}$$

- $\epsilon_{f_0(x)}(\xi) = 0$ if $\mathrm{mult}_{x=\xi}\, f_0(x)$ is even. In particular $\epsilon_{f_0(x)}(\xi) = 0$ if $f_0(\xi) \neq 0$.

---

[17] This fact must be familiar to all educated scientists in the late nineteenth and early twentieth century, often used freely without comments in mathematical writings at the time. This is the case for the proofs in [67, p. 557] and [53, p. 163] for the existence of $2m+1$ distinct real roots of the polynomial $l_m(B)$ corresponding to $2m+1$ Lamé functions for the equation

$$\frac{d^2w}{dz^2} - (m(m+1)\wp(z;\Lambda) + B)w = 0$$

when $\Lambda = \mathbb{Z} + \sqrt{-1}a\mathbb{Z}$ for some $a \in \mathbb{R}_{>0}$ and $m \in \mathbb{N}_{>0}$. However this then-well-known fact is no longer part of the general education for mathematicians today.

[18] The latter possibility is ruled out by condition (i).

[19] Here we used $f_i(\xi^+)$ to make sure that each term has a well-defined sign. In view of condition (ii), we could have used the sequence $(f_0(\xi^+), f_1(\xi), \ldots, f_{m-1}(\xi), f_m(\xi))$ in the definition, suppress zeros when counting the number of variations of signs in it.

[20] $f_1(\xi) \neq 0$ if $f_0(\xi) = 0$, by (i) and (ii).

[21] For a zero ξ of $f_0(x)$, the sign of $f_0(x)$ changes when x moves across ξ if and only if $\mathrm{mult}_{x=\xi}\, f_0(x)$ is odd.

(3) Define $Z_{f_0(x)}((a,b]) \in \mathbb{Z}$ by

$$Z_{f_0(x)}((a,b]) := \sum_{\xi \in (a,b]} \epsilon_{f_0(x)}(\xi).$$

This number $Z_{f_0(x)}((a,b])$ counts the number of zeros of $f_0(x)$ with odd multiplicity with a signed weight given by $\epsilon_{f_0(x)}$. It can be thought of as some sort of "total Lefschetz number" for $f_0(x)|_{(a,b]}$.

Proposition 3.3.3. *Let $f_0(x), f_1(x), \ldots, f_m(x)$ be a Sturm sequence on $(a,b]$. Then*

$$Z_{f_0(x)}((a,b]) = \sigma(a) - \sigma(b),$$

i.e. $\sigma(a) - \sigma(b)$ is the number of zeros of $f_0(x)$ in the half-open interval $(a,b]$ with odd *multiplicity, counted with the sign $\epsilon_{f_0(x)}$.*

Proof. Condition (ii) ensures that crossing a zero in $[a,b)$ of any of the internal members $f_1(x), \ldots, f_{m-1}(x)$ of the Sturm chain makes no contribution to changes of $\sigma(\xi)$. Each time a zero ξ_0 of $f_0(x)$ with odd multiplicity is crossed, $\sigma(\xi)$ decreases by $\epsilon_{f_0(x)}(\xi)$ as ξ moves from the left of ξ_0 to its right. On the other hand, moving across a zero of $f_0(x)$ with even multiplicity does not change the value of σ. So the $\sigma(b) - \sigma(a)$ is equal to the total number of zeros of $f_0(x)$ in $(a,b]$ with odd multiplicity, counted with the sign $\epsilon_{f_0(x)}$. □

Corollary 3.3.4. *Let $f_0(x), \ldots, f_m(x)$ be a Sturm sequence on $(a,b]$. Let $n \in \mathbb{N}$ be a non-negative integer. If $\sigma(a) - \sigma(b) = \pm n$ and $f_0(x)$ has at most n distinct real roots in $(a,b]$, then $f_0(x)$ has exactly n distinct real roots in the half-open interval $(a,b]$. In particular if $a = -\infty, b = \infty$,* $\deg(f_0(x)) = n$ *and $\sigma(-\infty) - \sigma(\infty) = \pm n$, then $f_0(x)$ has n distinct real roots.*

3.4. A proof of Theorem 3.2 (a). Let's start with any f as the quotient of two independent solutions of Lamé equation $L_{n+1/2,B}\, w = 0$ at $z = 0$ and consider $v(z) = \log f'(z)$. It is readily seen that

$$v'' - \frac{1}{2}(v')^2 = \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2 = S(f).$$

We remark that the function v satisfies the similar equation as u in (3.1.3), but v is analytic in nature while u is only a real function.

The indicial equation at $z = 0$ is given by $\lambda^2 - \lambda - \eta(\eta+1) = (\lambda - (\eta+1))(\lambda + \eta) = 0$. If there are logarithmic solutions, the fundamental solutions are given as

$$w_1(z) = z^{\eta+1} h_1(z), \quad w_2(z) = \xi w_1(z) \log z + z^{-\eta} h_2(z), \tag{3.4.1}$$

where $\xi \neq 0$ and h_1, h_2 are holomorphic and non-zero at $z = 0$. But then

$$f = \frac{a w_1 + b w_2}{c w_1 + d w_2}$$

is easily seen to be logarithmic as well if $ad - bc \neq 0$, thus the Lamé equation has no logarithmic solutions at $z = 0$ if and only if we have one nontrivial solution quotient f to be logarithmic free at $z = 0$.

Now suppose that the Lamé equation has no solutions with logarithmic term. Let f be a ratio of two independent solutions. Without lose of generality, we may assume that f is regular at 0. Since $\eta = n + \frac{1}{2}$ and

$$S(f) = -2((n + \tfrac{1}{2})(n + \tfrac{3}{2})\wp(z) + B), \tag{3.4.2}$$

to require that f is logarithmic free at $z = 0$ is equivalent to that $f(z) = c_0 + c_{2n+2}z^{2n+2} + \cdots$ near $z = 0$ with $c_0 \neq 0$.

Recall that

$$\wp(z) = \frac{1}{z^2} + \sum_{k \geq 1}(2k+1)G_{k+1}z^{2k}$$

where $G_k = \sum_{\omega \in \Lambda^*} 1/\omega^{2k}$ is the standard Eisenstein series of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$. It is customary to write $g_2 = 60G_2$ and $g_3 = 140G_3$. It is also well known that all G_k's are expressible as polynomials in g_2, g_3.

We will show that the solvability of the Schwarzian equation (3.4.2) for f being of the proposed form is equivalent to the statement that B satisfies $p_n(B) = 0$ for some universal polynomial $p_n(B, g_2, g_3)$ of degree $n+1$. Indeed, let

$$v := \log f' = \log c_{2n+2}(2n+2) + (2n+1)\log z + \sum_{j \geq 1} d_j z^j.$$

For convenience we set $e_j = (j+1)d_{j+1}$ for $j \geq 0$ and then

$$v' = \frac{2n+1}{z} + \sum_{j \geq 0} e_j z^j$$

The degree z^{-1} terms in

$$v'' - \frac{1}{2}(v')^2 = -2((n + \tfrac{1}{2})(n + \tfrac{3}{2})\wp(z) + B)$$

match by our choice. That there is no z^{-1} term in the right hand side of (3.4.2) shows that $e_0 = 0$. Then the constant terms give $e_1 - \frac{1}{2}2(2n+1)e_1 = -2B$, i.e. $ne_1 = B$. For $n = 0$, we must conclude $B = 0$. Thus we set $p_0(B) = B$.

Similarly, for $j \geq 1$, the degree j terms in the LHS give

$$(j+1)e_{j+1} - \tfrac{1}{2}2(2n+1)e_{j+1} - \tfrac{1}{2}\sum_{i=1}^{j-1} e_i e_{j-i}.$$

Since there is no odd degree terms in the right hand side of (3.4.2), by considering $j = 1, 3, 5, \ldots$ we first conclude inductively that $e_i = 0$ for i even.

Next we consider degree $j = 2, 4, 6, \ldots$ terms inductively. Write $E_k = e_{2k-1}$ for $k \geq 1$. Then $j = 2k$ leads to

$$2(k-n)E_{k+1} - \tfrac{1}{2}\sum_{i=1}^{k} E_i E_{k+1-i} = -2(n+\tfrac{1}{2})(n+\tfrac{3}{2})(2k+1)G_{k+1}. \tag{3.4.3}$$

We have just seen that $nE_1 = B$. If we assign degree k to G_k, then (3.4.3) shows inductively that $E_k = E_k(B, g_2, g_3)$ is a degree k polynomial in B which is homogeneous in B, g_2, g_3 of degree k up to $k \leq n$.

Now put $k = n$ in (3.4.3), the first term vanishes and we must have

$$\tilde{p}_n(B, g_2, g_3) := \sum_{i=1}^{n} E_i E_{n+1-i} - 8(n+\tfrac{1}{2})^2(n+\tfrac{3}{2})G_{n+1}$$

vanishes too. Up to a multiplicative constant, this $\tilde{p}_n(B)$ is the degree $n+1$ polynomial in B we search for. Indeed, by our inductive construction through (3.4.3), the leading coefficients c_n of $\tilde{p}_n(B)$ depends only on n. Hence $p_n(B, g_2, g_3) := c_n^{-1}\tilde{p}_n(B, g_2, g_3)$ is monic in B and homogeneous of degree $n+1$ in B, g_2, g_3.

Conversely, if $\tilde{p}_n(B) = 0$, then $E_1, \ldots, E_n$ can be solved by (3.4.3) up to $k = n-1$. For $k = n-1$, $\tilde{p}_n(B) = 0$ is equivalent to (3.4.3) at $k = n$. By assigning any value to E_{n+1}, we can use (3.4.3) for $k \geq n+1$ to find E_j, $j \geq n+2$. Thus this f is a solution to the Schwarzian equation (3.4.2) and is free from logarithmic terms. The proof is complete. □

Remark 3.4.1. Notice that $E_{n+1} = e_{2n+1} = (2n+2)d_{2n+2}$ is a free parameter. All E_k's are determined by B and E_{n+1}. For any B with $p_n(B) = 0$, the three constants c_0, c_{2n+2} and E_{n+1} provide the three dimensional freedom for f due to the freedom of $\mathrm{SL}_2(\mathbb{C})$ action on f.

Remark 3.4.2. We have seen that the type I developing map $f(z)$ is even. This also follows form our proof of Theorem 3.2 since we do not assume the a priori evenness during the proof.

To apply Theorem 3.2 to study mean field equations for $\rho = 4\pi(2n+1)$, the essential point is the following theorem.

Theorem 3.5 (= Theorem 0.4.1)**.** *Let n be a non-negative integer. The projective monodromy group of the Lamé equation $L_{n+(1/2),B}\, w = 0$ is isomorphic to Klein's four-group $(\mathbb{Z}/2\mathbb{Z})^2$ if and only if there exists two meromorphic solutions w_1, w_2 on $\mathbb{C}$ of the above Lamé equations such that $\frac{w_1}{w_2}$ is a type I developing map of a solution of the mean field equation $\triangle u + e^u = 4\pi(2n+1)\delta_0$. Moreover, each such value of the accessary parameter B with the above property gives rise to exactly one type I solution.*

Proof. Let u be a type I solution of the mean field equation $\triangle u + e^u = \rho\,\delta_0$ on $\mathbb{C}/\Lambda$ and let f be a normalized developing map of u satisfying the type I transformation rules (1.3.2). We know from Theorem 2.2 that there exists a non-negative integer n such that $\rho = 4\pi(2n+1)$, and we have seen that there exists a complex number B such that the Schwarzian derivative $S(f)$ of f is equal to $-2\big((n+\frac{1}{2})(n+\frac{3}{2})\wp(z;\Lambda)+B\big)$. Then local solutions of the Lamé equation $L_{n+1/2,B}\,w=0$ are free of logarithmic solutions, and there exists two solutions w_1, w_2 over $\mathbb{C}$ such that $f=\frac{w_1}{w_2}$. The projective monodromy group of the equation $L_{n+1/2,B}\,w=0$ is canonically isomorphic to the monodromy group of the meromorphic function $\frac{w_1}{w_2}$, which is a Klein four group K_4 by the type I transformations (1.3.2). We have proved the "only if" part of Theorem 0.4.1.

Conversely, suppose that the projective monodromy group of a Lamé equation $L_{n+1/2,B}\,w=0$ is a Klein-four group. Then all local solutions of this Lamé equation are free of logarithmic singularities, and there are for two linearly independent solutions w_1, w_2 of this equation which are meromorphic functions over $\mathbb{C}$. It is easy to check from basic theory of linear ODE's with regular singularities that the holomorphic map $\frac{w_1}{w_2}:\mathbb{C}\to\mathbb{P}^1(\mathbb{C})$ has no critical point outside Λ, and has multiplicity $2n+2$ at points of Λ.

Let $\rho:\Lambda\to \mathrm{GL}_2(\mathbb{C})$ be the monodromy representation of the differential equation $L_{n+1/2,B}\,w=0$ attached to the basis w_1, w_2 of solutions of $L_{n+1/2,B}\,w=0$. Let $\bar{\rho}:\Lambda\to \mathrm{PSL}_2(\mathbb{C})$ be the composition of ρ with the canonical projection $\mathrm{GL}_2(\mathbb{C})\to\mathrm{PSL}_2(\mathbb{C})$. Because $\mathrm{PSU}(2)$ is a maximal compact subgroup of $\mathrm{PSL}_2(\mathbb{C})$, the finite subgroup $\mathrm{Im}(\bar{\rho})$ of $\mathrm{PSL}_2(\mathbb{C})$ is a conjugate of a subgroup of $\mathrm{PSU}(2)$, i.e. there exists an element $S_1\in\mathrm{GL}_2(\mathbb{C})$ such that $S_1\cdot\mathrm{Im}(\bar{\rho})\cdot S_1^{-1}\subset\mathrm{PSU}(2)$. By Corollary 1.3.2, there exists an element $S_2\in\mathrm{PSU}(2)$ such that $S_2\cdot S_1\cdot\bar{\rho}(\omega_1)\cdot S_1^{-1}\cdot S_2^{-1}$ and $S_2\cdot S_1\cdot\bar{\rho}(\omega_2)\cdot S_1^{-1}\cdot S_2^{-1}$ are the image in $\mathrm{PSU}(2)$ of $\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}$ and $\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}$ respectively. Write $S_2\cdot S_1=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Then $f_1:=\dfrac{aw_1+cw_2}{bw_1+dw_2}$ is a developing map of a solution of $\triangle u+e^u=4\pi(2n+1)\delta_0$, by Lemma 1.2.6, and it is normalized of type I by construction. We have proved the "if" part of Theorem 0.4.1. The uniqueness assertion in the last sentence of Theorem 0.4.1 is clear from the correspondence we have established, between solutions of the mean field equation $\triangle u+e^u=4\pi(2n+1)\delta_0$ and Lamé equations $L_{n+1/2,B}\,w=0$ such that no solution has logarithmic singularity. □

Corollary 3.5.1. *On any flat torus $\mathbb{C}/\Lambda$, the mean field equation $\triangle u+e^u=4\pi(2n+1)\,\delta_0$ at most $n+1$ solutions. It has exactly $n+1$ solutions except for a finite number of conformal isomorphism classes of flat tori.*

Proof. This is an immediate consequence of theorems 3.2, 3.5 and 0.4.1. □

Corollary 3.5.2. *For $\eta=n+\frac{1}{2}$, the monodromy group M of $L_{n+1/2,B}\,w=0$ on an elliptic curve $\mathbb{C}/\Lambda$ is finite if and only if it corresponds to a type I solution of the mean field equation $\triangle u+e^u=4\pi(2n+1)\delta_0$ on $\mathbb{C}/\Lambda$ as in Theorem* 3.5.

Proof. It was shown in [4, Thm. 2.3] that the monodromy group of the Lamé equation $L_{n+(1/2),B}\, w = 0$ is finite if and only if no solution of $L_{n+(1/2),B}\, w = 0$ has logarithmic singularity, and if so the projective monodromy group $L_{n+(1/2),B}\, w = 0$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. □

Example 3.6. By (3.4.3), it is easy to determine $p_n(B)$. For example,

$$p_1(B) = B^2 - \tfrac{3}{4} g_2,$$
$$p_2(B) = B^3 - 7 g_2 B + 20 g_3.$$

For $\rho = 12\pi$, the two solutions to the mean field equation collapse to the same one precisely when $p_1(B)$ has multiple roots. This is the case if and only if $g_2 = 0$, which means that $\tau = e^{\pi i/3}$.

To see this from Example 2.5 is a little bit trickier. We may solve

$$(e_3 - e_1)^2 + 16(e_1 - e_2)(e_3 - e_2) = 0$$

in terms of the modular function

$$\lambda(\tau') = \frac{e_3 - e_2}{e_1 - e_2}$$

where $\tau' = \omega_2'/\omega_1' = 2\omega_2/\omega_1 = 2\tau$. A simple calculation leads to

$$\frac{(\lambda - 1)^2}{\lambda} = -16, \quad \text{i.e.} \quad \lambda^2 + 14\lambda + 1 = 0.$$

Then the corresponding j invariant is

$$j(\tau') := 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2} = -2^8 15^3 \frac{\lambda}{(\lambda - 1)^2} = 2^4 3^3 5^3.$$

In general it would be difficult to determine τ' from j. Fortunately the value $j = 2^4 3^3 5^3$ appears in the famous list of elliptic curves with complex multiplications (see e.g. [30]) and it is known that

$$\tau' \equiv \sqrt{-3} \pmod{\mathrm{SL}_2(\mathbb{Z})}.$$

Take τ in the fundamental region, then there is a unique choice of τ, namely $\tau = \frac{1}{2}(1 + \sqrt{-3}) = e^{\pi\sqrt{-1}/3}$, which gives rise to

$$2\tau = 1 + \sqrt{-3} \equiv \sqrt{-3} = \tau' \pmod{\mathrm{SL}_2(\mathbb{Z})}.$$

4. Singular Liouville equations with $\rho = 4\pi$ and modular forms

In §2, we discussed how to find all type I solutions by solving a system of polynomial equations which depends holomorphically on the moduli parameter τ of the torus $E_\tau = \mathbb{C}/\Lambda_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, where τ varies in the upper-half plane $\mathbb{H}$. In this section, we consider the simplest case

$$\triangle u + e^u = 4\pi \delta_0 \quad \text{in } E_\tau, \tag{4.0.1}$$

and show that certain modular forms of level 4 are naturally to the solutions of (4.0.1) as τ varies. The general case with multiple singular sources will be considered in a subsequent work.

4.1. Notation.

- Let $\mathbb{H}$ be the upper-half plane. The group $\mathrm{SL}_2(\mathbb{R})$ operates transitively on $\mathbb{H}$ through the usual formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$.
- Let $j(\gamma, \tau)$ be the 1-cocycle of $\mathrm{SL}_2(\mathbb{R})$ for its action on $\mathbb{H}$, defined by $j(\gamma; \tau) = c\tau + d$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and any $\tau \in \mathbb{H}$.
- Denote by K_4 the subgroup of $\mathrm{PSU}(2) \subset \mathrm{PSL}_2(\mathbb{C})$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, consisting of the image in $\mathrm{PSU}(2)$ of the four matrices
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
We know from Lemma 1.3.1 (1b) that the centralizer subgroup of K_4 in $\mathrm{PSL}_2(\mathbb{C})$ is equal to itself.
- Let $\mathrm{N}(K_4)$ be the normalizer subgroup of K_4 in $\mathrm{PSU}(2)$, which is also equal to the normalizer subgroup of K_4 in $\mathrm{PSL}_2(\mathbb{C})$. We know from Lemma 1.3.1 (1d) that $\mathrm{N}(K_4)$ is a semi-direct product of K_4 with S_3 and $N(K_4)$ is isomorphic to S_4. Moreover the conjugation action induces an isomorphism from $\mathrm{N}(K_4)/K_4$ to the permutation group of the three non-trivial elements of K_4.

Proposition 4.2. (a) *For any $\tau \in \mathbb{H}$, there exists a unique normalized developing map $f(z;\tau)$ for the unique solution $u(z)$ of the equation* (4.0.1) *which has the following properties.*

$$\operatorname{ord}_{z=a} f(z,\tau) = 0 \quad \forall a \not\equiv \tfrac{1}{2} \pmod{\Lambda} \tag{4.2.1}$$

$$\tfrac{d}{dz} f(z;\tau)\big|_{z=0} = 0, \quad \tfrac{d^2}{dz^2} f(z;\tau)\big|_{z=0} \in \mathbb{C}^\times. \tag{4.2.2}$$

$$\textit{The holomorphic map } f(z;\tau) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \textit{ is etale outside } \Lambda_\tau. \tag{4.2.3}$$

$$f(z+1;\tau) = -f(z), \quad f(z+\tau;\tau) = 1/f(z;\tau) \quad \forall z \in \mathbb{C}. \tag{4.2.4}$$

$$f(-z;\tau) = f(z;\tau) \quad \forall z \in \mathbb{C}. \tag{4.2.5}$$

$$f(\tfrac{1}{2}\tau;\tau) = 1 \quad \forall z \in \mathbb{C}. \tag{4.2.6}$$

$$\operatorname{ord}_{z=1/2} f(z;\tau) = 1 \tag{4.2.7}$$

$$\operatorname{ord}_{z=(1/2)+\tau} f(z;\tau) = -1 \tag{4.2.8}$$

(b) *The function $f(z;\tau)$ in* (a) *is characterized by properties* (4.2.1), (4.2.4), (4.2.6) *and* (4.2.7), *i.e. if $h(z)$ is a meromorphic function on $\mathbb{C}$ which satisfies* (4.2.1), (4.2.4), (4.2.6) *and* (4.2.7) *for an element $\tau \in \mathbb{H}$, then $h(z) = f(z;\tau)$ for all $z \in \mathbb{C}$.*

(c) *The function $f(z;\tau)$ can be expressed in terms of Weierstrass elliptic functions:*

$$\begin{aligned} f(z;\tau) &= -e^{\frac{1}{4}\eta(\tau;\Lambda_\tau)\cdot(1+\tau)} \cdot \frac{\sigma(\frac{z}{2}-\frac{1}{4};\Lambda_\tau)\cdot\sigma(\frac{z}{2}+\frac{1}{4};\,\Lambda_\tau)}{\sigma(\frac{z}{2}-\frac{1}{4}-\frac{\tau}{2};\Lambda_\tau)\cdot\sigma(\frac{z}{2}+\frac{1}{4}+\frac{\tau}{2};\Lambda_\tau)} \\ &= -e^{\frac{1}{4}\eta(\tau;\Lambda_\tau)\cdot(1+\tau)}\cdot\frac{\sigma^2(\frac{1}{4};\Lambda_\tau)}{\sigma^2(\frac{1}{4}+\frac{\tau}{2};\Lambda_\tau)}\cdot\frac{\wp(\frac{z}{2};\Lambda_\tau)-\wp(\frac{1}{4};\Lambda_\tau)}{\wp(\frac{z}{2};\Lambda_\tau)-\wp(\frac{1}{4}+\frac{\tau}{2};\Lambda_\tau)} \\ &= \frac{\wp(\frac{\tau}{4};\,\Lambda_\tau)-\wp(\frac{1}{4}+\frac{\tau}{2};\,\Lambda_\tau)}{\wp(\frac{\tau}{4};\,\Lambda_\tau)-\wp(\frac{1}{4};\,\Lambda_\tau)}\cdot\frac{\wp(\frac{z}{2};\Lambda_\tau)-\wp(\frac{1}{4};\Lambda_\tau)}{\wp(\frac{z}{2};\Lambda_\tau)-\wp(\frac{1}{4}+\frac{\tau}{2};\Lambda_\tau)} \end{aligned} \tag{4.2.9}$$

Proof. **(a)** For any $\tau \in \mathbb{H}$, we have proved that in §2 that equation (4.0.1) has a unique solution $u(z;\tau)$, $z \in E_\tau$ and there exists a normalized type I developing map $f(z;\tau)$ for $u(z;\tau)$. Because the centralizer subgroup of K_4 in $\mathrm{PSU}(2)$ is K_4 itself, normalized type I developing maps consists are of the form $\gamma\cdot f$ with $\gamma \in K_4$. Properties (4.2.1)–(4.2.5) are satisfied by all 4 normalized developing maps. The first part of (4.2.4) and (4.2.6) implies that $f(z;\tau)$ has either a zero or a pole at $z=\frac{1}{2}$. Changing f_1 to $\left(\begin{smallmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0\end{smallmatrix}\right)\cdot f$ if necessary, we may assume that $f_1(\frac{1}{2};\tau)=0$. Then $f(z;\tau)$ has a simple zero at $z=\frac{1}{2}$ by (4.2.1), and properties (4.2.7)–(4.2.8) hold for f. Similarly properties (4.2.1), (4.2.4) and (4.2.5 for f imply that $f(\frac{\tau}{2};\tau)=\pm 1$. Changing f to $\left(\begin{smallmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1}\end{smallmatrix}\right)\cdot f$ if necessary, we have produced a normalized developing map satisfying (4.2.1)–(4.2.8).

(b) Suppose that $h(z)$ is a meromorphic function which satisfies properties (4.2.1), (4.2.4), (4.2.6) and (4.2.7). Then $h(z)$ descends to a meromorphic function on $\mathbb{C}/2\Lambda_\tau$ which has simple zeros at $\pm\frac{1}{2} \bmod 2\Lambda_\tau$, simples poles at $\pm\frac{1}{2}+\tau \bmod 2\Lambda_\tau$ and no zeros or poles elsewhere just like $f(z,\tau)$. Therefore $h(z)=c\cdot f(z;\tau)$ for some $c\in\mathbb{C}^\times$. This constant c is equal to 1 by (4.2.7).

(c) For the first equality in (4.2.9), it suffices to show that the function

$$e^{\frac{1}{4}\eta(\tau;\Lambda_\tau)\cdot(1+\tau)}\cdot\frac{\sigma(\frac{z}{2}-\frac{1}{4};\Lambda_\tau)\cdot\sigma(\frac{z}{2}+\frac{1}{4};\,\Lambda_\tau)}{\sigma(\frac{z}{2}-\frac{1}{4}-\frac{\tau}{2};\Lambda_\tau)\cdot\sigma(\frac{z}{2}+\frac{1}{4}+\frac{\tau}{2};\Lambda_\tau)}$$

satisfies conditions (4.2.1), (4.2.4), (4.2.6) and (4.2.7) according to (b). The properties (4.2.1), (4.2.4) and (4.2.7) follows quickly from the transformation law for the Weierstrass σ-function $\sigma(z;\Lambda_\tau)$ and the fact that the entire function $\sigma(z;\Lambda_\tau)$ has simple zeros at points of Λ_τ does not vanish elsewhere. The condition (4.2.6) is equivalent to

$$\frac{\sigma(\frac{\tau}{4}-\frac{1}{4};\Lambda_\tau)}{\sigma(-\frac{3\tau}{4}-\frac{1}{4};\Lambda_\tau)} = -e^{-\frac{1}{4}\eta(\tau;\Lambda_\tau)\cdot(1+\tau)},$$

which follows from the transformation law of $\sigma(z;\Lambda_\tau)$ with respect to the element $\tau\in\Lambda_\tau$. We have proved that the first equality

$$f(z;\,\tau)=e^{\frac{1}{4}\eta(\tau;\Lambda_\tau)\cdot(1+\tau)}\cdot\frac{\sigma(\frac{z}{2}-\frac{1}{4};\Lambda_\tau)\cdot\sigma(\frac{z}{2}+\frac{1}{4};\,\Lambda_\tau)}{\sigma(\frac{z}{2}-\frac{1}{4}-\frac{\tau}{2};\Lambda_\tau)\cdot\sigma(\frac{z}{2}+\frac{1}{4}+\frac{\tau}{2};\Lambda_\tau)}$$

in (4.2.9). The second equality in (4.2.9) follows from the classical formula

$$\wp(u;\Lambda)-\wp(v;\Lambda)=-\frac{\sigma(u+v;\Lambda)\cdot\sigma(u-v;\Lambda)}{\sigma^2(u;\Lambda)\cdot\sigma^2(v;\Lambda)}\,. \tag{4.2.10}$$

The last equality in (4.2.9) is equivalent to

$$\frac{\wp(\frac{\tau}{4};\,\Lambda_\tau)-\wp(\frac{1}{4}+\frac{\tau}{2};\,\Lambda_\tau)}{\wp(\frac{\tau}{4};\,\Lambda_\tau)-\wp(\frac{1}{4};\,\Lambda_\tau)}=-e^{\frac{1}{4}\eta(\tau;\Lambda_\tau)\cdot(1+\tau)}\cdot\frac{\sigma^2(\frac{1}{4};\Lambda_\tau)}{\sigma^2(\frac{1}{4}+\frac{\tau}{2};\Lambda_\tau)}\,, \tag{4.2.11}$$

which is easily verified using (4.2.9) and the transformation law of the Weierstrass σ-function $\sigma(z;\Lambda_\tau)$ with respect to the lattice Λ_τ. We have proved part (c) of Proposition 4.2. $\square$

Proposition 4.3. *Let $f(z;\tau)$ be the developing map specified in Proposition* 4.2.
(a) *There exists a unique group homomorphism $\psi:\mathrm{SL}_2(\mathbb{Z})\to\mathrm{N}(K_4)$ such that*

$$f\big(j(\gamma,\tau)^{-1}\cdot z;\,\gamma\cdot\tau\big)=\psi(\gamma)\cdot f(z;\tau)\qquad\forall\,z\in\mathbb{C},\;\forall\,\tau\in\mathbb{H}. \tag{4.3.1}$$

Here $\psi(\gamma)\cdot f(z;\tau)=\frac{af(z;\tau)+b}{-\bar{b}f(z;\tau)+\bar{a}}$ if $\psi(\gamma)$ is the image of $\begin{pmatrix}a&b\\-\bar{a}&\bar{b}\end{pmatrix}$ in $\mathrm{PSU}(2)$.

(b) *The homomorphism ψ is* surjective. *The kernel $\mathrm{Ker}(\psi)$ of ψ is equal to the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by $\pm\mathrm{I}_2$ and the principal congruence subgroup $\Gamma(4)$ of level* 4, *consisting of all $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ with $\gamma\equiv\mathrm{I}_2\pmod 4$. The inverse image $\psi^{-1}(K_4)$ of K_4 under ψ is the principal congruence subgroup $\Gamma(2)$. (In other words ψ induces an isomorphism $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})/\{\pm\mathrm{I}_2\}\xrightarrow{\sim}\mathrm{N}(K_4)$, and also an isomorphism $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})\xrightarrow{\sim}\mathrm{N}(K_4)/K_4\cong S_3$.)*

(c) $\psi(\begin{pmatrix}1&1\\0&1\end{pmatrix})=$ *the image in* $\mathrm{PSU}(2)$ *of the unitary matrix* $\begin{pmatrix}e^{\pi\sqrt{-1}/4}&0\\0&e^{-\pi\sqrt{-1}/4}\end{pmatrix}$, *and* $\psi(\begin{pmatrix}0&1\\-1&0\end{pmatrix})=$ *the image in* $\mathrm{PSU}(2)$ *of* $\frac{\sqrt{-1}}{\sqrt{2}}\begin{pmatrix}-1&1\\1&1\end{pmatrix}$.

Proof. **(a)** It is easily checked that for each $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, $f\big(j(\gamma,\tau)^{-1}\cdot z;\,\gamma\cdot\tau\big)$ is a developing map of the unique solution of (4.0.1), and that for each $\omega\in\Lambda_\tau$ we have $f\big(j(\gamma,\tau)^{-1}\cdot z+\omega;\,\gamma\cdot\tau\big)=\pm f(z;\tau)^{\pm1}$. Since $f\big(j(\gamma,\tau)^{-1}\cdot z+\omega;\,\gamma\cdot\tau\big)$ and $f(z;\tau)$ are developing maps for the same solution of (4.0.1), there exists an unique element $\psi(\gamma)\in\mathrm{PSU}(2)$ such that the equality (4.3.1) holds. The fact that $f\big(j(\gamma,\tau)^{-1}\cdot z+\omega;\,\gamma\cdot\tau\big)=\pm f(z;\tau)^{\pm1}$ for each $\omega\in\Lambda_\tau$ means that $\psi(\gamma)\in\mathrm{N}(K_4)$.

For all $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned} f(j(\gamma_1\gamma_2, \tau)^{-1}z;\, \gamma_1\gamma_2\cdot\tau) &= f(j(\gamma_1, \gamma_2\cdot\tau)^{-1} \cdot j(\gamma_2, \tau)^{-1}\cdot z;\, \gamma_1\cdot(\gamma_2\cdot\tau)) \\ &= \psi(\gamma_1)\cdot f(j(\gamma_2, \tau)^{-1}\cdot z;\, \tau) \\ &= \psi(\gamma_1)\cdot\psi(\gamma_2)\cdot f(z;\, \tau), \end{aligned}$$

therefore $\psi(\gamma_1\gamma_2) = \psi(\gamma_1)\cdot \psi(\gamma_2)$. We have proved statement (a).

