TATE-LINEAR FORMAL VARIETIES

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Contents

Abstract. A Tate-linear structure on a smooth noetherian local formal scheme T over a field κ of characteristic p is an isomorphism $T \xrightarrow{\sim} \mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ of sheaves on the fpqc site of $\mathrm{Spec}(\kappa)$, where $\mathbf{N}$ is an fpqc sheaf of torsion free nilpotent on $\mathrm{Spec}(\kappa)$ which admits a central series $\mathbf{N} = \mathbf{N}_1 \supsetneqq \mathbf{N}_2 \supsetneqq \cdots \supsetneqq \mathbf{N}_{c+1} = (1)$ such that each subquotient $\mathbf{N}_i/\mathbf{N}_{i+1}$ is the Tate $\mathbb{Z}_p$-module attached to a p-divisible group over κ, and $\mathbf{N}_{\mathbb{Q}}$ is the Mal'cev completion of $\mathbf{N}$. A smooth formal scheme over κ with a Tate-linear structure is called a Tate-linear formal variety over κ. Examples of Tate-linear formal varieties include p-divisible formal groups, biextensions of p-divisible formal groups, and formal completions at closed points of central leaves in Siegel modular varieties in characteristic p. Tate-linear structures have a remarkable rigidity property: if a reduced irreducible closed formal subscheme W of a Tate linear formal variety T is stable under the action of a group of Tate-linear automorphisms of T which operates strongly nontrivially on T, then W is a Tate-linear formal subvariety. Proofs of statements in this survey article can be found in chapters 5–6 and 10–11 of [**?**].

1. What are Tate-linear structures

1.1. This is a survey of *Tate linear structures* on smooth formal varieties associated to *Tate unipotent groups*. In every Tate-linear structure, there is a prime number implicitly referred to, which will be denoted by p. This prime number p is fixed throughout this article.

A few soundbites may serve as lead-ins.

(a) A Tate unipotent group is analogous to the Tate $\mathbb{Z}_p$-modules of an abelian varieties or a p-divisible group. It is a sheaf of unipotent groups on the big fpqc site of the base field such that each graded piece of its ascending central series is the limit of a *projective* system attached to a p-divisible group.

(b) A Tate-linear formal variety T, or a formal variety T with a Tate-linear structure, is assembled from finitely many p-divisible formal groups $\big(X_i\big)$ through a family of torsors of p-divisible formal groups over Tate-linear subquotients of T. The assembly instruction is contained in a Tate unipotent group, which determines the Tate-linear structure.

(c) The relation between a Tate-linear formal variety to its associated Tate unipotent group is akin to the relation between a p-divisible group and its associated Tate $\mathbb{Z}_p$-module. From a parallel group-theoretic perspective, Tate-linear formal varieties are analogous to compact nilmanifolds. Under this analogy, the Tate unipotent group associated to a Tate-linear formal variety corresponds to the fundamental group of a compact nilmanifold.

(d) The formal completion $\mathcal{C}^{/x_0}$ at a closed point x_0 of a *central leaf* $\mathcal{C}$ of a Siegel modular variety $\mathscr{A}_{g,d,n,\overline{\mathbb{F}}_p}$, which classifies polarized g-dimensional abelian varieties of polarization degree d plus level-n structures, in characteristic p, has a natural Tate-linear structure. The same is true for formal completions of central leaves of Shimura subvarieties of $\mathscr{A}_{g,d,n,\overline{\mathbb{F}}_p}$.

(e) Tate-linear structures are remarkably rigid: Suppose that T is a Tate-linear formal variety assembled from a family $\big(X_i\big)$ of p-divisible groups, and G is a closed subgroup of the compact p-adic Lie group $\mathrm{Aut}_{\mathrm{TL}}(T)$ consisting of all Tate-linear automorphisms of T, such that G operates *strongly nontrivially* on T, in the sense that among all Jordan–Hölder components of the $\mathrm{Lie}(G)$-modules $\mathbb{D}_*(X_i)$ attached to the G-equivariant p-divisible groups X_i, none is the trivial representation of the Lie algebra $\mathrm{Lie}(G)$ of G. Then every reduced irreducible closed formal subscheme of T stable under the action of G is a Tate-linear formal subvariety of T.

1.2. To explain 1.1 (a)–(c), let's consider a p-divisible group X over a field K. It's Tate $\mathbb{Z}_p$-module $\mathbf{T}_p(X)$ is the projective limit $\varprojlim_n X[p^n]$, and $\mathbf{V}_p(X) := \mathbf{T}_p(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ is it's Tate $\mathbb{Q}_p$-module. The p-divisible group X is canonically isomorphic to $\mathbf{V}_p(X)/\mathbf{T}_p(X)$. This is well-known if p is invertible in K, and is still true when p is equal to the characteristic of K if the limit $\varprojlim_n X[p^n]$ is taken in the category of sheaves of abelian groups on the big fpqc site of $\mathrm{Spec}(K)$.

Recall that every torsion free nilpotent group N has an associated uniquely divisible nilpotent group $N_{\mathbb{Q}}$, called the Mal'cev completion of N, which is a minimal element among all uniquely divisible nilpotent group containing N. Moreover for every element $x \in N_{\mathbb{Q}}$, there exists a non-zero integer n such that $x^n \in N$. When N is commutative, its Mal'cev completion $N_{\mathbb{Q}}$ is the familiar localization $N \otimes_{\mathbb{Z}} \mathbb{Q}$ of N.

As indicated in 1.1 (a), a Tate unipotent group $\mathbf{N}$ over a field κ of characteristic p is sheaf of unipotent groups on the big fpqc site of $\mathrm{Spec}(\kappa)$ which is a successive extension of Tate $\mathbb{Z}_p$-modules attached to p-divisible groups over κ. The Mal'cev completion of a Tate unipotent group $\mathbf{N}$ over κ is an fpqc sheaf of uniquely divisible groups $\mathbf{N}_{\mathbb{Q}}$ on $\mathrm{Spec}(\kappa)$ containing $\mathbf{N}$. The quotient sheaf $\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ is represented by a smooth formal scheme $\mathrm{TL}(\mathbf{N})$ over κ. This formal scheme $\mathrm{TL}(\mathbf{N})$ is, by definition, the Tate-linear formal variety attached to the Tate unipotent group $\mathbf{N}$. Put differently, a Tate-linear structure on a formal scheme X over κ is an isomorphism $\xi : X \cong \mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ for a Tate unipotent group $\mathbf{N}$ over κ.

We remark that the inclusion $\mathbf{N} \hookrightarrow \mathbf{N}_{\mathbb{Q}}$ gives rise to a co-filtered family of finite locally free covers $\pi_{\mathrm{TL}(\mathbf{N},\mathrm{TL}(\mathbf{N}'))} : \mathrm{TL}(\mathbf{N}') \to \mathrm{TL}(\mathbf{N})$ of $\mathrm{TL}(\mathbf{N})$, where $\mathbf{N}'$ is a Tate unipotent subgroup of $\mathbf{N}$ such that $\mathbf{N}'_{\mathbb{Q}} = \mathbf{N}_{\mathbb{Q}}$ and $\pi_{\mathrm{TL}(\mathbf{N},\mathrm{TL}(\mathbf{N}')}$ is the canonical projection. The sheaf $\mathbf{N}_{\mathbb{Q}}$ operates on the left of this projective tower $\big(\mathrm{TL}(\mathbf{N}') \to \mathrm{TL}(\mathbf{N})\big)_{\mathbf{N}'}$, and gives rise to a family of finite algebraic correspondences on $\mathrm{TL}(\mathbf{N})$.

Every Tate unipotent group $\mathbf{N}$ over a field κ of characteristic p carries three filtrations: those given by the the ascending and descending central series, plus the slope filtration. Each one gives rise to a family of torsors of p-divisible groups over Tate-linear formal varieties which accomplishes the assembly task for the Tate-linear formal variety $\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ mentioned in (b).

So far we have explained 1.1 (a)–(b) and the first part of 1.1 (c). The analogy with compact nilmanifolds refers to the fact that every connected compact nilmanifold M is isomorphic to a homogeneous space G/Γ of a connected and simply connected nilpotent Lie group G, where Γ is a discrete cocompact torsion free subgroup of G such that the isolator subgroup

$$I(\Gamma, G) = \{x \in G \mid \exists\, n \in \mathbb{N}_{>0} \text{ s.t. } x^n \in \Gamma\}$$

of Γ in G is uniquely divisible and dense in G.

1.3. We gather some properties of Tate-linear structures below.

(1) Examples of Tate-linear formal varieties include
   (a) p-divisible formal groups over fields of characteristic p,
   (b) biextensions of p-divisible formal groups over fields of characteristic p,
   (c) sustained deformation spaces of (polarized) p-divisible groups over fields of characteristic p,
   (d) formal completions at closed points of central leaves [?] in modular varieties of PEL type,
   (e) reduced irreducible formal subschemes of a Tate-linear formal variety T stable under the *strongly nontrivial* action of a subgroup of the group $\mathrm{Aut}_{\mathrm{TL}}(T)$ of all Tate-linear automorphisms of T.

   Examples (c), (d) guided us to the definition of Tate-linear formal varieties.

(2) Given any Tate-linear formal variety T over a field κ of characteristic p, there exists an isogeny $\alpha : T_1 \to T$ of Tate-linear formal varieties and a Tate-linear embedding $\beta : T_1 \hookrightarrow \mathcal{D}ef\,(X)_{\mathrm{sus}}$ of T_1 into the sustained deformation space of a p-divisible group X over κ.

(3) For every Tate-linear formal variety T over a *perfect* field κ of characteristic p, there is a non-zero element $v_{\mathrm{Euler},T}$ of the Lie algebra of $\mathrm{Aut}_{\mathrm{TL}}(T)$ canonically attached to T. A one-parameter subgroup $\rho_{\mathrm{Euler},T,n} : p^n\mathbb{Z}_p \to \mathrm{Aut}_{\mathrm{TL}}(T)$ of the form

$$\rho_{\mathrm{Euler},T,n}(t) = \exp_{\mathrm{Aut}_{\mathrm{TL}}(T)}(t \cdot v_{\mathrm{Euler},T}) \quad \forall\, t \in p^n\mathbb{Z}_p$$

for some natural number n is called an *Euler flow* on T. Every Euler flow operates strongly nontrivially on T. Moreover

$$f \circ \exp_{\mathrm{Aut}_{\mathrm{TL}}(T_1)}(t) = \exp_{\mathrm{Aut}_{\mathrm{TL}}(T_2)}(t) \circ f \quad \forall\, t \in p^n\mathbb{Z}_p,\ \forall\, n \geq n_0.$$

for any Tate-linear morphism $f : T_1 \to T_2$ between Tate-linear formal varieties.

Careful readers likely have noticed that the above properties "almost determine" the class of Tate-linear formal subvarieties of sustained deformation spaces $\mathcal{D}ef\,(X)_{\mathrm{sus}}$. There aren't many good choices if the conditions (1)–(3) are imposed.

1.4. The motivation of the definition of Tate-linear formal varieties goes back to the Serre–Tate local coordinates on deformation spaces of abelian varieties and p-divisible groups; see [**?**], [**?**, Appendix], [**?**].

The relation between the Serre–Tate formal tori and Shimura subvarieties of Siegel modular varieties were investigated in [**?**], [**?**], [**?**]. Their results say that the formal completion at a point of the ordinary locus of a Shimura subvariety of a Siegel modular variety is a formal subtorus of the Serre–Tate formal torus. In [**?**] Moonen defined a notion of $[p]$-*ordinary p-divisible groups with prescribed endomorphisms*, and showed that the deformation space of a $[p]$-ordinary p-divisible group with prescribed endomorphisms has a natural structure as a *cascade*: they are assembled from a family of biextensions, with p-divisible groups as the basic building blocks.

Shortly after the notation of *central leaves* was introduced by Oort in [**?**], it was observed that the formal completion $\mathcal{C}^{/x_0}$ at a closed point x_0 of a central leaf $\mathcal{C}$ in $\mathscr{A}_{g,1,n,\overline{\mathbb{F}}_p}$ has a natural structure as an isoclinic p-divisible group with hight $g(g+1)/2$, if x_0 corresponds to a principally polarized abelian variety with exactly 2 slopes. Similarly the central leaf in the deformation space of a p-divisible group with exactly two slopes has a natural structure as an isoclinic p-divisible groups; see [**?**]. It follows that the central leaf in the deformation space of a p-divisible group *whose slope filtration splits* carries a natural *cascade* structure. The message was clear: the formal completion at a closed point of a central leaf in a Siegel modular variety should carry a natural "Tate-linear structure"; similarly the central leaf in the equi-characteristic p deformation space of any p-divisible group should have a natural "Tate-linear structure". But a precise definition codifying the general idea that Tate-linear formal variety are put together from p-divisible formal groups was not pinned down.

Part of the difficulty was that the notion of central leaves relies on the "pointwise" concept of *geometrically fiberwise constant* p-divisible groups, and it was unclear what a geometrically fiberwise constant p-divisible group over an artinian local ring should mean. This difficulty was removed by the notion of *sustained p-divisible groups*, a scheme-theoretic upgrade of the notion of geometrically fiberwise constant p-divisible groups; see [**?**], [**?**]. Analysis of sustained deformation spaces of p-divisible groups revealed the central role of the projective system of stabilized Aut group schemes $\mathscr{A}ut^{\mathrm{st}}(X)$ (respectively $\mathscr{A}ut^{\mathrm{st}}(Y,\lambda)$) of a p-divisible group X (respectively a polarized p-divisible group (Y,λ)): A sustained p-divisible group modeled on X is a right $\mathscr{A}ut^{\mathrm{st}}(X)$-torsor, while a sustained polarized p-divisible group modeled on (Y,λ) is a right $\mathscr{A}ut^{\mathrm{st}}(Y,\lambda)$-torsor. Moreover the sustained deformation space $\mathcal{D}ef\,(X)_{\mathrm{sus}}$ is canonically isomorphic to $\mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(X))/\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X))$, where the fpqc sheaf $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X))$ of nilpotent groups is the limit of the projective system $\mathscr{A}ut^{\mathrm{st}}(X)$, and $\mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(X))$ is the Mal'cev completion of $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X))$. Similarly the sustained deformation space $\mathcal{D}ef\,(Y,\lambda)_{\mathrm{sus}}$ is isomorphic to $\mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(Y,\lambda))/\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(Y,\lambda))$. These examples quickly led to a tentative formulation of the notion of Tate-linear formal subschemes of sustained deformation spaces in [**?**, 6.2], followed by the general notion of Tate unipotent groups and Tate-linear formal varieties in 2.1, 2.2, 3.2 and [**?**, Ch. 11].

The rigidity property of Tate-linear structures was first observed in the case of p-divisible formal groups; see [**?**, §6], [**?**, §8] and [**?**]. The rigidity of p-divisible formal groups traces back to [**?**, Larsen's Example, p. 443] and [**?**, Prop. 4, p. 471], and was used in the proof [**?**] of the Hecke orbit conjecture for Siegel modular varieties and Hida's theorem [**?**] on the vanishing of the Iwasawa μ-invariant of p-adic Hecke L-functions. For a long time it was unclear whether the rigidity property indicated in 1.1 (e) holds for other formal schemes which are assembled from p-divisible groups, for instance biextensions of p-divisible formal groups over algebraically closed fields of characteristic p. There were difficulties in adapting the proof of rigidity in [**?**, §6] and [**?**] to the case of biextensions. On the other hand no counter-examples were found. Eventually those technical obstacles were overcome through the notion of *tempered perfections*. The resulting proof of the rigidity of biextensions also works for general Tate-linear formal varieties. We refer to [**?**, Ch. 10–11] for more information about tempered perfections and the method of *hypocotyl elongation* in these non-noetherian complete local domains.

Tao Song has generalized the proof of orbital rigidity of biextensions of p-divisible formal groups in [**?**, Ch. 10], and proved the orbital rigidity of the sustained deformation space $\mathcal{D}ef\,(X)_{\mathrm{sus}}$ of a p-divisible group X with at most 4 slopes in his 2022 Penn thesis [**?**]. D'Addezio and van Hoften [**?**] have defined a class of Tate-linear formal varieties over perfect fields of characteristic p, under the assumption that p is strictly bigger than the nilpotency class of the Tate unipotent Lie algebra in question. They proved orbital rigidity of these Tate-linear formal varieties using the method of hypocotyl elongation in tempered perfections in an earlier draft of [**?**, Ch. 10]. This rigidity result linearizes the Hecke orbit problem, so that their results on monodromy of linear p-adic differential equations can be brought to bear.

1.5. The rest of this article is organized as follows. The definition and basic properties of Tate unipotent groups and Tate unipotent Lie algebras are explained in §2, which also contains a summary 2.3 on localizations of nilpotent groups and the Mal'cev completion. Basic properties of Tate-linear formal varieties are indicated in §3. The two families of examples of Tate-linear formal varieties mentioned in 3 (1), biextensions of formal groups and sustained deformation spaces of (polarized) p-divisible groups, are explained in 3.6 and 3.7 respectively. The definition of Euler flows, which are "universal automorphisms" of Tate-linear formal varieties, is in 2.15. The proof of the main rigidity theorem 4.1 of Tate-linear formal varieties is sketched in 6. The two ingredients of the proof, tempered virtual functions and hypocotyl elongation in tempered perfections, are explain in §5. A number of open questions are collected in §7.

1.6. The author would like to acknowledge his intellectual debts to Mumford's beautiful paper [**?**]. Biextensions of p-divisible groups, introduced in [**?**], provide an ideal testing ground for the validity of orbital rigidity of Tate-linear formal varieties. In addition, the explicit construction of the Weil pairings as structural cocycles of biextensions in [**?**, §5] was enormously helpful during the conception of notion of tempered virtual formal morphisms $E \dashrightarrow Z$ attached to one-parameter groups of automorphisms of a biextension E of (X, Y)

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2. Tate unipotent groups and Tate unipotent Lie algebras

Definition 2.1. Let κ be a field of characteristic p, and let $S = \mathrm{Spec}(\kappa)$. Let $\mathbf{N}$ be a sheaf of groups with respect to the fpqc topology on the category of all schemes over S. We say that $\mathbf{N}$ is a *Tate unipotent group over S* if there exist

- a natural number c,
- a central series $(1) = \mathbf{N}_0 \subseteq \mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots \subseteq \mathbf{N}_c = \mathbf{N}$ of fpqc sheaves of normal subgroups $\mathbf{N}_i$ of $\mathbf{N}$ with $[\mathbf{N}_i, \mathbf{N}]_{\mathrm{grp}} \subseteq \mathbf{N}_{i-1}$ for $i = 1, \ldots, c$, where $[\ ,\ ]_{\mathrm{grp}}$ is the group commutator $(x, y) \mapsto x^{-1}y^{-1}xy$,
- p-divisible groups $X_1, \ldots, X_c$ over S, and
- isomorphisms
$$\mathbf{N}_i/\mathbf{N}_{i-1} \cong \mathbf{T}_p(X_i) := \varprojlim_n X_i[p^n] \quad i = 1, \ldots, c,$$
where the transition maps $X_i[p^{n+1}] \to X_i[p^n]$ in the projective system $\big(X_i[p^n]\big)_{n\geq 1}$ are induced by $[p]_{X_i}$, multiplication by p on X_i, and the projective limit $\mathbf{T}_p(X_i)$ is taken in the 2-category of sheaves of abelian groups on the big fpqc site S_{fpqc} of S.

The minimum of all c's satisfying the above conditions is called the *nilpotency class* of $\mathbf{N}$. The fpqc sheaf $\mathbf{T}_p(X_i)$ above is called the Tate $\mathbb{Z}_p$-module of X_i. The p-divisible group X_i, as an fpqc sheaf of abelian groups, is canonically isomorphic to $(\mathbf{T}_p(X_i) \otimes_{\mathbb{Z}} \mathbb{Q})/\mathbf{T}_p(X_i)$.

Every Tate unipotent group over a field κ of characteristic p admits a slope filtration. More precisely, (2.1) is equivalent to the alternative definition (2.2) below.

Definition 2.2. Let κ be a field of characteristic p. A *Tate unipotent group* over κ is a sheaf of nilpotent groups $\mathbf{N}$ for the fpqc topology on the category $\mathfrak{Sch}_\kappa$ of all schemes over κ, together with a decreasing filtration $\big(\mathrm{Fil}_{\mathrm{sl}}^{\bullet}\mathbf{N}\big)_{\bullet>0}$ by sheaves of normal subgroups indexed by non-negative real numbers, with the following properties.

- $\mathrm{Fil}_{\mathrm{sl}}^0\mathbf{N} = \mathbf{N}$, and $\mathrm{Fil}_{\mathrm{sl}}^s\mathbf{N} = (1)$ for all $s > 1$.
- $[\mathrm{Fil}_{\mathrm{sl}}^{s_1}\mathbf{N}, \mathrm{Fil}_{\mathrm{sl}}^{s_2}\mathbf{N}]_{\mathrm{grp}} \subseteq \mathrm{Fil}_{\mathrm{sl}}^{s_1+s_2}\mathbf{N}$ for all $s_1, s_2 > 0$, where $[\ ,\ ]_{\mathrm{grp}}$ denotes the group commutator $(x, y) \mapsto x^{-1}y^{-1}xy$.
- There exists a *finite* subset $\mathrm{slope}(\mathbf{N}) \subseteq (0, 1] \cap \mathbb{Q}$ such that $\mathrm{gr}^s_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N} \neq (0)$ if and only if $s \in \mathrm{slope}(\mathbf{N})$, where $\mathrm{gr}^s_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N} := \mathrm{Fil}_{\mathrm{sl}}^s\mathbf{N}/\mathrm{Fil}_{\mathrm{sl}}^{>s}\mathbf{N}$.
- For every $t \in \mathrm{slope}(\mathbf{N})$, there exists a non-trivial p-divisible group Y_t over κ such that
$$\mathrm{gr}^t_{\mathrm{sl}}\mathbf{N} \;\cong\; \varprojlim_n Y_t[p^n],$$
where $\varprojlim_n Y_t[p^n]$ is the projective limit of the projective system $\big(Y_t[p^n]\big)_{n\geq 1}$ with transition map induced by $[p]_{Y_t}$, and the limit is taken in the category of sheaves of abelian groups on $\mathfrak{Sch}_\kappa$ for the fpqc topology.

Remark. (i) Definition 2.2 is used in [**?**, Ch. 11]. In 2.2, the slope filtration $\mathrm{Fil}^{\bullet}_{\mathrm{sl}}\mathbf{N}$ of $\mathbf{N}$ is uniquely determined by the group structure of $\mathbf{N}$: The Lie algebra $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ of the Mal'cev completion of $\mathbf{N}$, which is a sheaf of Lie $\mathbb{Q}_p$-algebras on the big fpqc site of $\mathfrak{Sch}_{\kappa}$, admits a slope filtration. The slope filtration on $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ gives rise to a slope filtration on $\mathbf{N}_{\mathbb{Q}}$ via the Mal'cev correspondence. The slope filtration on $\mathbf{N}$ is induced from the slope filtration on $\mathbf{N}_{\mathbb{Q}}$.
(ii) That every Tate unipotent group over κ in the sense of 2.2 satisfies the conditions in 2.1 is straight forward. To show that every Tate unipotent group $\mathbf{N}$ over the spectrum of a field κ of characteristic p in the sense of 2.1 carries a slope filtration with the properties required in 2.2, one uses the Mal'cev correspondence and the fact that every p-divisible group κ admits a unique slope filtration, similar to the argument indicated in (i). See remark 2.4.
(iii) The slopes of a Tate unipotent group $\mathbf{N}$ over κ together with their multiplicities form a multiset, uniquely determined by $\mathbf{N}$, called the *slope sequence* of $\mathbf{N}$. In definition 2.2, this multiset is the union of the slope sequences of the p-divisible groups X_i's; in definition 2.2 it is the union of the slope sequences of the isoclinic p-divisible groups Y_t's.

Remark. It is easy to see that every Tate unipotent group $\mathbf{N}$ is torsion free and uniquely ℓ-divisible for every prime number ℓ different from p.

2.3. **Localization of nilpotent groups.**

Let P be a subset of the set Φ of all prime numbers, and let ${}^{c}P$ be the complement of P in Φ. A P-number is a non-zero integer all of whose prime divisors are contained in P. A non-zero integer is said to be prime to P if and only if it is a ${}^{c}P$-number.

A group G is *P-torsion free* (respectively *P-divisible*, respectively *uniquely P-divisible*) if the self map $x \to x^n$ of G is injective (respectively surjective, respectively bijective) for every P-number n. When $P = \Phi$, we say that G is *torsion free* (respectively *divisible*, respectively *uniquely divisible*).

Let $P \subseteq \Phi$ and ${}^{c}P = \Phi \smallsetminus P$ as before. For every *nilpotent* group N, there exists a group homomorphism $\epsilon_{N,P} : N \to N_P$, uniquely determined up to unique isomorphism, such that N_P is uniquely P^c-divisible and the map $\epsilon^*_{N,P} : \mathrm{Hom}(N_P, H) \to \mathrm{Hom}(N, H)$ induced by $\epsilon_{N,P}$ is bijective for every uniquely ${}^{c}P$-divisible group H. The assignment $N \rightsquigarrow N_P$ defines a functor Loc_P from the category of all nilpotent groups to the category of all uniquely ${}^{c}P$-divisible nilpotent groups.

The localization functor Loc_P preserves short exact sequences. More precisely, the localization α_P of a homomorphism $\alpha : N_1 \to N_2$ between nilpotent groups is injective (respectively surjective) if and only if the order of every element of $\mathrm{Ker}(\alpha)$ is finite of order prime to P (respectively for every element $x_2 \in N_2$, there exists an element $x_1 \in N_1$ and a non-zero integer n prime tode P such that $x_2^n = \alpha(x_1)$).

2.4. **Mal'cev completion and Mal'cev correspondence.**

The *Mal'cev completion* $\mathrm{MC}(N)$ of a nilpotent group N is, by definition, the localization of N with respect to the empty subset $\emptyset$ of Φ. In other words, $\mathrm{MC} := \mathrm{Loc}_{\emptyset}$. The universal homomorphism $\epsilon_{N,\emptyset} : N \to \mathrm{MC}(N)$ is characterized by the following properties.