(b) We get from (a) that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ we have

$$f(j(\gamma, \tau)^{-1}z + u\,\gamma\cdot\tau + v;\, \gamma\cdot\tau) = \psi(\gamma)\cdot f(z + (ua+vc)\tau + (ub+vd);\, \tau) \tag{4.3.2}$$

for all $u, v \in \mathbb{Q}$. For any given $\gamma \in \Gamma(2)$, we have

$$(ua+vc)\tau + (ub+vd) \equiv u\tau + v \pmod{2\Lambda_\tau} \quad \forall\, (u, v) \in \mathbb{Z}^2,$$

so the equality (4.3.2) for all $(u, v) \in \mathbb{Z}^2$ implies that $\psi(\gamma)$ commutes with every element of K_4. Hence $\psi(\gamma) \in K_4$ for any $\gamma \in \Gamma(2)$.

Suppose that $\gamma \in \Gamma(4)$. Then

$$(ua+vc)\tau + (ub+vd) \equiv u\tau + v \pmod{2\Lambda_\tau} \quad \forall\, (u, v) \in \tfrac{1}{2}\mathbb{Z}^2,$$

and the equality (4.3.2) with $z = 0$ implies that

$$(\psi(\gamma)\cdot f)(u\tau + v;\, \tau) = f(u\gamma\cdot\tau + v;\, \gamma\cdot\tau) \quad \forall\, (u, v) \in \tfrac{1}{2}\mathbb{Z}^2.$$

Because we already know that $\psi(\gamma) \in K_4$, the last equality implies that $\psi(\gamma) = \mathrm{I}_2$. We have proved that $\Gamma(4) \subset \mathrm{Ker}(\psi)$.

Suppose that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Ker}(\psi)$. As before we have

$$f(j(\gamma, \tau)^{-1}z + u\,\gamma\cdot\tau + v;\, \gamma\cdot\tau) = f(z + (ua+vc)\tau + (ub+vd);\, \tau) \tag{4.3.3}$$

for all $(u, v) \in \mathbb{Q}^2$. The transformation law (4.2.4) for $f(z; \tau)$ and the above equality for $(u, v) \in \mathbb{Z}^2$ imply that

$$(ua + vc)\tau + (ub + vd) \equiv u\tau + v \pmod{2\Lambda_\tau} \quad \forall\, (u, v) \in \mathbb{Z}^2,$$

therefore $\gamma \in \Gamma(2)$. The equation (4.3.3) with $z = 0$ and $u, v \in \frac{1}{2}\mathbb{Z}$ tells us that

$$f(u\,\gamma\cdot\tau + v;\, \gamma\cdot\tau) = f((ua+vc)\tau + (ub+vd);\, \tau) \qquad \forall\, (u, v) \in \tfrac{1}{2}\mathbb{Z}^2.$$

The properties (4.2.4), (4.2.6) and (4.2.7) imply that

$$(ua + vc)\tau + (ub + vd) \equiv \pm(u\tau + v) \pmod{2\Lambda_\tau} \quad \forall\, (u, v) \in \tfrac{1}{2}\mathbb{Z}^2,$$

therefore $\gamma \in \{\pm\mathrm{I}_2\}\cdot\Gamma(4)$. We have proved the statement (b).

(c) We want to compute $T := \psi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $S := \psi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. The defining relation for T is

$$f(z;\,\tau+1) = T\cdot f(z;\tau) \quad \forall\, z\in\mathbb{C},\ \forall\,\tau\in\mathbb{H}. \tag{4.3.4}$$

Substituting z by $z+\omega$ in (4.3.4) with $\omega\in\Lambda_\tau$ gives us two equalities

$$T\cdot\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix} = \begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}\cdot T \quad\text{and}\quad \begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}\cdot T = T\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

in $\mathrm{PSU}(2)$. A easy computation with the above equalities reveals that $\gamma\in\begin{pmatrix}e^{\pi\sqrt{-1}/4} & 0\\ 0 & e^{-\pi\sqrt{-1}/4}\end{pmatrix}\cdot K_4$, i.e. $f(z;\,\tau+1) = \pm\sqrt{-1}\cdot f(z;\,\tau)^{\pm 1}$. Since $f(\tfrac12;\,\tau) = f(\tfrac12;\,\tau+1) = 0$, the possibilities narrow down to $f(z;\,\tau+1) = \pm\sqrt{-1}\cdot f(z;\,\tau)$. It remains to determine the sign, which amounts to computing $f(\tfrac{\tau+1}{2};\,\tau)$ From the first equality in (4.2.9) we get

$$\begin{aligned} f(\tfrac{\tau+1}{2};\tau) &= -e^{\frac14\eta(\tau;\Lambda_\tau)\cdot(1+\tau)}\cdot\frac{\sigma(\frac{\tau}{4};\Lambda_\tau)\cdot\sigma(\frac{\tau+2}{4};\,\Lambda_\tau)}{\sigma(-\frac{\tau}{4};\Lambda_\tau)\cdot\sigma(\frac{3\tau+2}{4};\Lambda_\tau)}\\ &= e^{\frac14[\eta(\tau;\Lambda_\tau)-\eta(1;\Lambda_\tau)]}\\ &= -\sqrt{-1}\end{aligned}$$

The second equality in the displayed equation above follows from the transformation law of the Weierstrass σ-function, while the last equality follows from the Legendre relation $\eta(1;,\Lambda_\tau)\tau - \eta(\tau;\,\Lambda_\tau) = 2\pi\sqrt{-1}$. We conclude that

$$f(z;\,\tau+1) = \sqrt{-1}\cdot f(z;\,\tau),$$

which gives the formula for $T = \psi\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)$.

Finally let's compute S. The defining relation for S is

$$f\big(-\tfrac{1}{\tau}z;\,-\tfrac{1}{\tau}\big) = S\cdot f(z;\,\tau)$$

The functional equation for $z\to z+\omega$ with $\omega\in\Lambda_\tau$ gives us two equalities

$$S\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot S \quad\text{and}\quad \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot S = S\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

in $\mathrm{PSU}(2)$. A straight-forward computation gives four possible solutions of S, which translates into

$$f(-\tfrac{1}{\tau}z;\,-\tfrac{1}{\tau}) = \pm\frac{f(z;\,\tau)+1}{f(z;\,\tau)-1} \quad\text{or}\quad \pm\frac{f(z;\,\tau)-1}{f(z;\,\tau)+1}$$

The requirement that $f(-\frac{1}{2\tau};\,-\frac{1}{\tau}) = 1$ eliminates two possibilities: the two $\pm$ signs above are both -1. The requirement that $f(\frac12;\,-\frac{1}{\tau}) = 0$ shows that

$$f\big(-\tfrac{1}{\tau}z;\,-\tfrac{1}{\tau}\big) = -\frac{f(z;\,\tau)-1}{f(z;\,\tau)+1}.$$

We have proved the statement (c). $\square$

Recall that the quotient $\Gamma(4)\backslash\mathbb{H}$ has a natural structure as (the $\mathbb{C}$-points of) a smooth affine algebraic curve $Y(4)$. The compactified modular curve $X(4)$ is the smooth compactification $X(4)$ of $Y(4)$. As a topological space $X(4)$ is naturally identified as the quotient by $\Gamma(4)$ of $\mathbb{H}\sqcup\mathbb{P}^1(\mathbb{Q})$; the topology of the latter is described in [56, p. 10]. The complement $X(4)\smallsetminus Y(4)$, called the cusps of $X(4)$, is a set with 6 elements naturally identified with $\Gamma(4)\backslash\mathbb{P}^1(\mathbb{Q})$, or equivalently the set $\mathbb{P}^1(\mathbb{Z}/4\mathbb{Z})$ of $\mathbb{Z}/4\mathbb{Z}$-valued points of the scheme $\mathbb{P}^1$ over $\mathbb{Z}$. It is well-known that $X(4)$ has genus zero; c.f. [56, (1.6.4), p. 23].

The general discussion in §2, of which the present situation is the special case $\rho = 4\pi$, implies that the function $\tau \mapsto f(0;\tau)$ is holomorphic on $\mathbb{H}$. Proposition 4.3 implies that the holomorphic function $\tau \mapsto f(0;\tau)$ on $\mathbb{H}$ descends to a holomorphic function $h_{X(4)}$ on the open modular curve $Y(4)$. The next corollary says that $h_{X(4)}$ is a *Hauptmodul* for $X(4)$.

Corollary 4.4. (a) *The holomorphic function $h_{X(4)}$ on $Y(4)$ is a meromorphic function on $X(4)$ which defines a biholomorphic isomorphism $h^*_{X(4)}$ from $X(4)$ to $\mathbb{P}^1(\mathbb{C})$. This isomorphism $h^*_{X(4)}$ is equivariant with respect to ψ, for the action of $\mathrm{SL}_2(\mathbb{Z})/\{\pm\mathrm{Id}_2\}\cdot\Gamma(4)$ on $X(4)$ and the action of $\mathrm{N}(K_4)$ on $\mathbb{P}^1(\mathbb{C})$.*
(b) *We have explicit formulas for $h_{X(4)}(\tau) = f(0,\tau)$:*

$$h(\tau) = -e^{\frac{1}{4}\eta(\tau;\Lambda_\tau)\cdot(1+\tau)}\cdot\frac{\sigma^2(\frac{1}{4};\Lambda_\tau)}{\sigma^2(\frac{1}{4}+\frac{\tau}{2};\Lambda_\tau)} = \frac{\wp(\frac{\tau}{4};\,\Lambda_\tau)-\wp(\frac{1}{4}+\frac{\tau}{2};\,\Lambda_\tau)}{\wp(\frac{\tau}{4};\,\Lambda_\tau)-\wp(\frac{1}{4};\,\Lambda_\tau)} \tag{4.4.1}$$

(c) *The isomorphism $h^*_{X(4)}$ sends (the image of) the standard cusp "$\infty\cdot\sqrt{-1}$", that is the point $\infty\in\mathbb{P}^1(\mathbb{Q})$, to the point $0\in\mathbb{P}^1(\mathbb{C})$.*

Proof. The formula (4.4.1) in (b) follows immediately from the formulas (4.2.9) for $f(z;\tau)$.

There are two ways to see that $h_{X(4)}$ is a meromorphic function on $X(4)$. One can use either of the two formulas in (b) and classical results on Weierstrass elliptic functions. The other way is to use Picard's theorem: we know that $f(0,\tau)\neq 0$ for all $\tau\in\mathbb{H}$. The ψ-equivariance of $f(0,\tau)$ implies that

$$f(0,\tau)\notin\{\pm 1,\pm\sqrt{-1},0,\infty\}\quad\forall\,\tau\in\mathbb{H},$$

for the set $\{\pm 1,\pm\sqrt{-1},0,\infty\}$ is the orbit of $\mathrm{N}(K_4)$ on $\mathbb{P}^1(\mathbb{C})$. So $h_{X(4)}$ cannot have essential singularities at any of the cusps.

From the meager information in the previous paragraph we can already conclude that the holomorphic map $h_{X(4)}$ from $X(4)$ to $\mathbb{P}^1(\mathbb{C})$ has degree 1: Because $X(4)$ has exactly 6 cusps, the map $h_{X(4)}$ is totally ramified over the six points of $\{\pm 1,\pm\sqrt{-1},0,\infty\}\subset\mathbb{P}^1$, and the Hurwitz formula forces the degree of $h_{X(4)}$ to be 1.

The fact that $h_{X(4)}$ sends the standard cusp $\infty\cdot\sqrt{-1}$ to 0 can be seen by an easy computation, using the formula (4.4.1) and the q-expansion of the

Weierstrass $\wp$-function

$$\frac{1}{(2\pi\sqrt{-1})^2}\,\wp(z;\Lambda_\tau) = \frac{1}{2} + \frac{q_z}{(1-q_z)^2} + \sum_{m,n\geq 1} n\, q_\tau^{mn}(q_z^n + q_z^{-n}) - \sum_{m,n\geq 1} n\, q_\tau^{mn}$$

in the range $|q_\tau| < |q_z| < |q_\tau|^{-1}$, where $q_\tau = e^{2\pi\sqrt{-1}\tau}$ and $q_z = e^{2\pi\sqrt{-1}z}$ for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$. □

Remark 4.4.1. (a) The fact that $f(0;\tau)$ is a Hauptmodul for the principal congruence subgroup $\Gamma(4)$ is classical; see [58, p. 176]. We have not been able to locate in the literature the transformation formula in Proposition 4.3, but formula is not difficult to prove starting from the formula (4.2.9) for $f(z;\tau)$. Perhaps the only new thing here is the phenomenon that type I solutions of the Liouville equation $\triangle u + e^u = 4\pi(2n+1)\delta_0$ on elliptic curves produce modular forms for $\Gamma(4)$ in an organized way.

(b) Clearly the function $f(0;\tau)$ is a *modular unit* in the sense that it is a unit in the integral closure of $\mathbb{C}[j]$ in $\mathbb{C}(X(4))$, where j is the j-invariant and $\mathbb{C}(X(4))$ is the function field of the modular curve $X(4)$ over $\mathbb{C}$. It turns out that $f(0;\tau)$ is actually a unit of the integral closure of $\mathbb{Q}[j]$ in the function field of the modular curve $X(4)_{\mathbb{Q}(\sqrt{-1})}$ over $\mathbb{Q}(\sqrt{-1})$; see [38, Thm. 1, p. 189].

Corollary 4.5. *For $k = 0, 1, 2, \ldots \in \mathbb{N}$, let $a_k(\tau)$ be the holomorphic function on $\mathbb{H}$ defined by*

$$f(z;\tau) = \sum_{k=0}^{\infty} a_k(\tau)\, z^k, \qquad z \in \mathbb{C},\ \tau \in \mathbb{H}, \tag{4.5.1}$$

where $f(z;\tau)$ is the developing map in Proposition 4.2. For each $k \geq 0$, $a_k(\tau)$ is a holomorphic function on $\mathbb{H}$ and defines a modular form of weight k for the congruence subgroup $\Gamma(4)$ in the sense that

$$a_k(\gamma\cdot\tau) = j(\gamma,\tau)^k \cdot a_k(\tau) \qquad \forall \gamma \in \Gamma(4).$$

Moreover $a_k(\tau)$ is meromorphic at the cusps of $X(4)$ for every $k \in \mathbb{N}$.

Proof. The transformation formula (4.5.1) follows immediately from Proposition 4.3. The fact that $f(z;\tau)$ is holomorphic on $\mathbb{C}\times\mathbb{H}$ implies that $a_k(\tau)$ is a holomorphic function on $\mathbb{H}$. The last assertion that $a_k(\tau)$ is meromorphic at the cusps is most easily seen from the explicit formula (4.2.9) for $f(z;\tau)$. □

4.6. Generalization to $\rho = (2n+1)4\pi$. The considerations leading to the transformation formula (4.3.1) with respect to $\mathrm{SL}_2(\mathbb{Z})$ for the normalized developing map $f(z;\tau)$ for the unique solution of $\triangle u + e^u = 4\pi\cdot\delta_0$ on $\mathbb{C}/\Lambda_\tau$ specified in Proposition 4.2 can be extended for all type I cases. In 4.6.1–4.6.5 below we formulate the basic geometric structures which lead to a generalization of (4.3.1), and ends with an unsolved irreducibility and monodromy question in 4.6.6.

Definition 4.6.1. Let n be a non-negative integer.

(1) Let $\mathbb{M}_n$ be the set of all pairs $(u(z), \tau)$, where $\tau \in \mathbb{H}$ and $u(z)$ is a solution of the mean field equation $\triangle u + e^u = 4(2n+1)\pi \cdot \delta_0$ on the elliptic curve $\mathbb{C}/\Lambda_\tau$.

Let $\pi_n : \mathbb{M}_n \to \mathbb{H}$ be the map which sends a typical element $(u(z), \tau)$ in $\mathbb{M}_n$ to the point τ of the upper-half plane $\mathbb{H}$. Note that π_n is surjective according to Theorem 0.4.

(2) Let $\mathbb{D}_n$ be the set of all $(f(z), \tau)$, where τ is an element of $\mathbb{H}$ and $f(z)$ is a developing map of a solution of $\triangle u + e^u = 4(2n+1)\pi \cdot \delta_0$ on the elliptic curve $\mathbb{C}/\Lambda_\tau$ whose monodromy group is equal to the standard Klein's four subgroup $K_4 \subset \mathrm{PSU}(2)$ in the notation of 4.1.

Let $p_n : \mathbb{D}_n \to \mathbb{M}_n$ be the map which sends a typical element $(f(z), \tau) \in \mathbb{D}_n$ to the element $\left(\log \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}, \tau\right)$ of $\mathbb{M}_n$, and let $\tilde{\pi}_n = \pi_n \circ p_n : \mathbb{D}_n \to \mathbb{H}_n$ be the natural projection map which sends each element $(f(z), \tau) \in \mathbb{D}_n$ to τ.

(3) Let $\mathbb{D}'_n$ be the subset of $\mathbb{D}_n$ consisting of all pairs $(f(z), \tau) \in \mathbb{D}'$ such that and $f(z)$ is a *normalized* type I developing map for an element $(u(z), \tau) \in \mathbb{M}_n$ satisfying the monodromy condition that $f(z+1) = -f(z)$ and $f(z+\tau) = f(z)^{-1}$ for all $z \in \mathbb{C}$. Let $p'_n : \mathbb{D}' \to \mathbb{M}_n$ be the restriction to $\mathbb{D}'_n$ of p_n, and let $\tilde{\pi}'_n = \pi \circ p'_n : \mathbb{D}' \to \mathbb{H}_n$ be the restriction to $\mathbb{D}'_n$ of π_n.

(4) Define $\phi_n : \mathbb{M}_n \to \mathbb{D}'_n$ be the map which sends a typical element $(u(z), \tau) \in \mathbb{M}_n$ to the element $(f(z), \tau) \in \mathbb{D}'_n$ such that $f(\frac{1}{2}) = 0$ and $f(\frac{\tau}{2}) = 1$.

This map ϕ_n is well-defined because for each normalized type I developing map $(f(z), \tau) \in \mathbb{D}'_n$, we have

$$f(\tfrac{1}{2}) = 0 \text{ or } \infty, \qquad f(\tfrac{\tau}{2}) = \pm 1;$$

these four possibilities are permuted simply transitively by the action of K_4 through fractional linear transformations.

Lemma 4.6.2. *Let $\mathbb{M}_n, \mathbb{D}_n, \mathbb{D}'_n$ be as in Definition 4.6.1 above.*

(1) *Each of the three sets $\mathbb{M}_n, \mathbb{D}_n$ and $\mathbb{D}'_n$ has a natural structure as a one-dimensional complex manifold such that the maps $\pi_n : \mathbb{M}_n \to \mathbb{H}$, $\tilde{\pi}_n : \mathbb{D}_n \to \mathbb{H}$ and $\tilde{\pi}'_n : \mathbb{D}'_n \to \mathbb{H}$ are finite surjective holomorphic maps. Moreover there exists a discrete subset $R_n \subset \mathbb{H}$ which is stable under the natural action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ with $|\mathrm{SL}_2(\mathbb{Z})\backslash R_n| < \infty$, such that $\pi_n, \tilde{\pi}_n$ and $\tilde{\pi}'_n$ are unramified over the complement $\mathbb{H} \smallsetminus R_n$ of R_n.*

(2) *The action of the finite group $\mathrm{N}(K_4)$ on $\mathbb{D}_n$ via linear fraction transformations is holomorphic, making $p_n : \mathbb{D}_n \to \mathbb{M}_n$ an unramified Galois cover with group $\mathrm{N}(K_4)$. Similarly the map $p'_n : \mathbb{D}'_n \to \mathbb{M}_n$ is a holomorphic unramified Galois cover for the action of the standard Klein's four group K_4 in $\mathrm{PSU}(2)$.*

(3) *The map* $\phi_n : \mathbb{M}_n \to \mathbb{D}'_n$ *is a holomorphic section of* $p'_n : \mathbb{D}'_n \to \mathbb{M}_n$. *Consequently* $p_n : \mathbb{D}_n \to \mathbb{M}_n$ *is a trivial Galois cover with group* $\mathrm{N}(K_4)$ *and* $p'_n : \mathbb{D}'_n \to \mathbb{M}_n$ *is a trivial Galois cover with group* K_4.

Proof. The statement (1) for $\pi_n : \mathbb{M}_n \to \mathbb{H}$ follows from theorems 0.4 and 3.2. The part of statement (1) for $\tilde{\pi}_n : \mathbb{D}_n \to \mathbb{H}$ and $\tilde{\pi}'_n : \mathbb{D}'_n \to \mathbb{H}$ is a consequence of theorems 0.4 and 3.2 and the group-theoretic lemmas 1.3.1 and 1.3.2.

The action of $\mathrm{N}(K_4)$ on $\mathbb{D}_n$ is easily seen to be continous and is simply transitive on every fiber of $p_n : \mathbb{D}_n \to \mathbb{M}_n$. The first part of statement (2) follows. The second part of (2) is proved similarly.

The fact that $p'_n \circ \phi_n = \mathrm{id}_{\mathbb{M}_n}$ is immediate from the definition. It is not difficult to see that ϕ_n is continuous, which implies that ϕ_n is holomorphic. The first statement in (3) is proved; the rest of (3) follows. □

Definition 4.6.3. Define compatible actions of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{M}_n$ and $\mathbb{D}_n$ as follows. For any element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, any element $(u(z), \tau) \in \mathbb{M}_n$, and any element $(f(z), \tau) \in \mathbb{D}_n$ such that $p_n((f(z), \tau)) = (u(z), \tau)$,

- γ sends $(u(z), \tau) \in \mathbb{M}_n$ to the element
$$\big(u(j(\gamma,\tau)\cdot z) + \log(|j(\gamma,\tau)|^2),\ \gamma\cdot\tau\big) \in \mathbb{M}_n,$$
- and γ sends $(f(z), \tau) \in \mathbb{D}_n$ to the element
$$\big(f(j(\gamma,\tau)\cdot z),\ \gamma\cdot\tau\big) \in \mathbb{D}_n.$$

It is easy to check that $p_n : \mathbb{D}_n \to \mathbb{M}_n$ is $\mathrm{SL}_2(\mathbb{Z})$-equivariant, i.e.
$$p_n\big(\gamma \cdot (f(z), \tau)\big) = \gamma \cdot p_n\big((f(z), \tau)\big)$$
for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and every element $(f(z), \tau) \in \mathbb{D}_n$.

Lemma 4.6.4. (1) *The actions of* $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathbb{D}_n$ *and* $\mathbb{M}_n$ *defined in* 4.6.3 *are holomorphic.*

(2) *The holomorphic maps*
$$p_n : \mathbb{D}_n \to \mathbb{M}_n, \quad \pi_n : \mathbb{M}_n \to \mathbb{H} \quad \text{and} \quad \tilde{\pi}_n = \pi_n \circ p_n : \mathbb{D}_n \to \mathbb{H}$$
are equivariant for the $\mathrm{SL}_2(\mathbb{Z})$*-actions on* $\mathbb{D}_n, \mathbb{M}_n$ *and* $\mathbb{H}$.

(3) *The actions of* $\mathrm{SL}_2(\mathbb{Z})$ *and* $\mathrm{N}(K_4)$ *on* $\mathbb{D}_n$ *commute.*

(4) *The submanifold* $\mathbb{D}'_n \subset \mathbb{D}_n$ *is stable under the action of the principal congruence subgroup* $\Gamma(2)$ *of level* 2 *in* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. That the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{D}_n$ and $\mathbb{M}_n$ is continuous can be verified without difficulty, from which (1) follows. The proofs of statements (2)–(4) are easy and omitted. □

Corollary 4.6.5. *Suppose that* $\mathbb{M}_n$ *is* connected.

(a) *There exists a group homomorphism*

$$\psi_n : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{N}(K_4)$$

such that

$$\phi_n\big(\gamma \cdot (f(z), \tau)\big) = \psi_n(\gamma) \cdot \phi_n\big((f(z), \tau)\big)$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *and all elements* $(f(z), \tau) \in \mathbb{M}_n$.

(b) *The homomorphism* ψ_n *in* (a) *satisfies*

$$\Gamma(4) \subseteq \mathrm{Ker}(\psi_n) \quad \text{and} \quad \psi_n(\Gamma(2)) \subseteq K_4.$$

4.6.6. Questions. (a) *Is* $\mathbb{M}_n$ *connected?*[22]

(b) *Suppose that* $\mathbb{M}_n$ *is connected. What is the Galois group of the ramified cover* $\pi_n : \mathbb{M}_n \to \mathbb{H}_n$*? Is it the symmetric group* S_{n+1}*?*

5. Type II solutions: Evenness and Green's functions

In this section we give a proof of Theorem 0.6 concerning type II solutions. By Proposition 1.5.1, we may assume that $\rho = 8n\pi$ ($l = 2n$). Let u be a solution of (0.1.3), and f be a developing map of u. We recall that u_λ in (0.2.6) is a one parameter family of solutions of (0.1.3).

5.1. Evenness of solutions for $\rho = 8n\pi$.

Theorem 5.2. *There is a unique even solution within each normalized type II family of solutions of the singular Liouville equation* $\triangle u + e^u = 8n\pi\,\delta_0$ *on a torus* E, n, *where* n *is a positive integer. In other words for any normalized type II developing map* f *of a solution* u *of the above equation, there exists a unique* $\lambda \in \mathbb{R}$ *such that the solution*

$$u_\lambda(z) = \log \frac{e^{2\lambda}|f'(z)|^2}{(1 + e^{2\lambda}|f(z)|^2)^2}$$

of the same equation satisfies $u_\lambda(-z) = u_\lambda(z)\ \forall\, z \in \mathbb{C}$.

Proof. Let f be a normalized type II developing map of a solution u of (0.1.3). It is enough to show that there exists a unique $\lambda \in \mathbb{R}$ such that $f_\lambda(z) := e^\lambda \cdot f(z)$ satisfies $f_\lambda(-z) = c/f_\lambda(z)$ for a constant c with $|c| = 1$.

Let $g := f'/f$, the logarithmic derivative of f; it is a meromorphic function on $E = \mathbb{C}/\Lambda$ because f is normalized of type II. It suffice to show that g is even, for then

$$f(-z) = f(0) \exp \int_0^{-z} g(w)\, dw = \frac{f(0)^2}{f(z)},$$

and the unique solution of λ is given by $\lambda = -\log|f(0)|$.

[22]We think the answer is very likely "yes", but we don't have a proof.

We know from Lemma 1.3.7 that f is a local unit at points of Λ and g is a meromorphic function on E which has a zero of order $2n$ at $0 \in E$, no other zeros and $2n$ simple poles on E. Moreover the residue of g is equal to 1 at n of the simple poles of g, and equal to -1 at the other n simple poles.

Denote by $P_1, \ldots, P_n$ the n simples poles of g with residue 1 on $E = \mathbb{C}/\Lambda$, corresponding to zeros of the developing map f, and let $Q_1, \ldots, Q_n$ be the n-simple poles of g with residue -1, corresponding to simple poles of f. Let $p_1, \ldots, p_n \in \mathbb{C}$ be representatives of $P_1, \ldots, P_n \in \mathbb{C}/\Lambda$; similarly let $q_1, \ldots, q_n \in \mathbb{C}$ be representatives of $Q_1, \ldots, Q_n \in \mathbb{C}/\Lambda$. The condition on the poles of g allows us to express g in terms of the Weierstrass ζ-function:

$$g(z) = \sum_{i=1}^{n} \zeta(z - p_i) - \sum_{i=1}^{n} \zeta(z - q_i) + \sum_{i=1} \zeta(p_i) - \sum_{i=1} \zeta(q_i) \tag{5.2.1}$$

for a unique constant c, because $g(z) - \sum_{i=1}^{n} \zeta(z - p_i) + \sum_{i=1}^{n} \zeta(z - q_i)$ is a meromorphic holomorphic function on E. Of course the constant c is completely determined by the elements $p_1, \ldots, p_n; q_1, \ldots, q_n \in \mathbb{C}$:

$$c = \sum_{i=1} \zeta(p_i) - \sum_{i=1} \zeta(q_i).$$

It remains to analyze the condition that $g(z)$ has a zero of order $2n$ at $0 \in E$, i.e.

$$0 = -g^{(r)}(0) = -\left.\frac{d^r g}{dz^r}\right|_{z=0} = \sum_{i=1}^{n} \wp^{(r-1)}(p_i) - \sum_{i=1}^{n} \wp^{(r-1)}(q_i) \tag{5.2.2}$$

for $r = 1, \ldots, 2n - 1$ because

$$-g^{(r)}(z) = \sum_{i=1}^{n} \wp^{(r-1)}(z - p_i) - \sum_{i=1}^{n} \wp^{(r-1)}(z - q_i).$$

Only the conditions that $g^{(2s+1)}(0) = 0$ for $s = 0, 1, \ldots, n - 1$ will be used for the proof of Theorem 5.2. The vanishing of the even-order derivatives of g will be explored in the proof of Theorem 5.6.

By Lemma 5.4 below, relations (5.2.2) for $r = 1, 3, 5, \ldots, 2n - 1$ imply that the sets $\{\wp(p_1), \ldots, \wp(p_n)\}$ and $\{\wp(q_1), \ldots, \wp(q_n)\}$ are equal as sets with multiplicities. Because $P_1, \ldots, P_n$; $Q_1, \ldots, Q_n$ are $2n$ distinct points on E, it follows that $\wp(p_i) \neq \wp(p_j)$ whenever $i \neq j$ and $\{Q_1, \ldots, Q_n\} = \{-P_1, \ldots, -P_n\}$ as subsets of $E \smallsetminus \{0\}$ with n elements. From the expression (5.2.1) of g and the fact that $\zeta(z)$ is an even function on $\mathbb{C}$ one sees that $g(z)$ is even. Theorem 5.2 is proved modulo the elementary Lemma 5.4. $\square$

We record the following statements from the proof of Theorem 5.2.

Corollary 5.3. *Let f be a normalize type II developing map of a solution u of the equation $\triangle u + e^u = 8n\pi\, \delta_0$ for a positive integer n. Then the zeros p_i's of f modulo Λ correspond to n elements $P_1, \ldots, P_n \in E \smallsetminus \{0\}$ and the poles q_i's of f modulo Λ correspond to n elements $Q_1, \ldots, Q_n \in E \smallsetminus \{0\}$. Moreover the following statements hold.*

(a) $\{Q_1, \ldots, Q_n\} = \{-P_1, \ldots, -P_n\}$ *as subsets of* $E \smallsetminus \{0\}$ *with* n *elements.*
(b) $\wp'(p_i) \neq 0$, *or equivalently* P_i *is not a* 2*-torsion point of* E, *for* $i = 1, \ldots, n$.
(c) $\wp(p_i) \neq \wp(p_j)$ *for any* $i, j = 1, \ldots, n$ *such that* $i \neq j$.

Lemma 5.4. (a) *For each positive integer* j, *there exists a polynomial* $h_j(X) \in \mathbb{C}[X]$ *of degree* $j+1$ *such that*

$$\wp^{(2j)}(z) := \Big(\frac{d}{dz}\Big)^{2j} \wp(z) = h_j(\wp(z))$$

as meromorphic functions on $\mathbb{C}$.

(b) *For every symmetric polynomial* $P(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]$, *there exists a polynomial* $Q(W_1, \ldots, W_n) \in \mathbb{C}[W_1, \ldots, W_n]$ *such that*

$$P(\wp(z_1), \ldots, \wp(z_n)) = Q\Big(\sum_{i=1}^{n} \wp(z_i), \sum_{i=1}^{n} \wp^{(2)}(z_i), \sum_{i=1}^{n} \wp^{(4)}(z_i), \ldots, \sum_{i=1}^{n} \wp^{(2n-2)}(z_i)\Big)$$

as meromorphic functions on $\mathbb{C}^n$.

Proof. Taking the derivative of the Weierstrass equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

and divide both sides by $2\wp'(z)$, we get

$$\wp^{(2)}(z) = 6\wp(z)^2 - \frac{1}{2}\, g_2.$$

An easy induction shows that $\wp^{(2j)}(z)$ is equal to $h_j(\wp(z))$ for a polynomial $h_j(X) \in \mathbb{C}[X]$ of degree $j+1$ and the coefficient a_j of X^{j+1} is positive. In fact one sees that $a_j = (2j+1)!$ when one compares the coefficient of z^{-2j-2} in the Laurent series expansion of $\wp^{(2j)}(z)$ and $h_j(\wp(z))$. We have proved the statement (a).

The statement (b) follows from the fact that every symmetric polynomial in $\mathbb{C}[X_1, \ldots, X_n]$ is a polynomial of the Newton polynomials $p_1(X_1, \ldots, X_n)$, $\ldots$, $p_n(X_1, \ldots, X_n)$, where $p_j(X_1, \ldots, X_n) := \sum_{i=1}^{n} X_i^j$ for $j = 1, \ldots, n$. □

5.5. Green/algebraic system for $\rho = 8n\pi$. Let u be a type II even solution of the singular Liouville equation $\triangle u + e^u = 8n\pi\, \delta_0$ on a torus $E = \mathbb{C}/\Lambda$, where n is a positive integer. As before let f be a normalized developing map of u and let $g = (\log f)' = f'/f$. Let $p_1, \ldots, p_n \in \mathbb{C}$ be the simple zeros of f modulo Λ and let $-p_1, \ldots, -p_n$ be the simple poles of f modulo Λ as in Corollary 5.3. Let $P_1 = p_1 \bmod \Lambda, \ldots, P_n = p_n \bmod \Lambda$; they are exactly the blow-up points of the scaling family u_λ of solutions of $\triangle u + e^u = 8n\pi\, \delta_0$.

5.5.1. *Two approaches to the configuration of blow-up points.* We will investigate the constraints on the configuration of the points $P_1, \ldots, P_n$ with two approaches outlined below. The plans are executed in §5.6 and §5.7 respectively.

A. In the first approach we have the relations (5.2.2) for $r = 2, 4, \ldots, 2n-2$ and also the condition that the period integrals of the meromorphic differential $g(z)\,dz$ on E are purely imaginary; the latter comes directly from the assumption that the image of the monodromy of the type II developing map lies in the diagonal maximal torus of PSU(2). The first $n-1$ conditions translate into a system of polynomial equations in the coordinates $\wp(p_1), \ldots, \wp(p_n), \wp'(p_1), \ldots, \wp'(p_n)$ of the n points $P_1, \ldots, P_n$:

$$\wp'(p_1)\wp^r(p_1) + \cdots + \wp'(p_n)\wp^r(p_n) = 0, \quad \forall\, r = 0, \ldots, n-2, \tag{5.5.1}$$

while the monodromy constraint becomes

$$\sum_{i=1}^{n} \frac{\partial G}{\partial z}(p_i) = 0 \tag{5.5.2}$$

where G is the Green's function on E as in (0.1.1).

The method used in this approach also shows that the n equations (5.5.1) and (5.5.2) are also sufficient: if $P_1 = p_1 \bmod \Lambda, \ldots, P_n = p_n \bmod \Lambda$ are n elements in E with distinct x-coordinates $\wp_1(p_1), \ldots, \wp(p_n)$ satisfying equations (5.5.1) and (5.5.2) and none of $P_1, \ldots, P_n$ is a 2-torsion point of E, then there exists a normalized type II developing map f for a solution of $\triangle u + e^u = 8n\pi\,\delta_0$ such that $p_1, \ldots, p_n$ is a set of representatives of the zeros of f modulo Λ.

B. In the second approach, results on blow-up solutions of a mean field equation on a Riemann surface provides the following constraints

$$n\frac{\partial G}{\partial z}(p_i) = \sum_{1 \le j \le n,\, j \ne i} \frac{\partial G}{\partial z}(p_i - p_j), \qquad \text{for } i = 1, 2, \ldots, n, \tag{5.5.3}$$

on the blow-up points $P_1, \ldots, P_n$.

5.5.2. We will see in 5.7 that the system of equations (5.5.3) is equivalent to the combination of (5.5.2) and the following system of equations

$$\sum_{j \ne i} \frac{\wp'(p_i) + \wp'(p_j)}{\wp(p_i) - \wp(p_j)} = 0, \qquad \text{for } i = 1, \ldots, n. \tag{5.5.4}$$

Moreover for elements $(P_1, \ldots, P_n) \in (E \smallsetminus E[2])^n$ with *distinct* x-coordinates $\wp(p_1), \ldots, \wp(p_n)$, the two systems of equations (5.5.1) and (5.5.4) are equivalent.[23] This means that among elements of the subset $\mathrm{Bl}_n \subset E^n$ consisting of all n-tuples $(P_1, \ldots, P_n) \in (E \smallsetminus \{0\})^n$ satisfying the constraints (5.5.2) for blow-up points, those satisfying the non-degeneracy condition

$$\wp'(p_i) \ne 0 \quad \text{and} \quad \wp(p_i) \ne \wp(p_j) \ \text{ whenever } i \ne j \quad \forall 1 \le i, j \le n \tag{5.5.5}$$

[23]For every $m \in \mathbb{N}$, $E[m]$ denote the subgroup of m-torsion points on E.

are indeed blow-up points of the scaling family $u_\lambda(z)$ of a type II solution of the singular Liouville equation $\triangle u + e^u = 8n\pi\,\delta_0$ on E.

We recall some properties about period integrals and Green's functions in lemmas 5.5.3–5.5.4 below, before returning to the first approach outlined in 5.5.1.

Lemma 5.5.3. *For any $y \in \mathbb{C}$ and any $\omega \in \Lambda = \mathrm{H}_1(E;\mathbb{Z})$, the ω-period of the meromorphic differential $\dfrac{\wp'(y)\,dz}{\wp(z)-\wp(y)}$ on $E = \mathbb{C}/\Lambda$ is given by*

$$\int_{L_\omega} \frac{\wp'(y)}{\wp(z)-\wp(y)}dz \equiv 2\omega\cdot\zeta(y) - 2\eta(\omega)\cdot y \pmod{2\pi\sqrt{-1}\mathbb{Z}}. \tag{5.5.6}$$

Here $L_\omega : [0,1] \to \mathbb{C}$ is any piecewise smooth path on $\mathbb{C}$ such that $L_\omega(1) - L_\omega(0) = \omega$ and $\wp(z) \neq \wp(y)$ for all $z \in L_\omega([0,1])$.

Proof. This is a reformulation of [44, Lemma 2.4]. Note that meromorphic differential $\frac{\wp'(y)\,dz}{\wp(z)-\wp(y)}$ on E has poles at 0 and $\pm y \pmod{\Lambda}$, with residues 0 and ± 1 respectively, therefore the period integral $I_\omega(y) := \int_{L_\omega}\frac{\wp'(y)\,dz}{\wp(z)-\wp(y)}$ is well-defined modulo $2\pi\sqrt{-1}\mathbb{Z}$. The addition formula for $\wp(z)$ gives

$$\frac{\wp'(y)\,dz}{\wp(z)-\wp(y)} = \frac{\wp'(z)\,dz}{\wp(z)-\wp(y)} - 2\zeta(z+y;\Lambda)dz + 2\zeta(z;\Lambda)dz + 2\zeta(y;\Lambda)dz$$

The lemma follows after an easy calculation, using the functional equation for $\zeta(z;\Lambda)$ and the fact that $\frac{d}{dz}\log\sigma(z) = \zeta(z)$ and; see [44, Lemma 2.4] for details.

Alternatively, one computes

$$\frac{d}{dy}I_\omega(y) = \int_{L_\omega}(2\wp(z+y) - 2\wp(y))dz = -2\wp(y) + 2\eta(\omega)$$

and determine the constant of integration up to $2\pi\sqrt{-1}\mathbb{Z}$ by evaluation at 2-torsion points of E. □

Lemma 5.5.4. *Let G be the Green's function on $E = \mathbb{C}/\Lambda$ as in* (0.1.1). *The following formulas hold.*

$$G(z) = -\frac{1}{2\pi}\log\left|\Delta(\Lambda)^{\frac{1}{12}}\cdot e^{-z\,\eta(z;\Lambda)/2}\cdot\sigma(z;\Lambda)\right| \qquad \text{on } E, \tag{5.5.7}$$

$$-4\pi\frac{\partial G}{\partial z}(z) = \zeta(z;\Lambda) - \eta(z;\Lambda) \qquad \forall\, z \in \mathbb{C}. \tag{5.5.8}$$

In the first formula (5.5.7)*, $\eta(z;\Lambda)$ is the quasi-period and $\Delta(\Lambda)$ is the non-zero cusp form of weight* 12 *for* $\mathrm{SL}_2(\mathbb{Z})$ *given by the formula*

$$\Delta(\Lambda) = g_2(\Lambda)^3 - 27g_3(\Lambda)^2 = \frac{(2\pi\sqrt{-1})^{12}}{(\omega_2)^{12}}\cdot q_\tau\cdot\prod_{m=1}^{\infty}(1-q_\tau^n)^{24},$$

where $q_\tau = e^{2\pi\sqrt{-1}\tau}$, $\tau = \omega_2/\omega_1$ with $\mathrm{Im}(\tau) > 0$.

Remark 5.5.5. (a) An equivalent form of 5.5.4 (a) is

$$G(z;\Lambda_\tau) = -\frac{1}{2\pi}\log\left|e^{-\frac{\pi\,\mathrm{Im}(z)^2}{\mathrm{Im}(\tau)}}\cdot\Delta(\mathbb{C}/\Lambda_\tau)^{-\frac{1}{12}}\cdot\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](z;\tau)\right|$$

for the Green's function $G(z;\Lambda_\tau)$ on the elliptic curve $\mathbb{C}/\Lambda_\tau$, where τ is an element of the upper-half plane and $\Lambda_\tau = \mathbb{Z}+\mathbb{Z}\cdot\tau)$). Here we have used the general notation for theta functions with characteristics

$$\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](z;\tau) := \sum_{m\in\mathbb{Z}} e^{\pi\sqrt{-1}\,\tau(m+a)^2}\cdot e^{2\pi\sqrt{-1}\,(m+a)(z+b)}.$$

The equivalence of the two formulas follows from the formulas

$$\sigma(z,\Lambda_\tau) = -\frac{1}{\pi}\,e^{\eta(1,\Lambda_\tau)}\cdot\frac{\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](z;\tau)}{\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](z;\tau)\theta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right](z;\tau)\theta\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right](z;\tau)}$$

and

$$\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](z;\tau)\theta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right](z;\tau)\theta\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right](z;\tau) = 2\,\eta_{\text{Dedekind}}(\tau)^3 = 2\left(q_\tau^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q_\tau^m)\right)^3.$$

(b) The function $Z(z;\Lambda) := \zeta(z;\Lambda)-\eta(z;\Lambda) = -4\pi\frac{\partial G}{\partial z}$ appeared in [28, p. 452]; we will call it the *Hecke form*. For any integers a,b and $N\geq 1$ such that $\gcd(a,b,N)=1$ and $(a,b)\not\equiv(0,0) \pmod N$, $Z(\frac{a}{N}+\frac{b}{N}\tau;\,\mathbb{Z}+\mathbb{Z}\tau)$ is a modular form of weight one and level N, equal to the Eisenstein series

$$\begin{aligned}&-N\cdot E_1^N(\tau,s;a,b)\Big|_{s=0}\\ &\quad= -N\cdot\mathrm{Im}(\tau)^s\cdot\sideset{}{'}\sum_{(m,n)\equiv(a,b)\bmod N}(m\tau+n)^{-1}\cdot|m\tau+n|^{-2s}\Big|_{s=0}.\end{aligned}$$

See [28, p. 475].

Proof of Lemma 5.5.4. The formula (5.5.7) is proved in [40, II, §5]. The equivalent formula in 5.5.5 (a) is proved in [23, p. 417–418]. See also [43, §7] and [44, §2]. The formula (5.5.8) follows from (a) by an easy computation. ☐

Theorem 5.6. *Let n be a positive integer. Let $P_i = p_i \bmod \Lambda$, $i=1,\dots,n$ be n distinct points on $E=\mathbb{C}/\Lambda$ such that $\{P_1,\dots,P_n\}\cap\{-P_1,\dots,-P_n\}=\emptyset$. In other words $\wp(p_1),\dots,\wp(p_n)$ are mutually* distinct *and none of the P_i's is a 2-torsion point of E. There exists a normalized type II developing map f for a solution u of $\triangle u+e^u = 8\pi n\delta_0$ on E such that $f(p_1)=\cdots=f(p_n)=0$ if and only if*

$$\sum_{i=1}^{n}\frac{\partial G}{\partial z}(p_i) = 0 \tag{5.6.1}$$

and

$$\wp'(p_1)\wp^r(p_1)+\cdots+\wp'(p_n)\wp^r(p_n) = 0 \quad \text{for } r=0,\dots,n-2. \tag{5.6.2}$$

Notice that (5.6.1) is the same as (5.5.2) and (5.6.2) is the same as (5.5.1).

Proof. We use the notation in the proof of Theorem 5.2 and continue with the argument there. The logarithmic derivative $g' = f'/f$ of a normalized type II developing map has simple poles at the $2n$ points $\pm P_1, \ldots, \pm P_n$ and is holomorphic elsewhere on E. Moreover the residues of g at P_i (resp. at $-P_i$) is 1 (resp. -1) for each i. Therefore

$$g(z) = \frac{\wp'(p_1)}{\wp(z) - \wp(p_1)} + \cdots + \frac{\wp'(p_n)}{\wp(z) - \wp(p_n)}. \tag{5.6.3}$$

because $g(0) = 0$. We know two more properties of g: (a) $g(z)$ has a zero of order $2n$ at $z = 0$, and (b) for any $\omega \in \Lambda$ and any piecewise smooth path $L_\omega : [0,1] \to \mathbb{C}$ such that $L(1) - L(0) = \omega$ and

$$L([0,1]) \cap \left[\left(\textstyle\bigcup_{i=1}^n p_i + \Lambda\right) \cup \left(\textstyle\bigcup_{i=1}^n -p_i + \Lambda\right)\right] = \emptyset,$$

we have

$$\int_{L_\omega} g\, dz \in \sqrt{-1}\mathbb{R}.$$

To see what property (a) means, we expand $g(z)$ at $z = 0$ as a power series in $\wp(z)$:

$$g(z) = \sum_{j=1}^n \frac{\wp'(p_j)}{\wp(z)\,(1 - \wp(p_j)/\wp(z))} = \sum_{m=0}^\infty \left(\sum_{j=1}^n \wp'(p_j)\, \wp(p_j)^m\right) \cdot \wp(z)^{-m-1}$$

Because g has exactly $2n$ simple poles and is holomorphic elsewhere on E, we see that $\mathrm{order}_{z=0}\, g(z) = 2n$ if and only if all $n-1$ equations in (5.6.2) hold.

We know that $\eta(z;\Lambda)y - z\eta(y;\Lambda) \equiv 0 \pmod{\sqrt{-1}\mathbb{R}}$ for all $y, z \in \mathbb{C}$ because the left-hand side is $\mathbb{R}$-bilinear, and we know from the Legendre relation that the statement holds when y, z are both in Λ. By Lemma 5.5.3,

$$\begin{aligned}\int_{L_\omega} g(z)\, dz &\equiv 2\sum_{j=1}^n \left(\omega\zeta(p_j) - \eta(\omega)p_j\right) \pmod{2\pi\sqrt{-1}\,\mathbb{Z}}\\ &\equiv 2\omega \cdot \sum_{j=1}^n \left(\zeta(p_j) - \eta(p_j)\right) \pmod{\sqrt{-1}\mathbb{R}}\end{aligned}$$

for all $\omega \in \Lambda$. Therefore property (b) holds for g given by (5.6.3) if and only (5.6.1) holds. We have proved the "only if" part of Theorem 5.6.

Conversely suppose that equations (5.6.2) and (5.6.1) hold. We have seen that the meromorphic function $g(z)$ given by (5.6.3) has a zero of order $2n$ at $z = 0$ and the period integrals of $g\, dz$ are all purely imaginary. Therefore $f(z) = \exp \int_0^z g(w)\, dw$ is a type II developing map for a solution of the singular Liouville equation $\triangle u + e^u = 8\pi n\, \delta_0$. □

Remark 5.6.1. The property (a) that the order of the meromorphic function

$$\frac{\wp'(p_1)}{\wp(z) - \wp(p_1)} + \cdots + \frac{\wp'(p_n)}{\wp(z) - \wp(p_n)}$$

on E at $z = 0$ is equal to $2n$ is also equivalent to: $\exists\, C \in \mathbb{C}^\times$ such that

$$\sum_{j=1}^{n} \wp'(p_j) \prod_{i \neq j} (\wp(z) - \wp(p_i)) = C. \tag{5.6.4}$$

5.7. Analytic approach to the configuration of the blow-up set.

5.7.1. We may also study the set $\{p_1, \ldots, p_n\}$ from the analytic point of view. As we have already seen, $\{p_i\}$ represents the blow-up set of the family of solutions u_λ as $\lambda \to \infty$. The equations which determine the position of blow-up points are of fundamental importance in the study of *bubbling solutions of semi-linear equations* such as mean field equations, Chern–Simons–Higgs equation, Toda system in two dimension, or scalar curvature equation in higher dimensions. Hence we will derive these equations from the analytic perspective.

We recall the definition of blow-up points for a sequence of solutions u_k, $k \in \mathbb{N}$, to the mean field equation

$$\triangle u_k + e^{u_k} = \rho_k\, \delta_0 \qquad \text{on } E \tag{5.7.1}$$

with possibly varying singular strength ρ_k such that $\rho_k \to \rho = 8\pi n$ for some $n \in \mathbb{N}$. If $\rho_k = 8\pi n$ for all k, this goes back to the situation u_λ in (0.2.6) as has been discussed. It is important to consider blow-up phenomenon from a sequence of solutions u_k with $\rho_k \to \rho$. (It is known that if $\rho \notin 8\pi\mathbb{N}$ then there is no blow-up phenomenon [12].)

Definition 5.7.2. A subset $S = \{P_1, \ldots, P_m\} \subset E = \mathbb{C}/\Lambda$ is called the *blow-up set* of the sequence of solutions $(u_k)_{n\in\mathbb{N}}$ of (5.7.1) with $\rho_k \to 8\pi n$ if for all i

$$u_k(P_i) \to +\infty \quad \text{as} \quad k \to \infty,$$

while if $P \notin S$ then

$$u_k(P) \to -\infty \quad \text{as} \quad k \to \infty.$$

Points P_i in the blow-up set are called *blow-up points* of the sequence of solutions (u_k).

It is also shown in [12] that $m = n$ and the configuration of the blow-up points $\{P_1, \ldots, P_n\}$ satisfies the following equations:

$$nG_z(P_i) = \sum_{j=1, \neq i}^{n} G_z(P_i - P_j), \qquad i = 1, 2, \ldots, n, \tag{5.7.2}$$

where z is the coordinate for $\mathbb{C}$ and $G_z = \frac{\partial G}{\partial z}$. Notice that the system of equations (5.7.2) is the same as the equations (5.5.3). Summing the n equations in (5.7.2) from $i = 1, \ldots, n$, we get

$$\sum_{i=1}^{n} G_z(P_i) = 0, \tag{5.7.3}$$

since $\frac{\partial G}{\partial z}$ is an odd function. The last equation (5.7.3) is the same as the Green equation (5.5.2) and (5.6.1).

Lemma 5.7.3. *Let* $\{P_1, \ldots, P_n\}$ *be a set of* n *mutually distinct points in* $E \smallsetminus \{0\} = \mathbb{C}/\Lambda \smallsetminus \{[0]\}$*, and let* $p_1, \ldots, p_n$ *be elements of* $\mathbb{C}$ *such that* $P_i = [p_i] := p_i \bmod \Lambda$ *for* $i = 1, \ldots, n$*. The system of equations* (5.7.2) *for the set* $\{P_1, \ldots, P_n\}$ *is equivalent to the combination of the Green equation* (5.7.3) *and the following system of equations*

$$\sum_{1 \le j \le n,\ j \ne i} \big(\zeta(p_i - p_j; \Lambda) + \zeta(p_j; \Lambda) - \zeta(p_i; \Lambda)\big) = 0, \qquad i = 1, \ldots, n. \tag{5.7.4}$$

Notice that for each i, the sum in the left-hand side of (5.7.4) is independent of the choice of representatives $p_1, \ldots, p_n \in \mathbb{C}$ of $P_1, \ldots, P_n \in \mathbb{C}/\Lambda$.

Proof. We have seen that the n equations in (5.7.2) imply the Green equation (5.5.2). It suffices to show that under (5.7.3), the system of equations (5.7.2) is equivalent to the system of equations (5.7.4).

We know from (5.5.8) that $G_z(P_i) = \zeta(p_i; \Lambda) - \eta(p_i; \Lambda)$ for each i. So the Green equation (5.7.3) means that $\sum_{i=1}^n \zeta(p_i; \Lambda) = \sum_{i=1}^n \eta(p_i; \Lambda)$. For each i the i-th equation in (5.7.2) becomes

$$n \cdot \big[\zeta(p_i; \Lambda) - \eta(p_i; \Lambda)\lambda\big] = \sum_{1 \le j \le i,\ j \ne i} \big(\zeta(p_i - p_j; \Lambda) - \eta(p_i; \Lambda) + \eta(p_j; \Lambda)\big)$$

which is equivalent to the i-th equation in (5.7.4) because $\sum_{i=1}^n \zeta(p_i; \Lambda) = \sum_{i=1}^n \eta(p_i; \Lambda)$. □

Remark 5.7.4. Part of the condition for the blow-up set $\{P_1, \ldots, P_n\}$ of a sequence of solutions (u_k) as in Definition 5.7.2 is that

$$P_i \ne P_j \quad \text{for} \quad i \ne j, \tag{5.7.5}$$

instead of the stronger property

$$\{P_1, \ldots, P_n\} \cap \{-P_1, \ldots, -P_n\} = \emptyset \tag{5.7.6}$$

which is satisfied when $\{p_1, \ldots, p_n\}$ are zeros of a normalized developing map of a solution of $\triangle u + e^u = 8\pi n\, \delta_0$.

5.8. Equivalence of algebraic systems (5.6.2) and (5.7.4) under (5.7.6)

5.8.1. In light of Theorem 0.6, the analytic discussion in §5.7 suggests that the system of equations (5.5.2) + (5.6.2) may be *equivalent* to the system of equations (5.5.2) + (5.7.4) under the constraint that $P_i \ne P_j$ whenever $i \ne j$ and $P_i \ne -P_j$ for all i, j.

Since the Green equation (5.5.2) is the only non-holomorphic equation shared by both systems, one might optimistically ask

> *Are the two holomorphic systems of* $n - 1$ *equations* (5.6.2) and (5.7.4) *equivalent?*

Note that the sum of the n equations in (5.7.4) is zero, hence we may remove one equation from (5.7.4).

5.8.2. To answer this question, we recall the addition formula

$$\frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)} = \zeta(z+u)-\zeta(z)-\zeta(u).$$

Thus (5.7.4) with additional constraint $p_i \neq \pm p_j$ for $i \neq j$ is equivalent to

$$\sum_{j\neq i}\frac{\wp'(p_i)+\wp'(p_j)}{\wp(p_i)-\wp(p_j)} = 0, \qquad i=1,\ldots,n. \tag{5.8.1}$$

Let $(x_i, y_i) := (\wp(p_i), \wp'(p_i))$. As points on E they are related by the defining cubic curve equation $y_i^2 = p(x_i) = 4x_i^3 - g_2x_i - g_3$. Then (5.8.1) and (5.6.2) can be written as the following systems respectively:

$$\sum_{j=1,\neq i}^{n}\frac{y_i+y_j}{x_i-x_j} = 0, \quad i=1,\ldots,n, \tag{5.8.2}$$

where $x_i \neq x_j$ for $i \neq j$ is imposed, and

$$\sum_{i=1}^{n} x_i^l y_i = 0, \quad l = 0,\ldots,n-2. \tag{5.8.3}$$

Both systems of equations (5.8.2) and (5.8.3) are linear in the y_i's, and in fact we can prove their equivalence in a general context:

Proposition 5.8.3. *For a given set of mutually distinct elements $x_1, \ldots, x_n \in \mathbb{C}$, the linear systems of equations*

$$\sum_{1\le j\le n,\ j\neq i}\frac{Y_i+Y_j}{x_i-x_j} = 0, \quad \forall\, i=1,\ldots,n \tag{5.8.4}$$

and

$$\sum_{i=1}^{n} x_i^l \cdot Y_i = 0, \quad \forall\, l=0,\ldots,n-2 \tag{5.8.5}$$

in variables $Y_1, \ldots, Y_n$ are equivalent.

Proof. The system (5.8.4) corresponds to the $n \times n$ matrix

$$A_n = \begin{pmatrix} \sum\limits_{k=2}^{n}\frac{1}{x_1-x_k} & \frac{1}{x_1-x_2} & \frac{1}{x_1-x_3} & \cdots & \frac{1}{x_1-x_n} \\ \frac{1}{x_2-x_1} & \sum\limits_{k=1,\neq 2}^{n}\frac{1}{x_2-x_k} & \frac{1}{x_2-x_3} & \cdots & \frac{1}{x_2-x_n} \\ \vdots & \vdots & \ddots & & \vdots \\ \frac{1}{x_n-x_1} & \frac{1}{x_n-x_2} & \cdots & \cdots & \sum\limits_{k=1}^{n-1}\frac{1}{x_n-x_k} \end{pmatrix},$$

that is, $A_n = (a_{ij}) \in M_n(\mathbb{Q}(x_1,\ldots,x_n))$, where $a_{ij} = \frac{1}{x_i-x_j}$ if $j \neq i$, and $a_{ij} = \sum_{k=1,\neq i}^{n}\frac{1}{x_i-x_k}$ if $j = i$, which is the sum of all the other entries in the same row. Note that the sum of all rows in A_n is the zero row vector. In particular, $\det A_n = 0$.

The system of equations (5.8.5) corresponds to the $(n-1) \times n$ matrix

$$B_n := \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_1^{n-2} & \cdots & x_n^{n-2} \end{pmatrix}.$$

Let $b = (b_1, \ldots, b_n)$, where b_i is the determinant of the $(n-1) \times (n-1)$ minor of B_n without the i-th column. Then b_i's are given by the Vandermonde determinant:

$$b_i = \prod_{1 \le l < k \le n;\, l,k \ne i} (x_k - x_l).$$

Let C_n be the $(n-1) \times n$ matrix consisting of the first $n-1$ rows of A_n, and define $c = (c_1, \ldots, c_n)$ similarly.

We want to prove that $c \ne 0$, which implies that rank $A_n = n-1$ and the kernel of A_n is spanned by c. Then the equivalence of these two linear systems simply means that b and c are proportional to each other.

We claim that

$$c_i = \frac{(-1)^{n+i}(n-1)!}{\prod_{k \ne i}(x_k - x_i)}, \qquad i = 1, \ldots, n. \tag{5.8.6}$$

Due to symmetry, it is enough to consider the case $i = n$. We will show that the order of c_n along the divisor $x_k - x_l$ is non-negative for all $k, l \ne n$. This will imply that c_n is a constant times $\prod_{k=1}^{n-1}(x_k - x_n)^{-1}$.

Again by symmetry, it is enough to check the case $k = 1$, $l = 2$. The only terms which may contribute poles along $x_k - x_l$ are a_{11}, a_{12}, a_{21} and a_{22}. If we subtract the second row by the first row, and then add the resulting first column into the second column, we get the following $(n-1) \times (n-1)$ matrix

$$\begin{pmatrix} \frac{1}{x_1 - x_2} + r & -r & * & \cdots & * \\ r & (x_1 - x_2)* & * & \cdots & * \\ * & (x_1 - x_2)* & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & (x_1 - x_2)* & * & \cdots & * \end{pmatrix},$$

where $r = \sum_{j=3}^{n} \frac{1}{x_1 - x_j}$, as well as all entries labeled by $*$, does not have pole along the divisor $x_1 - x_2$. This shows that $\det(a_{ij})_{1 \le i,j \le n-1}$ has non-negative order along $x_1 - x_2$. So there exists an element $d_n \in \mathbb{C}$ such that

$$c_n = \frac{d_n}{\prod_{k<n}(x_k - x_n)}. \tag{5.8.7}$$

By Lemma 5.8.4 below we have $d_n = (n-1)! \ne 0$. Then (5.8.6) holds and we have $c \ne 0$. Now we note that

$$b_i = \frac{(-1)^{n-i}\prod_{1 \le l < k \le n}(x_k - x_l)}{\prod_{k \ne i}(x_k - x_i)} = \frac{1}{(n-1)!}\prod_{1 \le l < k \le n}(x_k - x_l)c_i,$$

i.e. b is parallel to c. Hence the equivalence is proved. $\square$

Lemma 5.8.4. *The constant d_n in (5.8.7) is $(n-1)!$.*

We offer two proofs.

The first/analytic proof. It is easy to see that $d_1 = 1$. If we may show that $d_n = (n-1)d_{n-1}$ for $n \geq 2$ then we are done. We observe that

$$\frac{d_n}{\prod_{2\leq k<n}(x_k - x_n)} = \lim_{x_1\to\infty} \frac{d_n x_1}{\prod_{k<n}(x_k - x_n)}$$

which, by the definition of c_n and (5.8.7), is equal to

$$\lim_{x_1\to\infty} \begin{vmatrix} \sum_{k=2}^{n} \frac{x_1}{x_1-x_k} & \frac{x_1}{x_1-x_2} & \frac{x_1}{x_1-x_3} & \cdots & \frac{x_1}{x_1-x_{n-1}} \\ \frac{1}{x_2-x_1} & \sum_{k=1,\neq 2}^{n} \frac{1}{x_2-x_k} & \frac{1}{x_2-x_3} & \cdots & \frac{1}{x_2-x_{n-1}} \\ \vdots & \vdots & \ddots & & \vdots \\ \frac{1}{x_{n-1}-x_1} & \frac{1}{x_{n-1}-x_2} & \cdots & \cdots & \sum_{k=1,\neq n-1}^{n} \frac{1}{x_{n-1}-x_k} \end{vmatrix}.$$

Since

$$\lim_{x_1\to\infty} \sum_{k=2}^{n} \frac{x_1}{x_1 - x_k} = n-1,$$

by evaluating the limit, the determinant becomes

$$(n-1) \begin{vmatrix} \sum_{k=3}^{n} \frac{1}{x_2-x_k} & \frac{1}{x_2-x_3} & \frac{1}{x_2-x_4} & \cdots & \frac{1}{x_2-x_{n-1}} \\ \frac{1}{x_3-x_2} & \sum_{k=2,\neq 3}^{n} \frac{1}{x_3-x_k} & \frac{1}{x_3-x_4} & \cdots & \frac{1}{x_3-x_{n-1}} \\ \vdots & \vdots & \ddots & & \vdots \\ \frac{1}{x_{n-1}-x_2} & \frac{1}{x_{n-1}-x_3} & \cdots & \cdots & \sum_{k=2,\neq n-1}^{n} \frac{1}{x_{n-1}-x_k} \end{vmatrix}$$

$$= (n-1)\frac{d_{n-1}}{\prod_{2\leq k<n}(x_k - x_n)}.$$

Thus $d_n = (n-1)d_{n-1}$ as expected. $\square$

The second/algebraic proof. It is enough to consider the specialization $x_i = \zeta^i$ for $i = 1, \ldots, n$, where $\zeta = e^{2\pi i/n}$ is the n-th primitive root of unity. Let $A' = (a'_{ij})$ be the specialized matrix and $A'' = (a''_{ij})$ the Hermitian matrix with

$$a''_{ij} := \zeta^i a'_{ij} = \begin{cases} \dfrac{1}{1-\zeta^{j-i}} & \text{if } i \neq j, \\ \displaystyle\sum_{k=1}^{n-1} \frac{1}{1-\zeta^k} = \frac{n-1}{2} & \text{if } i = j. \end{cases}$$

Here the diagonal entries $a''_{ii} = \frac{1}{2}(n-1)$ follows from the fact that

$$\frac{1}{1-\zeta^k} + \frac{1}{1-\zeta^{n-k}} = \frac{1-\zeta^{n-k}+1-\zeta^k}{1-\zeta^k-\zeta^{n-k}+1} = 1.$$

Let V be the underlying vector space of the group ring $\bar{\mathbb{Q}}[\mathbb{Z}/n\mathbb{Z}] = \bigoplus_{j\in\mathbb{Z}/n\mathbb{Z}} \bar{\mathbb{Q}}\cdot[j]$. Then $(A'')^t$ is the matrix representation of the following operator $\mathbf{T}$ on V with respect to the basis $[\bar{1}], [\bar{2}], \cdots [\bar{n}]$:

$$\mathbf{T} = \frac{n-1}{2} + \sum_{j=1}^{n-1} \frac{1}{1-\zeta^j}[j].$$

We put the Hermitian inner product on V so that $[i]$'s are orthonormal. It is easy to diagonalize A''. Indeed, for $a \in \mathbb{Z}/n\mathbb{Z}$, let

$$x_a := \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \zeta^{-ia}[i] \in \bar{\mathbb{Q}}[\mathbb{Z}/n\mathbb{Z}].$$

Then V is also the orthogonal direct sum of the one dimensional subspaces $V_a := \bar{\mathbb{Q}}\cdot x_a$. It is easily seen that $[j]\cdot x_a = \zeta^{ja}x_a$. Hence x_a's are eigenvectors of $\mathbf{T}$ with eigenvalues

$$\lambda_a = \frac{n-1}{2} + \sum_{j=1}^{n-1} \frac{\zeta^{ja}}{1-\zeta^j}.$$

In fact $\lambda_a = a-1$ for $a = 1, \ldots, n$. To see this, we rewrite λ_a as

$$\lambda_a = (n-1) - \sum_{j=1}^{n-1} \frac{1-\zeta^{ja}}{1-\zeta^j} = (n-1) - \sum_{j=1}^{n-1}\sum_{k=0}^{a-1} \zeta^{jk}.$$

By changing the order of summation, for $k=0$ we get $n-1$, while for $k = 1, \ldots, a-1$ we get $\sum_{j=1}^{n-1}\zeta^{jk} = -1$. Hence $\lambda_a = a-1$ as expected.