- $\mathrm{Ker}(\epsilon_{N,\emptyset}) = N_{\mathrm{tor}}$, the subgroup of N consisting of all elements of N of finite order.

- $\mathrm{MC}(N)$ is a uniquely divisible nilpotent group.
- For every $y \in \mathrm{MC}(N)$, there exists a non-zero integer n and an element $x \in N$ such that $y^n = \epsilon_{N,\emptyset}(x)$.

The *Mal'cev correspondence* asserts that there is an equivalence between the category of uniquely divisible nilpotent groups and the category of nilpotent Lie $\mathbb{Q}$-algebras. If a uniquely divisible nilpotent group N corresponds to a nilpotent Lie $\mathbb{Q}$-algebra $\mathfrak{n}$, then the nilpotency class of N is equal to the nilpotency class of $\mathfrak{n}$, and there are mutually inverse bijections

$$\exp_N : \mathfrak{n} \to N, \quad \log_N : N \to \mathfrak{n},$$

such that the function

$$\mathfrak{n} \times \mathfrak{n} \longrightarrow \mathfrak{n}, \quad (x, y) \mapsto \log_N(\exp_N(x + y))$$

from $\mathfrak{n} \times \mathfrak{n}$ to $\mathfrak{n}$ is given by the Baker-Campbell-Hausdorff (BCH) formula. Recall that the BCH formula is a specific element of the completion of the free Lie $\mathbb{Q}$-algebra in variables $\{X, Y\}$ with respect to its descending lower central series; see [?, Part I, Ch.II §8] and [?, Ch.II §6] for Dynkin's explicit form of the BCH formula. Note that for each uniquely divisible nilpotent group N corresponding to a niopotent Lie $\mathbb{Q}$-algebra $\mathfrak{n}$, the infinite series $\log_N(\exp_N(x + y))$ is a *finite* sum.

There is an "integral version" of the Mal'cev correspondence, due to Lazard, but restricted to the case when the nilpotency class is "small". Let c be a positive integer, and let $P_{\leq c}$ be the set of all prime numbers not exceeding c. The *Lazard correspondence* asserts that there is an equivalence of categories between the category of all uniquely $P_{\leq c}$-divisible nilpotent groups of class at most c, and the category of nilpotent $\mathbb{Z}[1/c!]$-algebras of class at most c.

We refer to [?], [?], [?], [?], [?] for general information about nilpotent groups and their localizations, [?], [?], [?, Ch. 2], [?], [?] for the Mal'cev completion and Mal'cev correspondence, and [?] for the Lazard correspondence.

Definition 2.5. Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p.

(a) Let $\mathrm{MC}(\mathbf{N})$ be the presheaf on $\mathfrak{Sch}_\kappa$ whose value on any κ-scheme S is $\mathrm{MC}(\mathbf{N}(S))$, the Mal'cev completion of $\mathbf{N}(S)$. Here $\mathfrak{Sch}_\kappa$ is the category of κ-schemes.

(b) Denote by $\mathbf{N}_\mathbb{Q}$ the sheafification of the presheaf $\mathrm{MC}(\mathbf{N})$ on $\mathfrak{Sch}_\kappa$ with respect to the fpqc topology.

(c) The fpqc sheaf $\mathbf{N}_\mathbb{Q}$ of uniquely divisible nilpotent groups corresponds, under the Mal'cev correspondence, to an fpqc sheaf of Lie $\mathbb{Q}$-algebras. The Lie $\mathbb{Q}$-algebras structure of the latter sheaf extends naturally to an fpqc sheaf of Lie $\mathbb{Q}_p$-algebras, denoted by $\mathfrak{Lie}\mathbf{N}_\mathbb{Q}$. We call $\mathfrak{Lie}\mathbf{N}_\mathbb{Q}$ the *Tate unipotent Lie* $\mathbb{Q}_p$*-algebra* attached to $\mathbf{N}$.

(d) Let $\mathbf{N}'$ be another Tate unipotent group over κ.

  (d1) A *κ-homomorphism up to isogeny* from $\mathbf{N}$ to $\mathbf{N}'$ is a κ-homomorphism from $\mathbf{N}_\mathbb{Q}$ to $\mathbf{N}'_\mathbb{Q}$.

  (d2) A κ-isogeny from $\mathbf{N}$ to $\mathbf{N}'$ is a κ-homomorphism from $\mathbf{N}$ to $\mathbf{N}'$ which induces an isomorphism from $\mathbf{N}_\mathbb{Q}$ to $\mathbf{N}'_\mathbb{Q}$. A quasi-isogeny over κ from $\mathbf{N}$ to $\mathbf{N}'$ is a κ isomorphism from $\mathbf{N}_\mathbb{Q}$ to $\mathbf{N}'_\mathbb{Q}$.

Lemma 2.6. *Let $\mathbf{N}, \mathbf{N}'$ be Tate unipotent groups over a field κ of characteristic p, and let $\alpha : \mathbf{N}_{\mathbb{Q}} \to \mathbf{N}'_{\mathbb{Q}}$ be a κ-homomorphism up to isogeny from $\mathbf{N}$ to $\mathbf{N}'$. There exists a κ-homomorphism $\beta : \mathbf{N} \to \mathbf{N}'$ such that the homomorphism from $\mathbf{N}_{\mathbb{Q}}$ to $\mathbf{N}'_{\mathbb{Q}}$ induced by β is equal to $p^n\alpha$ for a positive integer $n > 0$.*

Definition 2.7. Let κ be a field of characteristic p.

(a) A *Tate unipotent Lie $\mathbb{Z}_p$-algebra* over κ is an fpqc sheaf $\mathfrak{N}_{\mathbb{Z}_p}$ of Lie $\mathbb{Z}_p$-algebras on $\mathfrak{Sch}_\kappa$ such there exists a p-divisible group N over κ and an isomorphism $\mathbf{T}_p(N) \cong \mathfrak{N}_{\mathbb{Z}_p}$ of fpqc sheaves of $\mathbb{Z}_p$-modules on $\mathfrak{Sch}_\kappa$. Here $\mathbf{T}_p(N) = \varprojlim_n N[p^n]$ is the Tate $\mathbb{Z}_p$-module of N.

(b) A *Tate unipotent Lie $\mathbb{Q}_p$-algebra* over κ is an fpqc sheaf of Lie $\mathbb{Q}_p$-algebras on $\mathfrak{Sch}_\kappa$, isomorphic to $\mathfrak{N}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for some Tate unipotent Lie $\mathbb{Z}_p$-algebra $\mathfrak{N}_{\mathbb{Z}_p}$ over κ.

Remark. (i) In 2.7(a), under the isomorphism $\mathbf{T}_p(N) \cong \mathfrak{N}_{\mathbb{Z}_p}$, the Lie algebra structure on $\mathfrak{N}_{\mathbb{Z}_p}$ corresponds to a projective family $\big([\ ,\ ]_{N,n} : N[p^n] \times N[p^n] \to N[p^n]\big)_{n\geq 1}$ of Lie algebra structures on $N[p^n]$, compatible with the transition maps $N[p^{n+1}] \twoheadrightarrow N[p^n]$. Moreover the slope filtration on N corresponds to a filtration on $\mathfrak{N}_{\mathbb{Z}_p}$, also called the slope filtration on $\mathfrak{N}_{\mathbb{Z}_p}$.

(ii) Let $\mathbf{N}$ be a Tate unipotent group over κ in the sense of definition 2.1. It is easy to see that the fpqc sheaf $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ of Lie $\mathbb{Q}_p$-algebras on $\mathrm{Spec}(\kappa)$ is a Tate unipotent Lie $\mathbb{Q}_p$-algebra over κ in the sense of 2.7. The slope filtration on $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ induces a slope filtration on $\mathbf{N}_{\mathbb{Q}}$ via the Mal'cev correspondence, hence $\mathbf{N}$ is a Tate unipotent group in the sense of 2.2. This shows that the definitions 2.1 and 2.2 are compatible; c.f. 2.

Lemma 2.8. *Let $\mathfrak{N}_{\mathbb{Q}_p}$ be a Tate unipotent Lie $\mathbb{Q}_p$-algebra over a field κ of characteristic p. There exists a Tate unipotent group $\mathbf{N}$ over κ and an isomorphism $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}} \cong \mathfrak{N}_{\mathbb{Q}_p}$. Note that $\mathbf{N}_{\mathbb{Q}}$ is determined by $\mathfrak{N}_{\mathbb{Q}_p}$ up to unique isomorphism, according to the Mal'cev correspondence.*

Definition 2.9. The fpqc sheaf uniquely divisible nilpotent groups $\mathbf{N}_{\mathbb{Q}}$ in 2.8 is called the *Tate unipotent group up to isogeny* attached to the Tate unipotent Lie $\mathbb{Q}_p$-algebra $\mathfrak{N}_{\mathbb{Q}_p}$.

2.10. We explain two families of examples of Tate unipotent groups and Tate unipotent Lie algebras, from the theory of sustained p-divisible groups. See [?], [?], [?, Ch. 5] for more information.

(a) Let X be a p-divisible group over a field κ of characteristic p. For each positive integer n, the stabilized End group scheme $\mathscr{E}nd^{\mathrm{st}}(X)_n$ at level-n attached to X is, by definition, the schematic image

$$\mathrm{Image}\big(r_{n,n+N} : \mathscr{E}nd(X[p^{n+N}]) \to \mathscr{E}nd(X[p^n])\big), \qquad N >> 0$$

of the restriction homomorphisms $r_{n,n+N}$ for N sufficiently large, where $\mathscr{E}nd(X[p^n])$ is the ring scheme over κ whose S-points are in functorial bijection with S-endomorphisms of $X[p^n]$, for all κ-schemes S. The ring scheme $\mathscr{E}nd^{\mathrm{st}}(X)_n$ is finite over κ for every n. Restricting endomorphisms of $X[p^{n+1}]$ to $X[p^n]$ gives epimorphisms $\mathscr{E}nd^{\mathrm{st}}(X)_{n+1} \twoheadrightarrow \mathscr{E}nd^{\mathrm{st}}(X)_n$, and we get a projective system $\big(\mathscr{E}nd(X[p^n])\big)_{n\geq 1}$ of ring schemes over κ. Note that we also have natural monomorphisms $(\mathscr{E}nd^{\mathrm{st}}(X)_n, +) \hookrightarrow (\mathscr{E}nd^{\mathrm{st}}(X)_{n+1}, +)$; the resulting *inductive system* is a p-divisible group over κ. However these monomorphism do not respect multiplication.

Let $\mathscr{E}nd^{\mathrm{st}}(X)_n^0$ be the neutral component of $\mathscr{E}nd^{\mathrm{st}}(X)_n$; it is a nilpotent ring scheme without unity. Again we have a projective system $\big(\mathscr{E}nd^{\mathrm{st}}(X)_n^0\big)_{n\geq 1}$ of nilpotent ring schemes. Let $\mathscr{A}ut^{\mathrm{st}}(X)_n := (\mathscr{E}nd^{\mathrm{st}}(X)_n)^\times$, the group of units of $\mathscr{E}nd^{\mathrm{st}}(X)_n$. Clearly the neutral component $\mathscr{A}ut^{\mathrm{st}}(X)_n^0$ of $\mathscr{A}ut^{\mathrm{st}}(X)_n$ is equal to $1 + \mathscr{E}nd^{\mathrm{st}}(X)_n^0$.

Consider the projective system $\mathscr{A}ut^{\mathrm{st}}(X)^0 := \big(\mathscr{A}ut^{\mathrm{st}}(X)_n^0\big)_{n\geq 1}$ of nilpotent group schemes over κ, and let

$$\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0) := \varprojlim_n \mathscr{A}ut^{\mathrm{st}}(X)_n^0$$

be the projective limit of $\mathscr{A}ut^{\mathrm{st}}(X)$ as an fpqc sheaf on $\mathrm{Spec}(\kappa)$. We also have an fpqc sheaf of nilpotent $\mathbb{Z}_p$-algebras

$$\mathbf{T}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0) := \varprojlim_n \mathscr{E}nd^{\mathrm{st}}(X)_n^0.$$

Formally adding 1 to the sheaf of nilpotent $\mathbb{Z}_p$-algebras $\mathbf{T}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$ gives us an fpqc sheaf of groups $1 + \mathbf{T}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$, naturally isomorphic to $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$.

The sheaf $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$ is a Tate unipotent group over κ. From the natural action of $\mathbf{T}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$ on $\mathbf{T}_p(X)$, it is easy to see that if X has r distinct slopes, then both $\mathbf{T}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$ and $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$ have $\frac{(r+1)(r-2)}{2}$ slopes. On the other hand, the r-th power of the ideal $\mathbf{T}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$ of $\mathbf{T}_p(\mathscr{E}nd^{\mathrm{st}}(X))$ is (0), and $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$ is nilpotent of class $r-1$.

Define the Tate $\mathbb{Q}_p$-algebra $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$ of $\mathscr{E}nd^{\mathrm{st}}(X)^0$ by

$$\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0) := \mathbf{T}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0) \otimes_{\mathbb{Z}} \mathbb{Q},$$

an fpqc sheaf of nilpotent $\mathbb{Q}_p$-algebras on $\mathrm{Spec}(\kappa)$.