The diagonalization in terms of matrices reads as

$$CA'' = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & n-1 \end{pmatrix} C,$$

where the a-th row vector of $C = (z_{ij})_{1\le i,j\le n}$, $z_{ij} := \zeta^{-ij}$ corresponds to x_a.

Now we work on $\Lambda^{n-1}V$ and $\Lambda^{n-1}\mathbf{T} \in \mathrm{End}(\Lambda^{n-1}V)$. For a square matrix B, $\Lambda^{n-1}B = \mathrm{adj}(B)^t$ is the "non-transposed" cofactor matrix. It has the covariant property that $\Lambda^{n-1}B_1B_2 = (\Lambda^{n-1}B_1)(\Lambda^{n-1}B_2)$. We find

$$\Lambda^{n-1}A'' = \Lambda^{n-1}C^{-1} \begin{pmatrix} (n-1)! & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \Lambda^{n-1}C.$$

Hence

$$(\Lambda^{n-1}A'')_{ni} = (n-1)!(\Lambda^{n-1}C^{-1})_{n1}(\Lambda^{n-1}C)_{1i}. \tag{5.8.8}$$

To compute the right hand side, from $C.\bar{C}^t = nI_n$ and $C.(\Lambda^{n-1}C)^t = (\det C)I_n$, we get

$$\Lambda^{n-1}C = n^{-1}(\det C)\bar{C}.$$

Also $C^{-1} = n^{-1}\bar{C}^t$. The same reasoning implies that

$$\Lambda^{n-1}C^{-1} = n^{-(n-1)}\Lambda^{n-1}\bar{C}^t = n^{-n}(\det \bar{C}^t)C^t = (\det C)^{-1}C^t.$$

In particular, (5.8.8) becomes

$$(\Lambda^{n-1}A'')_{ni} = \frac{(n-1)!}{n}\zeta^i.$$

By definition of c_i, the equation (5.8.7) for d_n specialized to $x_i = \zeta^i$ reads as (notice that $\prod_{j=1}^{n-1}(1-\zeta^j) = n$)

$$(-1)^{n+i}(\Lambda^{n-1}A')_{ni} = \frac{(-1)^{n+i}d_n}{\prod_{k\neq i}(\zeta^k - \zeta^i)} = \frac{(-1)^{n+i}d_n\zeta^i(-1)^{n-1}}{n}.$$

Since $(-1)^{n+i}(\Lambda^{n-1}A')_{ni} = (-1)^{n+i}(-1)^{n-1}(\Lambda^{n-1}A'')_{ni}$, the above two expressions lead to $d_n = (n-1)!$. □

Remark. We found the algebraic proof first which gives the value $d_n = (n-1)!$. The shorter and more elementary analytic proof came much later which was inspired by the factorial nature of d_n. Then we were informed by Y. Zarhin that Lemma 5.8.4 appeared in [55, §1], with a different proof.

Corollary 5.8.5. *For $P_1, \ldots, P_n \in E$ satisfying $P_i \neq P_j$ for $i \neq j$ and $P_i \neq -P_j$ for all i, j, the system of equations* (5.6.2) *is equivalent to* (5.8.1)*, hence also equivalent to* (5.7.4).

Remark 5.8.6. We record two easy observations about the system of linear equations

$$\sum_{i=1}^{n} s_i^l \cdot Y_i = 0, \qquad l = 0, \ldots, n-2 \tag{5.8.9}$$

with $s_1, \ldots, s_n$ in $\mathbb{C}$.

(a) If $s_1, \ldots, s_n$ are mutually distinct, and $(y_1, \ldots, y_n)$ is a solution of (5.8.9) in which one y_i is 0, then all y_j's are equal to zero.

(b) If $s_1, \ldots, s_n$ are not mutually distinct, then (5.8.9) only has trivial solutions in the following sense: We have a set $\{t_1, \ldots, t_m\}$ consisting of mutually distinct numbers such that $\{s_1, \ldots, s_n\} = \{t_1, \ldots, t_m\}$. Suppose that $(y_1, \ldots, y_n)$ is a solution of (5.8.9), let $z_j := \sum_{\text{all } i \text{ s.t. } s_i = t_j} y_i$ for $i = 1, \ldots, m$. Then the system of linear equations for $y_1, \ldots, y_n$ becomes

$$\sum_{j=1}^{m} s_j^l \cdot z_j = 0 \qquad \text{for } l = 0, \ldots, n-2,$$

and $z_1 = \cdots = z_m = 0$ by the non-vanishing of the Vandermonde determinant.

6. Lamé for type II: Characterizations of X_n and Y_n

6.1. An overview for this section.

6.1.1. In §5, we have proved that for each positive integer n, for every solution u of the mean field equation

$$\triangle u + e^u = 8\pi n \cdot \delta_0 \qquad \text{on } \mathbb{C}/\Lambda \tag{6.1.1}$$

there exists a set $a = \{a_1, \ldots, a_n\}$ of n complex numbers which satisfies (5.5.2), (5.6.2) and (5.7.6) such that

$$f(z) = f_a(z) := \prod_{i=1}^{n} \exp \int_0^z \frac{\wp'(a_i)}{\wp(w) - \wp(a_i)}\, dw, \tag{6.1.2}$$

is a normalized type II developing map of u. Moreover every set $a = \{a_1, \ldots, a_n\}$ of complex numbers satisfying conditions (5.6.1), (5.6.2) and (5.7.6) gives rise to a solution of the above mean field equation.

6.1.2. In this section we will leave the Green equation (5.6.1) alone and consider those $a = \{a_1, \ldots, a_n\}$ satisfy only the equations (5.6.2) under the constraint (5.7.6), that is, we consider a in the set X_n defined in (0.6.6) in the introduction. We would like to characterize $a \in X_n$ in terms of certain Lamé equations.

6.1.3. We will make use of the following addition formulas freely:

$$\frac{\wp'(z)}{\wp(z) - \wp(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z), \tag{6.1.3}$$

$$\frac{\wp'(u)}{\wp(z) - \wp(u)} = \zeta(z-u) - \zeta(z+u) + 2\zeta(u), \tag{6.1.4}$$

$$\frac{1}{2}\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)} = \zeta(z+u) - \zeta(z) - \zeta(u), \tag{6.1.5}$$

$$\frac{1}{4}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2 = \wp(z+u) + \wp(z) + \wp(u). \tag{6.1.6}$$

Definition 6.1.4. Let Λ be a cocompact lattice in $\mathbb{C}$. Let $n \geq 1$ be a positive integer. Let $[a] = \{[a_1], \ldots, [a_n]\}$ be an unordered list of n elements in $(\mathbb{C}/\Lambda) \smallsetminus \{[0]\}$, possibly with multiplicity. Define a meromorphic function $f_{[a]}(z)$ on $\mathbb{C}$ by

$$f_{[a]}(z) = f_{[a]}(z;\Lambda) := \prod_{i=1}^{n} \exp \int_0^z \left(\zeta(w-a_i) - \zeta(w+a_i) + 2\zeta(a_i)\right) dw. \tag{6.1.7}$$

where a_i is a representative in $\mathbb{C}$ of $[a_i]$ for each $i = 1, \ldots, n$.

Note that $f_{[a]}$ depends only on the element $\{[a_1],\dots,[a_n]\}$ of the symmetric product $\mathrm{Sym}^n\big(\mathbb{C}/\Lambda \smallsetminus \{[0]\}\big)$ and not on the choice of representatives $a_i \in [a_i]$. Because $\zeta(z;\Lambda) = \frac{d}{dz}\log\sigma(z;\Lambda)$, we get from (6.1.4) an equivalent definition

$$f_{[a]}(z) := (-1)^n \cdot e^{2z\sum_{i=1}^n \zeta(a_i)} \cdot \prod_{i=1}^n \frac{\sigma(z-a_i)}{\sigma(z+a_i)}\,. \tag{6.1.8}$$

Note also that $f_{[a]}(0) = 1$ and $f_{[a]}(-z)\cdot f_{[a]}(z) = 1$ for all z.

Definition 6.1.5. Let $a = \{a_1,\dots,a_n\}$ be an unordered list of elements of $\mathbb{C} \smallsetminus \Lambda$. The Hermite-Halphen ansatz function $w_a(z)$ attached to the list a is the meromorphic function on $\mathbb{C}$ defined by

$$w_a(z) = w_a(z;\Lambda) := e^{z\sum\zeta(a_i)} \prod_{i=1}^n \frac{\sigma(z-a_i)}{\sigma(z)}\,. \tag{6.1.9}$$

Remark. (a) In classical literature the functions $w_a(z)$ arise as explicit solutions of the Lamé equation

$$w'' = \big(n(n+1)\wp(z) + B\big)w; \tag{6.1.10}$$

see [29, I–VII], [27, p. 495–497] and also [67, §23.7].

(b) Clearly we have

$$f_{[a]}(z) = \frac{w_a(z)}{w_{-a}(z)}\,,$$

where $-a$ is the list $\{-a_1,\dots,-a_n\}$ and $[a]$ is the list $\{[a_1],\dots,[a_n]\}$.

(c) If $b = \{b_1,\dots,b_n\}$ is a list such that $b_i - a_i \in \Lambda$ for all $i = 1,\dots,n$, then $\frac{w_b}{w_a} \in \mathbb{C}^\times$, a non-zero constant.

Lemma 6.1.6. *If a list* $[a] = \{[a_1],\dots,[a_n]\}$ *of* n *elements of* $(\mathbb{C}/\Lambda) \smallsetminus \{[0]\}$ *satisfies* (5.6.1), (5.6.2) *and the non-degeneracy condition* (5.7.6), *then there exists a constant* $B = B_{[a]}$ *such that the Schwarzian derivative of* $f_{[a]}$ *satisfies*

$$S(f_{[a]}) = -2\big(n(n+1)\wp(z;\Lambda) + B_{[a]}\big).$$

Proof. By Theorem 5.6, $f_{[a]}$ is a normalized developing map for the mean field equation (6.1.1), and the assertion follows from (3.1.4). □

6.1.7. The constant $B_{[a]}$ in Lemma 6.1.6 can be evaluated by a straightforward computation; the answer is

$$B_{[a]} = (2n-1)\sum_{i=1}^n \wp(a_i;\Lambda). \tag{6.1.11}$$

On the other hand there is a proof of the formula (6.1.11) for $B_{[a]}$ without resorting to messy computations, via Lamé's differential equation (6.1.10) because $f_{[a]}$ can be written as the ratio of two linearly independent solutions of (6.1.10). The idea is this: use the Hermite–Halphen ansatz functions

$w_a(z)$ to find solutions to Lamé equations, and the constant B can be computed from the ansatz solutions w_a. Then $f_{[a]}a = w_a/w_{-a}$ has the expected Schwarzian derivative by ODE theory.

We take this approach since it requires less computation to prove the formula (6.1.11) for B, and it leads to a characterization of the set Y_n defined in (0.5.2) as the set of all unordered lists $a = \{[a_1], \ldots, [a_n]\}$ of n elements in $(\mathbb{C}/\Lambda) \smallsetminus \{[0]\}$ such that $w_a(z;\Lambda)$ satisfies a Lamé equation (6.1.10) for some $B \in \mathbb{C}$, see Theorem 6.2. We then move back to characterize the set X_n defined in (0.6.6) as the set of all a's such that $\mathrm{ord}_{z=0} f'_{[a]}(z) = 2n$, which is the highest possible value of $\mathrm{ord}_{z=0} f_{[a]}(z)$; see Theorem 6.5. This leads to the important consequence that for $a \in Y_n$, $a \notin X_n$ if and only if $a = -a$, and a characterization of X_n via the Schwarzian derivative.

The following result is known in the literature, see e.g. [27]. We reproduce its proof here for the sake of completeness.

Theorem 6.2 (Characterization of Y_n)**.** *Let $n \geq 1$ be a positive integer. Let $a = \{a_1, \ldots, a_n\}$ be an unordered list of n elements in $\mathbb{C} \smallsetminus \Lambda$. Let w_a be defined as in* (6.1.9)*. Let $[a]$ be the unordered list $\{[a_1], \ldots, [a_n]\}$, where $[a_i] := a_i \bmod \Lambda \in \mathbb{C}/\Lambda$ for each i.*

(1) *There exists a constant $B \in \mathbb{C}$ such that the meromorphic function w_a on $\mathbb{C}$ satisfies the Lamé equation* (6.1.10) *if and only if the following conditions hold.*
   - *$[a_i] \neq [a_j]$ whenever $i \neq j$, and*
   - *the a_i's satisfy*

$$\sum_{j \neq i} \big(\zeta(a_i - a_j;\Lambda) - \zeta(a_i;\Lambda) + \zeta(a_j;\Lambda)\big) = 0, \qquad i = 1, \ldots, n. \tag{6.2.1}$$

   In other words the necessary and sufficient condition is that a is a point of the variety Y_n in the notation of (0.6.6)*.*

(2) *If the system of equations* (6.2.1) *holds for a, then w_a satisfies the Lamé equation* (6.1.10) *whose accessary parameter B of the equation is*

$$B = B_{[a]} = (2n-1)\sum_{i=1}^{n} \wp(a_i;\Lambda).$$

Proof. If there are two indices $i_1 \neq i_2$ such that $[a_{i_1}] = [a_{i_2}]$, then $w_a(a_{i_1}) = w'_a(a_{i_1}) = 0$. If $w_a(z)$ is a solution of the second order linear ODE (6.1.10), then all higher derivatives of $w_a(z)$ vanish at $z = a_{i_1}$, so $w_a(z)$ is identically zero, a contradiction. We have shown that the $[a_i]$'s must be mutually distinct if $w_a(z)$ is a solution of (6.1.10).

The logarithmic derivative

$$\frac{w'_a(z;\Lambda)}{w_a(z;\Lambda)} = \sum_i \big(\zeta(a_i;\Lambda) + \zeta(z - a_i;\Lambda) - \zeta(z\Lambda)\big)$$

of w_a is an elliptic function on $\mathbb{C}/\Lambda$. Applying $\frac{d}{dz}$ again, we get

$$\begin{aligned}
\frac{w_a''}{w_a} &= \Big(\frac{w_a'}{w_a}\Big)' + \Big(\frac{w_a'}{w_a}\Big)^2 \\
&= \sum \big(\wp(z) - \wp(z-a_i)\big) + \sum \big(\zeta(a_i) + \zeta(z-a_i) - \zeta(z)\big)^2 \\
&\quad + \sum_{i\neq j} \big(\zeta(a_i) + \zeta(z-a_i) - \zeta(z)\big)\big(\zeta(a_j) + \zeta(z-a_j) - \zeta(z)\big) \\
&= 2n\wp(z) + \sum_i \wp(a_i) \\
&\quad + \sum_{i\neq j} \big(\zeta(a_i) + \zeta(z-a_i) - \zeta(z)\big)\big(\zeta(a_j) + \zeta(z-a_j) - \zeta(z)\big)
\end{aligned} \tag{6.2.2}$$

where we have used the consequence

$$\big(\zeta(a_i) + \zeta(z-a_i) - \zeta(z)\big)^2 = \wp(z) + \wp(a_i) + \wp(z-a_i).$$

of (6.1.5) and (6.1.6) to add up the first two sums after the second equality sign in (6.2.2) to get the last expression of $\frac{w_a''}{w_a}$ in (6.2.2).

The sum in the last line of (6.2.2) is an elliptic function on $\mathbb{C}/\Lambda$ with a double pole at $z=0$ with Laurent expansion $\frac{n^2}{z^2} + O(1)$; denote this function by $F_a(z)$. Therefore w_a satisfies a Lamé equation (6.1.10) for some $B \in \mathbb{C}$ if and only if $F_a(z)$ has no pole outside of $[0] \in \mathbb{C}/\Lambda$.

Suppose $[a_{i_0}]$ appears in the list $[a] = \{[a_1], \ldots, [a_n]\}$ r times with $r \geq 2$ for some $i_0 \in \{1, \ldots, n\}$. Then $F_a(z)$ has a double pole at $z = a_{i_0}$, where it has a Laurent expansion

$$F_a(z) = r(r-1)(z-a_{i_0})^{-2} + O\big((z-a_{i_0})^{-1}\big).$$

We have shown that if $F_a(z)$ is holomorphic outside Λ, then $[a_i] \neq [a_j]$ whenever $i \neq j$.

Under the assumption that $[a_1], \ldots, [a_n]$ are mutually distinct, the function $F_a(z)$ is holomorphic on $(\mathbb{C}/\Lambda) \smallsetminus \{[z_1], \ldots, [a_n]\}$ and has at most simple poles at $[z_1], \ldots, [z_n]$. Therefore $F_a(z)$ is holomorphic outside Λ if and only if its residue at $z = a_i$ is zero for $i = 1, \ldots, n$, which means that

$$\sum_{j\neq i} \big(\zeta(a_j;\Lambda) + \zeta(a_i - a_j;\Lambda) - \zeta(a_i;\Lambda)\big) = 0, \quad \forall\, 1 \leq i \leq n.$$

This proves the statement (1) of Theorem 6.2.

We know that there a constants $B_1 \in \mathbb{C}$ such that

$$F_a(z) = n(n-1)\wp(z;\Lambda) + B_1, \tag{6.2.3}$$

because $F_a(z)$ is holomorphic on $\mathbb{C}/\Lambda \smallsetminus \{[0]\}$ and its Laurent expansion at $z = 0$ is $n(n-1)\cdot z^{-2} + O(1)$. To determine B_1, we need to compute its Laurent expansion at $z = 0$ modulo $O(z)$. From

$$\zeta(z-a_i;\Lambda) = -\zeta(a_i;\Lambda) - \wp(a_i;\Lambda)z + O(z^2),$$

we get

$$F_a(z) = \sum_{i\neq j}\Big(-\frac{1}{z} - \wp(a_i;\Lambda)z + O(z^2)\Big)\Big(-\frac{1}{z} - \wp(a_j;\Lambda)z + O(z^2)\Big)$$
$$= n(n-1)\frac{1}{z^2} + 2(n-1)\sum_i \wp(a_i;\Lambda) + O(z).$$

In particular $B_1 = 2(n-1)$. From (6.2.2) we get

$$\frac{w_a''}{w_a} = n(n+1)\wp(z;\Lambda) + (2n-1)\sum_{i=1}^n \wp(a_i;\Lambda).$$

We have proved the statement (2). □

Remark 6.2.1. (a) Clearly that the necessary and sufficient condition in Theorem 6.2 (1), which defines the variety Y_n, depends only on the list $[a] = \{[a_1], \ldots, [a_n]\}$ of elements in $(\mathbb{C}/\Lambda) \smallsetminus \{[0]\}$ determined by a.

(b) It is also clear that a list $a = \{a_1, \ldots, a_n\}$ satisfies the condition in 6.2 (1) if and only if the list $-a = \{-a_1, \ldots, -a_n\}$ does.

Proposition 6.3. *Let $a = \{a_1, \ldots, a_n\}$ be an unordered list of elements in $\mathbb{C} \smallsetminus \Lambda$, $n \geq 1$.*

(1) *The function $w_a(z)$ is a common eigenvector for the translation action by elements of Λ:*

$$\frac{w_a(z+\omega)}{w_a(z)} = e^{\omega\cdot\sum_{i=1}^n \zeta(a_i;\Lambda) - \eta(\omega;\Lambda)\cdot\sum_{i=1}^n a_i} \qquad \forall\omega \in \Lambda.$$

This "eigenvalue package" attached to w_a is the homomorphism

$$\chi_a : \Lambda \to \mathbb{C}^\times, \quad \omega \mapsto e^{\omega\cdot\sum_{i=1}^n \zeta(a_i;\Lambda) - \eta(\omega;\Lambda)\cdot\sum_{i=1}^n a_i} \quad \forall\omega \in \Lambda,$$

which depends only on the list $[a] = \{[a_1], \ldots, [a_n]\}$.

(2) *If w_a satisfies a Lamé equation* (6.1.10)*, then so does w_{-a}*

(3) *For any unordered list $b = \{b_1, \ldots, b_n\}$ of elements in $\mathbb{C} \smallsetminus \Lambda$, the functions w_a and w_b are linearly dependent if and only if either $[b] = [a]$ or $[b] = [-a]$, where $[-a]$ is the unordered list $\{[-a_1], \ldots, [-a_n]\}$ of elements in $\mathbb{C}/\Lambda$.*

(4) *The homomorphisms χ_a and χ_{-a} are equal if and only if there exists an element $\omega \in \Lambda$ such that*

$$\sum_{i=1}^n \zeta(a_i;\Lambda) = \frac{\eta(\omega;\Lambda)}{2} \quad \text{and} \quad \sum_{i=1}^n a_i = \frac{\omega}{2}, \tag{6.3.1}$$

in which case $\mathrm{Im}(\chi_a) \subseteq \{\pm 1\}$.

(5) *Suppose that w_a and w_{-a} are two solutions of a Lamé equation* (6.1.10)*, and $[a] \neq [-a]$. Then $\chi_a \neq \chi_{-a}$. Moreover $\mathbb{C}\cdot w_a$ and $\mathbb{C}\cdot w_{-a}$ are characterized by the monodromy representation of* (6.1.10) *as the two one-dimensional subspaces of solutions which are stable under the monodromy.*

Proof. The statement (1) is immediate from the transformation formula for the Weierstrass σ function. The statements (2) and (3) are obvious and easy respectively. The statement (4) is a consequence of the Legendre relation for the quasi-periods.

Suppose that $[a] \neq [-a]$ and $\chi_a = \chi_{-a}$. By (3) the monodromy representation of the Lamé equation (6.1.10) is isomorphic to the direct sum $\chi_a \oplus \chi_b$, and the character χ_a has order at most 2 by (4). Consider the algebraic form

$$p(x)\frac{d^2y}{dx^2} + \frac{1}{2}p'(x)\frac{dy}{dx} - \big(n(n+1)x + B\big)y = 0 \tag{6.3.2}$$

of the Lamé equation (6.1.10). The monodromy group M of (6.3.2) contains the monodromy group (6.1.10) as a normal subgroup of index at most 2, therefore M is a finite abelian group of order dividing 4. In particular the monodromy representation of the algebraic Lamé equation (6.3.2) is completely reducible. However one knows from [65, Thm. 4.4.1] or [4, Thm. 3.1] that the monodromy representation of (6.3.2) is not completely reducible, a contradiction. We have proved the first part of (5). The second part of (5) follows from the first part of (5). □

Proposition 6.4. *Suppose that $[a] = \{[a_1], \ldots, [a_n]\}$ and $[b] = \{[b_1], \ldots, [b_n]\}$ are two points of Y_n, $n \geq 1$. If $\sum_{i=1}^n \wp(a_i;\Lambda) = \sum_{i=1}^n \wp(b_i;\Lambda)$, the either $[a] = [b]$ or $[a] = [-b]$.*

Proof. Pick representatives $a_i \in [a_i]$ and $b_i \in [b_i]$ for each $i = 1, \ldots, n$. Suppose that $[b] \neq [a]$ and $[b] \neq [-a]$. The functions $w_a(z)$ and $w_b(z)$ are linearly independent by Proposition 6.3 (3) because $[b] \neq [a]$, and they satisfy the same Lamé differential equation because $B_{[a]} = B_{[b]}$. By either [65, Thm. 4.4.1] or [4, Thm. 3.1], that image of the monodromy representation of the Lamé equation $\frac{d^2w}{d^2z} - \big(n(n+1)\wp(z;\Lambda) + B_{[a]}\big)w = 0$ is not contained in $\mathbb{C}^\times \mathrm{I}_2$, for otherwise the monodromy group of the algebraic form of the above Lamé equation on $\mathbb{P}^1(\mathbb{C})$ is contained in the product of $\mathbb{C}^\times \mathrm{I}_2$ with a subgroup of order two in $\mathrm{GL}_2(\mathbb{C})$. So $\mathbb{C}\cdot w_a$ and $\mathbb{C}\cdot w_b$ are the two distinct common eigenspaces of the monodromy representation of the above Lamé equation on $\mathbb{C}/\Lambda$. If follows that $\mathbb{C}\cdot w_{-a} = \mathbb{C}\cdot w_a$ and $\mathbb{C}\cdot w_{-b} = \mathbb{C}\cdot w_{-b}$, i.e. $[a] = [-a]$ and $[b] = [-b]$. Therefore the cardinality of the monodromy group of the above Lamé equation divides 4, and the cardinality of the monodromy group of the algebraic form of the same Lamé equation divides 8, which again contradicts [65, Thm. 4.4.1] and [4, Thm. 3.1]. □

Theorem 6.5 (Characterization of X_n by $\mathrm{ord}_{z=0} f'_{[a]}(z)$)**.** *Let $n \geq 1$ be a positive integer. Let $a = \{[a_1], \ldots, [a_n]\}$ be an unordered list of n non-zero points on the elliptic curve $\mathbb{C}/\Lambda$. Let $a_1, \ldots, a_n$ be representatives of $[a_1], \ldots, [a_n]$ in $\mathbb{C} \smallsetminus \Lambda$.*

(0) *$f_{[a]}$ is a constant if and only if $[a] = [-a]$, where $[-a]$ is the unordered list $\{[-a_1], \ldots, [-a_n]\}$ of n non-zero elements in the elliptic curve $\mathbb{C}/\Lambda$.*
(1) *If $[a] \neq [-a]$, then $\mathrm{ord}_{z=0} f'_{[a]}(z) \leq 2n$.*

(2) *Assume that* $[a] \neq [-a]$. *Then* $\mathrm{ord}_{z=0} f'_{[a]}(z) = 2n$ *if and only if the Weierstrass coordinates* $(\wp(a_i;\Lambda), \wp'(a_i;\Lambda))$ *of* $[a_1], \ldots, [a_n]$ *in* $\mathbb{C}/\Lambda$ *satisfy the following system of polynomial equations.*

$$\sum_{i=1}^{n} \wp'(a_i;\Lambda)\cdot \wp^k(a_i;\Lambda) = 0, \quad \text{for } k = 0,\ldots,n-2. \tag{6.5.1}$$

Moreover if the above equivalent conditions hold, then
- $\wp'(a_i;\Lambda) \neq 0$ *for all* $i = 1,\ldots,n$, *and*
- $\wp(a_i;\Lambda) \neq \wp(a_j;\Lambda)$ *whenever* $i \neq j$.

In other words $[a]$ *is a point of the variety* X_n *defined in* (0.6.6).

Proof. The divisor $\mathrm{div}(f_{[a]})$ of the meromorphic function $f_{[a]}$ on $\mathbb{C}$ is stable under translation by Λ, and $\mathrm{div}(f_{[a]}) \bmod \Lambda$ is the formal sum (or 0-cycle)

$$\sum_{i=1}^{n}[a_n] - \sum_{i=1}^{n}[-a_n]$$

of points of $\mathbb{C}/\Lambda$. So if $f_{[a]}$ is a constant, the above formal sum is 0, meaning that $[a] = [-a]$. Conversely if $[a] = [-a]$, then

$$\sum_{i=1}^{n}\zeta(a_i;\Lambda) = -\sum_{i=1}^{n}\zeta(a_i;\Lambda) = 0$$

and $f_{[a]}$ is equal to the constant function 1. We have proved statement (0).

Let $(x_i, y_i) := (\wp(a_i;\Lambda), \wp'(a_i;\Lambda))$. We have

$$f'_{[a]} = f_{[a]}\cdot \sum_{i=1}^{n} \frac{\wp'(a_i;\Lambda)}{\wp(z;\Lambda) - \wp(a_i;\Lambda)} \tag{6.5.2}$$

and

$$\sum_{i=1}^{n} \frac{y_i}{\wp(z;\Lambda) - x_i} = \sum_{i=1}^{n} \frac{y_i\,\wp(z;\Lambda)^{-1}}{1 - x_i\,\wp(z;\Lambda)^{-1}} = \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{n} y_i x_i^k\Big)\,\wp(z;\Lambda)^{-k-1}.$$

We conclude that
- $\mathrm{ord}_{z=0} f_{[a]}(z) > 2n$ if and only if $\sum_{i=1}^n y_i x_i^k = 0$ for $k = 0,\ldots,n-1$,
- $\mathrm{ord}_{z=0} f_{[a]}(z) = 2n$ if and only if $\sum_{i=1}^n y_i x_i^k = 0$ for $0 \leq k \leq n-2$ while $\sum_{i=1}^n y_i x_i^{n-1} \neq 0$.

Suppose that $\mathrm{ord}_{z=0} f'_{[a]}(z) > 2n$, i.e. $\sum_{i=1}^n y_i x_i^k = 0$ for $k = 0,\ldots,n-1$. If $x_1,\ldots,x_n$ are distinct, we get from the non-vanishing of the Vandermonde determinant that $y_1 = \cdots = y_n = 0$, meaning that $[a_1],\ldots,[a_n]$ are all 2-torsion points. That contradicts the assumption that $[a] \neq [-a]$. So we know that $x_1,\ldots,x_n$ are not all distinct. Apply the argument in Remark 5.8.6 (b): let $\{s_1,\ldots,s_m\} = \{x_1,\ldots,x_n\}$ as sets without multiplicity, let

$$z_j := \sum_{\text{all } i \text{ s.t. } s_i = t_j} y_i \qquad \text{for } i = 1,\ldots m,$$

and we have $z_1 = \cdots = z_m = 0$. Note that for each j, the y_i's which appear in the sum defining z_j differ from each other at most by a sign ± 1, so that the sum z_j is either 0 or a non-zero multiple of a y_i. Note that we have cancelled a number of pairs $([a_{i_1}], [-a_{i_2}])$ in forming the reduced system of equations

$$\sum_{j=1}^{m} z_j s_j^k = 0 \qquad \text{for } k = 0, \ldots, n-1.$$

That $z_1 = \cdots = z_m = 0$ means that, after removing a number of pairs $([a_{i_1}], [-a_{i_2}])$ from the unordered list $[a]$, we are left with another unordered list $[b]$ with $[b] = [-b]$. So again we have $[a] = [-a]$, a contradiction with proves the statement (1). The first part of statement (2) follows.

It remains to prove the second part of (2). We are assuming that $[a] \neq [-a]$ and $\sum_{i=1}^{n} y_i x_i^k = 0$ for $k = 0, \ldots, n-2$. If there exists i_1, i_2 between 1 and n such that $x_{i_1} = x_{i_2}$, the same argument in the previous paragraph produces a contradiction that $[a] = [-a]$. Therefore $x_1, \ldots, x_n$ are mutually distinct. If there is an i_0 such that $y_{i_0} = 0$, then $y_1 = \cdots = y_n = 0$ by Remark 5.8.6 (a), contradicting the assumption that $[a] \neq [-a]$. □

We have seen in Proposition 5.8.5 that $X_n \subset Y_n$, where X_n is defined in (0.6.6) and Y_n is defined in (0.5.2). The following proposition, which is a consequence of Theorem 6.5, describes the complement of X_n in Y_n.