- The Mal'cev completion of $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$, denoted by $\mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$, is

  $$\mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0) = \mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)_{\mathbb{Q}} = 1 + \mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0),$$

  the sheaf of groups obtained from the sheaf of nilpotent $\mathbb{Q}_p$-algebras $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$ by formally adding 1 to the latter.
- The slope filtrations on $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$ and $\mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$ are induced by the slope filtration on $\mathbf{T}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$, or equivalently the slope filtration on the p-divisible group whose p^n-torsion subgroup scheme is $\mathscr{E}nd^{\mathrm{st}}(X)_n^0$.
- The Tate unipotent Lie $\mathbb{Q}_p$-algebra $\mathfrak{Lie}\,\mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$ corresponding to the sheaf $\mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$ of uniquely divisible nilpotent groups under the Mal'cev correspondence is the sheaf of Lie $\mathbb{Q}_p$-algebras underlying the nilpotent associative $\mathbb{Q}_p$-algebra $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$.
- The exponential map

  $$\exp : \mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0) \to \mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$$

  is given by the truncated exponential series

  $$z \mapsto 1 + \sum_{n=1}^{r-1} \frac{z^n}{n!}\,,$$

while the logarithm map

$$\log : \mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0) \to \mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$$

is given by the truncated logarithm series

$$u \mapsto \sum\nolimits_{n=1}^{r-1} (-1)^{n-1}\, \tfrac{(u-1)^n}{n!}\,,$$

where r is the number of distinct slopes of the p-divisible group X.

(b) For every polarized p-divisible group (Y,λ) over a field κ of characteristic p, the polarization λ induces an involution τ on the fpqc sheaf $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)) := \varprojlim_n \mathscr{E}nd^{\mathrm{st}}(Y)_n$ of $\mathbb{Q}_p$-algebras on $\mathrm{Spec}(\kappa)$, and also on the sheaf $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0)$ of nilpotent associative $\mathbb{Q}_p$-algebras.

- The subsheaf $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0)^{\tau=-1}$ of $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0)$ is a Tate unipotent Lie subalgebra of $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0)$ over κ.
- The Tate unipotent group up to isogeny corresponding to $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0)^{\tau=-1}$ is the sheaf $\mathrm{U}\big(\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0),\tau\big)$ of unitary groups attached to the sheaf of nilpotent associative $\mathbb{Q}_p$-algebras with involution $\big(\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0),\tau\big)$, whose points consists of all functorial points z of $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0)$ such that $z+z^\tau+z\cdot z^\tau = 0 = z+z^\tau+z^\tau\cdot z$.
- Let $\mathscr{A}ut^{\mathrm{st}}(Y,\lambda)^0 = \big(\mathscr{A}ut^{\mathrm{st}}(Y,\lambda)^0_n\big)_{n\geq 1}$ be the projective system of connected stabilized Aut group scheme of (Y,λ), where

$$\mathscr{A}ut^{\mathrm{st}}(Y,\lambda)^0_n = \big(\mathrm{Image}\big(r_{n,n+N}\colon \mathscr{A}ut(Y[p^{n+N}],\lambda[p^{n+N}]) \to \mathscr{A}ut(Y[p^n],\lambda[p^n])\big)\big)^0,\ \ N\gg 0$$

  for each n. The projective limit $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(Y,\lambda)^0) := \varprojlim_n \mathscr{A}ut^{\mathrm{st}}(Y,\lambda)^0_n$ is a Tate unipotent group over κ, naturally isomorphic to the intersection of $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(Y)^0)$ and $\mathrm{U}\big(\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0),\tau\big)$ in $\mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(Y)^0)$:

$$\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(Y,\lambda)^0) \cong \mathrm{U}\big(\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0),\tau\big) \cap \mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(Y)^0).$$

- The Mal'cev completion $\mathbf{V}_p(\mathscr{A}ut^{\mathrm{st}}(Y,\lambda)^0)$ of $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(Y,\lambda)^0)$ is naturally isomorphic to $\mathrm{U}\big(\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0),\tau\big)$.
- The exponential map

$$\exp : \mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0)^{\tau=-1} \longrightarrow \mathrm{U}\big(\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0),\tau\big)$$

  and the logarithm map

$$\log : \mathrm{U}\big(\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0,\tau)\big) \longrightarrow \mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(Y)^0)^{\tau=-1}$$

  are given by the truncated exponential series

$$z \mapsto 1 + \sum\nolimits_{n=1}^{r-1} \tfrac{z^n}{n!}$$

  and the truncated logarithm series

$$u \mapsto \sum\nolimits_{n=1}^{r-1} (-1)^{n-1}\, \tfrac{(u-1)^n}{n}$$

  respectively, where r is the number of distinct sums of Y.

Proposition 2.11 below is an analog of Ado's theorem. It says that up to isogeny, every Tate unipotent group can be realized as a TL subgroup of the Tate module of the stabilized Aut group of a p-divisible group.

Proposition 2.11. *Let $\mathfrak{N}_{\mathbb{Q}_p}$ be a Tate unipotent Lie $\mathbb{Q}_p$-algebra over a field κ of characteristic p. There exists a p-divisible group X over κ and an embedding*

$$\mathfrak{N}_{\mathbb{Q}_p} \hookrightarrow \mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$$

of the fpqc sheaf $\mathfrak{N}_{\mathbb{Q}_p}$ of Lie $\mathbb{Q}_p$-algebras into the fpqc sheaf of Lie $\mathbb{Q}_p$-algebras underlying the sheaf of nilpotent associative $\mathbb{Q}_p$-algebras (without unity) $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$ on $\mathfrak{Sch}_\kappa$.

Lemma 2.12. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p.*

(a) *The group $\mathrm{Aut}(\mathbf{N}_\mathbb{Q})$ of automorphisms of the Mal'cev completion $\mathbf{N}_\mathbb{Q}$ of $\mathbf{N}$ has a natural structure as a p-adic Lie group. It is the group of $\mathbb{Q}_p$-points of a linear algebraic group over $\mathbb{Q}_p$.*
(b) *The group $\mathrm{Aut}(\mathbf{N})$ of automorphisms of $\mathbf{N}$ is a compact open subgroup of $\mathrm{Aut}(\mathbf{N}_\mathbb{Q})$.*
(c) *The Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathbf{N}_\mathbb{Q}))$ of $\mathrm{Aut}(\mathbf{N}_\mathbb{Q})$ is naturally isomorphic to the Lie $\mathbb{Q}_p$-algebra consisting of all $\mathbb{Q}_p$-linear derivations $\partial : \mathfrak{Lie}\mathbf{N}_\mathbb{Q} \to \mathfrak{Lie}\mathbf{N}_\mathbb{Q}$ of the Tate unipotent Lie $\mathbb{Q}_p$-algebra $\mathfrak{Lie}\mathbf{N}_\mathbb{Q}$ attached to $\mathbf{N}$.*

2.13. Suppose that κ is a perfect field of characteristic p. Then the *covariant* Dieudonné theory tells us that a Tate unipotent Lie $\mathbb{Z}_p$-algebra $\mathfrak{N}_{\mathbb{Z}_p}$ corresponds to a free module $\mathbb{D}_*(\mathfrak{N}_{\mathbb{Z}_p})$ of finite rank over the ring $\mathrm{W}(\kappa)$ of p-adic Witt vectors with entries in κ, together with semilinear operators $\mathtt{F}, \mathtt{V}$ on $\mathbb{D}_*(\mathfrak{N}_{\mathbb{Z}_p})$ and a skew symmetric $\mathrm{W}(\kappa)$-bilinear map

$$[\ ,\ ] = [\ ,\ ]_{\mathbb{D}_*(\mathfrak{N}_{\mathbb{Z}_p})} : \mathbb{D}_*(\mathfrak{N}_{\mathbb{Z}_p}) \times \mathbb{D}_*(\mathfrak{N}_{\mathbb{Z}_p}) \longrightarrow \mathbb{D}_*(\mathfrak{N}_{\mathbb{Z}_p}),$$

satisfying the following properties.

(i) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x, y, z \in \mathbb{D}_*(\mathfrak{N}_{\mathbb{Z}_p})$
(ii) $\mathtt{F}(a\cdot x) = \sigma(a)\cdot \mathtt{F}(x)$ and $\mathtt{V}(a\cdot x) = \sigma^{-1}(a)\cdot \mathtt{V}(x)$ for all $a \in \mathrm{W}(\kappa)$ and all $x \in \mathbb{D}_*(\mathfrak{N}_{\mathbb{Z}_p})$.
(iii) $[\mathtt{V}(x), \mathtt{V}(y)] = \mathtt{V}([x,y])$ for all $x, y \in \mathbb{D}_*(\mathfrak{N}_{\mathbb{Z}_p})$.
(iv) $[\mathtt{F}(x), y] = \mathtt{F}([x, \mathtt{V}(y)])$ and $[x, \mathtt{F}(y)] = \mathtt{F}([\mathtt{V}(x), y])$ for all $x, y \in \mathbb{D}_*(\mathfrak{N}_{\mathbb{Z}_p})$.

Here σ denotes the canonical lifting of Frobenius on $\mathrm{W}(\kappa)$.

Similarly a Tate unipotent Lie $\mathbb{Q}_p$-algebra $\mathfrak{N}_{\mathbb{Q}_p}$ over κ corresponds to a finite dimensional vector space $\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}_p})$ over $\mathrm{W}(\kappa)\otimes_\mathbb{Z}\mathbb{Q}$, together with operators $\mathtt{F}, \mathtt{V}$ and a skew bilinear map $[\ ,\ ] : \mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}_p}) \times \mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}_p}) \longrightarrow \mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}_p})$, satisfying conditions (i)–(iv) above.

2.14. Suppose that the base field κ is perfect. Let $\mathfrak{N}_{\mathbb{Q}_p}$ be a Tate unipotent Lie $\mathbb{Q}_p$-algebra. Then the slope filtration on $\mathfrak{N}_{\mathbb{Q}_p}$ splits, and we have a canonical decomposition

$$\mathfrak{N}_{\mathbb{Q}_p} \cong \oplus_{s\in \mathrm{slope}(\mathfrak{N}_{\mathbb{Q}_p})} \mathfrak{N}_{s,\mathbb{Q}_p},$$

where $\mathfrak{N}_{s,\mathbb{Q}_p} \cong \mathbf{V}_p(N_s)$ for some isoclinic p-divisible group N_s over κ, for each $s \in \mathrm{slope}(\mathfrak{N}_{\mathbb{Q}_p})$. Moreover

$$[\mathfrak{N}_{s,\mathbb{Q}_p}, \mathfrak{N}_{s',\mathbb{Q}_p}]_{\mathfrak{N}_{\mathbb{Q}_p}} \subseteq \mathfrak{N}_{s+s',\mathbb{Q}_p} \quad \forall\, s, s' \in \mathrm{slope}(\mathfrak{N}_{\mathbb{Q}_p}).$$

Definition 2.15. Let κ be a *perfect* field of characteristic p. Let

$$\mathfrak{N}_{\mathbb{Q}_p} = \oplus_{s\in \mathrm{slope}(\mathfrak{N}_{\mathbb{Q}_p})}\mathfrak{N}_{s,\mathbb{Q}_p}$$

be the slope decomposition of $\mathfrak{N}_{\mathbb{Q}_p}$ of a Tate unipotent Lie $\mathbb{Q}_p$-algebra.

(a) Denote by $\partial_{\mathrm{Euler}}$ the derivation of $\mathfrak{N}_{\mathbb{Q}_p}$ such that $\partial_{\mathrm{Euler}}(u_s) = su_s$ for all $s \in \mathrm{slope}(\mathfrak{N}_{\mathbb{Q}_p})$ and all functorial points u_s of $\mathfrak{N}_{s,\mathbb{Q}_p}$.

The element of the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{N}_{\mathbb{Q}_p}))$ of $\mathrm{Aut}(\mathfrak{N}_{\mathbb{Q}})$ corresponding to $\partial_{\mathrm{Euler}}$ will be called the *Euler vector field* on $\mathfrak{N}_{\mathbb{Q}_p}$, denoted again by $\partial_{\mathrm{Euler}}$.

(b) Let $\mathbf{N}$ be a Tate unipotent group over κ whose Mal'cev completion $\mathbf{N}_{\mathbb{Q}}$ is the Tate unipotent group up to isogeny attached to $\mathfrak{N}_{\mathbb{Q}_p}$. A closed subgroup of the p-adic Lie group $\mathrm{Aut}(\mathbf{N})$ of the form

$$\left\{ \exp_{\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})}(t\cdot \partial_{\mathrm{Euler}}) \,\middle|\, t\in U\right\}$$

for an open subgroup U of $(\mathbb{Z}_p,+)$ is called an *Euler flow* on $\mathbf{N}$.

Remark. (i) In the case when $\mathbf{N}$ is the Tate $\mathbb{Z}_p$-module of a p-divisible group X, the Euler flows correspond to the subgroups $\{[a]_X \mid a\in 1+p^n\mathbb{Z}_p\}$ of $\mathrm{Aut}(X)$, where n ranges through all inters $n\geq 1$ if $p\neq 2$, and all integers $n\geq 2$ if $p=2$.