Proposition 6.6. *Let $[a] = \{[a_1], \ldots, [a_n]\}$ be a point of Y_n, i.e. $[a_i] \neq [0]$ for each i, $[a_i] \neq [a_j]$ whenever $i \neq j$ and the equations* (6.2.1) *hold. Then $[a] \in X_n$ if and only if $[a] \neq [-a]$.*

Proof. The "only if" part is part of the definition of X_n. Assume that $[a] \neq [-a]$. We mush show that $a \in X_n$. We know from Theorem 6.2 and 6.5 (0) that w_a and w_{-a} are linearly independent solutions of the Lamé equation (6.1.10). If $a \notin X_n$, then the lists $[a]$ and $[-a]$ have common members. So either (A) there exist two indices i_1, i_2 such that $i_1 \neq i_2$ and $[a_{i_1}] = [-a_{i_2}]$, or (B) there exists an index i_3 such that $[a_{i_3}] = [-a_{i_3}]$. We start with a non-canonical process to reduce the length of the list $[a]$ while keeping the associated functions w_a and w_{-a} unchanged up to some non-zero constants: First remove all a_i's such that $[a_i] = [-a_i]$ from the list a. In the resulting reduced list, remove both a_{i_1} and a_{i_2} from the list if $i_1 \neq i_2$ and $[a_{i_1}] = [-a_{i_2}]$. Keep doing so until we get a sublist $b = \{b_1, \ldots, b_m\}$ of a such that $[b]$ and $[-b]$ have no common members, $m < n$, and there exists a non-zero constant $C \in \mathbb{C}^\times$ such that

$$f_{[b]} = \frac{w_b}{w_{-b}} = C \cdot \frac{w_a}{w_{-a}} = C \cdot f_{[a]}.$$

The Schwarzian derivative $S(f_{[a]})$ satisfies

$$S(f_{[a]}) = -2(n(n+1)\wp(z) + B_{[a]}). \tag{6.6.1}$$

Let

$$2\eta := \operatorname{ord}_{z=0} f'_{[a]}(z).$$

Theorem 6.5 (1) tells us that

$$2\eta = \operatorname{ord}_{z=0} f'_{[b]}(z) \le 2m < 2n.$$

But then

$$S(f_{[a]}) = \frac{f'''_{[a]}}{f'_{[a]}} - \frac{3}{2}\Big(\frac{f''_{[a]}}{f'_{[a]}}\Big)^2 = -2\eta(\eta+1)\frac{1}{z^2} + O(1),$$

which contradicts (6.6.1). Therefore $[a] \in X_n$. $\square$

The characterization of X_n in terms of the Schwarzian derivative follows similarly:

Corollary 6.7 (Characterization of X_n by $S(f)$)**.** *Let $n \ge 1$ be a positive integer. Let $a_1, \ldots, a_n$ be complex numbers in $\mathbb{C} \smallsetminus \Lambda$, let $[a]$ be the unordered list $\{[a_1], \ldots, [a_n]\}$ and let $[-a]$ be the unordered list $\{[-a_1], \ldots, [-a_n]\}$. Then $[a] \in X_n$ if and only if $[a] \ne [-a]$ and*

$$S(f_{[a]}) = -2\Big(n(n+1)\wp(z;\Lambda) + (2n-1)\sum_{i=1}^{n} \wp(a_i;\Lambda)\Big). \tag{6.7.1}$$

Proof. If $[a] \in X_n$ then $[a] \ne [-a]$ by definition, and $a \in Y_n$ because $X_n \subset Y_n$. It follows that $f_{[a]} = w_a/w_{-a}$ is a quotient of two linearly independent solutions of a Lamé equation (6.1.10) and the formula for $S(f_a)$ follows from Theorem 6.2 and the standard ODE theory.

Conversely, if (6.7.1) holds then $\operatorname{ord}_{z=0} f'_a(z) = 2n$. Hence $a \in X_n$ by Theorem 6.5. $\square$

Remark 6.8. We would like to point out the striking similarity between the solution w_a to the Lamé equation and the defining power series of *complex elliptic genera* in the Weierstrass form studied in [66]. In a certain context of *topological* field theory, complex elliptic genera serves as the *genus one partition function*. In contrast to it, the mean field equation studies local yet very precise *analytic* behavior of a genus one curve. It would be very interesting to uncover a good reason that will account for the similarity between these two theories.

7. Hyperelliptic geometry on $\bar{X}_n$

We have seen in Proposition 6.4 that the fibers of the map $\pi : Y_n \to \mathbb{C}$ which sends a typical point of Y_n represented by an unordered list $[a] = \{[a_1], \ldots, [a_n]\}$ of n elements in $\mathbb{C}/\Lambda \smallsetminus \{[0]\}$ to

$$\pi([a]) = B_{[a]} = (2n-1) \cdot \sum_{i=1}^{n} \wp(a_i;\Lambda)$$

are exactly the orbits of the involution $\iota : [a] \mapsto [-a]$ on Y_n. We have also seen that the complement $Y_n \smallsetminus X_n$ of $X_n \subset Y_n$ is the set of all points of Y_n

fixed by the involution ι. In turn the fact that both X_n and Y_n are locally the locus of common zeros of $n-1$ holomorphic functions on n-dimensional complex manifolds suggest that X_n and Y_n are both one-dimensional. The fact that there exists a two-to-one map from $X_n \to \mathbb{C}$ suggest that X_n is the unramified locus of a possibly singular hyperelliptic curve, Y_n is a partial compactification of X_n, ι is the hyperelliptic involution on Y_n, and $\pi : Y \to \mathbb{C}$ is the hyperelliptic projection. This section is devoted to the proof of Theorem 0.7, which asserts that the above guesses are indeed true, and provides more detailed information about this hyperelliptic curve. Due to its fundamental importance, we shall give two different treatments of this result, one based on the theory of ordinary differential equations and another one based on purely algebraic method.

The analytic method continues the train of ideas in §6, that points of Y_n corresponds to the ansatz solutions of Lamé equations with fixed index n but varying accessory parameters. With the analytic method it is easier to show that the closure $\bar{X}_n$ of X_n in the n-th symmetric product $\mathrm{Sym}^n E = E^n/S_n$ of $E = \mathbb{C}/\Lambda$ is $Y_n \cup \{\infty\}$, where "∞" stands for the point $\{0, \ldots, 0\}$ of $\mathrm{Sym}^n E$. Moreover one gets a recursively defined polynomial $\ell_n(B)$ of degree $2n+1$ in B, whose roots are the image of the ramification points of $\pi : Y_n \to \mathbb{C}$, i.e. fixed points of the involution ι on Y_n. With the algebraic method one gets not only the same polynomial $\ell_n(B)$, but also an explicit regular function C on $\bar{X}_n$ such that $C^2 = \ell_n(B)$. In particular, $\bar{X}_n$ has arithmetic genus n. The algebraic method also allows us to analyze the limiting equations at ∞ and prove that ∞ is a smooth point of $\bar{X}_n$.

We emphasize that a priori there is no definite reason that the compactification of Y_n in $\mathrm{Sym}^n E$ should agree with the *projective hyperelliptic model* of X_n defined in 7.6.1.e. Such an identification is one of the main statements we will establish; see Proposition 7.7.

7.1. Review of linear second order ODE. The starting point of this section is the following simple well-known observation on a second order ODE

$$w'' = Iw \tag{7.1.1}$$

Recall that the Wronskian

$$C := \begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix} = w_1 w_2' - w_2 w_1' = w_1 w_2 \cdot \frac{d}{dz} \log \frac{w_1}{w_2} \tag{7.1.2}$$

of two linearly independent solutions w_1, w_2 is obviously a non-zero constant since $C' = 0$. If the product $X = w_1 w_2$ is easier to get hold of, then we may express the solutions w_1, w_2 in terms of C and X: we have

$$\frac{X'}{X} = \frac{w_1'}{w_1} + \frac{w_2'}{w_2}, \qquad \frac{C}{X} = \frac{w_2'}{w_2} - \frac{w_1'}{w_1},$$

hence

$$\frac{w_1'}{w_1} = \frac{X' - C}{2X}, \qquad \frac{w_2'}{w_2} = \frac{X' + C}{2X}.$$

In particular,

$$w_1 = X^{1/2} \exp\Big(-\frac{C}{2}\int \frac{dz}{X}\Big), \qquad w_2 = X^{1/2} \exp\Big(\frac{C}{2}\int \frac{dz}{X}\Big). \tag{7.1.3}$$

From

$$\Big(\frac{X'+C}{2X}\Big)' = \Big(\frac{w_2'}{w_2}\Big)' = \frac{w_2''}{w_2} - \Big(\frac{w_2'}{w_2}\Big)^2 = I - \frac{(X'+C)^2}{4X^2},$$

one finds easily that

$$C^2 = X'^2 - 2X''X + 4IX^2. \tag{7.1.4}$$

Differentiating (7.1.4) we see that the product $X = w_1 \cdot w_2$ of any two solutions w_1, w_2 of the equation (7.1.1) satisfies

$$X''' - 4IX' - 2I'X = 0. \tag{7.1.5}$$

This third order ODE is known as the *second symmetric power* of the equation (7.1.1) and can easily be derived directly. In this way, (7.1.4) is simply the first integral of (7.1.5) with integration factor $-2X$.

Conversely suppose we have a non-trivial solution X of the second symmetric power (7.1.5) of (7.1.1). Then $X'^2 - 2X''X + 4IX^2$ is a constant[24] and the constant C is defined up to sign by (7.1.4), so we get a pair of functions w_1, w_2 defined by (7.1.3). It can be checked easily using (7.1.4) that w_1 and w_2 are indeed solutions of the equation (7.1.1).

Remark 7.2. The product $X = w_1 \cdot w_2$ has appeared implicitly in our previous discussions, in the sense that there exists a developing map for a solution of the mean field equation $\triangle u + e^u = 8\pi n\, \delta_0$ whose logarithmic derivative is equal to $-C/X$ for some non-zero constant C. To see this, recall that for any given solution of the above mean field equation on $\mathbb{C}/\Lambda$, there exist two independent solutions of a Lamé equation

$$\frac{d^2 w}{dz^2} - (n(n+1)\wp(z;\Lambda) + B)w = 0$$

such that $w_1/w_2 =: f$ is a developing map of u. Then

$$g := \frac{f'}{f} = \frac{(w_1'w_2 - w_1w_2')/w_2^2}{w_1/w_2} = \frac{-C}{w_1 w_2} = \frac{-C}{X}. \tag{7.2.1}$$

We start with the analytic approach of Hermite-Halphen; c.f. [27, p. 499] and [67, §23.7].

Theorem 7.3. *Let $n \geq 1$ be a positive integer.*

(i) *There exist polynomials $s_1(B), s_2(B), s_3(B), \ldots, s_n(B)$ in B with coefficients in $\mathbb{Q}[g_2(\Lambda), g_3(\Lambda)]$ with the following properties:*

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[24] Of course if this constant is zero, then the functions w_1, w_2 defined by (7.1.3) are linearly dependent.

- *for every element $[a] = \{[a_1], \ldots, [a_n]\} \in Y_n$ and any $i = 1, \ldots, n$, the i-th elementary polynomial of $\{\wp(a_1; \Lambda), \ldots, \wp(a_n; \Lambda)\}$ is equal to $s_i(B_{[a]})$, where $B_{[a]} := (2n-1)\sum_{i=1}^{n} \wp(a_i; \Lambda)$.*
- *$s_1(B) = (2n-1)^{-1}B$, and $s_i(B)$ is of degree i with leading coefficients in $\mathbb{Q}^\times$ for $i = 2, \ldots, n$.*
- *$s_i(B)$ is homogenous of weight i for $i = 2, \ldots, n$ if B, g_2, g_3 are given weights $1, 2, 3$ respectively.*

(ii) *The fibers of the map $\pi : Y_n \to \mathbb{C}$ defined by*

$$\pi : \{(x_i, y_i)\}_{i=1}^{n} \mapsto B_{[a]} = (2n-1) \sum_{i=1}^{n} x_i$$

are orbits of the involution $\iota : [a] \mapsto [-a]$ on Y_n. In other words

$$\pi^{-1}(B_{[a]}) = \{[a], [-a]\} \quad \forall\, [a] \in Y_n.$$

The subset $X_n \subset Y_n$ is the complement in Y_n of the fixed point set $(Y_n)^\iota$ of the involution ι; it is also equal to the subset of Y_n consisting of all elements $[a] = \{[a_1], \ldots, [a_n]\} \in Y_n$ such that $\wp'(a_i; \Lambda) \neq 0$ for all i and $\wp(a_i; \Lambda) \neq \wp(a_j; \Lambda)$ for all pairs (i, j) with $i \neq j$ and $1 \leq i, j \leq n$. Moreover X_n is a locally closed smooth one-dimensional complex submanifold of $\mathrm{Sym}^n(\mathbb{C}/\Lambda)$.

(iii) *The set $(Y_n)^\iota = Y_n \smallsetminus X_n$ is a finite subset of Y_n with at most $2n+1$ elements. Up to $\mathbb{C}^\times$ the set of all ansatz functions $w_a(z)$ with $[a] \in Y_n^\iota$ coincides with the set of all Lamé functions of index n. In other words $Y_n^\iota = Y_n \smallsetminus X_n$ is in natural bijection with the set of all Lamé functions of index n up to non-zero constants.*

(iv) *The closure $\bar{X}_n$ of X_n in $\mathrm{Sym}^n(\mathbb{C}/\Lambda)$ consists of Y_n and a single "point at infinity" $[0]^n := \{[0], \ldots, [0]\}$: $\bar{X}_n = Y_n \cup \{[0]^n\}$.*

(v) *The map $\pi : Y_n \to \mathbb{C}$ extends to a surjective continuous map $\bar{\pi} : \bar{X}_n \to \mathbb{P}^1(\mathbb{C})$ which sends the point $[0]^n$ to $\infty \in \mathbb{P}^1(\mathbb{C})$.*

7.3.1. PROOF OF THEOREM 7.3 (i). Consider the Weierstrass equation $y^2 = p(x) = 4x^3 - g_2 x - g_3$, where $(x, y) = (\wp(z), \wp'(z))$, and we set $(x_i, y_i) = (\wp(a_i), \wp'(a_i))$ for $[a] = \{[a_1], \ldots, [a_n]\} \in Y_n$. Pick $a_i \in [a_i]$ for $i = 1, \ldots, n$. Consider the following pair of ansatz solutions $\Lambda_a(z)$, $\Lambda_{-a}(z)$ the Lamé equation

$$\frac{d^2 w}{dz^2} - \big(n(n+1)\wp(z; \Lambda) + B\big) w = 0, \tag{7.3.1}$$

where

$$\Lambda_a(z) := \frac{w_a(z)}{\prod_{i=1}^{n} \sigma(a_i)} = e^{z \sum \zeta(a_i)} \prod_{i=1}^{n} \frac{\sigma(z - a_i)}{\sigma(z)\sigma(a_i)}. \tag{7.3.2}$$

Let $\tilde{X}_{[a]}(z) = \Lambda_a(z)\Lambda_{-a}(z)$ be the product of this pair. Note that if $[a] \notin X_n$ then $[a] = [-a]$ by Proposition 6.6 and $\Lambda_a = \Lambda_{-a}$. From the addition theorem we have

$$\tag{7.3.3} \begin{aligned} \tilde{X}_a(z) &= (-1)^n \prod_{i=1}^{n} \frac{\sigma(z+a_i;\Lambda)\sigma(z-a_i;\Lambda)}{\sigma(z;\Lambda)^2\sigma(a_i;\Lambda)^2} \\ &= (-1)^n \prod_{i=1}^{n} (\wp(z;\Lambda) - \wp(a_i;\Lambda)) = X(\wp(z;\Lambda)) \end{aligned}$$

That is $\tilde{X}_{[a]}(z) = X_{[a]}(\wp(z;\Lambda))$, where

$$X_{[a]}(x) = (-1)^n \prod_{i=1}^{n} (x - \wp(a_i;\Lambda)).$$

a polynomial of degree n in the variable x.

We know that $\tilde{X}_{[a]}(z)$ satisfies the second symmetric power of the Lamé equation (7.3.1)

$$\tag{7.3.4} \frac{d^3\tilde{X}}{dz^3} - 4(n(n+1)\wp(z;\Lambda) + B)\frac{d\tilde{X}}{dz} - 2n(n+1)\wp'(z,\Lambda)\tilde{X}(z) = 0,$$

it is thus a *polynomial solution* in the variable x, to the algebraic form

$$\tag{7.3.5} p(x)\frac{d^3X}{dx^3} + \frac{3}{2}\frac{dp}{dx}\cdot\frac{d^2X}{dx^2} - 4((n^2+n-3)x + B)\frac{dX}{dx} - 2n(n+1)X = 0,$$

of (7.3.4), where $p(x)$ is the cubic polynomial

$$p(x) = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$$

in the Weierstrass equation for $\mathbb{C}/\Lambda$. As a result, $X_{[a]}(x)$ will be determined by B and certain initial conditions.

Indeed, write $X_{[a]}(x) = (-1)^n(x^n - s_1x^{n-1} + \cdots + (-1)^n s_n)$, then (7.3.5) translates into a linear recursive relation for each $\mu = n-1, \ldots, 0$, where we set $s_0 = 1$ by convention:

$$\tag{7.3.6} \begin{aligned} 0 = 2(n-\mu)(2\mu+1)(n+\mu+1)s_{n-\mu} &- 4(\mu+1)Bs_{n-\mu-1} \\ + \tfrac{1}{2}g_2(\mu+1)(\mu+2)(2\mu+3)s_{n-\mu-2}& \\ - g_3(\mu+1)(\mu+2)(\mu+3)s_{n-\mu-3}.& \end{aligned}$$

Since $B = (2n-1)s_1$, the initial relation for $\mu = n-1$ is automatic. Let $s_1(B) := (2n-1)^{-1}B$. The recursive relations (7.3.6) with s_i substituted by $s_i(B)$ define polynomials $s_2(B), \ldots, s_n(B) \in \mathbb{Q}[g_2, g_3]$ which satisfy the first condition listed in Theorem 7.3 (i). Moreover we see from the recursive relations that s_j is a polynomial of degree j in B, and it is homogenous of weight n if B, g_2, g_3 are given weights $1, 2, 3$ respectively, for $j = 1, \ldots, n$, We have proved Theorem 7.3 (i). $\square$

7.3.2. PROOF OF THEOREM 7.3 (ii). The first sentence of Theorem 7.3 (ii) is a restatement of Proposition 6.4. We give another proof here more in line with the proof of (i). Suppose now we have two elements

$$[a] = \{[a_1], \ldots, [a_n]\},\ [b] = \{[b_1], \ldots, [b_n]\} \in Y_n$$

such that

$$\sum_{i=1}^{n} \wp(a_i; \Lambda) = \sum_{i=1}^{n} \wp(a_i'; \Lambda).$$

Let $X_{[a]}(x)$ be the polynomial in x of degree n such that $X_{[a]}(\wp(z;\Lambda)) = \Lambda_a(z)\Lambda_{-a}(z)$; similarly let $X_{[b]}$ be the polynomial of degree n such that $X_{[b]}(\wp(z;\Lambda)) = \Lambda_b(z)\Lambda_{-b}(z)$. Then $X_{[a]}$ and $X_{[b]}$ both satisfy the same equation (7.3.5), and we get from the recursive relations (7.3.6) that $X_{[a]} = X_{[b]}$, i.e.

$$\prod_{i=1}^{n}(x - \wp(a_i;\Lambda)) = \prod_{i=1}^{n}(x - \wp(b_i;\Lambda)).$$

Therefore $\{\wp(a_1;\Lambda), \ldots, \wp(a_n;\Lambda)\} = \{\wp(b_1;\Lambda), \ldots, \wp(b_n;\Lambda)\}$ as unordered lists.

We claim that either $[a] = [b]$ or $[a] = [-b]$ as unordered lists. Otherwise after renumbering the a_i's and the b_i's, there exist integers $r, s \geq 1$ with $r + s \leq n$ such that the following hold:

(i) $[a_i] = [b_i] \in \frac{1}{2}\Lambda/\Lambda$ for all i such that $r + s + 1 \leq i \leq n$,
(ii) $[a_i] \notin \frac{1}{2}\Lambda/\Lambda$ and $[b_i] \notin \frac{1}{2}\Lambda/\Lambda$, for all i such that $1 \leq i \leq r + s$,
(iii) $[a_i] = [-b_i]$ for $i = 1, \ldots, r$,
(iv) $[a_i] = [b_i]$ for $i = r + 1, \ldots, r + s$.
(v) $[a_i] \neq [-a_j]$ if $i \neq j$ and $1 \leq i, j \leq r$.

We know from Theorem 6.5 (0) that $w_a(z)$ and $w_b(z)$ are linearly independent because $[a] \neq [\pm b]$, and they satisfy the same Lamé equation with index n because $B_{[a]} = B_{[b]}$. So the Schwarzian derivative of w_a/w_b is

$$S(w_a/w_b) = 2\big(n(n+1)\wp(z;\Lambda) + B_{[a]}\big).$$

On the other hand conditions (i)–(iv) tells us that w_a/w_b is equal to a nonzero constant times the function $f_{[c]} = w_c/w_{-c}$, where $[c] = \{c_1, \ldots, c_r\}$, so $S(f_{[c]}) = 2\big(n(n+1)\wp(z;\Lambda) + B_{[a]}\big)$. The condition (v) above tells us that $[c] \neq [-c]$, so we get from Proposition 6.5 (1) that $\operatorname{ord}_{z=0} f_{[c]} \leq 2r \leq 2n-2$, which implies that $S(f_{[c]}) \neq 2\big(n(n+1)\wp(z;\Lambda) + B_{[a]}\big)$. We have proved the first sentence of (ii): if $B_{[a]} = B_{[b]}$, $[a], [b] \in Y_n$, then either $[a] = [b]$ or $[a] = [-b]$.

The second sentence of Theorem 7.3 (ii) is the content of Proposition 6.6. The argument below provides a different proof and also the rest of the statement (ii) at the same time. Suppose that $[a] = \{[a_1], \ldots, [a_n]\}$ is a given point of Y_n. As in §7.1, we know that

$$\begin{aligned}(7.3.7)\qquad &\left(\tfrac{d}{dz}X_{[a]}(\wp(z;\Lambda))\right)^2 - 2\,X_{[a]}(\wp(z;\Lambda))\,\tfrac{d^2}{dz^2}X_{[a]}(\wp(z;\Lambda))\\ &\quad+4\left(n(n+1)\wp(z;\Lambda)+B_{[a]}\right)X_{[a]}(\wp(z;\Lambda))^2\end{aligned}$$

is a constant because its derivative vanishes identically; write this constant as C^2. This constant C^2 can be evaluated by plugging in $z = a_i$ in equation (7.3.7), for any i with $1 \leq i \leq n$:

$$C^2 = \Big(\frac{dX_{[a]}}{dx}(\wp(a_i;\Lambda))\cdot\wp'(a_i;\Lambda)\Big)^2$$

for each $i = 1,\ldots,n$.

Suppose that $C^2 = 0$. We know from §7.1 that w_a and w_{-a} are linearly dependently, therefore $[a] = [-a]$. The above argument also tells us that $\pi^{-1}(B_{[a]})$ is the singleton $\{[a]\}$. In this case $w_a(z)$ is a Lamé function of index n: up to $\mathbb{C}^\times$ it is a square root of $X_{[a]}(\wp(z;\Lambda))$, a polynomial of degree n in $\wp(z;\Lambda)$. We also see that $[a] \notin X_n$, because for each i we know that either $\wp'(a_i;\Lambda) = 0$ or $\wp(a_i;\Lambda)$ is a multiple root of $X_{[a]}(x)$.

On the other hand, suppose that $C^2 \neq 0$. Then

$$\wp'(a_i;\Lambda) \neq 0 \quad\text{and}\quad \frac{dX_{[a]}}{dx}(\wp(a_i;\Lambda)) \neq 0 \quad\text{for}\ \ i = 1,\ldots,n.$$

Therefore $[a] \neq [-a]$, and $[a],[-a] \in X_n$. After making a choice of a square root C of C^2, one can "pick out" $[a]$ from the pair $\{[a],[-a]\}$ using C and $\{\wp(a_1;\Lambda),\ldots,\wp(a_n;\Lambda)\}$, by

$$C = \frac{dX_{[a]}}{dx}(\wp(a_i;\Lambda))\cdot\wp'(a_i;\Lambda) \tag{7.3.8}$$

The above formula shows that the map $\pi : Y_n \to \mathbb{C}$ is a local isomorphism near $[a]$ and $[-a]$. The procedure reviewed in §7.1 tells us that the pair of ansatz functions w_a, w_{-a} are determined up to $\mathbb{C}^\times$ by $X_{[a]}(x)$, so $\pi^{-1}(B_{[a]}) = \{[a],[-a]\}$ in this case. We have proved Theorem 7.3 (ii). □

Remark 7.3.3. The proofs of Theorem 7.3 (i) and (ii) employed the method in [27, pp. 498–500] and [67, pp. 570–572] which gives a recursive formula for the product of a pair of ansatz solutions w_a and w_{-a} in terms of the auxiliary parameter $B_{[a]}$, then bootstrap to find the ansatz pair w_a, w_{-a}.

The ansatz solutions parametrized by Y_n are eigenfunctions for the translation action of the lattice Λ, and they are also eigenfunction for the differential operator $\frac{d^2}{dz^2} - n(n+1)\wp(z;\Lambda)$. In this sense Y_n can be regarded as the *spectral curve* of this second order differential operator.

Theorem 7.3 (v) asserts that for every $B \in \mathbb{C}$ there exist an element $[a] \in Y_n$ such that $B_{[a]} = B$. We discuss the dichotomy whether $\pi : Y_n \to \mathbb{C}$ is ramified above B from the perspective of the translation action of Λ on the solution space of the Lamé equation $L_{n,B_{[a]}}$.

1. CASE $[a] \in X_n$, equivalently $[a] \neq [-a]$. In this case $\mathbb{C}\cdot w_a$ and $\mathbb{C}\cdot w_{-a}$ are one-dimensional spaces of solutions of the same Lamé equation $L_{n,B_{[a]}}$ but their eigenvalue packages for the translation action by Λ are different, hence the ansatz solutions $\mathbb{C}\cdot w_a$ and $\mathbb{C}\cdot w_{-a}$ are *intrinsic* to the Lamé equation $L_{n,B_{[a]}}$.

2. CASE $[a] \notin X_n$, equivalently $[a] = [-a]$. The assumption that $[a] \notin X_n$ is equivalent to $[a] = [-a]$. We have seen in the proof of 7.3 (ii) that up to a non-zero constant $w_a(z)$ is a square root of a polynomial of $\wp(z;\Lambda)$; in other words w_a is a *Lamé function*. In this case the action of Λ on the space of solutions of the Lamé equation $L_{n,B_{[a]}}$ is not diagonalizable, and the Lamé functions $\mathbb{C}\cdot w_a$ are the only Λ-eigenfunctions among the space of solutions of the Lamé equation $L_{n,B_{[a]}}$.

7.3.4. PROOF OF THEOREM 7.3 (iii). We have seen in the last paragraph of Remark 7.3.3 that $(Y_n)^\iota$ is in natural bijection with the set of all Lamé functions of index n up to $\mathbb{C}^\times$. One knows from classical literature that there exists a polynomial $\ell_n(B) \in \mathbb{Q}[g_2(\Lambda), g_3(\Lambda)]$ of degree $2n+1$ in the variable B, explicitly defined by recursion, whose roots are precisely the $B_{[a]}$'s with $[a] \in (Y_n)^\iota$; see Theorem B in §0. Theorem 7.3 (iii) follows.

The definition of this polynomial $\ell_n(B)$ will be reviewed in the proof of Theorem 7.4. It is known that $\ell_n(B)$ has $2n+1$ distinct real roots when the lattice has the form $\Lambda = \mathbb{Z} + \sqrt{-1}t\mathbb{Z}$ for some $t \in \mathbb{R}_{>0}$. This fact is stated on line 13, page 221 in Liouville's letter [47], where Liouville said that one can use Sturm's method to prove that the polynomial $\ell_n(B)$, written as $R(B)$ in *loc. cit.*, has $2n+1$ (real) roots and therefore there are $2n+1$ Lamé functions. The proof in [27, pp. 471–476] goes through a change of variable $u = 2v$, $y = \wp'(v;\Lambda)$, which has the advantage that every Lamé function is rationally expressible in terms of $\wp(v,\Lambda)$ and $\wp'(v;\Lambda)$; this proof is sketched in [53, p. 163]. In [67, §23.41] Lamé functions "of the third kind" (in the case when n is even) is discussed, with the other three cases left as exercises. Sturm's method, in the form of Corollary 3.3.4, was used in all references above. $\square$

7.3.5. PROOF OF THEOREM 7.3 (iv)–(v).

Suppose that $[a] = \{[a_1], \ldots, [a_n]\}$ is a point of $\bar{X}_n$. By definition there exists a sequence $[a]_k = \{[a_{k,1}], \ldots, [a_{k,n}]\} \in X_n$ which converges as $k \to \infty$. Let $B_k = B_{[a]_k} = (2n-1)\sum_{i=1}^n \wp(a_{k,i};\Lambda)$. Then $B := \lim_{k\to\infty} B_k$ exists as an element of $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$. Let $X_k(x) = X_{[a]_k}(x)$ be the polynomial of degree n in x determined by B_k through the recursive relation (7.3.6). We discuss the two cases (a) $B \in \mathbb{C}$ and (b) $B = \infty \in \mathbb{P}^1(\mathbb{C})$.

(a) Suppose that $B \neq \infty$. By (7.3.6) the coefficients $s_j(B_k)$ of $X_k(x)$ are uniformly bounded for $j = 1, \ldots, n$. Thus the roots $x_{k,i} = \wp(a_{k,i};\Lambda) \in \mathbb{C}$ of $X_k(x)$ are uniformly bounded as well. This implies that $a_i \notin \Lambda$ for $i = 1, \ldots, n$, $w_{a_k} \to w_a$, $[a]_k \to [a]$ in $\mathrm{Sym}^n(\mathbb{C}/\Lambda)$, and w_a is a non-trivial solution of the Lamé equation $w'' = (n(n+1)\wp(z;\Lambda) + B)w$. Notice that we must have $a_i \neq a_j$ whenever $i \neq j$. For otherwise $w_a(z)$ has multiplicity at least 2 at $z = a_i$, which implies that w_a is identically zero, a contradiction. We conclude that $[a] \in Y_n$.

(b) Suppose that $B = \infty$. We claim that $[a_{k,i}] \to [0]$ for all $i = 1, \ldots, n$. Change the variable to $t = x^{-1}$. Look at the polynomial

$$\begin{aligned} Y_k(y) &:= s_n(B_k)^{-1} \cdot y^n \cdot X_{[a]_k}(y^{-1}) \\ &= y^n - \tfrac{s_{n-1}(B_k)}{s_n(B_k)} y^{n-1} + \cdots + (-1)^{n-1} \tfrac{s_1(B_k)}{s_n(B_k)} + (-1)^n \tfrac{1}{s_n(B_k)} \end{aligned} \tag{7.3.9}$$

whose roots are $\{\wp(a_{k,1};\Lambda)^{-1}, \ldots, \wp(a_{k,n};\Lambda)^{-1}\}$. The assumption that $B_k \to \infty$ as $k \to \infty$ tells us that $Y_k(y) \to y^n$ as $k \to \infty$, which implies that all the roots $\wp(a_{k,i})^{-1}$ of $Y_k(y)$ go to 0 as $k \to \infty$. Therefore $a_{k,i} \to 0$ as $k \to \infty$ for all $i = 1, \ldots, n$.

Combining the cases (a) and (b), we draw the following conclusions.

- The map $\pi_n|_{X_n} : X_n \to \mathbb{C}$ extends to a continuous map $\bar{\pi}_n : \bar{X}_n \to \mathbb{P}^1(\mathbb{C})$.
- The inverse image $\bar{\pi}_n^{-1}(\infty)$ of the point $\infty \in \mathbb{P}^1(\mathbb{C})$ under $\bar{\pi}$ consists of a single point $[0]^n = \{[0], \ldots, [0]\}$.
- The inverse image $\bar{\pi}^{-1}(\mathbb{C})$ of $\mathbb{C}$ under $\bar{\pi}$ is contained in Y_n. In other words $\bar{X}_n \smallsetminus \{[0]^n\} \subseteq Y_n$.
- Because $\bar{\pi}_n$ is compact by definition, and we already know that $\pi_n(X_n)$ contains the complement of a finite subset of $\mathbb{C}$, therefore $\bar{\pi}_n$ is surjective.

We have proved Theorem 7.3 (v) and half of (iv).

To complete the proof of Theorem 7.3 (iv), we need to show that $(Y_n)^\iota \subset \bar{X}_n$. Let $[a] \in (Y_n)^\iota$ be a given element of $Y_n \smallsetminus X_n$. We know that there exists an element $[b] \in \bar{X}_n$ such that $B_{[b]} = B_{[a]}$, and have seen $[b] \in Y_n$. Theorem 7.3 (ii) tells us that either $[b] = [a]$ or $[b] = [-a]$. In either case we conclude that $[a] = [-a] = [b] \in \bar{X}_n$. We have proved Theorem 7.3. $\square$

Corollary 7.3.6. *Let $([a]_k)_{k\in\mathbb{N}} = (\{[a_{k,1}], \ldots, [a_{k,n}]\})_{k\in\mathbb{N}}$ be a sequence of elements in X_n indexed by $\mathbb{N}$. If there exists an i between 1 and n such that $[a_{k,i}] \to [0]$ in $\mathbb{C}/\Lambda$, then $[a_{k,i}] \to [0]$ in $\mathbb{C}/\Lambda$ for all $i = 1, \ldots, n$.*

Remark 7.3.7. The proof of Theorem 7.3 (i) has appeared in [27, p. 499–500]. The proof in [67, §23.7] is essentially the same, except that $X_{[a]}(x)$ is expressed as a polynomial in $x - e_2$ and recursive formula was given for the coefficients of powers of $x - e_2$. We may compare our argument with [67, §23.7] on such a polynomial solution X to (7.3.5), which is indeed the origin where $X(x)$ and Λ_a were first found during our study. Let $X = \sum_{r=0}^{\infty} c_r (x - e_2)^{n-r}$ be a solution of (7.3.5) in descending power. Since

$p(x) = 4(x-e_1)(x-e_2)(x-e_3)$, there is a recursive formula for c_r:

$$\begin{aligned}&4r(n+\tfrac{1}{2}-r)(2n+1-r)c_r\\&\quad=(n+1-r)\big(12e_2(n-r)(n-2-r)-4e_2(n^2+n-3)-4B\big)c_{r-1}\\&\qquad-4(n+1-r)(n+2-r)(n+\tfrac{3}{2}-r)(e_1-e_2)(e_2-e_3)c_{r-2}.\end{aligned}$$

The above recursive formula is slightly different from (7.3.6). Given $c_0 = 1$ and c_1, we can solve $c_2, \ldots, c_n$ and express them as polynomials in c_1 and B. The recursive formula forces $c_{n+1} = \cdots = c_{2n} = 0$. The next coefficient c_{2n+1} appears as another "free parameter", and the coefficients of higher order terms are expressed as polynomials of c_1. The condition that $X(x)$ is a polynomial is that $c_l = 0$ for all $l \geq n$. Thus $X(x)$ is a polynomial solution, which is determined by c_1 and B.