(ii) Every homomorphism $h:\mathfrak{N}_{\mathbb{Q}_p}\to\mathfrak{N}'_{\mathbb{Q}_p}$ between Tate unipotent Lie $\mathbb{Q}_p$-algebras respects the Euler vector fields on $\mathfrak{N}$ and $\mathfrak{N}'$, in the sense that $\partial_{\mathrm{Euler},\mathfrak{N}'_{\mathbb{Q}_p}}\circ h = h\circ\partial_{\mathrm{Euler},\mathfrak{N}_{\mathbb{Q}_p}}$. Similarly every homomorphism $\alpha:\mathbf{N}\to\mathbf{N}'$ between Tate unipotent groups respects Euler flows on $\mathbf{N}$ and $\mathbf{N}'$.

(iii) Let b_0 be the least common multiple of all denominators of $\mathrm{slope}(\mathfrak{N}_{\mathbb{Q}_p})$. The map ϕ^{b_0} from $\mathfrak{N}_{\mathbb{Q}_p}$ to itself such that $\phi^{b_0}(u_s) = p^{b_0 s}u_s$ for all functorial points u_s of $\mathfrak{N}_{s,\mathbb{Q}_p}$ is an endomorphism of the fpqc sheaf $\mathfrak{N}_{\mathbb{Q}_p}$ of Lie $\mathbb{Q}_p$-algebras on $\mathrm{Spec}(\kappa)$. It is an analog of the b_0-th iterate of the relative Frobenius of $\mathfrak{N}_{\mathbb{Q}_p}$.

3. Tate-linear formal varieties

Proposition 3.1. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. The fpqc sheaf $\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ on $\mathrm{Spec}(\kappa)$ is represented by a noetherian local formal scheme smooth over κ, isomorphic to the formal spectrum of a formal power series ring over κ in finitely many variables.*

Definition 3.2. Let κ be a field of characteristic p.

(a) Let T be a noetherian local formal scheme over κ. A *Tate-linear structure* on T is an isomorphism $\zeta: T\xrightarrow{\sim}\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ from the fpqc sheaf on $\mathrm{Spec}(\kappa)$ represented by T to the fpqc sheaf $\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$, where $\mathbf{N}$ is a Tate unipotent group over κ. We will denote the quotient $\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ by $\mathrm{TL}(\mathbf{N})$, and call it the *Tate-linear formal variety* attached to the Tate unipotent group $\mathbf{N}$.

(b) A Tate-linear formal variety over κ is pair $(T,\zeta: T\xrightarrow{\sim}\mathbf{N}_{\mathbb{Q}}/\mathbf{N})$ consisting of a formal κ-scheme T and a Tate-linear structure ζ on T.

Definition 3.3. Let $\mathrm{TL}(\mathbf{N}_1)$ and $\mathrm{TL}(\mathbf{N}_2)$ be Tate-linear formal varieties over a field κ of characteristic p.

(a) A formal morphism $f : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ over κ is *Tate-linear*, or a *TL* morphism, if there exists a homomorphism $h : \mathbf{N}_1 \to \mathbf{N}_2$ over κ which induces f. Note that such a homomorphism h is unique if it exists.

(b) A TL morphism $\mathrm{TL}(h) : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ induced by a homomorphism $h : \mathbf{N}_1 \to \mathbf{N}_2$ over κ is said to be an isogeny (respectively a quasi-isogeny) if h is.

Definition 3.4. Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. Let G be a closed subgroup of the group of all Tate-linear automorphisms of $\mathrm{TL}(\mathbf{N})$. We say that G operates *strongly nontrivially* on $\mathrm{TL}(\mathbf{N})$ if G corresponds to a closed subgroup of $\mathrm{Aut}(\mathbf{N})$ such that every Jordan–Hölder component of the $\mathrm{Lie}(G)$-module $\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}_p,\kappa^{\mathrm{alg}}})$ is non-trivial, where κ^{alg} is an algebraic closure of κ and $\mathfrak{N}_{\mathbb{Q}_p,\kappa^{\mathrm{alg}}}$ is the base change to κ^{alg} of the Lie $\mathbb{Q}_p$-algebra $\mathfrak{N}_{\mathbb{Q}_p}$ of $\mathbf{N}$.

Remark. Every Euler flow (2.15) on $\mathbf{N}$ induces a subgroup of TL-automorphisms of $\mathrm{TL}(\mathbf{N})$ acting strongly nontrivially on $\mathrm{TL}(\mathbf{N})$.

Lemma 3.5 below implies that a Tate-linear formal variety can be assembled from p-divisible formal groups through a finite collection of torsors for p-divisible formal groups.

Lemma 3.5. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. Let $\mathbf{Z}$ be a Tate unipotent subgroup of $\mathbf{N}$ contained in the center $Z(\mathbf{N})$ of $\mathbf{N}$, such that $\mathbf{N}/\mathbf{Z}$ is a Tate unipotent group. Then the map $\pi : \mathrm{TL}(\mathbf{N}) \to \mathrm{TL}(\mathbf{N}/\mathbf{Z})$ induced by the quotient map $\mathbf{N} \twoheadrightarrow \mathbf{N}/\mathbf{Z}$ has a natural structure as a $\mathrm{TL}(\mathbf{Z})$-torsor over $\mathrm{TL}(\mathbf{N}/\mathbf{Z})$.*

3.6. **Biextensions as Tate-linear formal varieties.**

Let X, Y, Z be p-divisible formal groups over a field κ of characteristic p. Let $\mathbf{X} = \mathbf{T}_p(X)$, $\mathbf{Y} = \mathbf{T}_p(Y)$, $\mathbf{Z} = \mathbf{T}_p(Z)$ be the Tate $\mathbb{Z}_p$-modules of X, Y, Z respectively. Let

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{N} \xrightarrow{q} \mathbf{X} \times \mathbf{Y} \longrightarrow 1$$

be a central extension of $\mathbf{X} \times \mathbf{Y}$ by $\mathbf{Z}$, which splits over $\mathbf{X}$ and also over $\mathbf{Y}$. Then $\mathbf{N}$ is a Tate unipotent group over κ. We choose and fix embeddings $\mathbf{X} \hookrightarrow \mathbf{N}$ and $\mathbf{Y} \hookrightarrow \mathbf{N}$ which splits the central extensions $\mathbf{Z} \rightarrowtail q^{-1}\mathbf{X} \twoheadrightarrow \mathbf{X}$ and $\mathbf{Z} \rightarrowtail q^{-1}\mathbf{Y} \twoheadrightarrow \mathbf{Y}$ respectively, and regard $\mathbf{X}$ and $\mathbf{Y}$ as sheaves of subgroups of $\mathbf{N}$.

Let $E := \mathrm{TL}(\mathbf{N})$ be the Tate-linear formal variety over κ attached to $\mathbf{N}$. By 3.5, we have a canonical translation action of Z on E and a projection map $\pi : E \to X \times Y$ such that E is a Z-torsor over $X \times Y$. We will explain below an enhancement of the Z-torsor structure on $E \to X \times Y$ to a biextension of (X, Y) by Z, and identify the family of Weil pairings of this biextension in terms of the group structure of $\mathbf{N}$.

Remark. Strictly speaking, the input data which produce the biextension structure on E include the embeddings $\mathbf{X} \hookrightarrow \mathbf{N}$ and $\mathbf{Y} \hookrightarrow \mathbf{N}$. This fine point is suppressed below.

(a) We know from the exactness of localization functors for nilpotent groups that the Mal'cev completion $\mathbf{N}_{\mathbb{Q}}$ of $\mathbf{N}$ is a central extension of $\mathbf{X}_{\mathbb{Q}} \times \mathbf{Y}_{\mathbb{Q}}$ by $\mathbf{Z}_{\mathbb{Q}}$. The group commutator

$$[\ ,\ ]_{\mathrm{grp},\mathbf{N}_{\mathbb{Q}}} : \mathbf{N}_{\mathbb{Q}} \times \mathbf{N}_{\mathbb{Q}} \longrightarrow \mathbf{N}_{\mathbb{Q}}, \quad [\mathbf{n}_1, \mathbf{n}_2]_{\mathrm{grp}} = \mathbf{n}_1^{-1}\mathbf{n}_2^{-1}\mathbf{n}_1\mathbf{n}_2 \ \ \forall\, \mathbf{n}_1, \mathbf{n}_2 \in \mathbf{N}_{\mathbb{Q}}$$

on $\mathbf{N}_{\mathbb{Q}}$ induces a skew-symmetric bilinear pairing

$$\langle\ ,\ \rangle_{\mathbf{N}_{\mathbb{Q}}} : (\mathbf{X}_{\mathbb{Q}} \times \mathbf{Y}_{\mathbb{Q}}) \times (\mathbf{X}_{\mathbb{Q}} \times \mathbf{Y}_{\mathbb{Q}}) \longrightarrow \mathbf{Z}_{\mathbb{Q}}$$

such that the diagram

$$\begin{tikzcd} \mathbf{N}_{\mathbb{Q}} \times \mathbf{N}_{\mathbb{Q}} \arrow[r, "{[\ ,\ ]_{\mathrm{grp},\mathbf{N}_{\mathbb{Q}}}}"] \arrow[d, "q_{\mathbb{Q}} \times q_{\mathbb{Q}}"] & \mathbf{N}_{\mathbb{Q}} \\ (\mathbf{X}_{\mathbb{Q}} \times \mathbf{Y}_{\mathbb{Q}}) \times (\mathbf{X}_{\mathbb{Q}} \times \mathbf{Y}_{\mathbb{Q}}) \arrow[r, "\langle\ ,\ \rangle_{\mathbf{N}_{\mathbb{Q}}}"] & \mathbf{Z}_{\mathbb{Q}} \arrow[u, hook] \end{tikzcd}$$

commutes.

(b) We will define relative group laws

$$+_1 : E \times_Y E \to E \quad \text{and} \quad +_2 : E \times_X E \to E$$

on E, which will give E a biextension structure. See [**?**] for the notion of biextensions.

(i) Given a functorial point y of Y, pick a functorial point $\mathbf{y}_0$ on $\mathbf{Y}_{\mathbb{Q}}$ lifting y. The fiber $(\mathrm{pr}_2 \circ \pi)^{-1}(y)$ of E over y consists of all elements of the form $[\mathbf{z} \cdot \mathbf{x} \cdot \mathbf{y}_0]$, where $\mathbf{z}$ is a functorial point of $\mathbf{Z}_{\mathbb{Q}}$, $\mathbf{x}$ is a functorial point of $\mathbf{X}_{\mathbb{Q}}$, and $[\mathbf{z} \cdot \mathbf{x} \cdot \mathbf{y}_0]$ is the image of $\mathbf{z} \cdot \mathbf{x} \cdot \mathbf{y}_0$ in $\mathrm{TL}(\mathbf{N}) = E$. Two functorial points $[\mathbf{z}_1 \cdot \mathbf{x}_1 \cdot \mathbf{y}_0]$ and $[\mathbf{z}_2 \cdot \mathbf{x}_2 \cdot \mathbf{y}_0]$ of E_y over the same κ-scheme are equal if and only

$$\mathbf{x}_1 - \mathbf{x}_2 \in \mathbf{X} \quad \text{and} \quad \langle \mathbf{x_1} - \mathbf{x}_2, \mathbf{y}_0 \rangle_{\mathbf{N}_{\mathbb{Q}}} + \mathbf{z}_1 - \mathbf{z}_2 \in \mathbf{Z}.$$

(ii) For any two functorial points $[\mathbf{z}_1 \cdot \mathbf{x}_1 \cdot \mathbf{y}_0]$ and $[\mathbf{z}_2 \cdot \mathbf{x}_2 \cdot \mathbf{y}_0]$ of E_y over the same κ-scheme, define their sum under $+_1$ by

$$[\mathbf{z}_1 \cdot \mathbf{x}_1 \cdot \mathbf{y}_0] +_1 [\mathbf{z}_2 \cdot \mathbf{x}_2 \cdot \mathbf{y}_0] := [(\mathbf{z}_1 + \mathbf{z}_2) \cdot (\mathbf{x}_1 + \mathbf{x}_2) \cdot \mathbf{y}_0].$$

This gives a well-defined morphism $+_1 : E \times_Y E$ to E.

Moreover for each functorial point y of Y, the sheaf of commutative groups $(\mathrm{pr}_2 \circ \pi)^{-1}(y)$ under the group law $+_1$ is the push-out of the top row by the vertical arrow ξ_y in the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{Z}_{\mathbb{Q}} \cdot \mathbf{X} \arrow[r] \arrow[d, "\xi_y"'] & \mathbf{Z}_{\mathbb{Q}} \cdot \mathbf{X}_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathbf{X}_{\mathbb{Q}}/\mathbf{X} = X \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & Z \arrow[r] & (\mathrm{pr}_2 \circ \pi)^{-1}(y) \arrow[r] & X \arrow[r] & 1\,. \end{tikzcd}$$

with exact rows. Here $\xi_y : \mathbf{Z}_{\mathbb{Q}} \cdot \mathbf{X} \to Z$ is the group homomorphism given by

$$\xi_y(\mathbf{z} \cdot \mathbf{x}) = \langle \mathbf{x}, \mathbf{y}_0 \rangle_{\mathbf{N}_{\mathbb{Q}}} \bmod \mathbf{Z} \qquad \forall\, \mathbf{z} \in \mathbf{Z}_{\mathbb{Q}},\ \forall\, \mathbf{x} \in \mathbf{X},$$

and $\mathbf{y}_0$ is a representative of y in $\mathbf{X}_{\mathbb{Q}}$.

(iii) Similarly, define a group law $+_2 : E \times_X E \to X$ relative to X by

$$[\mathbf{z}_1 \cdot \mathbf{y}_2 \cdot \mathbf{x}_0] +_2 [\mathbf{z}_1 \cdot \mathbf{y}_2 \cdot \mathbf{x}_0] := [(\mathbf{z}_1 + \mathbf{z}_2) \cdot (\mathbf{y}_1 + \mathbf{y}_2) \cdot \mathbf{x}_0].$$

for functorial points $\mathbf{x}_0 \in \mathbf{X}$, $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{Y}$, $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{Z}$ with values in the same κ-scheme.

For each functorial point x of X, the sheaf of commutative groups $(\mathrm{pr}_1\circ\pi)^{-1}(x)$ is an extension of Y by Z, which sits in the push-out diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{Z}_{\mathbb{Q}}\cdot\mathbf{Y} \arrow[r] \arrow[d, "\eta_x"'] & \mathbf{Z}_{\mathbb{Q}}\cdot\mathbf{Y}_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathbf{Y}_{\mathbb{Q}}/\mathbf{Y}=Y \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & Z \arrow[r] & (\mathrm{pr}_1\circ\pi)^{-1}(x) \arrow[r] & Y \arrow[r] & 1\,, \end{tikzcd}$$

where $\eta_x:\mathbf{Z}_{\mathbb{Q}}\cdot\mathbf{Y}\to Z$ is given by

$$\eta_x(\mathbf{z}\cdot\mathbf{y})=\langle\mathbf{y},\mathbf{x}_0\rangle \bmod \mathbf{Z}=-\langle\mathbf{x}_0,\mathbf{y}\rangle \bmod \mathbf{Z}\quad \forall\,\mathbf{z}\in\mathbf{Z}_{\mathbb{Q}},\ \forall\,\mathbf{y}\in\mathbf{Y}.$$

(c) A straight forward calculation shows that the family of Weil pairings

$$\beta_n: X[p^n]\times Y[p^n]\longrightarrow Z[p^n]$$

attached to the biextension $E\to X\times Y$ constructed in (B) above, with the sign convention in [**?**], is expressed in terms of the Lie bracket on $\mathbf{N}_{\mathbb{Q}}$ by

$$\beta_n(x_n,y_n)=p^n\langle\mathbf{x}_n,\mathbf{y}_n\rangle_{\mathbf{N}_{\mathbb{Q}}},$$

for all functorial points (x_n,y_n) of $X[p^n]\times Y[p^n]$ and liftings $(\mathbf{x}_n,\mathbf{y}_n)$ in $(p^{-n}\mathbf{X})\times(p^{-n}\mathbf{Y})$ of (x_n,y_n).

Since every biextension of p-divisible groups is determined up to isomorphism by its Weil pairings, the above formula for β_n identifies the biextension structure on $E=\mathrm{TL}(\mathbf{N})$. On the other hand, the skew symmetric pairing $\langle,\ \rangle_{\mathbf{N}_{\mathbb{Q}}}$ is easily recovered from the family $(\beta_n)_{n\geq 1}$ of Weil pairings. So the Tate unipotent group $\mathbf{N}$ is also determined by the biextension $E\to X\times Y$ up to isomorphism.

The formula for the Weil pairings also shows that every biextension of p-divisible formal groups arises from a Tate-linear formal variety associated to Tate unipotent group $\mathbf{N}$ satisfying the conditions in the first paragraph of 3.6. Thus every Tate-linear formal variety $\mathrm{TL}(\mathbf{N})$ over a perfect field attached to a Tate unipotent group $\mathbf{N}$ of nilpotency class at most 2 is *isogenous* to a biextension of p-divisible formal groups.

3.7. **Sustained deformation spaces are Tate-linear formal varieties.**

(a) Let X be a p-divisible group over a field κ of characteristic p. Let $\mathcal{D}\mathit{ef}\,(X)_{\mathrm{sus}}$ be the sustained deformation space of X, such that for every augmented commutative artinian local κ-algebra R, $\mathcal{D}\mathit{ef}\,(X)_{\mathrm{sus}}(R)$ is canonically identified with the set of all equivalence classes of sustained p-divisible groups over R whose closed fiber is X. It is shown in [**?**, Ch. 5] that $\mathcal{D}\mathit{ef}\,(X)_{\mathrm{sus}}$ is represented by a smooth formal scheme over κ, and there is a natural isomorphism

$$\zeta_X:\mathrm{TL}\big(\mathbf{T}_p(\mathscr{A}\mathit{ut}^{\mathrm{st}}(X)^0)\big)\ \xrightarrow{\sim}\mathcal{D}\mathit{ef}\,(X)_{\mathrm{sus}}\,.$$

The gist is as follows. Let R be an augmented artinian local κ-algebra.

- A sustained p-divisible group $\tilde{X}$ over R is strongly sustained modeled on X, because its closed fiber is X. Therefore $\tilde{X}$ corresponds to a right torsor for the projective system $\mathscr{A}\mathit{ut}^{\mathrm{st}}(X)$ of stabilized Aut group schemes.

- A (rigidified) right $\mathscr{A}ut^{\mathrm{st}}(X)$-torsor over R is induced by a right $\mathscr{A}ut^{\mathrm{st}}(X)^0$-torsor over R, unique up to isomorphism. The latter is the same as a right torsor for $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$.
- An R-valued point of $\mathrm{TL}\big(\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0\big)$ determines a right $\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)$-torsor. This defines a natural map $\zeta_X : \mathrm{TL}\big(\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(X)^0)\big) \xrightarrow{\sim} \mathcal{D}ef\,(X)_{\mathrm{sus}}$.
- The fact that ζ_X is an isomorphism is a special case of a more general statement on the deformation of torsors for Tate unipotent groups. A d'evisage argument reduces the latter statement to the case of a commutative Tate unipotent group, which is known.

(b) Let (Y, λ) be a polarized p-divisible group. Let $\mathcal{D}ef\,(Y, \lambda)_{\mathrm{sus}}$ be the sustained deformation functor of (Y, λ). Similar to (a) above, $\mathcal{D}ef\,(Y, \lambda)_{\mathrm{sus}}$ is represented by a smooth formal scheme over κ, and we have a canonical isomorphism

$$\zeta_{Y,\lambda} : \mathrm{TL}\big(\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(Y, \lambda)^0)\big) \xrightarrow{\sim} \mathcal{D}ef\,(Y, \lambda)_{\mathrm{sus}}\,.$$

It follows that for every $\overline{\mathbb{F}_p}$-point x_0 of a central leaf $\mathcal{C}$ in a moduli space $\mathscr{A}_{g,d,n,\overline{\mathbb{F}_p}}$ with $n \geq 3$ and $\gcd(n, p) = 1$, such that x_0 corresponds to a polarized abelian variety (A_0, λ_0) with level-n structure, the formal completion $\mathcal{C}^{/x_0}$ of $\mathcal{C}$ at x_0 has a natural Tate-linear structure

$$\zeta_{x_0} : \mathrm{TL}\big(\mathbf{T}_p(\mathscr{A}ut^{\mathrm{st}}(A_{x_0}[p^\infty], \lambda_{x_0}[p^\infty])^0)\big) \xrightarrow{\sim} \mathcal{C}^{/x_0}.$$

Such a group-theoretic description of the local structure of central leaves in Siegel modular varieties motivated the notion of Tate-linear formal varieties.

Proposition 3.8 below provides a "trivial estimate" of the actions of one-parameter subgroups of $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$ on a Tate-linear formal variety $\mathrm{TL}(\mathbf{N})$.

Proposition 3.8. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. Let $s \in (0, 1]$ be a real number such that $\max(\mathrm{slope}(\mathbf{N})) \leq s \leq 1$. Let U be a finitely generated $\mathbb{Z}_p$-submodule of the Lie algebra of the p-adic Lie group $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$. There exist constants $c_0, n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $B \in U$, the exponential $\exp_{\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})}(p^n B)$ of $p^n B$ is an element of $\mathrm{Aut}(\mathbf{N})$ which operates trivially on the infinitesimal neighborhood $\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{\lfloor n/s \rfloor - c_0}]$ of the base point of $\mathrm{TL}(\mathbf{N})$. Here $\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{\lfloor n/s \rfloor - c_0}]$ denotes the inverse image under the relative Frobenius morphism $\mathrm{Fr}^{\lfloor n/s \rfloor - c_0}_{\mathrm{TL}(\mathbf{N})} : \mathrm{TL}(\mathbf{N}) \longrightarrow \mathrm{TL}(\mathbf{N})^{(p^{\lfloor n/s \rfloor - c_0})}$ of the base point of $\mathrm{TL}(\mathbf{N})^{(p^{\lfloor n/s \rfloor - c_0})}$.*

4. Orbital rigidity of Tate-linear structures

Tate-linear formal varieties satisfy a strong rigidity property, theorem 4.1 below. The special case when the Tate unipotent group $\mathbf{N}$ is commutative, i.e. the Tate-linear formal variety $\mathrm{TL}(\mathbf{N})$ is a p-divisible formal group X over κ and $\mathbf{N} = \mathbf{T}_p(X)$, is the main result of [**?**].

Theorem 4.1 (Orbital rigidity of Tate-linear formal varieties)**.** *Let κ be a perfect field of characteristic p, and let $\mathbf{N}$ be a Tate unipotent group over κ. Let G be a closed subgroup of* $\mathrm{Aut}(\mathbf{N})$ *acting* strongly nontrivially *on $\mathbf{N}$. Let W be a reduced irreducible closed formal* *subscheme of* $\mathrm{TL}(\mathbf{N})$*. If* W *is stable under the action of* G *on* $\mathrm{TL}(\mathbf{N})$*, then* W *is a Tate-linear formal subvariety of* $\mathrm{TL}(\mathbf{N})$*. In other words, there exists a unique* cotorsion free *Tate unipotent subgroup* $\mathbf{N}'$ *of* $\mathbf{N}$ *such that* $W = \mathrm{TL}(\mathbf{N}')$*.*

Let λ_1 be the highest slope of $\mathbf{N}$ and let $\mathbf{Z} = \mathrm{Fil}_{\mathrm{sl}}^{\lambda_1}\mathbf{N}$ be the Tate unipotent subgroup of $\mathbf{N}$ isomorphic to $\mathrm{T}_p(Z)$ of an isoclinic p-divisible group Z over κ with slope λ_1, such that all slopes of $\mathbf{N}/\mathbf{Z}$ are strictly smaller than λ_1. An easy induction on the number of distinct slopes of $\mathbf{N}$ shows that theorem 4.1 follows from the case when $\mathbf{N}$ is isoclinic, which is a special case of [**?**], plus 4.2 and 4.3 below.

Theorem 4.2. *Notation and assumptions as in the above paragraph.*

(a) *The reduced closed subscheme* $(W \cap Z)_{\mathrm{red}}$ *of* Z *is a* p*-divisible subgroup of* Z*. Here* $(W \cap Z)_{\mathrm{red}}$ *is the largest reduced closed subscheme of* Z *contained in* W*.*

(b) *The formal subscheme* W *of* $\mathrm{TL}(\mathbf{N})$ *is stable under the action of the* p*-divisible subgroup* $Z' := (W \cap Z)_{\mathrm{red}}$ *of* Z*.*

Let $\mathbf{Z}'$ *be the Tate unipotent subgroup of* $\mathbf{Z}$ *corresponding to* Z'*. Let* $\mathbf{N}_1 := \mathbf{N}/\mathbf{Z}'$*. The quotient* $W_1 := W/Z'$ *is an irreducible closed formal subscheme of* $\mathrm{TL}(\mathbf{N}_1)$*, and is stable under the action of* G*. Let* $\bar{\pi} : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ *be the map associated to the quotient map* $\mathbf{N}_1 \twoheadrightarrow \mathbf{N}_2$*.*

(c) *The restriction* $\bar{\pi}\big|_{W_1} : W_1 \longrightarrow \mathrm{TL}(\mathbf{N}_2)$ *of* $\bar{\pi} : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ *to* W/Z_1 *is purely inseparable.*

Theorem 4.3. *Let* $\mathbf{N}_1$ *be a Tate unipotent group over a field* κ *of characteristic* p*. Let* G *be a* p*-adic Lie group acting strongly nontrivially on* $\mathbf{N}_1$*. Let* $\mathbf{Z}_1$ *be a Tate unipotent normal subgroup of* $\mathbf{N}_1$ *stable under the action of* G*, such that* $\mathrm{Min}(\mathrm{slope}(\mathbf{Z}_1)) > \mathrm{Max}(\mathrm{slope}(\mathbf{N}_2))$*, where* $\mathbf{N}_2 := \mathbf{N}_1/\mathbf{Z}_1$*.*

Suppose that the projection map $\pi : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ *admits a* G*-equivariant section* ξ*. Then* ξ *is a TL morphism, i.e. there exists a* G*-equivariant group homomorphism* $\psi : \mathbf{N}_2 \to \mathbf{N}_1$ *of fpqc sheaves on* $\mathfrak{Sch}_\kappa$ *which splits the quotient homomorphism* $\mathbf{N}_1 \twoheadrightarrow \mathbf{N}_2$*, such that* ξ *is equal to the TL morphism* $\mathrm{TL}(\psi) : \mathrm{TL}(\mathbf{N}_2) \to \mathrm{TL}(\mathbf{N}_1)$ *induced by* ψ*.*

Remark 4.4. (i) The proof of theorem 4.3 uses rigidity of p-divisible groups in [**?**], and is easier than theorem 4.2. Here is a sketch.