From $(-1)^n\Lambda_a\Lambda_{-a} = \prod_{i=1}^n((x-e_2)+(e_2-x_i)) = \sum_{r=0}^n c_r(x-e_2)^{n-r}$, we see that $c_1 = \sum_{i=1}^n(e_2-x_i) = ne_2 - \sum_{i=1}^n x_i = ne_2 - B$. Hence X is a polynomial in B.

Remark 7.3.8. The statement of Theorem 7.3 (iv), that $\bar{X}_n = Y_n \cup \{\infty\}$, does not seem to have appeared in the literature, but this fact must be known as it follows quickly from the method of recovering the ansatz pair w_a, w_{-a} from their product. The behavior of X_n or Y_n at $B = \infty$ is important, which will be discussed in Proposition 7.5).

We would like to rephrase Theorem 7.3 in purely algebraic terms without appealing to solutions of Lamé equations. It is given below, whose proof uses system (5.6.2) instead of (5.8.1).

Theorem 7.4. *Let $n \geq 1$ be a positive integer. Let $s_1, \ldots, s_n \in \mathbb{Q}[g_2, g_3][B]$ be the polynomials in Theorem* 7.3 (i) *defined recursively by the relation* (7.3.6).

(1) *The space X_n admits a natural projective compactification $\hat{X}_n$ as a possibly singular, hyperelliptic curve defined by the following equation in (B, C):*

$$\begin{aligned}C^2 &= \ell_n(B, g_2(\Lambda), g_3(\Lambda))\\&= 4B\, s_n^2 + 4g_3(\Lambda)\, s_{n-2}\, s_n - g_2(\Lambda)\, s_{n-1}\, s_n - g_3(\Lambda)\, s_{n-1}^2.\end{aligned}$$

(2) *The discriminant* $\mathrm{disc}_B(\ell_n(B)) \in \mathbb{Q}[g_2, g_3]$ *in the variable B of the polynomial $\ell_n(B)$ is a non-zero polynomial in two variables g_2, g_3; it is homogeneous of weight $2n(2n+1)$ if g_2, g_3 are given weights* 2 *and* 3 *respectively. In other words* $\mathrm{disc}_B(\ell_n(B)) \in \mathbb{Q}[g_2(\Lambda), g_3(\Lambda)]$ *is a non-zero modular form of weight $4n(2n+1)$ for the full modular group* $\mathrm{SL}_2(\mathbb{Z})$, *holomorphic on* $\mathbb{H}$ *and also on the cusps.*

Remark. The polynomial $\ell_n(B, g_2, g_3)$ has degree $2n+1$ in the variable B; it is homogeneous of weight $2n+1$ in B, g_2 and g_3 when B, g_2, g_3 are given weights $1, 2, 3$ respectively. The projective curve $\hat{X}_n$ has arithmetic genus n; it is smooth unless $\mathrm{disc}_B(\ell_n(B))(\Lambda) = 0$.

Proof. Let $p(x) = 4x^3 - g_2 x - g_3$ and let $q(x) = \prod_{j=1}^n (x - x_j)$. The set X_n is defined by $2n - 1$ polynomial equations

$$y_i^2 = p(x_i) \ \ \forall\, i = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^n x_i^k\, y_i = 0 \ \ \forall\, k = 0, 1, \ldots, n-2$$

in the $2n$ variables $x_1, \ldots, x_n; y_1, \ldots, y_n$ and $n(n+1)/2$ inequalities

$$x_i \neq x_j \ \ \forall i \neq j,\ 1 \leq i, j \leq 1 \quad \text{and} \quad y_i \neq 0 \ \ \forall i = 1, \ldots, n.$$

Applying Cramer's rule to the $n - 1$ linear equations $\sum_{i=1}^n x_i^k\, y_i = 0$ in y_i's, we conclude that there is a constant[25] $C \in \mathbb{C}^\times$ such that

$$y_i = \frac{C}{\prod_{j \neq i}(x_i - x_j)}, \qquad i = 1, \ldots, n. \tag{7.4.1}$$

Since $q'(x_i) = \prod_{j \neq i}(x_i - x_j)$, we get

$$p(x_i) q'(x_i)^2 = C^2, \quad i = 1, \ldots, n, \tag{7.4.2}$$

and so $q(x) | (p(x) q'(x)^2 - C^2)$. This implies that there are $h_1, \ldots, h_n, a, b \in \mathbb{C}$ such that

$$\frac{p(x) q'(x)^2 - C^2}{q(x)^2} = \sum_{i=1}^n \frac{h_i}{(x - x_i)} + ax + b. \tag{7.4.3}$$

It is easy to compute (e.g. using power series expansion of p, q at x_i)

$$\begin{aligned} h_i &= \mathrm{Res}_{x = x_i} p(x) \frac{q'(x)^2}{q(x)^2} - \mathrm{Res}_{x = x_i} \frac{C^2}{q(x)^2} \\ &= p'(x_i) + p(x_i) \frac{q''(x_i)}{q'(x_i)} + C^2 \frac{q''(x_i)}{q'(x_i)^3} \\ &= p'(x_i) + 2p(x_i) \frac{q''(x_i)}{q'(x_i)}. \end{aligned} \tag{7.4.4}$$

From (7.4.3) we get

$$p(x) q'(x)^2 - C^2 = \sum_{i=1}^n h_i \frac{q(x)^2}{(x - x_i)} + (ax + b) q^2(x). \tag{7.4.5}$$

Comparing coefficients of x^{2n+1} and x^{2n} on both sides, we get

$$a = 4n^2, \qquad b = 8n \sum_{i=1}^n x_i = 8n\, s_1 \tag{7.4.6}$$

(recall that $q(x) = x^n - s_1\, x^{n-1} + \cdots + (-1)^n s_n$ and so $s_1 = \sum_{i=1}^n x_i$).

[25] This "constant" C depends on n and the lattice Λ.

Now, in a similar and easier manner, we write

$$\begin{aligned} \frac{p'(x)q'(x)}{q(x)} &= \sum_{i=1}^{n} \frac{p'(x_i)}{(x-x_i)} + (12nx + 12s_1), \\ \frac{p(x)q''(x)}{q(x)} &= \sum_{i=1}^{n} \frac{p(x_i)}{(x-x_i)} \frac{q''(x_i)}{q'(x_i)} + (4n(n-1)x + 8(n-1)s_1). \end{aligned} \tag{7.4.7}$$

Then (7.4.4), (7.4.5), (7.4.6) and (7.4.7) lead to

$$pq'^2 - p'q'q - 2pq''q + 4\big(n(n+1)x + (2n-1)s_1\big)q^2 - C^2 = 0. \tag{7.4.8}$$

One more differentiation gives

$$\begin{aligned} 0 &= p'q'^2 + 2pq'q'' - p''q'q - p'q''q - p'q'^2 - 2p'q''q - 2pq'''q - 2pq''q' \\ &\quad + 4n(n+1)q^2 + 8(n(n+1)x + (2n-1)s_1)qq' \\ &= -2q\big(pq''' + \tfrac{3}{2}p'q'' - 4((n^2+n-3)x + B)q' - 2n(n+1)q\big), \end{aligned}$$

which is $(-2q)$ times the linear ODE (7.3.5), and so the same recursive relation (7.3.6) shows that q is determined by s_1.

Suppose we have two different points $\underline{x} = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ $\underline{x}' = \{(x_1', y_1'), \ldots, (x_n', y_n')\}$ in X_n such that $\pi_n(\underline{x}) = \sum_{i=1}^n x_i = \sum_{i=1}^n x_i' = \pi_n(\underline{y})$, by rearrangement we have $x_i = x_i'$ for all i and then $y_i' = \pm y_i$ for all i. If $y_i = y_i'$ for some i, then by (7.4.1),

$$\frac{C'}{\prod_{j\neq i}(x_i' - x_j')} = \frac{C}{\prod_{j\neq i}(x_i - x_j)},$$

which implies that $C = C'$ and $y_j = y_j'$ for all j, a contradiction. Hence $y_i = -y_i'$ for all i. We have shown that if two different points $\underline{x}, \underline{x}'$ of X_n have the same image in $\mathbb{C}$ under the map π_n, then $\underline{x}' = \iota(\underline{x})$, where ι is the involution on X_n defined by "multiplication by -1" on $\mathbb{C}/\Lambda$.

The constant terms in formula (7.4.8) leads to

$$C^2 = \ell_n(B) = 4B\, s_n^2 + 4g_3\, s_{n-2}\, s_n - g_2\, s_{n-1}\, s_n - g_3\, s_{n-1}^2, \tag{7.4.9}$$

where $s_k = s_k(B)$ is a polynomial of degree k and $B = (2n-1)s_1$. Thus $\deg \ell_n = 2n+1$. Equation (7.4.9) provides a natural algebraic (hyperelliptic) compactification $\hat{X}_n$ of X_n.

To make this precise, we show that X_n is mapped onto those $B \in \mathbb{C}$ with $C^2 = \ell_n(B) \neq 0$. Indeed we define s_k by $s_k(B)$ and x_i's by $q(x) = x^n - s_1 x^{n-1} + \cdots + (-1)^n s_n = \prod_{i=1}^n (x - x_i)$. Then (7.4.8) holds, and by substituting $x = x_i$ we get $p(x_i)q'(x_i)^2 = C^2$ as in (7.4.2).

If $C \neq 0$, we get $p(x_i) \neq 0$ and $q'(x_i) \neq 0$ which give the non-degenerate conditions. Now we define

$$y_i := \frac{C}{q'(x_i)} = \frac{C}{\prod_{j\neq i}(x_i - x_j)} \neq 0, \qquad i = 1, \ldots, n.$$

Then $y_i^2 = p(x_i)$ and $\{(x_i, y_i)\}$ solves the system of equations

$$\sum_{i=1}^{n} x_i^k y_i = 0 \qquad k = 0, \ldots, n-2,$$

hence gives rise to a point in X_n.

If $C = 0$, we have either $p(x_i) = 0$ or $q'(x_i) = 0$ for all $i = 1, \ldots, n$. Let $x_i = \wp(a_i)$. In the former case $a_i = -a_i$ is a half period and $y_i = 0$. In the latter case $x_i = x_j$ for some $j \neq i$. Notice that $\{(x_i, y_i)\}$ still satisfies the equations $\sum_{i=1}^{n} x_i^k y_i = 0$ for $k = 0, \ldots, n-2$ since they define a closed set. The same argument in the proof of Theorem 6.5 shows that $[a] = [-a]$, where $[a] = \{[a_1], \ldots, [a_n]\}$.

If $B \to \infty$, then the first n elementary symmetric polynomials for the unordered list $x_1^{-1}, \ldots, x_n^{-1}$ all go to 0, because the i-th elementary symmetric polynomial in $x_1^{-1}, \ldots, x_n^{-1}$ is $\frac{s_{n-i}}{s_n}$ for $i = 1, \ldots, n$. Since $x_i = \wp(a_i)$, we get $a_i \to 0$ for all i. That is, $\bar{\pi}^{-1}(\infty) = (0, \ldots, 0)$.

We have proved Theorem 7.4 (1) at this point. The statement of Theorem 7.4 (2) is a consequence of the second paragraph of 7.3.4 in the proof of Theorem 7.3 (iii). There we recalled that for a rectangular lattice Λ_τ with $\tau \in \sqrt{-1}\,\mathbb{R}_{>0}$, the polynomial $\ell_n(B; \Lambda_\tau)$ in B has $2n+1$ distinct real roots, and gave references for this fact. Clearly this fact implies that the discriminant of $\ell_n(B)$ is not identically zero. Theorem 7.4 (2) follows. $\square$

Example 7.4.1. For $n = 1$, $s_0 = 1$, $s_1 = B$ and then

$$C^2 = \ell_1(B) = 4B^3 - g_2 B - g_3$$

which is exactly the equation for E, since $\bar{X}_1 \cong E$.

For $n = 2$, $s_0 = 1$, $s_1 = \frac{1}{3}B$, $s_2 = \frac{1}{9}B^2 - \frac{1}{4}g_2$, and then

$$\begin{aligned} C^2 = \ell_2(B) &= \tfrac{4}{81}B^5 - \tfrac{7}{27}g_2 B^3 + \tfrac{1}{3}g_3 B^2 + \tfrac{1}{3}g_2^2 B - g_2 g_3 \\ &= \tfrac{1}{81}(B^2 - 3g_2)(4B^3 - 9g_2 B + 27g_3). \end{aligned}$$

In terms of s_1, it is $C^2 = \ell_2(3s_1) = (3s_1^2 - g_2)(4s_1^3 - g_2 s_1 + g_3)$.

For $n = 3$, $s_0 = 1$, $s_1 = \frac{1}{5}B$, $s_2 = \frac{2}{75}B^2 - \frac{1}{4}g_2$, $s_3 = \frac{1}{3^2 5^2}B^3 - \frac{1}{3\cdot 5}g_2 B + \frac{1}{4}g_3$, and then

$$\begin{aligned} C^2 &= \ell_3(B) \\ &= \tfrac{1}{2^2 3^4 5^4} B(16B^6 - 504 g_2 B^4 + 2376 g_3 B^3 \\ &\qquad + 4185 g_2^2 B^2 - 36450 g_2 g_3 B + 91125 g_3^2 - 3375 g_2^3) \\ &= s_1\big(\tfrac{500}{81}s_1^6 - \tfrac{70}{9}g_2 s_1^4 + \tfrac{22}{3}g_3 s_1^3 + \tfrac{31}{12}g_2^2 s_1^2 - \tfrac{9}{2}g_2 g_3 s_1 + \tfrac{9}{4}g_3^2 - \tfrac{1}{12}g_2^3\big). \end{aligned}$$

Remark. The referee has kindly informed us that the curve $C^2 = \ell_2(B)$ appeared in [22, p. 63] as a hyperelliptic curve $\hat{C}$ whose affine coordinates (z, w) are related to (B, C) here by $z = -B$ and $w = \sqrt{-1}\frac{9C}{2}$.

The paper [22] is based on the general construction of spectral curves Γ_n introduced in [35, §1]. Note that the factors $e^{\zeta(a_i)z}\frac{\sigma(z-a_i)}{\sigma(z)}$ of the ansatz function $w_a(z)$ in Definition 6.1.5 appeared in [35, p. 284] up to a factor $-\sigma(a_i)$: the function $\Phi(x,\alpha)$ in [35] is $-e^{\zeta(\alpha)z}\frac{\sigma(z-\alpha)}{\sigma(z)\sigma(\alpha)}$.

Explicit examples of Riemann surfaces associated to (finite gap) Lamé potentials and Treibich–Verdier potentials can be found in [62].

Remark 7.4.2 (Meaning of the parameter C)**.** We have introduced the same notation C in various places. Indeed they are all equivalent: The constant C in (7.4.1) coincides with the constant C in (5.6.4) by setting $w = x_i = \wp(a_i)$ in (5.6.4). It also coincide with the Wronskian C defined in (7.1.2) up to sign by comparing (7.2.1) with the expression of $g(z)$ in (5.6.3) (using (5.6.4) and (7.3.3)). These equivalences allow us to study the hyperelliptic curve Y_n from both the analytic and algebraic point of views at the same time.

Remark 7.4.3 (Relation to KdV theory)**.** There are several methods for computing compute $\ell_n(B)$ in the literature, e.g. [20, 65]. It is also interesting to note that the hyperelliptic curve $\hat{X}_n$ also appears in the study of KdV equations, where it is known as the *spectral curve*.

Indeed in KdV theory, a differential operator P_{2n+1} of order $2n+1$ is constructed by

$$P_{2n+1} = \sum_{l=0}^{n} (f'_{n-l}(z) - \tfrac{1}{2} f_{n-l}(z)) L^l,$$

where $L = -d^2/dz^2 + u(z)$, $f_0(z) = 1$ and $f_k(z)$ satisfies the recursive relation

$$f'_{k+1} = -\tfrac{1}{4} f'''_k + u f'_k + \tfrac{1}{2} u' f_k, \qquad k = 0, 1, 2, \cdots. \tag{7.4.10}$$

Using the recursion (7.4.10), we have

$$[P_{2n+1}, L] = 2 f'_{n+1}.$$

A potential $u(z)$ is called a stationary solution to an n-th KdV hierarchy equation if $f'_{n+1} = 0$. Let

$$F(z; E) = \sum_{l=0}^{n} f_{n-l}(z) E^l.$$

Then $F(z; E)$ satisfies

$$F''' - 4(u - E)F' - 2u'F = 0. \tag{7.4.11}$$

Conversely, if $F(z; E)$ is a monic polynomial in E of degree n and satisfies (7.4.11), write $F(z; E) = \sum_{l=0}^{n} f_{n-l}(z) E^l$. Then $f_k(z)$ satisfies (7.4.10) with $f_k = 0$ for $k \geq n+1$. By integrating (7.4.11), we obtain

$$\tfrac{1}{2} F'' F - \tfrac{1}{4} (F'')^2 - (u - E) F^2 = R_{2n+1}(E), \tag{7.4.12}$$

where $R_{2n+1}(E)$ is independent of z and is a monic polynomial in E of degree $2n+1$. The spectral curve for the potential u, if u is a stationary solution of the n-th KdV hierarchy, is by definition the hyperelliptic curve

$$y^2 = R_{2n+1}(E);$$

it parametrizes one-dimensional eigenspaces of the commutator subring of the differential operator L in the space of ordinary differential operators.

If $u(z) = n(n+1)\wp(z)$ is the Lamé potential and $B = -E$, then (7.4.11) is identical to (7.3.4) with $F(z; E) = X(z)$. As we have seen already, $X(z)$ is also a polynomial in $\wp(z)$. By using $x = \wp(z)$, (7.4.8) is identical to (7.4.12) (with $C^2 = 4R_{2n+1}(E)$ and $E = -B$). By this adjustment, the curve (7.4.9) is identical to the spectral curve in KdV theory. For more details see [24, Ch.1 §2].

The Lamé potential is a very special type of *finite gap potentials*. There is an extensive literature. The readers may consult [31, 21, 32, 50, 48, 34, 35, 36, 57, 62, 37, 59, 60, 61].

The Lamé potential is also a special case of *Picard potential* [26]; the system of equations (5.7.4) (i.e. equations for Y_n (0.5.3)) appeared in [25, (3.8) in p.82]. According to [25, Rmk. 3.3], that was the first time after [8] when (5.7.4) reappeared in mathematical publications. However in a comment in [25, p.83] the authors said that *the conditions* (5.7.4) *appear to be too difficult to be handled directly*, so they turned to develop another method to compute the spectral curve.

The following proposition arises from the study of the process $B \to \infty$. When $x_i \to \infty$, we have $y_i \to \infty$ too. Asymptotically $(x_i, y_i) \sim (t_i^2, 2t_i^3)$ hence $\sum x_i^k y_i \sim 2\sum t_i^{3+2k}$. The uniqueness of $\bar{\pi}^{-1}(\infty)$ suggests the uniqueness of solutions of the limiting equations up to permutations. It turns out to be true and can be proved along the similar reasoning as above.

Proposition 7.5. *Consider the following system of $n-1$ homogeneous equations in $\mathbb{P}^{n-1}(\mathbb{C})$ ($n \geq 2$) with coordinates $t_1, \ldots, t_n$:*

$$\sum_{i=1}^{n} t_i^{2k+1} = 0, \qquad k = 1, 2, \ldots, n-1, \tag{7.5.1}$$

subject to the non-degeneracy conditions $\prod_{i=1}^{n} t_i \neq 0$ and $\prod_{i<j}(t_i + t_j) \neq 0$. Then the solution exists uniquely up to permutations.

Proof. When $B \to \infty$, by either (7.3.9) or (7.4.2) we see that all t_i's have the same order $|B|^{1/2}$. Since the polynomial system in t_i's comes from the leading order terms of the original system $\sum x_i^k y_i = 0$, by passing to a subsequence if necessary, in the limit $B \to \infty$ we get a point $[t_1 : \cdots : t_n] \in \mathbb{P}^{n-1}$ solving the limiting equations. In fact $[t] \in \mathbb{P}(T_0(\bar{X}_n)) \subset \mathbb{P}(T_0(\mathrm{Sym}^n E))$.

However a more careful argument is needed to verify the nondegeneracy conditions. We recall that for $a \in X_n$, $\wp(a_i)$'s are the roots of the polynomial $X(x)$ where the coefficients $s_j(B)$'s satisfy the recursive relation (7.3.6). Thus $\wp(a_i)/B$ tends to the roots of the limiting polynomial X_∞:

$$X_\infty(x) = x^n - \bar{s}_1 x^{n-1} + \bar{s}_2 x^{n-2} + \cdots + (-1)^n \bar{s}_n,$$

where we set $\bar{s}_0 = 1$ and

$$\bar{s}_k = \frac{2(n-k+1)}{k(2n-2k+1)(2n-k+1)}\bar{s}_{k-1}, \qquad k = 1, \ldots, n. \tag{7.5.2}$$

To prove $(\wp(a_i) - \wp(a_j))/B \not\to 0$ as $B \to \infty$ is equivalent to showing that X_∞ has n distinct roots, a statement which does not seem to be obvious. Instead, we use (7.4.1) in its analytic form

$$C = \wp'(a_i)\prod_{j\neq i}(\wp(a_i) - \wp(a_j)). \tag{7.5.3}$$

Obviously $|C| \sim |B|^{n+1/2}$ and $|\wp'(a_i)| \sim |\wp(a_i)|^{3/2} \sim |B|^{3/2}$. Thus if there is some j such that $|\wp(a_i) - \wp(a_j)| = o(1)|B|$ as $|B| \to \infty$ then (7.5.3) yields

$$|B|^{n+1/2} \sim |C| \le o(1)|B|^{n+1/2},$$

which is a contradiction. Therefore we have

$$\lim_{B\to\infty}\frac{\wp(a_i)}{B} \neq \lim_{B\to\infty}\frac{\wp(a_j)}{B}$$

for $i \neq j$. Now we write $(\wp(a_i), \wp'(a_i)) = (x_i, y_i) \sim (t_i^2, 2t_i^3)$. Then the leading term of $\sum_i x_i^k y_i$ is $2\sum_i t_i^{2k+3}$ for $k = 0, \ldots, n-2$. By passing to $B \to \infty$, the limit of $t_i/|B|^{1/2}$ (still denoted by t_i) then satisfies

$$\sum_{i=1}^n t_i^{2k+1} = 0, \qquad 1 \le k \le n-1,$$

and $t_i + t_j \neq 0$ for any i, j. This proves the existence of solutions.

The remaining task is to prove the uniqueness. While it may be possible to prove this by working harder on the asymptotic equations, however, owing to its elementary nature, we will offer a purely elementary argument using only basic algebra.

Before we proceed, notice that the loci $\prod t_i = 0$ or $\prod_{i<j}(t_i + t_j) = 0$ provide positive dimensional solutions to the system. Thus it is crucial to analyze the non-degenerate conditions. By a Vandermonde-like determinant argument, it is easy to see that under the assumption $t_i \neq 0$ for all i, we have $t_i \neq -t_j$ for all $i \neq j$ if and only if $t_i^2 \neq t_j^2$ for all $i \neq j$.

Let $q(t) = \prod_{j=1}^n (t - t_j) = \sum_{i=0}^n (-1)^i s_i t^{n-i}$, where s_i is the i-th elementary symmetric polynomial in t_j's, and $p_l = \sum_{i=1}^n t_i^l$ being the Newton symmetric polynomial for all $l \ge 0$. Then

$$\frac{q'(t)}{q(t)} + \frac{q'(-t)}{q(-t)} = \sum_{i=1}^n \frac{1}{t-t_i} - \frac{1}{t+t_i} = 2\sum_{m\ge1} p_{2m-1}t^{-2m}.$$

The conditions $p_3 = p_5 = \cdots = p_{2n-1} = 0$ imply that (comparing degrees)

$$\frac{q'(t)}{q(t)} + \frac{q'(-t)}{q(-t)} = \frac{2p_1}{t^2} + \frac{(-1)^n 2p_{2n+1}}{t^2 q(t)q(-t)}.$$

Denote by $u = t^2$, $u_i = t_i^2$, $G(u) = q(t)q(-t) = \prod_{i=1}^n (u - u_i)$, this then could be regarded as an equality in $\mathbb{C}(u)$ as

$$\sum_{i=1}^n \frac{t_i}{u - u_i} = \frac{p_1}{u} + \frac{(-1)^n p_{2n+1}}{uG(u)}.$$

From now on we denote $' = d/du$, then

$$t_i = \mathrm{Res}_{u=u_i} = \frac{(-1)^n p_{2n+1}}{u_i G'(u_i)}.$$

In particular, $u_i^3 G'(u_i)^2 = C^2$ is independent of i, where $C = (-1)^n p_{2n+1}$. So $G(u) \mid u^3 G'(u)^2 - C^2$ and we may perform division to write

$$\frac{u^3 G'(u)^2 - C^2}{G(u)^2} = \sum_{i=1}^n \frac{h_i}{u - u_i} + n^2 u + 2n\tau_1$$

for some $h_i \in \mathbb{C}$ and $\tau_1 = \sum_{i=1}^n u_i$. Using series expansion in $u - u_i$ we calculate

$$\mathrm{Res}_{u=u_i} \frac{u^3 G'(u)^2 - C^2}{G(u)^2} = 3u_i^2 + \frac{2u_i^3 G''(u_i)}{G'(u_i)^2},$$

hence there are $a, b \in \mathbb{C}$ such that

$$\frac{u^3 G'(u)^2 - C^2}{G(u)^2} = \frac{3u^2 G'(u)}{G(u)} + \frac{2u^2 G''(u)}{G(u)} + au + b.$$

By division again, it is clear that

$$2u^3 G''(u) = 2(n(n-1)u + 2(n-1)\tau_1)G(u) + \cdots$$

and $3u^2 G'(u) = 3(nu + \tau_1)G(u) + \cdots$. Hence

$$a = -n(n+1), \qquad b = -(2n+1)\tau_1,$$

$$u^3 G'^2 - 3u^2 G'G - 2u^2 G''G - (au + b)G^2 = C^2.$$

Differentiation and simplification lead to $-G(u)$ times the equation

$$2u^3 G''' + 9u^2 G'' - 2((n^2 + n - 3)u + (2n-1)\tau_1)G' - n(n+1)G = 0.$$

Write $G(u) = \sum_{i=0}^n (-1)^{n-i} \tau_{n-i} u^i$ (so $\tau_0 = 1$ and $\tau_k = 0$ if $k < 0$ or $k > n$ by convention), the above linear third order ODE translates into the recursive relation:

$$(i-n)(2i+1)(i+n+1)\tau_{n-i} = -2(i+1)(2n-1)\tau_1 \tau_{n-i-1}.$$

This rather short recursion (instead of four terms) is due to the fact that the ODE has an irregular singularity at $u = 0$. It is consistent for $i = n$ ($0 = \tau_0 \tau_{-1}$) and for $i = n-1$ ($\tau_1 = \tau_1$), and then all τ_k, $k \geq 2$, are completely determined by τ_1. This proves the uniqueness of solution up to permutations. $\square$

Remark. The non-degeneracy conditions in Proposition 7.5 are essential: when $n \geq 4$ the set of all degenerate solutions has a natural structure as a positive dimensional algebraic variety.

7.5.1. Question. Let $(b_1 : \ldots : b_n) \in \mathbb{P}^{n-1}(\mathbb{C})$ be a non-degenerate solution of equation (7.5.1). Let K_n be the smallest subfield of $\mathbb{C}$ which contains $b_2/b_1, \ldots, b_n/b_1$. Is $[K_n : \mathbb{Q}] = n!$?[26]

Corollary 7.5.2. *The curve $\bar{X}_n$ is smooth at the infinity point $[0]^n$.*

Proof. The idea is that the solutions sought in Proposition 7.5 describe the tangent directions of $\bar{X}_n$ at 0^n, in the sense that the projectivized tangent cone of $\bar{X}_n$ at $[0]^n$ is the affine open subset projective spectrum of the ring

$$\mathcal{R} = \mathbb{C}\left[t_1, \ldots, t_n\right] \Big/ \left(\textstyle\sum_{i=1}^n t_i^{2k+1}\right)_{1\le k\le n-1};$$

associated by localization to the homogeneous element

$$\prod_{i=1}^n t_i \cdot \prod_{1\le i<j\le n} (t_i + t_j)$$

of $\mathcal{R}$. Once we know this then the existence and uniqueness statement in Proposition 7.5 is equivalent to the smoothness of $\bar{X}_n$ at 0^n. However the above description of the projectivized tangent cone of $\bar{X}_n$ at $[0]^n$ is not self-evident from the definition of $\bar{X}_n$ as the closure of X_n in $\mathrm{Sym}^n(\mathbb{C}/\Lambda)$. So we proceed slightly differently.

Let $(r_1, \ldots, r_n)$ be a non-degenerate solution of the system of equations in Proposition 7.5. From the non-vanishing of the Vandermonde determinant one sees that $\sum_{i=1}^n r_i \neq 0$. From Hensel's lemma one sees that there exists a morphism α from the spectrum of a formal power series ring $\mathbb{C}[[t]]$ to the inverse image in $\mathrm{Sym}^n(\mathbb{C}/\Lambda)$ of $\bar{X}_n$ which sends the closed point of $\mathrm{Spec}\,\mathbb{C}[[t]]$ to $[0]^n$, such that

$$\frac{x_i}{y_i} \mapsto r_i\, t + O(t^2) \quad \forall\, i = 1, \ldots, n.$$

The condition that $r_1 + \cdots + r_n \neq 0$ tells us that α induces an isomorphism between $\mathbb{C}[[t]]$ and the completed local ring of $\bar{X}_n$ at the point $[0]^n$. $\square$

7.6. Comparing the compactifications $\bar{X}_n$ and $\hat{X}_n$ of X_n.

7.6.1. At this point we have two compactifications of the smooth algebraic curve X_n. We summarize the situation.

7.6.1.a. By definition X_n is a locally closed algebraic subvariety of the symmetric product $\mathrm{Sym}^n(\mathbb{C}/\Lambda)$. The first compactification $\bar{X}_n$ of X_n is the closure of X_n in $\mathrm{Sym}^n(\mathbb{C}/\Lambda)$. We have seen that $\bar{X}_n$ contains the closed subvariety Y_n of $\mathrm{Sym}^n(\mathbb{C}/\Lambda \smallsetminus \{[0]\})$. The latter variety Y_n classifies all ansatz solutions modulo $\mathbb{C}^\times$ to Lamé equations of index $n \in \mathbb{N}_{>0}$.

[26]The answer is likely "yes", but we don't have a proof.

7.6.1.b. The map "multiplication by -1" on $\mathbb{C}/\Lambda$ defines an involution $\tilde{\iota}$ on $\mathrm{Sym}^n(\mathbb{C}/\Lambda)$. The subvarieties $X_n, Y_n, \bar{X}_n$ of $\mathrm{Sym}^n(\mathbb{C}/\Lambda)$ are stable under the involution $\tilde{\iota}$. The restriction of $\tilde{\iota}$ to $\bar{X}_n$ is an involution $\bar{\iota}$ on $\bar{X}_n$. It turned out that X_n is the complement in $\bar{X}_n$ of the fixed point set $(\bar{X}_n)^{\bar{\iota}}$ of the involution $\bar{\iota}$. One of the fixed points of $\bar{\iota}$ is the point $[0]^n = \{[0], \ldots, [0]\}$ of $\mathrm{Sym}^n(\mathbb{C}/\Lambda)$; the rest are all in Y_n. In particular $\bar{X}_n \smallsetminus Y_n = \{[0]^n\}$. It is known that $\#(\bar{X}_n)^{\bar{\iota}} \leq 2n+2$, and the equality $\#(\bar{X}_n)^{\bar{\iota}} = 2n+2$ holds for all Λ outside of a finite number of homothety classes of lattices in $\mathbb{C}$.