The slope decompositions of $\mathfrak{Lie}(\mathbf{N}_i)_{\mathbb{Q}}$ for $i = 1, 2$ provide a canonical isomorphism $\mathfrak{Lie}((\mathbf{N}_1)_{\mathbb{Q}}) \cong \mathfrak{Lie}\mathbf{Z}_{\mathbb{Q}} \oplus \mathfrak{Lie}((\mathbf{N}_2)_{\mathbb{Q}})$. To prove 4.3, it suffices to show that the existence of a G-equivariant section of $\pi : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ implies that the sheaf $\mathfrak{Lie}((\mathbf{N}_1)_{\mathbb{Q}})$ of $\mathbb{Q}_p$-submodules of $\mathfrak{Lie}((\mathbf{N}_2)_{\mathbb{Q}})$ is stable under the Lie bracket of $\mathfrak{Lie}((\mathbf{N}_1)_{\mathbb{Q}})$, or equivalently,

$$\big[\mathbb{D}_*\big(\mathfrak{Lie}((\mathbf{N}_2)_{\mathbb{Q}})\big)_s,\, \mathbb{D}_*\big(\mathfrak{Lie}((\mathbf{N}_2)_{\mathbb{Q}})\big)_{s'}\big]_{\mathbb{D}_*(\mathfrak{Lie}(\mathbf{N}_1)_{\mathbb{Q}})} = (0)$$

for all slopes s, s' of $\mathbf{N}_2$ such that $s + s' > \mathrm{Max}(\mathrm{slope}(\mathbf{N}_2))$, where $\mathbb{D}_*\big(\mathfrak{Lie}((\mathbf{N}_2)_{\mathbb{Q}})\big)_t$ denotes the slope-t component of the Dieudonné module $\mathbb{D}_*\big(\mathfrak{Lie}((\mathbf{N}_2)_{\mathbb{Q}})\big)$ of $\mathfrak{Lie}((\mathbf{N}_2)_{\mathbb{Q}})$, regarded as a submodule of $\mathbb{D}_*\big(\mathfrak{Lie}((\mathbf{N}_1)_{\mathbb{Q}})\big)$, for every $t \in \mathrm{slope}(\mathbf{N}_2)$.

An easy induction reduces this assertion to the case when Z_1 is isoclinic. By functoriality of the slope decomposition and the Lie bracket structures of $\mathfrak{Lie}((\mathbf{N}_i)_{\mathbb{Q}})$ further reduces the assertion to the special case when $\mathbf{N}_2$ has at most 2 slopes. In this situation $\mathrm{TL}(\mathbf{N}_1)$ is isogenous to a biextension, and the desired conclusion is deduced from the orbital rigidity of p-divisible formal groups applied to suitable maps produced from the section ξ and the biextension structure of $\mathrm{TL}(\mathbf{N}_1)$.

(ii) The proof of 4.2 uses the method of hypocotyl prolongation in tempered perfections, discussed in §5. An outline of the proof of 4.2 is given in 6.

Remark 4.5. For every Tate unipotent subgroup $\mathbf{N}_1$ of a Tate unipotent group $\mathbf{N}$ over a field κ of characteristic p, there exists an Euler flow on the Tate-linear formal variety $\mathrm{TL}(N)$ which sends the Tate-linear formal subvariety $\mathrm{TL}(N)_1$ to itself. So the definition of Tate-linear formal varieties is tightly constrained by the orbital rigidity property 4.1, the requirement that the sustained deformation spaces $\mathcal{D}ef\,(X)_{\rm sus}$ of p-divisible groups over κ are Tate-linear formal varieties and proposition 2.11.

5. Tempered virtual functions and hypocotyl elongation

The proof of orbital rigidity for Tate-linear formal varieties 4.1 uses the the general strategy of the proof of orbital rigidity of p-divisible formal groups [**?**]: To show that a formal power series $f(u_1,\dots,u_a,v_1,\dots,v_a)$ over κ in $2a$ variables is identically zero, it suffices to produce a infinite sequence of congruence relations

$$f(x_1,\dots,x_a,x_1^{q^n},\dots,x_a^{q^n})\equiv 0 \ \big(\mathrm{mod}(x_1,\dots,x_a)^{d_n}\big),$$

where q is a power of p, and $(d_n)_{n\geq n_0}$ is a sequence of positive integers such that

$$\lim_{n\to\infty}\frac{q^n}{d_n}=0.$$

We call this the method of *hypocotyl elongation*. See 5.3 for a slightly more general formulation.

5.1. **Tempered virtual functions.** The notion of *tempered virtual functions* on noetherian local formal schemes in characteristic p was discovered during the investigation of the rigidity phenomenon of biextensions of p-divisible formal groups. We explain this in the simplest nontrivial case, a biextension $\pi: E\to X\times Y$ of (X,Y) by Z, where Z,X,Y are isoclinic p-divisible formal groups over a perfect field κ of characteristic p, with slopes $\lambda_1,\lambda_2,\lambda_3$ respectively, $\lambda_1=\lambda_2+\lambda_3$, and the family of Weil pairings $\big(\beta_n: X[p^n]\times Y[p^n]\to Z[p^n]\big)_{n\geq 1}$ is non-trivial. If one naively follows the method of hypocotyl elongation 5.3, one would attempt to find a projection morphism $\tilde{\mathfrak{z}}: E\to Z$ from E to Z, and use it to construct a "first order asymptotic approximation" of the action on E of any given one-parameter subgroup of $\mathrm{Aut}_{\rm biext}(E)$. However this is a non-starter, for there is no projection morphism $E\to Z$ which is natural in any sense.

Next one tries to find, for each given one-parameter subgroup $\exp(tv)$ of $\mathrm{Aut}_{\rm biext}(E)$, where v is an element of $\mathrm{Lie}(\mathrm{Aut}_{\rm biext}(E))$ and t ranges through an open subgroup $p^m\mathbb{Z}_p$ of $(\mathbb{Z}_p,+)$, a morphism $\tilde{\mathfrak{z}}[v]: E\to Z$ which interpolates the action of $\exp(tv)$ on E. This attempt fails again. However if one analyzes this failure more closely, one sees that the root cause is that the ring of all formal functions on E is "too small". If one enlarges the affine coordinate ring R_E of E to a suitable extension ring $R_E^\flat$ contained in the completion of the perfection of R_E, one gets a "virtual morphism $\tilde{\mathfrak{z}}[v] : E \dashrightarrow Z$ with coefficients in $R_E^\flat$" which has the desired properties. Here a virtual morphism $E \dashrightarrow Z$ is interpreted as a continuous κ-linear homomorphism from the affine coordinate ring of Z to $R_E^\flat$. See steps 2–3 of 6 for a description of the process of producing such a virtual $\tilde{\mathfrak{z}}[v]$ in the more general setting of Tate-linear formal varieties.

Tempered virtual functions on the formal spectrum $\mathrm{Spf}(R)$ of a complete noetherian integral domain $(R, \mathfrak{m})$ over a perfect field κ of characteristic p are elements contained in *tempered perfections* of $(R, \mathfrak{m})$. There are several filtered inductive families of tempered perfections of $(R, \mathfrak{m})$. These families are cofinal with each other, so their inductive limits focoincide. In 5.2 we give a summary of tempered perfections of complete augmented local domains over perfect fields of characteristic p, and refer to [**?**, Ch. 10.7] for more information.

5.2. **Tempered perfections.** Let κ be a perfect field of characteristic p, and let $(R, \mathfrak{m})$ be a complete augmented noetherian local integral domain over κ, i.e. structural homomorphism $\kappa \to R$ of the κ-algebra R induces an isomorphism $\kappa \xrightarrow{\sim} R/\mathfrak{m}$.

(a) There is a family

$$\big(\,(R, \mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}\,\big)_{r,s,i_0}$$

of non-noetherian complete augmented local domains over κ, sandwiched between $(R, \mathfrak{m})$ and the completion $((R, \mathfrak{m})^{\mathrm{perf}})^\wedge$ of its perfection $(R, \mathfrak{m})^{\mathrm{perf}}$, with integer parameters r, s, i_0 satisfying

$$0 < r < s, \quad i_0 \geq 0.$$

Recall that the perfection R^{perf} of $(R, \mathfrak{m})$ has a decreasing filtration $\mathrm{Fil}^\bullet_{\deg} R^{\mathrm{perf}}$ indexed by real numbers, defined by

$$\mathrm{Fil}^u_{\deg} R^{\mathrm{perf}} := \begin{cases} \big\{x \in R^{\mathrm{perf}} \mid \exists j \in \mathbb{N} \text{ s.t. } x^{p^j} \in \mathfrak{m}^{\lceil u \cdot p^j \rceil}\big\} & \text{if } \ u \geq 0 \\ R^{\mathrm{perf}} & \text{if } \ u \leq 0 \end{cases}$$

The completed perfection $((R, \mathfrak{m})^{\mathrm{perf}})^\wedge$ of R is the completion of R^{perf} with respect to the above filtration. The filtration $\mathrm{Fil}^\bullet_{\deg}$ on R^{perf} induces a decreasing filtration on $((R, \mathfrak{m})^{\mathrm{perf}})^\wedge$, denoted again by $\mathrm{Fil}^\bullet_{\deg}$.

By definition, $(R, \mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}$ is the completion of the subring

$$\sum_{n \geq 1} \phi^{-nr}\big(\mathfrak{m}^{ns - i_0}\big)$$

of R^{perf} with respect to the filtration given by powers of the ideal generated by $\mathfrak{m}$, where ϕ is the Frobenius endomorphism $x \mapsto x^p$ of R^{perf}. Every closed subring of the completed perfection of R which lies between R and $(R, \mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}$ for some parameters (r, s, i_0) is said to be a *tempered perfection* of $(R, \mathfrak{m})$.

Clearly each $(R, \mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}$ is a tempered perfection of $(R, \mathfrak{m})$. This family of complete augmented local domains over κ is *filtered* in the following sense: given any two rings R_1, R_2

in this family, there is a third ring R_3 in the family which contains both R_1 and R_2. The union

$$(R,\mathfrak{m})^{\mathrm{tmp\,perf}} := \bigcup_{r,s,i_0} (R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$$

is a subring of $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$ which contains R^{perf}, but strictly smaller than $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$. Elements of $(R,\mathfrak{m})^{\mathrm{tmp\,perf}}$ will be called *tempered* elements of the completed perfection $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$ of R. The filtration $\mathrm{Fil}^\bullet_{\mathrm{deg}}$ on the completed perfection $(R^{\mathrm{perf}})^\wedge$ induces a filtration on each tempered perfection of $(R,\mathfrak{m})$.

(b) There are other versions of families

$$\big((R,\mathfrak{m})^{\mathrm{perf},\#}_{s:\phi^r;[i_0]}\big)_{r,s,i_0},\quad \big((R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\big)_{A,b,d} \quad\text{and}\quad \big((R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\big)_{A,b,d}$$

of tempered perfections of $(R,\mathfrak{m})$, indexed by parameters (r,s,i_0) and (A,b,d) respectively. Each of these three families is cofinal with the family $\big((R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}\big)_{r,s,i_0}$. For instance each ring $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$ is contained in $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$ for a suitable (A,b,d), and each $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$ is contained in a $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$. Therefore the union of rings in each of these families is equal to the subring $(R,\mathfrak{m})^{\mathrm{tmp\,perf}}$ consisting of all tempered elements of $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$. Any ring in one of the above families is a *tempered perfection* of $(R,\mathfrak{m})$. It is instructive to regard elements of $(R,\mathfrak{m})^{\mathrm{tmp\,perf}}$ as a sort of "tempered generalized functions" on the formal scheme $\mathrm{Spf}(R)$. We will call elements of $(R,\mathfrak{m})^{\mathrm{tmp\,perf}}$ *tempered virtual functions* on $\mathrm{Spf}(R)$.

(c) For fixed parameters (r,s,i_0), the assignment

$$(R,\mathfrak{m}) \rightsquigarrow (R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$$

is functorial in $(R,\mathfrak{m})$. Moreover the continuous κ-algebra homomorphism

$$h^\flat : (R_1,\mathfrak{m}_1)^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]} \to (R_2,\mathfrak{m}_2)^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$$

induced by a continuous κ-algebra homomorphism

$$h : (R_1,\mathfrak{m}_1) \to (R_2,\mathfrak{m}_2)$$

is surjective (respectively injective) if h is surjective (respectively injective). The same is true for the formations of $(R,\mathfrak{m})^{\mathrm{perf},\#}_{s:\phi^r;[i_0]}$, $\big((R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\big)_{A,b,d}$ and $\big((R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\big)_{A,b,d}$.