7.6.1.c. The map $\pi_n : Y_n \to \mathbb{C}$ which sends a point $[a] \in Y_n$ to the accessary parameter $B_{[a]}$ of the Lamé equation satisfied by the ansatz function w_a is an algebraic morphism from Y_n to the affine line $\mathbb{A}^1$ over $\mathbb{C}$. The morphism $\pi_n : Y_n \to \mathbb{A}^1$ extends to a morphism $\bar{\pi}_n : \bar{X}_n \to \mathbb{P}^1$. This projection morphism $\bar{\pi}_n$ is compatible with the involution ι in the sense that $\bar{\pi}_n = \bar{\pi}_n \circ \bar{\iota}$, and $\bar{\pi}_n(\underline{x}) = \bar{\pi}_n(\underline{x}')$ for two points $\underline{x}, \underline{x}' \in \bar{X}_n$ if and only if either $\underline{x} = \underline{x}'$ or $\bar{\iota}(\underline{x}) = \underline{x}'$.

In particular $\bar{\pi}_n$ induces a bijection from the fixed point set $(\bar{X}_n)^{\bar{\iota}}$ to a finite subset $\bar{\Sigma}_n \subset \mathbb{P}^1(\mathbb{C})$. This ramification locus $\bar{\Sigma}_n$ for $\bar{\pi}_n$ is the disjoint union of $\{\infty\}$ with a finite subset $\Sigma_n \subset \mathbb{C}$. The restriction $\pi_n|_{X_n}$ of π_n to X_n makes X_n an unramified double cover of the complement $\mathbb{A}^1 \smallsetminus \Sigma_n$ of Σ_n in $\mathbb{A}^1$.

7.6.1.d. The ramification locus Σ_n is the set of roots of a polynomial $\ell_n(B)$

$$\ell_n(B) = 4Bs_n^2 + 4g_3(\Lambda)s_{n-2}s_n - g_2(\Lambda)s_{n-1}s_n - g_3(\Lambda)s_{n-1}^2$$

of degree $2n+1$ in the variable B with coefficients in $\mathbb{Q}[g_2, g_3]$, where the polynomials $s_n, s_{n-1}, s_{n-2} \in \mathbb{Q}[g_2, g_3][B]$ are defined recursively by equations (7.3.6), starting with $s_0 = 1$ and $s_1 = (2n-1)^{-1}B$. The recursive relation (7.3.6) implies that $\ell_n(B, g_2, g_3)$ is homogenous of weight $2n+1$ if g_2, g_3 are given weights $1, 2, 3$ respectively; the coefficient of B^{2n+1} in $\ell_n(B)$ is a positive rational number.[27]

7.6.1.e. The polynomial $\ell_n(B)$ gives rise to another compactification $\hat{X}_n$ of X_n. Let X_n^* be the zero locus of the homogeneous polynomial

$$F_n(\hat{A}, \hat{B}, \hat{C}) := \hat{C}^2 \hat{A}^{n-1} - \hat{A}^{2n+1} \ell_n(\hat{B}/\hat{A})$$

in the projective plane $\mathbb{P}^2$ with projective coordinates $(\hat{A} : \hat{B} : \hat{C})$. By definition $\hat{X}_n$ is the partial desingularization of X_n^*, changing the local structure near the singular point $(0 : 0 : 1)$ by replacing the structure sheaf near $(0 : 0 : 1)$ by its normal closure in the field of fractions. More explicitly we replace a small Zariski open neighborhood of the point $(0 : 0 : 1)$ in X_n^* by the corresponding open neighborhood of the curve

$$v^2 = u \cdot (u^{2n+1} \ell_n(1/u))$$

[27]The coefficient of B^i in $s_i(B)$ is
$\frac{2^i}{2n-1} \cdot \frac{n(n-1)(n-2)\cdots(n-i+1)}{[(2n)(2n-1)(2n-2)\cdots(2n-i+1)]\cdot[(2n-1)(2n-3)(2n-5)\cdots(2n-2i+1)]}$ for $i = 1, \ldots, n$.

near $(u,v)=(0,0)$; the coordinates are related by

$$\frac{\hat{B}}{\hat{A}} = B = \frac{1}{u}, \ \frac{\hat{C}}{\hat{A}} = C = \frac{v}{u^{n+1}}.$$

The natural morphism $\hat{X}_n \to X_n^*$ is a homeomorphism, and is a local isomorphism outside the point $\hat{\infty}$ which maps to the point $(0:0:1) \in X_n^*$. The projective curve $\hat{X}_n$ is reduced, irreducible and has arithmetic genus n; we call it the "hyperelliptic model" of X_n.

We have a "hyperelliptic involution" $\hat{\iota}$ on $\hat{X}_n$, given by

$$\hat{\iota}\colon\ (\hat{A}:\hat{B}:\hat{C}) \mapsto (\hat{A}:\hat{B}:-\hat{C})$$

in projective coordinates. The X_n is the complement in $\hat{X}_n$ of the fixed point set $(\hat{X}_n)^{\hat{\iota}}$ of the hyperelliptic involution. We also have a morphism $\hat{\pi}_n : \hat{X}_n \to \mathbb{P}^1$, defined by $\hat{\pi}_n\colon (\hat{A}:\hat{B}:\hat{C}) \mapsto (\hat{A}:\hat{B})$ over the open subset of $\hat{X}_n$ where $\hat{A}$ is invertible, and $\hat{\pi}_n\colon (\hat{A}:\hat{B}:\hat{C}) \mapsto \left(\frac{\hat{A}}{\hat{C}} : \frac{\hat{B}}{\hat{C}}\right)$ over the open subset of $\hat{X}_n$ where $\hat{C}$ is invertible. One of the fixed points $\hat{\iota}$ is $\hat{\infty}$. The map $\hat{\pi}_n$ induces a bijection from $(\hat{X}_n)^{\hat{\iota}}$ to $\bar{\Sigma}_n$.

The hyperelliptic projection $\hat{\pi}_n$ is compatible with the hyperelliptic involution $\hat{\iota}$ on $\hat{X}_n$, in the sense that $\hat{\pi}_n(P) = \hat{\pi}_n(P')$ if and only either $P' = P$ or $P' = \hat{\iota}(P)$, for any two points $P, P' \in \hat{X}_n$. The restriction of the triple $(\hat{X}_n, \hat{\pi}_n, \hat{\iota})$ to the open subset $X_n \subset \hat{X}_n$ is naturally identified with the restriction to X_n of the triple $(\bar{X}_n, \bar{\pi}_n, \bar{\iota})$.

7.6.2. A natural and inevitable question is:

> *Is there an isomorphism between the two triples $(\bar{X}_n, \bar{\pi}_n, \bar{\iota})$ and $(\hat{X}_n, \hat{\pi}_n, \hat{\iota})$ which extends the natural isomorphism between the complements of the fixed point sets of $\bar{\iota}$ and $\hat{\iota}$?*

The parallel properties of the two compactifications reviewed in 7.6.1 suggest that the answer is likely "yes". Since both $\bar{X}_n$ and $\hat{X}_n$ are reduced and irreducible, to answer this question affirmatively, we need to show that the natural identification of the "common open" dense subset X_n of both sides extends to an isomorphism.

We will see in Lemma 7.6.4 that methods in the previous part of this section already shows that the identity map on X_n extends to a *morphism* $\phi : \bar{X}_n \to \hat{X}_n$ of algebraic varieties. That ϕ is a morphism at $\bar{\infty} = [0]^n$ is a consequence of (and equivalent to) Corollary 7.5.2.

The following properties of the morphism $\phi : \bar{X}_n \to \hat{X}_n$ between reduced irreducible complete algebraic curves are easily deduced from previous arguments:

(a) ϕ is bijective on points, i.e. ϕ is a *homeomorphism*.
(b) ϕ is an isomorphism over $X_n = \bar{X}_n \setminus (\bar{X}_n)^{\bar{\iota}}$.
(c) ϕ is an isomorphism near the point $\bar{\infty} = [0]^n$. This is a rather trivial case of Zariski's Main Theorem and is easily verified directly.

So we are left with showing that ϕ is a local isomorphism at each point of $(Y_n)^\iota$, ramification points "at finite distance".

7.6.3. The properties in 7.6.2 (a)–(c) of the morphism $\phi : \bar{X}_n \to \hat{X}_n$ do *not* formally imply that ϕ is an isomorphism: it may happen that there exists a point $P \in Y_n \subset \bar{X}_n$ such that the *injection*

$$\phi^* : \mathcal{O}_{\hat{X}_n, \phi(P)} \to \mathcal{O}_{\bar{X}_n, P}$$

induced by ϕ between the stalks of the structure sheaves of $\bar{X}_n$ and $\hat{X}_n$ at P and $\phi(P)$ is not an isomorphism. If this "undesirable" phenomenon happens at one ramification point $P \in Y_n$, the arithmetic genus of $\bar{X}_n$ will be strictly smaller than n, the arithmetic genus of $\hat{X}_n$. To put it differently, the fact that $\phi : \bar{X}_n \to \hat{X}_n$ is a bijective morphism tells us that the hyperelliptic model $\hat{X}_n$ can "only be more singular" than $\bar{X}_n$.

If we can show that the arithmetic genus of $\bar{X}_n$ is n, it will follow that ϕ is an isomorphism. This approach may well be possible, but we will take an easier route: We have seen that the discriminant $\mathrm{disc}(\ell_n(B))$ of $\ell_n(B)$ is a non-zero holomorphic modular form for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. If the discriminant $\mathrm{disc}(\ell_n(B))$ does not vanish when evaluated at the lattice $\Lambda \subset \mathbb{C}$ in question, then the polynomial $\ell_n(B; \Lambda)$ has $2n+1$ distinct roots in $\mathbb{C}$ and $\hat{X}_n$ is smooth, which forces the morphism $\phi : \bar{X}_n \to \hat{X}_n$ to be an isomorphism.

Suppose now that the elliptic curve we are given is $\Lambda_{\tau_0} = \mathbb{Z} + \mathbb{Z}\tau_0$ for some $\tau_0 \in \mathbb{H}$ such that $\mathrm{disc}(\ell_n(B))(\Lambda_{\tau_0}) = 0$. The idea now is to embed the given situation in a one-parameter family such that the morphism ϕ is an isomorphism outside the central fiber, then use purity (Hartog's theorem):

> *Let τ vary in a small open disk $D \subset \mathbb{H}$ containing τ_0 such that* $\mathrm{disc}(\ell_n(B))(\Lambda_\tau) \neq 0$ *for all $\tau \in D$ and get a family of maps $\left(\phi_\tau : \bar{X}_{n,\tau} \to \hat{X}_{n,\tau}\right)_{\tau \in D}$ parametrized by D. Use the fact that ϕ_τ is an isomorphism for all τ in the punctured disk $D^* = D \smallsetminus \{\tau_0\}$ and that ϕ_{τ_0} is an isomorphism outside a finite subset of $\bar{X}_{n,\tau_0}$ to show that ϕ_{τ_0} itself is an isomorphism.*

Details will be carried out in Proposition 7.7

Lemma 7.6.4. *The identity map* $\mathrm{id}_{X_n} : X_n \to X_n$ *on X_n extends uniquely to a* morphism $\phi : \bar{X}_n \to \hat{X}_n$.

Proof. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the Weierstrass coordinates of the product $E^n = E \times \cdots \times E$. The affine coordinate B of X_n^* is given by $B = (2n-1)\sum_{i=1}^n x_i$. From the proofs of Theorems 7.3 and 7.4 we see that the other affine coordinate C of X_n^* can be expressed by polynomials in the x_i's and y_i's: $C = y_i \cdot \prod_{j \neq i}(x_i - x_j)$ for all $i = 1, \ldots, n$. It follows that the birational map ϕ from $\bar{X}_n$ to $\hat{X}_n$ is a morphism at every point of $\bar{X}_n \smallsetminus \{\bar{\infty}\}$. Corollary 7.5.2 implies that ϕ is a morphism at $\bar{\infty}$ as well. □

Proposition 7.7. *The morphism* $\phi : \bar{X}_n \to \hat{X}_n$ *is an isomorphism.*

Proof. We may and do assume that the given lattice Λ is Λ_{τ_0} for an element $\tau_0 \in \mathbb{H}$. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the Weierstrass coordinates of E^n as in the proof of Lemma 7.6.4, where $E = \mathbb{C}/\Lambda$. It suffices to prove the following

Claim: *For every monomial* $h(\underline{x}, \underline{y})$ *in* $x_1, \ldots, x_n, y_1, \ldots, y_n$*, the restriction to* X_n *of its* S_n*-symmetrization*

$$\Pi_{\mathrm{S}_n}(h) = (n!)^{-1} \sum_{\sigma \in \mathrm{S}_n} h(\sigma(\underline{x}), \sigma(\underline{y}))$$

is the pull-back under ϕ *of a polynomial in* B *and* C.

We have seen in the proofs of Theorems 7.3 and 7.4 that the restriction to X_n of every symmetric polynomial in $x_1, \ldots, x_n$ can be expressed as a polynomial in B. Because

$$\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \Big/ \left(y_i^2 - 4x_i^3 + g_2 x_i + g_3\right)_{1 \le i \le n}$$

is a free module over the ring of symmetric polynomials $\mathbb{C}[x_1, \ldots, x_n]^{\mathrm{S}_n}$ with basis given by monomials of the form

$$\underline{x}^{\underline{r}} \, \underline{y}^{\underline{s}} = \prod_{i=1}^{n} x_i^{r_i} \cdot \prod_{i=1}^{n} y^{s_i}$$

with $0 \le r_i \le n - i$ and $0 \le s_i \le 1$ for all $i = 1, \ldots, n$, and the symmetrization operator Π_{S_n} is linear over the ring of symmetric polynomials in $x_1, \ldots, x_n$, it suffices to prove the claim in the case when $h(\underline{x}, \underline{y})$ is one of the above basis elements $\underline{x}^{\underline{r}} \, \underline{y}^{\underline{s}}$. In principle one should be able to show directly that the symmetrization $\Pi_{\mathrm{S}_n}(\underline{x}^{\underline{r}} \, \underline{y}^{\underline{s}})$ of $\underline{x}^{\underline{r}} \, \underline{y}^{\underline{s}}$ is a polynomial in B and C. Here we take an easier way out using purity as indicated in 7.6.3.

Let τ vary over $\mathbb{H}$, and consider the fiber product over $\mathbb{H}$ of n copies of the universal elliptic curve whose fiber over $\tau \in \mathbb{H}$ is $E_\tau = (\mathbb{C}/\Lambda_\tau)$. Over the open subset of the n-time fiber product of $E \smallsetminus \{[0]\}$, we have affine coordinates $x_1, y_1, \ldots, x_n, y_n$ as before. Let $\mathcal{X}_n$ be the relative affine spectrum over $\mathbb{H}$ of

$$\mathcal{O}_{\mathbb{H}}\left[\begin{matrix} x_1, \ldots, x_n, y_1, \ldots, y_n, \\ \left(\prod_{1 \le i < j}(x_i - x_j)\right)^{-1} \end{matrix}\right] \Big/ \left(\begin{matrix} \sum_{j=1}^{n} x_j^l y_j, \quad {\scriptstyle l = 0, 1, \ldots, n-2,} \\ y_i^2 - 4x_i^3 + g_2(\Lambda_\tau) x_i + g_3(\Lambda_\tau), \quad {\scriptstyle 1 \le i \le n} \end{matrix}\right)$$

Let $\mathcal{X}_n^*$ be the relative projective curve over $\mathcal{O}_{\mathbb{H}}$ defined by the homogeneous polynomial

$$F_n(\hat{A}, \hat{B}, \hat{C}) = \hat{C}^2 \hat{A}^{n-1} - \hat{A}^{2n+1} \ell_n(\hat{B}/\hat{A})$$

as in 7.6.1.e, where $g_2 = g_2(\Lambda_\tau)$ and $g_3 = g_3(\Lambda_\tau)$ in the definition of $\ell_n(B)$. Let $\Phi : \bar{\mathcal{X}}_n \to \mathcal{X}_n^*$ be the morphism extending the identity map on $\mathcal{X}_n$. Let $\mathcal{U}$ be the complement in $\mathcal{X}_n^*$ of the set of all ramification points over those τ's where the discriminant $\operatorname{disc}(\ell_n(B))$ vanishes, so that $\mathcal{U}$ is the complement of the union of the section $\bar{\infty}$ and a discrete set of points in $\mathcal{X}_n^*$. We know that Φ is an isomorphism over $\mathcal{U}$. Notice that the two-dimensional variety $\mathcal{X}_n^*$ is *normal* outside the zero locus of $\hat{A}$, because it is regular in codimension one and Cohen–Macaulay (in codimension two).

For any symmetrized monomial $\Pi_{S_n}(\underline{x}^{\underline{r}}\,\underline{y}^{\underline{s}})$ of $\underline{x}^{\underline{r}}\,\underline{y}^{\underline{s}}$ considered earlier, we know that its restriction to $\mathcal{U}$ is equal the pull-back of the restriction to the a regular function on the open subset $\Phi(\mathcal{U})$ of codimension at least 2. By purity (or Hartog's theorem for normal analytic spaces) this regular function on $\Phi(\mathcal{U})$ extends to a regular function $h_{\underline{r},\underline{s}}$ on the complement in $\mathcal{X}_n^*$ of the section $(0:0:1)$ "at infinity". Restricting to the fiber over τ_0 and the Claim follows. We have proved the ϕ is an isomorphism for every elliptic curve of the form $E_\tau = \mathbb{C}/\Lambda_\tau$, for any element τ in the upper-half plane $\mathbb{H}$. □

Remark 7.7.1. There is a variant of the proof following the same idea, but uses Zariski's Main Theorem instead of purity: take the closure $\bar{\mathcal{X}}_n$ of $\mathcal{X}_n$ in the n-th symmetric product of the universal elliptic curve. Apply Zariski's Main Theorem to the map from the normalization of $\bar{\mathcal{X}}_n$ to $\hat{\mathcal{X}}_n^*$. One needs to be careful when it comes to the operation of taking the closure, for in general this operation does not commute with the operation of passing to a fiber. Details are left to the interested readers.

8. Deformations via blow-up sequences

8.1. In this section we will prove Theorem 0.7.5 concerning the *blow-up set* of a *blow-up sequence* u_k to the mean field equation $\triangle u_k + e^{u_k} = \rho_k\delta_0$ with $\rho_k \to 8\pi n$ on $E = \mathbb{C}/\Lambda$. Recall that the assumption that u_k is a blow-up sequence means that the subset $S \subset E$ consisting of all elements $P \in E$ such that $\lim_{k\to\infty} u_k(P) = \infty$ is a non-empty finite subset of E; this subset S is called the blow-up set of the blow-up sequence (u_k).

8.1.1. The following facts are known; see for instance [6, Thm. 3, p. 1237], [41, p. 1256].

(i) $\lim_{k\to\infty} u_k(x) = -\infty$ for all $x \in E\smallsetminus S$, uniformly on compact subsets of $E\smallsetminus S$,

(ii) There exists an $\big(8\pi\mathbb{N}_{\geq 1}\big)$-valued function $P \mapsto \alpha_P$ on S such that the limit $\lim_{k\to\infty} e^{u_k}|_E$ converges to the measure $\sum_{P\in S}\alpha_P\,\delta_P$ on E, where δ_P denotes the delta-measure at P for all $P \in S$.

Note that $\sum_{P\in S}\alpha_P = 8\pi n$ because $\int_E e^{u_k} = \rho_k$ for all k, and this sequence converges to $8\pi n$ by assumption.

Clearly the blow-up set of a blow-up sequence does not change if we pass to a subsequence, therefore we may assume either (1) $\rho_k \neq 8\pi n$ for all k, or (2) $\rho_k = 8\pi n$ for all k. Theorem 0.7.5 asserts that the blow-up set S is an element of Y_n and $\alpha_P = 8\pi$ for all $P \in S$; moreover $S \in X_n$ in case (1) and $S \in Y_n\smallsetminus X_n$ in case (2).

8.1.2. Part (2) of the Theorem 0.7.5, namely the case $\rho_k = 8\pi n$, follows easily from results in §5, 6 and 7. Suppose that n is a positive integer $(u_k)_{k\in\mathbb{N}}$ is a blow-up sequence of solutions of $\triangle u + e^u = 8\pi\delta_0$ on $E = \mathbb{C}/\Lambda$. By Theorems 5.2, 5.6, Proposition 5.8.3 and Theorem 6.5, for each k there exists an element $[a^{(k)}] \in X_n$ and a real number $\lambda_k \in \mathbb{R}$ such that

$$u_k(z) = \log \frac{8\, e^{2\lambda_k}\, |\, f'_{[a^{(k)}]}\, |^2}{(1 + e^{2\lambda_k}|f_{[a^{(k)}]}(z)|^2)^2} \qquad \forall\, k,$$

where $f_{[a^{(k)}]}(z) = w_{[a^{(k)}]}(z)/w_{[-a^{(k)}]}(z)$ is the quotient of the ansatz functions as in Definitions 6.1.4 and 6.1.5. The curve $\bar{X}_n$ being projective, after passing to a subsequence we may and do assume that the sequence $[a^{(k)}] \in X_n$ converges to a point $[x_0] \in \bar{X}_n$. We claim that $[x_0] \in X_n$, i.e. x_0 is not a ramification point of $\bar{\pi}_n : \bar{X}_n \to \mathbb{P}^1(\mathbb{C})$. For otherwise the sequence of functions $f_{[a^{(k)}]}(z)$ converges to the constant function 1, uniformly on compact subsets of E, which implies that u_k cannot be a blow-up sequence, a contradiction which proves the claim.

From the fact that $[x_0] \in X_n$ it follows that the sequence $f_{[a^{(k)}]}(z)$ converges to $f_{[x_0]}$. Therefore the sequence λ_k goes to ∞, and $S = [x_0] \in X_n$. We have proved Theorem 0.7.5 (2). The remaining case (1) when $\rho_k \neq 8\pi n$ for all k will be proved in Theorem 8.4 and Corollary 8.6. $\square$

8.2. The setup. Let n be a positive integer. Write $\rho = 8\pi\eta$, $\eta \in \mathbb{R}^+$. Consider a solution u be of

$$\triangle u + e^u = 8\pi\eta\, \delta_0 \tag{8.2.1}$$

on $E = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where the parameter $\eta \in \mathbb{R}_{>0}$ satisfies $|\eta - n| < \frac{1}{2}$. When the parameter η satisfies $n - \frac{1}{2} < \eta < n + \frac{1}{2}$, the topological Leray–Schauder degree of the equation (8.2.1) is non-zero, hence it has solution. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of solutions of (MFE-eta) with parameters η_k in the above range, and assume that $\lim_{k\to\infty} \eta_k = n$. We are interested in knowing the behavior of this sequence (u_k).

8.2.1. The natural map $\mathbb{C}\smallsetminus\Lambda \to E\smallsetminus\{[0]\}$ is a Galois covering space with group Λ. We know that $\mathbb{C}\smallsetminus\Lambda$ has a universal covering isomorphic to the upper-half plane $\mathbb{H}$. Let $z : \mathbb{H} \to \mathbb{C}\smallsetminus\Lambda$ be a universal covering map; here we have abused the notation and use the same symbol "z" for both the coordinate function on $\mathbb{C}$ and this covering map. Denote by $[z]$ the composition of z with the natural projection map $\mathbb{C}\smallsetminus\Lambda \to E\smallsetminus\{[0]\} =: E^\times$, so that $[z] : \mathbb{H} \to E^\times$ is a universal covering map of $E^\times$.

8.2.2. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be the discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ consisting of all deck transformations of the covering map $[z] : \mathbb{H} \to E^\times$, so that Γ is naturally isomorphic to the fundamental group of $E^\times$ Let $\Delta \subset \Gamma$ be the group of all deck transformations of the covering map $z : \mathbb{H} \to \mathbb{C}\smallsetminus\Lambda$. We know that Δ is a normal subgroup of Γ, and the quotient Γ/Δ is naturally isomorphic to Λ, the Galois group of the Galois cover $\mathbb{C}\smallsetminus\Lambda \to E^\times$, therefore Δ is equal to the subgroup $[\Gamma,\Gamma]$ of Γ generated by all commutators.

The fundamental group of $E\smallsetminus\{[0]\}$ is a free group in two generators, so Γ is a finitely generated Fuchsian subgroup of $\mathrm{PSL}_2(\mathbb{R})$. The Fuchsian subgroup $\Delta \subseteq \Gamma$ is not finitely generated; it is a free group with a set of free generators indexed by Λ.

Let $\mathbb{H}^*$ be the union of $\mathbb{H}$ and the set of all cusps[28] with respect to Γ, with the usual topology as defined in [56, pp. 8–10] which is compatible with the Γ-action. Note that $\mathbb{H}^*$ is contractible.

Lemma 8.2.3. (1) *The map $z : \mathbb{H} \to \mathbb{C}\smallsetminus\Lambda$ extends uniquely to a continuous Δ-invariant map $z^* : \mathbb{H}^* \to \mathbb{C}$, which lifts the continuous Γ-invariant map $[z]^* : \mathbb{H}^* \to E$ extending $[z] : \mathbb{H} \to E^\times$.*

(2) *The maps z^* and $[z]^*$ induce homeomorphisms $\Delta\backslash\mathbb{H}^* \xrightarrow{\sim} \mathbb{C}$ and $\Gamma\backslash\mathbb{H}^* \xrightarrow{\sim} E$ respectively.*

Proof. The existence of the latter map $[z]^* : \mathbb{H}^* \to E$ is well-known, and the existence of the former map $z^* : \mathbb{H}^* \to \mathbb{C}$ follows from the existence of the latter because H^* is contractible. We have proved (1). The statement (2) follows from the statement (1). $\square$

8.2.4. Choose and fix free generators $\tilde\gamma_1, \tilde\gamma_2$ of Γ such that the image of $\tilde\gamma_i$ in $\Gamma/[\Gamma,\Gamma] \cong \Lambda$ is ω_i for $i = 1,2$. Let $\mathfrak{c}_0 \in \mathbb{H}^*$ be the unique cusp such that $z^*(\mathfrak{c}_0) = 0$ and the stabilizer subgroup $\mathrm{Stab}_\Gamma(\mathfrak{c}_0)$ of $\mathfrak{c}_0$ in Γ is equal to the cyclic subgroup generated by the commutator $[\tilde\gamma_1,\tilde\gamma_2] := \tilde\gamma_1\tilde\gamma_2\tilde\gamma_1^{-1}\tilde\gamma_2^{-1}$. It follows that the inverse image $z^*(0)$ of $0 \in \mathbb{C}$ under $z^* : \mathbb{H}^* \to \mathbb{C}$ is equal to $[\Gamma,\Gamma]\cdot\mathfrak{c}_0$.

8.2.5. According to Proposition 1.1.2, for each k there exists a meromorphic function $\mathbf{f}_k(\xi)$ on $\mathbb{H}$ such that

$$u_k \circ z = \log \frac{8|\frac{d}{dz}\mathbf{f}_k|^2}{(1+|\mathbf{f}_k|^2)^2}. \tag{8.2.2}$$

The Schwarzian derivative

$$S(\mathbf{f}_k) = \frac{\frac{d^3}{dz^3}\mathbf{f}_k}{\frac{d}{dz}\mathbf{f}_k} - \frac{3}{2}\left(\frac{\frac{d^2}{dz^2}\mathbf{f}_k}{\frac{d}{dz}\mathbf{f}_k}\right)^2$$

of Φ is equal to

$$S(\mathbf{f}_k) = \frac{d^2u_k}{dz^2}\circ z - \frac{1}{2}\Big(\frac{du_k}{dz}\Big)^2 = -2(\eta_k(\eta_k+1)\wp(z;\Lambda) + B_k)$$

[28] Recall that a *cusp* with respect to Γ is an element of $\mathbb{P}^1(\mathbb{R}) = \mathbb{R}\cup\{\infty\}$ which is fixed by a parabolic element in Γ.

for some constant B_k. Thus $\mathbf{f}_k$ can be written as a ratio of two independent solutions of the Lamé equation

$$\frac{d^2w}{dz^2} = \big(\eta_k(\eta_k+1)\wp(z;\Lambda) + B_k\big)w.$$

8.2.6. Choose and fix a branch of $\log z$ on $\mathbb{H}$, i.e. a holomorphic function on $\mathbb{H}$ whose exponential is equal to the function $z : \mathbb{H} \to \mathbb{C}\smallsetminus\Lambda$; we abuse the notation and denote this function again by $\log z$.

The indicial equation of the Lamé equation above is given by

$$\lambda^2 - \lambda - \eta_k(\eta_k+1) = (\lambda - (\eta_k+1))(\lambda + \eta_k) = 0.$$

The difference $\eta_k + 1 - (-\eta_k) = 2\eta_k + 1$ of the two roots of the above indicial equation is *not* an integer because of the assumption that $|\eta_k - n| < \frac{1}{2}$. Hence there exist two linearly independent solutions $w_{k,1}, w_{k,2}$ on $\mathbb{H}$ which near $\mathfrak{c}_0$ are of the form

$$w_{k,1} = e^{(\eta_k+1)\log z}\cdot(h_{k,1}\circ z), \qquad w_{k,2} = e^{-\eta_k \log z}\cdot(h_{k,2}\circ z),$$

where $h_{k,1}(z)$ and $h_{k,2}(z)$ are holomorphic functions in an open neighborhood of $z=0$ with the property that $h_{k,1}(0) = 1 = h_{k,2}(0)$. The quotient

$$\mathfrak{f}_k := \frac{w_{k,1}}{w_{k,2}} \tag{8.2.3}$$

is a meromorphic function $\mathfrak{f}_k$ on $\mathbb{H}$ such that

$$\mathfrak{f}_k(\mathfrak{c}_0) := \lim_{\xi\to\mathfrak{c}_0} \mathfrak{f}_k(\xi) = 0 \tag{8.2.4}$$

and

$$\mathfrak{f}_k(\tilde{\gamma}_2^{-1}\tilde{\gamma}_1^{-1}\tilde{\gamma}_2\tilde{\gamma}_1\cdot\xi) = e^{4\pi\sqrt{-1}\eta_k}\mathfrak{f}_k(\xi) \qquad \forall\xi\in\mathbb{H}. \tag{8.2.5}$$

Lemma 8.2.7. *Let $T_k \in \mathrm{PSL}_2(\mathbb{C})$ be the linear fractional transformation such that $\mathbf{f}_k = T_k\cdot\mathfrak{f}_k$. The limit*

$$\lim_{\xi\to\mathfrak{c}_0} \mathbf{f}_k(\xi) =: \mathbf{f}_k(\mathfrak{c}_0)$$

exists in $\mathbb{P}^1(\mathbb{C})$ and is equal to $T_k\cdot 0$, the image of $0\in\mathbb{P}^1(\mathbb{C})$ under the linear fractional transformation T_k.

Lemma 8.2.8. (a) *If $\mathbf{f}_k(0) = 0$, then there exists a constant $A_k \in \mathbb{C}^\times$ such that $\mathbf{f}_k = A_k\cdot\mathfrak{f}_k$.*
(b) *If $\mathbf{f}_k(0) = \infty$, then there exists a constant $A_k\in\mathbb{C}^\times$ such that then $\mathbf{f}_k = A_k/\mathfrak{f}_k$.*

Proof. Let T_k be the linear fractional transformation such that $\mathbf{f}_k = T_k\cdot\mathfrak{f}_k$ as in the proof of Lemma 8.2.7.

If $\mathbf{f}_k(\mathfrak{c}_0) = 0 = \mathfrak{f}_k(0)$, then there exists a unique element $A_k\in\mathbb{C}^\times$ such that T_k is the image of $\begin{pmatrix} A_k & 0 \\ 0 & 1\end{pmatrix}$; the statement (a) follows. If $\mathbf{f}_k(\mathfrak{c}_0) = \infty$, then there exist a unique element $A_k\in\mathbb{C}^\times$ such that T_k is the image of $\begin{pmatrix} 0 & A_k \\ 1 & 0\end{pmatrix}$. We have proved (b). $\square$

8.3. Normalizing $\mathbf{f}_k$'s through monodromy.

8.3.1. Let $\rho_{\mathbf{f}_k} : \Gamma \to \mathrm{PSU}(2)$ be the monodromy representation attached to the developing map $\mathbf{f}_k$ of the solution u_k, defined by

$$\mathbf{f}_k(\gamma \cdot \xi) = \rho_{\mathbf{f}_k}(\gamma) \cdot \mathbf{f}_k(\xi) \qquad \forall \gamma \in \Gamma,\ \forall \xi \in \mathbb{H}.$$

Note that

$$\rho_{\mathbf{f}_k}(\gamma_1 \cdot \gamma_2) = \rho_{\mathbf{f}_k}(\gamma_1) \cdot \rho_{\mathbf{f}_k}(\gamma_2) \qquad \forall\, \gamma_1, \gamma_2 \in \Gamma.$$

Let $S_{k,i} = \rho_{\mathbf{f}_k}(\tilde{\gamma}_i) \in \mathrm{PSU}(2)$ for $i = 1, 2$. Then we have

$$\mathbf{f}_k(\tilde{\gamma}_i \cdot \xi) = S_{k,i} \cdot \mathbf{f}_k(\xi) \qquad \text{for } i = 1, 2,\ \forall\, \xi \in \mathbb{H}. \tag{8.3.1}$$

Let

$$\beta_k := S_{k,2} \cdot S_{k,1} \cdot S_{k,2}^{-1} \cdot S_{k,1}^{-1} = \rho_{\mathbf{f}_k}([\tilde{\gamma}_2, \tilde{\gamma}_1]) \in \mathrm{PSU}(2). \tag{8.3.2}$$

8.3.2. So far we have not imposed any restriction on the developing map $\mathbf{f}_k$ of the solution u_k of $\triangle u + e^u = \rho_k \cdot \delta_0$. Modifying $\mathbf{f}_k$ by a suitable element of $\mathrm{PSU}(2)$, we may and do assume that $S_{k,1} = \rho_{\mathbf{f}_k}(\tilde{\gamma}_1)$ lies in the diagonal maximal torus of $\mathrm{PSU}(2)$, i.e. there exists $\theta_k \in \mathbb{R}/2\pi\sqrt{-1}\mathbb{Z}$ and $a_k, b_k \in \mathbb{C}$ with $|a_k|^2 + |b_k|^2 = 1$ such that

$$S_{k,1} = \begin{pmatrix} e^{\sqrt{-1}\cdot\theta_k} & 0 \\ 0 & e^{-\sqrt{-1}\cdot\theta_k} \end{pmatrix} \quad \text{and} \quad S_{k,2} = \begin{pmatrix} a_k & -b_k \\ \overline{b_k} & \overline{a_k} \end{pmatrix}, \tag{8.3.3}$$

in matricial notation. Note that we have

$$\beta_k := [S_{k,2}, S_{k,1}] = \begin{pmatrix} a_k\overline{a_k} + e^{-2\sqrt{-1}\theta_k} b_k\overline{b_k} & a_k b_k(-1 + e^{2\sqrt{-1}\theta_k}) \\ \overline{a_k}\,\overline{b_k}(1 - e^{-2\sqrt{-1}\theta_k}) & b_k\overline{b_k} e^{2\sqrt{-1}\theta_k} + a_k\overline{a_k} \end{pmatrix} \tag{8.3.4}$$

in $\mathrm{PSU}(2)$.

Lemma 8.3.3. *Recall that we have assumed that $S_{k,1}\mathbf{f}_k = e^{2i\theta_k}\mathbf{f}_k$, $n - \frac{1}{2} < \eta_k := \rho_k/8\pi < n + \frac{1}{2}$ and $\eta_k \neq n$. Suppose in addition that $n - \frac{1}{4} < \eta_k < n + \frac{1}{4}$, then $\mathbf{f}_k(\mathfrak{c}_0) \in \mathbb{C}^\times$.*

Proof. We need to show that $\mathbf{f}_k(\mathfrak{c}_0)$ is not equal to 0 nor to ∞. Suppose first that $\mathbf{f}_k(\mathfrak{c}_0) = 0$. By Lemma 8.2.8 (a) there exists a constant $A_k \in \mathbb{C}^\times$ such that $\mathbf{f}_k = A_k \cdot \mathfrak{f}_k$. The monodromy relation (8.2.5) for $\mathfrak{f}_k$ implies that

$$[S_{k,2}, S_{k,1}] = \rho_{\mathbf{f}_k}([\tilde{\gamma}_2, \tilde{\gamma}_1]) = \begin{pmatrix} e^{2\pi\sqrt{-1}\eta_k} & 0 \\ 0 & e^{2\pi\sqrt{-1}\eta_k} \end{pmatrix}$$

in $\mathrm{PSU}(2)$. Comparing with (8.3.4), we get $a_k \cdot b_k \cdot (e^{2\sqrt{-1}\theta_k} - 1) = 0$. But we know that $b_k \neq 0$ and $e^{2\sqrt{-1}\theta_k} \neq 1$, for otherwise $S_{k,1}$ would commute with $S_{k,2}$, contradicting the assumption on η_k. We conclude that $a_k = 0$. In other words becomes

$$\mathbf{f}_k(\tilde{\gamma}_1\xi) = e^{2i\theta_k}\mathbf{f}_k(\xi), \quad \mathbf{f}_k(\tilde{\gamma}_2\xi) = -\frac{b_k^2}{\mathbf{f}_k(\xi)} \quad \forall\, \xi \in \mathbb{H}. \tag{8.3.5}$$

Therefore the logarithmic derivative

$$\mathbf{g}_k := \frac{d}{dz}(\log \mathbf{f}_k) = \frac{\frac{d\mathbf{f}_k}{dz}}{\mathbf{f}_k}$$

of $\mathbf{f}_k$ descends to a meromorphic function g_k on the elliptic curve $E' := \mathbb{C}/\Lambda'$, where $\Lambda' = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$. Moreover we know that g_k has a simple pole at $0 \bmod \Lambda'$ with residue $2\eta_k + 1$, and the equation (8.3.5) tells us that g_k has a simple at $\omega_2 \bmod \Lambda'$ as well. On the other hand because $\mathbf{f}_k$ is a developing map of a solution u_k to the equation $\triangle u + e^u = 8\pi\eta_k\,\delta_0$, the meromorphic function $\mathbf{f}_k$ has multiplicity 1 at all points above $[0] = 0 \bmod \Lambda$. So the meromorphic function g_k on E' has two simple poles but no zero, a contradiction. We have proved that $\mathbf{f}_k(\mathfrak{c}_0) \neq 0$.

Suppose that $\mathbf{f}_k(\mathfrak{c}_0) = \infty$. By Lemma (8.2.8) (b) there exists a constant $A_k \in \mathbb{C}^\times$ such that $\mathbf{f}_k = A_k/\mathfrak{f}_k$. The same argument as in the previous case shows that the logarithmic derivative of $\mathbf{f}_k$ descends to a meromorphic function on $\mathbb{C}/\Lambda'$ which has at least two simple poles but no zero, a contradiction again. □

Lemma 8.3.4. *Notation and assumptions as in* 8.3.2 *and Lemma* 8.3.3. *Let* $\gamma_k \in \mathrm{PSU}(2)$ *be the element*

$$\gamma_k := \begin{pmatrix} e^{2\pi\sqrt{-1}\eta_k} & 0 \\ 0 & e^{2\pi\sqrt{-1}\eta_k} \end{pmatrix} \tag{8.3.6}$$

in $\mathrm{PSU}(2)$. *Let* p_k, q_k *be elements of* $\mathbb{C}^\times$ *such that*

$$|p_k|^2 + |q_k|^2 = 1 \quad \text{and} \quad \mathbf{f}_k(\mathfrak{c}_0) = \frac{q_k}{p_k}.$$

Let

$$T_k := \begin{pmatrix} p_k & -q_k \\ \overline{q_k} & \overline{p_k} \end{pmatrix} \tag{8.3.7}$$

(1) *The equality*

$$[S_{k,2}, S_{k,1}] = T_k^{-1} \cdot \gamma_k \cdot T_k \tag{8.3.8}$$

holds in $\mathrm{PSU}(2)$.

(2) *Explicitly, the equality in* (1) *means that either*

$$\begin{aligned} [1 - (|p_k|^2 e^{-2\sqrt{-1}\eta_k} + |q_k|^2 e^{2\sqrt{-1}\eta_k})]a_k &= \overline{p_k} q_k (e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k})\overline{b_k} \\ [e^{2\sqrt{-1}\theta_k} - (|p_k|^2 e^{-2\sqrt{-1}\eta_k} + |q_k|^2 e^{2\sqrt{-1}\eta_k})]b_k &= -\overline{p_k} q_k (e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k})\overline{a_k} \end{aligned} \tag{8.3.9}$$

or

(8.3.10)

$$\begin{aligned} [1 + (|p_k|^2 e^{-2\sqrt{-1}\eta_k} + |q_k|^2 e^{2\sqrt{-1}\eta_k})]a_k &= -\overline{p_k} q_k (e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k})\overline{b_k} \\ [e^{2\sqrt{-1}\theta_k} + (|p_k|^2 e^{-2\sqrt{-1}\eta_k} + |q_k|^2 e^{2\sqrt{-1}\eta_k})]b_k &= \overline{p_k} q_k (e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k})\overline{a_k} \end{aligned}$$

In both cases we have $a_k \cdot b_k \neq 0$, *for all* k.

(3) *If* (8.3.9) *holds, then*

$$|b_k|^2 = \frac{|(1-|p_k|^2 e^{-2\sqrt{-1}\eta_k} - |q_k|^2 e^{2\sqrt{-1}\eta_k})|^2}{|p_k|^2|q_k|^2\,|(e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k})|^2 + |(1-|p_k|^2 e^{-2\sqrt{-1}\eta_k} - |q_k|^2 e^{2\sqrt{-1}\eta_k})|^2}\,,$$

$$|a_k|^2 = \frac{|p_k|^2|q_k|^2\,|(e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k})|^2}{|p_k|^2|q_k|^2\,|(e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k})|^2 + |(1-|p_k|^2 e^{-2\sqrt{-1}\eta_k} - |q_k|^2 e^{2\sqrt{-1}\eta_k})|^2}\,,$$

and

$$e^{2\sqrt{-1}\theta_k} = -\frac{1-|p_k|^2 e^{-2\sqrt{-1}\eta_k} - |q_k|^2 e^{2\sqrt{-1}\eta_k}}{1-|p_k|^2 e^{2\sqrt{-1}\eta_k} - |q_k|^2 e^{-2\sqrt{-1}\eta_k}}\,,$$

so $|a_k|^2, |b_k|^2$ *and* $e^{2\sqrt{-1}\theta_k}$ *are all determined by* $e^{2\sqrt{-1}\eta_k}$ *and* $\mathbf{f}_k(\mathfrak{c}_0)$. *In addition* $a_k \cdot b_k$ *is also determined by* $e^{2\sqrt{-1}\eta_k}$ *and* $\mathbf{f}_k(\mathfrak{c}_0)$.

(4) *If* (8.3.10) *holds, then*

$$|b_k|^2 = \frac{|(1+|p_k|^2 e^{-2\sqrt{-1}\eta_k} + |q_k|^2 e^{2\sqrt{-1}\eta_k})|^2}{|p_k|^2|q_k|^2\,|(e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k})|^2 + |(1+|p_k|^2 e^{-2\sqrt{-1}\eta_k} + |q_k|^2 e^{2\sqrt{-1}\eta_k})|^2}\,,$$

$$|a_k|^2 = \frac{|p_k|^2|q_k|^2\,|(e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k})|^2}{|p_k|^2|q_k|^2\,|(e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k})|^2 + |(1+|p_k|^2 e^{-2\sqrt{-1}\eta_k} + |q_k|^2 e^{2\sqrt{-1}\eta_k})|^2}\,,$$

and

$$e^{2\sqrt{-1}\theta_k} = -\frac{1+|p_k|^2 e^{-2\sqrt{-1}\eta_k} + |q_k|^2 e^{2\sqrt{-1}\eta_k}}{1+|p_k|^2 e^{2\sqrt{-1}\eta_k} + |q_k|^2 e^{-2\sqrt{-1}\eta_k}}\,,$$

so $|a_k|^2, |b_k|^2$ *and* $e^{2\sqrt{-1}\theta_k}$ *are all determined by* $e^{2\sqrt{-1}\eta_k}$ *and* $\mathbf{f}_k(\mathfrak{c}_0)$. *In addition* $a_k \cdot b_k$ *is also determined by* $e^{2\sqrt{-1}\eta_k}$ *and* $\mathbf{f}_k(\mathfrak{c}_0)$.

Proof. (1) Clearly $\lim_{\xi\to\mathfrak{c}_0}(T_k\cdot\mathbf{f}_k)(\xi) = 0$. By Lemma 8.2.8 there exists a constant $c_k \in \mathbb{C}^\times$ such that $T_k \cdot \mathbf{f}_k = c_k\,\mathfrak{f}_k$. We have

$$T_k[S_{k,2}, S_{k,1}]\mathbf{f}_k(\xi) = T_k\mathbf{f}_k([\tilde\gamma_2,\tilde\gamma_1]\,\xi) = c_k\,\mathfrak{f}_k([\tilde\gamma_2,\tilde\gamma_1]\,\xi) = c_k\,\gamma_k\cdot\mathfrak{f}_k(\xi)$$

for all $\xi \in \mathbb{H}$, therefore

$$T_k[S_{k,2}, S_{k,1}]\mathbf{f}_k = c_k\,\gamma_k\cdot\mathfrak{f}_k = \gamma_k\cdot(c_k\mathfrak{f}_k) = \gamma_k\cdot T_k\cdot\mathbf{f}_k.$$

So $T_k[S_{k,2}, S_{k,1}] = \gamma_k\cdot T_k$ in $\mathrm{PSU}(2)$. We have proved the statement (1).

(2) The equality $[S_{k,2}, S_{k,1}] = T_k^{-1}\cdot\gamma_k\cdot T_k$ in $\mathrm{PSU}(2)$ is equivalent to the equality

$$S_{k,1}\cdot S_{k,2}\cdot S_{k,1}^{-1} = \pm S_{k,2}^{-1}\cdot T_k^{-1}\cdot\gamma_k\cdot T_k$$

when both sides are regarded as elements of $\mathrm{SU}(2)$ with $S_{k,1}, S_{k,2}$ given by (8.3.3) and T_k given by (8.3.7). A straightforward calculation shows that

$$T_k^{-1}\cdot\gamma_k\cdot T_k = \begin{pmatrix} |p_k|^2\,e^{2\sqrt{-1}\eta_k} + |q_k|^2\,e^{-2\sqrt{-1}\eta_k} & -\overline{p_k}\,q_k(e^{2\sqrt{-1}\eta_k} - e^{-2\sqrt{-1}\eta_k}) \\ p_k\overline{q_k}(e^{-2\sqrt{-1}\eta_k} - e^{2\sqrt{-1}\eta_k}) & |p_k|^2 e^{-2\sqrt{-1}\eta_k} + |q_k|^2 e^{2\sqrt{-1}\eta_k} \end{pmatrix}$$

and

$$S_{k,1}\cdot S_{k,2}\cdot S_{k,1}^{-1} = \begin{pmatrix} \overline{a_k} & b_k\,e^{2\sqrt{-1}\theta_k} \\ -\overline{b_k}\,e^{-2\sqrt{-1}\theta_k} & a_k \end{pmatrix}$$

The statement (2) follows: the equation (8.3.9) corresponds to the case $S_{k,1} \cdot S_{k,2} \cdot S_{k,1}^{-1} = S_{k,2}^{-1} \cdot T_k^{-1} \cdot \gamma_k \cdot T_k$, while the equation (8.3.10) corresponds to the case $S_{k,1} \cdot S_{k,2} \cdot S_{k,1}^{-1} = -S_{k,2}^{-1} \cdot T_k^{-1} \cdot \gamma_k \cdot T_k$.

The formulas in (3) and (4) for $|a_k|$ and $|b_k|$ follow from (8.3.9) and (8.3.10) respectively, through routine calculations which are omitted here. Notice that these formulas imply that $\cdot b_k \neq 0$. Recall that we already know that $a_k \neq 0$, for otherwise $S_{k,2}$ would commute with $S_{k,1}$. We also know that $p_k \cdot q_k \neq 0$, for $\mathbf{f}_k(\mathfrak{c}_0) \neq 0, \infty$. The assumptions on η_k implies that $\left||p_k|^2 e^{-2\sqrt{-1}\eta_k} + |q_k|^2 e^{2\sqrt{-1}\eta_k}\right| < 1$, so we know from (8.3.9) and (8.3.10) that $a_k \neq 0$ in both cases.

The formulas in (3) and (4) for $e^{2\sqrt{-1}\theta_k}$ follows from (8.3.9) and (8.3.10) respectively. For instance if we multiply the second equation in (8.3.9) by the complex conjugate of the first equation in (8.3.9), cancel out the non-zero factor $\overline{a_k} b_k$ on both sides, then we get the formula for $e^{2\sqrt{-1}\theta_k}$ in (3).

These formulas clearly show that in both cases $|a_k|, |b_k|$ and $e^{2\pi\sqrt{-1}\theta_k}$ are determined by $e^{2\pi\sqrt{-1}\eta_k}$ and $\mathfrak{f}_k(\mathfrak{c}_0)$ and not on the choices of p_k and q_k. That $a_k \cdot b_k$ is also determined by $e^{2\pi\eta_k}$ and $\mathfrak{f}_k(\mathfrak{c}_0)$ in each of the two cases follows quickly from (8.3.9) and (8.3.10). $\square$

We are ready to prove the remaining case (1) of Theorem 0.7.5. Let's recall the situation. We are given a $(u_k)_k$ is a blow-up sequence of solutions such that $\triangle u_k + e^u = 8\pi\eta_k \delta_0$ on $E = \mathbb{C}/\Lambda$ for each k, $\lim_{k\to\infty} \eta_k = n$ with $\{P_1, \ldots, P_m\}$ as the blow-up set of this blow-up sequence. We may and do assume that $n - \frac{1}{4} < \eta_k < n + \frac{1}{4}$ and $\eta_k \neq n$ for all k. We know from [12] that $m = n$, $P_i \neq [0]$ for each $i = 1$, and $\lim_{k\to\infty} u_k(P) \to -\infty$ $P \in E \smallsetminus \{P_1, \ldots, P_n\}$, uniformly on compact subsets of $\smallsetminus\{P_1, \ldots, P_n\}$.

Theorem 8.4. *There is a constant $A \in \mathbb{C}^\times$ such that* $\lim_{k\to\infty} \mathbf{f}_k(\xi) \to A$ *uniformly on compact subsets of the inverse image $\mathbb{H} \smallsetminus [z]^{-1}(\{[0], P_1, \ldots, P_n\})$ of $E \smallsetminus \{[0], P_1, \ldots, P_n\}$. Furthermore we have $\{P_1, \ldots, P_n\} = \{-P_1, \ldots, -P_n\}$.*

8.5. Proof of Theorem 8.4.

8.5.1. We first note that there exists a constant $B \in \mathbb{C}$ such that the Lamé equations

$$w'' = \big(\eta_k(\eta_k+1)\wp + B_k\big)w \tag{8.5.1}$$

converge to

$$w'' = \big(n(n+1)\wp + B_k\big)w \tag{8.5.2}$$

because $\lim_{k\to\infty} \eta_k = n$ and $\lim_{k\to n} B_k = B$. This is a consequence of the fact that $u_{k,z}$ and $u_{k,zz}$ converge uniformly on compact sets in $E \smallsetminus \{p_1, \ldots, p_n\}$.

8.5.2. For each k choose a normalized developing map $\mathbf{f}_k$ of u_k as in 8.3.2, and choose $p_k, q_k \in \mathbb{C}^\times$ such that $|p_k|^2 + |q_k|^2 = 1$ and $\mathbf{f}_k(\mathfrak{c}_0) = q_k/p_k$. By Lemma 8.2.8 there exists for each k a constant $c_k \in \mathbb{C}^\times$ such that

$$\frac{p_k \mathbf{f}_k - q_k}{\bar{q}_k \mathbf{f}_k + \bar{p}_k} = c_k \mathfrak{f}_k,$$

where $\mathfrak{f}_k = w_{k,1}/w_{k,2}$ is a solution of the Lamé equation (8.5.1) on the universal covering $\mathbb{H}$ of $E \smallsetminus \{[0]\}$ defined in 8.2.6. Equivalently,

$$\mathbf{f}_k = \frac{\overline{p_k} c_k \mathfrak{f}_k + q_k}{-\overline{q_k} c_k \mathfrak{f}_k + p_k}. \tag{8.5.3}$$

The convergence of the Lamé equations (8.5.1) implies that after passing to a subsequence if necessary, there exist solutions w_1, w_2 of (8.5.2) on $\mathbb{H}$ such that $\lim_{k\to\infty} w_{k,i} = w_i$ for $i = 1, 2$ and $\lim_{k\to\infty} \mathfrak{f}_k \to \mathfrak{f} := w_1/w_2$, uniformly on compact set in E away from the discrete set of poles and zeros of w_1 and w_2. Clearly

$$w_i'' = (n(n+1)\wp + B) w_i,$$

and locally near $\mathfrak{c}_0$ the w_i's can be written in the form

$$w_1(\xi) = e^{(n+1)\log z} \cdot (h_1 \circ z), \quad w_2(\xi) = e^{-n \log z} \cdot (h_2 \cdot \circ z)$$

where h_1, h_2 holomorphic functions in a neighborhood of $\mathfrak{c}_o$ such that

$$\lim_{\xi \to \mathfrak{c}_0} h_i(\xi) = 1 \qquad \text{for } i = 1, 2.$$

Most of our analysis will be based on the limiting behavior of (8.5.3) as $k \to \infty$.

Again passing to a subsequence if necessary, we may and do assume that there exist $p, q \in \mathbb{C}$ with $|p|^2 + |q|^2 = 1$ such that $p_k \to p$ and $q_k \to q$ as $k \to \infty$. Similarly we may and do assume that there exist $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$ such that $a_k \to a$ and $b_k \to b$ as $k \to \infty$. Let $A := q/p \in \mathbb{P}^1(\mathbb{C})$. Clearly

$$\lim_{k\to\infty} \mathbf{f}_k(0) = \lim_{k\to\infty} q_k/p_k = A.$$

We may and do assume moreover that the limit $\lim_{k\to\infty}$ exists in $\mathbb{P}^1(\mathbb{C})$; let

$$c := \lim_{k\to\infty} c_k.$$

8.5.3. Our first claim is that c is either 0 or ∞. Suppose that $c \in \mathbb{C}^\times$. Let

$$\mathbf{f} := \frac{\bar{p} c \mathfrak{f} + q}{-\bar{q} c \mathfrak{f} + p}.$$

Then $\mathbf{f}_k(\xi) \longrightarrow \mathbf{f}(\xi)$ for all ξ outside some discrete subset Σ of $\mathbb{H}$. Hence

$$u_k \circ z \to u \circ z = \log \frac{8|f'(z)|^2}{(1 + |f(z)|^2)^2}$$

uniformly on compact sets outside some discrete set of $\mathbb{H}$. This contradicts to our assumption that u_k blows up as $k \to \infty$, We have proved that either $c = 0$ or $c = \infty$.

8.5.4. Next we claim that

$$A \in \mathbb{C}^\times \qquad \text{and} \qquad c = 0.$$

This claim and the equation (8.5.3) imply that $\lim_{n\to\infty} \mathbf{f}_k(\xi) = A$ for all ξ outside of some discrete subset of $\mathbb{H}$, and the first statement of Theorem 8.4 follows.

8.5.5. We will show that $A \neq \infty$. Suppose to the contrary that $A = \infty$, i.e. $p = 0$ and $|q| = 1$.

Our first step is to show that $a := \lim_{k\to\infty} a_k = 0$. Write $\alpha_k := e^{2\pi\sqrt{-1}\eta_k}$; clearly $\lim_{k\to\infty} \alpha_k = 1$. Passing to a subsequence if necessary, we may and do assume that either (8.3.9) holds for all k or (8.3.10) holds for all k. In the case when (8.3.10) holds for all k, taking the limit of both sides of the first equation in (8.3.10), we see immediately that $\lim_{k\to\infty} a_k = 0$. In the case when (8.3.9) holds for all k, substitute $|p_k|^2$ by $1 - |q_k|^2$ in the first equation of (8.3.9), then divide both sides by $\alpha_k - \alpha_k^{-1}$, we get

$$\left(\frac{1}{\alpha_k + 1} + |q_k|^2\right) a_k = \overline{p_k}\, q_k\, \overline{b_k}.$$

Taking the limit of the above equality as $k \to \infty$, again we conclude that $a = \lim_{k\to\infty} a_k = 0$, so $|b| = 1$.

We will analyse the two possibilities of c separately and show that both lead to contradictions. Suppose first that $c = 0$. From (8.5.3) it is clear that

$$\mathfrak{f}(\xi) \neq \infty \quad \Longrightarrow \quad |\mathbf{f}_k(\xi)| \longrightarrow \infty.$$

However, if we select ξ so that $\mathfrak{f}(\xi) \neq \infty$ and $\mathfrak{f}(g_2\xi) \neq \infty$ (such ξ certainly exists), then

$$\mathbf{f}_k(\tilde{\gamma}_2\xi) = S_{k,2}\mathbf{f}_k(\xi) = \frac{a_k\mathbf{f}_k(\xi) - b_k}{\overline{b_k}\mathbf{f}_k(\xi) + \overline{a_k}} \longrightarrow 0, \tag{8.5.4}$$

which contradicts the previous conclusion that $|\mathbf{f}_k(\tilde{\gamma}_2\xi)| \to \infty$. So $c \neq 0$.

Suppose next that $c = \infty$. Again by (8.5.3) we have

$$\mathfrak{f}(\xi) \neq 0 \quad \Longrightarrow \quad \mathbf{f}_k(\xi) \longrightarrow 0,$$

and convergence is uniform on compact sets outside the zeros of $\mathfrak{f}$. But then $\mathbf{f}_k(\tilde{\gamma}_2\xi) \to \infty$, which contradicts $\mathbf{f}_k(\tilde{\gamma}_2\xi) \to 0$ provided that $\mathfrak{f}(\tilde{\gamma}_2\xi) \neq 0$. We have shown that the assumption $A = \infty$ leads to assumption for both possible values of c. We have proved that $A \neq \infty$, i.e. $A \in \mathbb{C}$.

8.5.6. Now we know that $\mathbf{f}_k(0) \to A$ with $A \in \mathbb{C}$. Substituting $\mathfrak{f}_k \circ z = z^{2\eta_k+1} + O(|z|^{2\eta_k+2})$ near $\mathfrak{c}_0$ into (8.5.3), we get

$$\mathbf{f}_k \circ z = \frac{q_k}{p_k} + \frac{c_k}{p_k^2} z^{2\eta_k+1}(1 + O(|z|)).$$

On the other hand we know from general facts about blow-up sequences that the regular part of $u_k(z)$ at $z = 0$ tends to $-\infty$, i.e.

$$\begin{aligned} &\left(u_k \circ z - 4\eta_k \log|z|\right)\Big|_{z=0} \\ &= \log \frac{8|\frac{d}{dz}\mathbf{f}_k|^2|z|^{-4\eta_k}}{(1+|\mathbf{f}_k \circ z|^2)^2}\Big|_{z=0} = \log \frac{8|c_k|^2(2\eta_k+1)^2}{|p_k|^2(1+|\mathbf{f}_k(0)|^2)^2} \longrightarrow -\infty, \end{aligned}$$

which implies that $c_k \to c = 0$.

8.5.7. We still need to exclude the possibility that $A = 0$. Suppose to the contrary that $A = 0$, or equivalently $|p| = 1$ and $q = 0$. The same argument used at the beginning of 8.5.5 shows that $a = 0$ and $|b| = 1$. Since $c = 0$, by (8.5.3),

$$\mathfrak{f}(\xi) \neq \infty \quad \Longrightarrow \quad \mathbf{f}_k(\xi) \longrightarrow 0.$$

Then by the expression (8.5.4) we have $\mathbf{f}_k(\gamma_2\xi) \to \infty$ whenever $\mathfrak{f}(\xi) \neq \infty$, hence $\mathfrak{f}(\gamma_2\xi) = \infty$ by the above implication. But as before we may select ξ so that both $\mathfrak{f}(\xi) \neq \infty$ and $\mathfrak{f}(\gamma_2\xi) \neq \infty$. This is a contradiction. We have proved that $A \in \mathbb{C}^\times$.

8.5.8. It remains to show that the blow-up set is stable under $[-1]_E$, i.e. $\{P_1, \ldots, P_n\} = \{-P_1, \ldots, -P_n\}$. From (3.1.3), we see that

$$S(\mathbf{f}_k(-z)) = S(\mathbf{f}_k)(-z) = -2(\eta(\eta+1)\wp(-z) + B_k) = S(\mathbf{f}_k(z)),$$

hence $\mathbf{f}_k(-z)$ is also a developing map for u_k. Since $\mathbf{f}_k(z)$ is normalized in the sense of 8.3.2, $\mathbf{f}_k(-z) = \mathbf{f}_k \circ (-z)$ is also normalized, so our results applies to $\mathbf{f}_k(-z)$ as well. We conclude that $f_k(-z) \to A$ outside $\{P_1, \ldots, P_n\}$. This implies that this set coincides with $\{-P_1, \ldots, -P_n\}$. We have prove Theorem 8.4. $\square$

Corollary 8.6. *Let u_k be a blow-up sequence of solutions to* (0.1.3) *with $\rho_k = 8\pi\eta_k \to 8\pi n$, $\eta_k \neq n$ for all k. Let $S = \{P_1, \ldots, P_n\}$ be the blow-up set. Then S is a finite branch point of the hyperelliptic curve $C^2 = \ell_n(B)$.*

Proof. The blow-up set S satisfies equations (5.7.2), or equivalently (5.7.4), thus $S \in Y_n$. Now Theorem 8.4 implies that $\{P_i\}_{i=1}^n = \{-P_i\}_{i=1}^n$, hence $S \in Y_n \smallsetminus X_n$, i.e. it is a branch point of $\bar{\pi} : \bar{X}_n \to \mathbb{P}^1(\mathbb{C})$. $\square$

8.7. Further remarks.

8.7.1. Consider the regular part of the Green function

$$\tilde{G}(z,q) := G(z-q) + \frac{1}{2\pi}\log|z-q|.$$

Let $S = \{P_1, \ldots, P_n\} \subset E$ be a set of n distinct points on $E = \mathbb{C}/\Lambda$; pick representatives $p_1, \ldots, p_n \in \mathbb{C}$ of $P_1, \ldots, P_n$. For $i = 1, \ldots, n$ we define

$$\begin{aligned} f_{p_i}(z) &:= 8\pi(\tilde{G}(z,p_i) - \tilde{G}(p_i,p_i)) \\ &\qquad + \sum_{j\neq i}(G(z-p_j) - G(p_i - p_j)) - 8\pi n(G(z) - G(p_i)), \\ \mu_i &:= \exp(8\pi(\tilde{G}(p_i,p_i) + \sum_{j\neq i} G(p_j - p_i)) - 8\pi n G(p_i)). \end{aligned}$$

Then we have the important global quantity associated to the set S:

$$D(S) := \lim_{r\to 0}\sum_{i=1}^{n}\mu_i\left(\int_{\Omega_i\setminus B_r(p_i)}\frac{e^{f_{p_i}(z)}-1}{|z-p_i|^4} - \int_{\mathbb{R}^2\setminus\Omega_i}\frac{1}{|z-p_i|^4}\right),$$

where Ω_i is any open neighborhood of p_i in E such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n}\overline{\Omega_i} = E$.

Under the hypothesis of Corollary 8.6, it was shown in [16] for $n = 1$ and in [45] for general $n \in \mathbb{N}$, but in a slightly different context, that

$$\rho_k - 8\pi n = (D(S) + o(1))\exp(-\max_T u_k).$$

In general it is difficult to compute $D(S)$ even for $n = 1$. In the case $n = 1$, the hyperelliptic curve is the torus E and the branch points consist of the three half-periods. In the very special case that T is a rectangular torus, the sign of $D(\frac{1}{2}\omega_i)$ has been calculated: $D(\frac{1}{2}\omega_3) < 0$ and $D(\frac{1}{2}\omega_i) > 0$ for $i = 1, 2$. Furthermore $D(\frac{1}{2}\omega_i) < 0$ if and only if $\frac{1}{2}\omega_i$ is a minimal point [16].

It is clear that when $D(S) \neq 0$, its sign provides important information when we study bubbling solutions (blow-up sequence) with $\rho \neq 8\pi n$ (e.g. if $D(S) > 0$ then the bubbling may only occur from the right hand side). Also in the case $\rho_k = 8\pi n$ for all k, if a blow-up family u_λ exists then $D(S) = 0$ trivially for S begin the blow-up set.

In particular, we pose the following

Conjecture 8.7.2. *For rectangular tori, there are n branch points on the associated hyperelliptic curve with $D(S) < 0$, and $n+1$ branch points with $D(S) > 0$.*

Conjecture 8.7.2 is known only when $n = 1$ as mentioned above.

8.7.3. Theorem 8.4 provides a connection between singular Liouville equations and the branch points of the associated hyperelliptic curve. We expect the phenomenon to hold true for other related equations. For example we might ask the following question on Chern–Simons–Higgs equation:

Suppose that u_ϵ is a sequence of bubbling solutions of the Chern–Simons–Higgs equation

$$\triangle u_\epsilon + \frac{1}{\epsilon^2} e^{u_\epsilon}(1 - e^{u_\epsilon}) = 8\pi n \delta_0 \quad \textit{in } E.$$

Is the bubbling set $\{p_1, \ldots, p_n\}$, as $\epsilon \to 0$, a branch point of the hyper elliptic curve $C^2 = \ell_n(B)$?

This has recently been answered affirmatively for $n = 1$ and for E a rectangular torus [45, 46].

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