(d) We illustrate the general idea of tempered virtual functions with the family

$$\big(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}\big)_{E,c,d}$$

of tempered perfections of power series ring $\kappa[[t_1,\dots,t_m]]$, depending on parameters (E,c,d), where E,c,d are real numbers, $E>0, C>0, d\geq 0$. As we have already mentioned, this family of tempered perfections of $\kappa[[t_1,\dots,t_m]]$ is cofinal with each of the four families of tempered perfections of $\kappa[[t_1,\dots,t_m]]$.

By definition $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,\flat}$ consists of all formal power series of the form

$$\sum\nolimits_{I\in\mathrm{supp}(m:\flat:E;C,d)} b_I\,\underline{t}^I$$

such that $b_I\in\kappa$ for all $I\in\mathrm{supp}(m:\flat:E;C,d)$. Here

- $\mathrm{supp}(m:\flat:E;C,d)$ is the sub-semigroup of $(\mathbb{N}[\frac{1}{p}]^m,+)$ given by
$$\mathrm{supp}(m:\flat:E;C,d):=\left\{\,I\in\mathbb{N}[\tfrac{1}{p}]^m\;\middle|\;|I|_p\le\max\bigl(C\cdot(|I|_\sigma+d)^E,1\bigr)\,\right\}.$$
- $\mathbb{N}[\frac{1}{p}]^m$ is the sub-semigroup of $(\mathbb{Z}[\frac{1}{p}]^m,+)$ consisting of all m-tuples $(i_1,\dots,i_m)$ in $\mathbb{Z}[\frac{1}{p}]^m$ with all entries $i_j\ge 0$.
- For each $I=(i_1,\dots,i_m)\in\mathbb{N}[\frac{1}{p}]^m$, $|I|_p$ is the usual p-adic norm of I given by
$$|I|_p:=p^{-\mathrm{ord}_p(\max(i_1,\dots,i_m))},$$
while $|I|_\sigma$ is the archimedean norm of I given by
$$|I|_\sigma:=i_1+\cdots+i_m.$$

In particular the p-adic norm of I is bounded by a polynomial $f_{E,c,d}(|I|_\sigma)$ of the archimedean norm of I, for all I in $\mathrm{supp}(m:\flat:E;C,d)$.

(e) If we replace the archimedean norm $|I|_\sigma$ on $\mathbb{N}[\frac{1}{p}]^m$ by the max norm

$$|I|_\infty:=\max(i_1,\dots,i_p),$$

we get a ring $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,\#}$. The resulting family

$$\bigl(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,\#}\bigr)_{E,c,d}$$

of tempered perfections $\kappa[[t_1,\dots,t_m]]$ is cofinal with the family $\bigl(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,\flat}\bigr)_{E,c,d}$ as well.

Proposition 5.3 below, called *hypocotyl elongation* in commutative noetherian local domains over perfect fields of characteristic p, is the main input of the proof of rigidity of p-divisible formal groups in [**?**]. It follows from propositions 2.1 and 3.1 of [**?**], and provides a way to establish power series relations $f(u_1,\dots,u_a,v_1,\dots,v_b)$ between functions on a product formal scheme $\mathrm{Spf}(R)\times\mathrm{Spf}(R)$ of the form $\mathrm{pr}_1^*g_1,\dots,\mathrm{pr}_1^*g_a$ and $\mathrm{pr}_2^*h_1,\dots,\mathrm{pr}_2^*h_b$.

Proposition 5.3 (Hypocotyl elongation in complete noetherian local domains). *Let κ be a perfect field of characteristic p. Let $\mathbf{u}=(u_1,\dots,u_a)$, $\mathbf{v}=(v_1,\dots,v_b)$ be two tuples of variables, and let $f(\mathbf{u},\mathbf{v})\in\kappa[[\mathbf{u},\mathbf{v}]]$ be a formal power series in variables $u_1,\dots,u_a,v_1,\dots,v_b$ with coefficients in κ. Let $(R,\mathfrak{m})$ be an augmented noetherian complete local domain R over κ such that $\kappa\xrightarrow{\sim}R/\mathfrak{m}$. Let $g_1,\dots,g_a,h_1,\dots,h_b$ be elements of the maximal ideal $\mathfrak{m}$. Let $n_0\in\mathbb{N}$ and let $(d_n)_{n\ge n_0}$ be a sequence of positive integers and let q be a power of p such that $\lim_{n\to\infty}\frac{q^n}{d_n}=0$. Suppose that*

$$(\star)\qquad f(g_1,\dots,g_a,h_1^{q^n},\dots,h_b^{q^n})\equiv 0\pmod{\mathfrak{m}^{d_n}}$$

for all $n \geq n_0$. Then

$$f(g_1 \otimes 1, \ldots, g_a \otimes 1, 1 \otimes h_1, \ldots, 1 \otimes h_b) = 0$$

in the completed tensor product $R \hat{\otimes}_k R$, where $R \hat{\otimes}_k R$ is the formal completion of the local domain $R \otimes_\kappa R$.

Proposition 5.4 extends 5.3 to tempered perfections, and allows $g_1, \ldots, g_a, h_1, \ldots, h_b$ to be tempered virtual functions on $\mathrm{Spf}(R)$. See [**?**, Ch. 10 §5] for a proof.

Proposition 5.4 (Hypocoptyl elongation in tempered perfections). *Let $(R, \mathfrak{m})$ be an augmented complete Noetherian local domain over a perfect field κ of characteristic p.*

- *Let $g_1, \ldots, g_m, h_1, \ldots, h_{m'}$ be elements of the maximal ideal of $(R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d}$.*
- *Let $f(u_1, \ldots, u_m, v_1, \ldots, v_{m'})$ be an element of*
$$\kappa\langle\langle u_1^{p^{-\infty}}, \ldots, u_m^{p^{-\infty}}, v_1^{p^{-\infty}}, \ldots, v_{m'}^{p^{-\infty}}\rangle\rangle^{E, \flat}_{C;\, d}$$
which lies in the closure of the image of
$$\kappa\langle\langle \underline{u}^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d} \otimes_\kappa \kappa\langle\langle \underline{v}^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d} \longrightarrow \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}.$$
- *Let $q = p^r$ be a power of p for some positive integer r. Let $(d_n)_{n \in \mathbb{N},\, n \geq n_0}$ be a sequence of positive integers such that $\lim_{n\to\infty} \frac{q^n}{d_n} = 0$.*

Suppose that

$$(\star^\flat) \qquad f(g_1, \ldots, g_m, h_1^{q^n}, \ldots, h_{m'}^{q^n}) \equiv 0 \ \left(\mathrm{mod}\ \mathrm{Fil}^{d_n}_{\deg}(R, \mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'}\right)$$

in $(R, \mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'}$ for all sufficiently large natural numbers n. Then

$$f(g_1 \otimes 1, \ldots, g_m \otimes 1, 1 \otimes h_1, \ldots, 1 \otimes h_{m'}) = 0$$

in the completed tempered perfection $\left(R \hat{\otimes}_\kappa R, \mathfrak{m}_{R\hat{\otimes}_\kappa R}\right)^{\mathrm{perf},\flat}_{A',b';d'}$ of $R \hat{\otimes}_\kappa R$.

Remark 5.5. (a) The condition in 5.4 that the relation $f(u_1, \ldots, u_a, v_1, \ldots, v_b)$ lies in the closure of the image of $\kappa\langle\langle \underline{u}^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d} \otimes_\kappa \kappa\langle\langle \underline{v}^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d} \longrightarrow \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}$ may seem to be too stringent at first sight. However some subtleties in tensor products of tempered perfections are to be expected, if one recalls analogous situations in the theory of distributions, such as the Schwartz kernel theorem.

(b) The proof of orbital rigidity of Tate-linear formal varieties uses the special case of proposition 5.4 when the relation $f(\mathbf{u}, \mathbf{v})$ is an element of $\kappa[[u_1, \ldots, u_a, v_1, \ldots, v_b]]$.

(c) In 5.3, f is a formal function on a product formal scheme $X \times Y$, where $X = \mathrm{Spf}(\kappa[[\mathbf{u}]])$, $Y = \mathrm{Spf}(\kappa[[\mathbf{v}]])$, and the tuple $(g_1, \ldots, g_a)$ (respectively $(h_1, \ldots, h_b)$) represents a formal κ-morphism from $\mathrm{Spf}(R)$ to X (respectively Y). Coordinates are similarly involved in the statement of proposition 5.4. Proposition 5.6 below is a "coordinate-free" formulation of 5.4. The special case when S_2 is the affine coordinate ring of an isoclinic p-divisible group Z of slope λ_1 such that $Z[p^{\lambda_1 r_1}] = Z[\mathrm{Fr}^q]$ with $q = p^{r_1}$ is used in the proof of orbital rigidity of Tate-linear formal varieties. The condition $Z[p^{\lambda_1 r_1}] = Z[\mathrm{Fr}^q]$ implies that there is a coordinate system on Z in which the endomorphism $[p^{\lambda_1 r_1}]_Z$ corresponds to "raising all coorinates to the p^{r_1}-th power".

Proposition 5.6 (Hypocoptyl elongation in tempered perfections reformulated). *Let κ be a perfect field of characteristic p which contains a finite field with $q = p^r$ elements. Let $(R,\mathfrak{m})$, $(S_1,\mathfrak{m}_1)$ and $(S_2,\mathfrak{m}_2)$ be augmented commutative noetherian local domains over κ. Assume that S_2 has an $\mathbb{F}_q$-model $S_{2,\mathbb{F}_q}$, i.e. an augmented noetherian local subring $S_{2,\mathbb{F}_q}$ over $\mathbb{F}_q$ such that the natural map $S_{2,\mathbb{F}_q}\hat{\otimes}_{\mathbb{F}_q}\kappa \to S_2$ is an isomorphism.*

- *Let $\phi = \phi_q : S_2 \to S_2$ be the κ-linear continuous ring endomorphism of S_2 which sends every element $x \in S_{2,\mathbb{F}_q}$ to x^q.*
- *Let $g_1 : S_1 \to (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$ and $g_2 : S_2 \to (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$ be continuous κ-linear ring homomorphisms from S_i to $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$, $i = 1, 2$.*
- *Let f be an element of the completed tensor product $(S_1,\mathfrak{m}_1)^{\mathrm{perf},\flat}_{A_1,b_1;d_1} \hat{\otimes}_\kappa (S_2,\mathfrak{m}_2)^{\mathrm{perf},\flat}_{A_2,b_2;d_2}$ for parameters (A_1, b_1, d_1) and (A_2, b_2, d_2).*
- *Let $(d_n)_{n\in\mathbb{N},\, n\geq n_0}$ be a sequence of positive integers such that $\lim_{n\to\infty} \frac{q^n}{d_n} = 0$.*
- *Let (A', b', d') be suitable parameters such that the homomorphisms g_i extends to a continuous κ-linear homomorphism $g_i^\flat : (S_i,\mathfrak{m}_i)^{\mathrm{perf},\flat}_{A_i,b_i;d_i} \to (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'}$ for $i = 1, 2$.*

Let $(g_1^\flat \cdot g_2^\flat) \circ (1 \otimes (\phi_q^\flat)^n)$ be the composition

$$(S_1,\mathfrak{m}_1)^{\mathrm{perf},\flat}_{A_1,b_1;d_1} \hat{\otimes}_\kappa (S_2,\mathfrak{m}_2)^{\mathrm{perf},\flat}_{A_2,b_2;d_2} \xrightarrow{1\otimes(\phi_q^\flat)^n} (S_1,\mathfrak{m}_1)^{\mathrm{perf},\flat}_{A_1,b_1;d_1} \hat{\otimes}_\kappa (S_2,\mathfrak{m}_2)^{\mathrm{perf},\flat}_{A_2,b_2;d_2} \xrightarrow{g_1^\flat\cdot g_2^\flat} (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'} .$$

Suppose that

$$(\dagger) \qquad \big((g_1^\flat \cdot g_2^\flat) \circ (1 \otimes (\phi_q^\flat)^n)\big)(f) \equiv 0 \pmod{\mathrm{Fil}^{d_n}_{\deg}(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'}}$$

for all sufficiently large natural numbers n. Then

$$(g_1^\flat \otimes g_2^\flat)(f) = 0 \quad \text{in} \quad (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'} \hat{\otimes}_\kappa (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'} ,$$

where $g_1^\flat \otimes g_2^\flat$ denotes the composition

$$(S_1,\mathfrak{m}_1)^{\mathrm{perf},\flat}_{A_1,b_1;d_1} \hat{\otimes}_\kappa (S_2,\mathfrak{m}_2)^{\mathrm{perf},\flat}_{A_2,b_2;d_2} \xrightarrow{g_1^\flat\otimes g_2^\flat} (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'} \hat{\otimes}_\kappa (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'} .$$

Proof. This is an easy consequence of 5.4. $\square$

6. An outline of the proof of theorem 4.2

Let $Z := \mathbf{V}_p(\mathbf{Z})/\mathbf{T}_p(\mathbf{Z})$, an isoclinic p-divisible group. Let λ_1 be the slope of Z, let $r_0 \in \mathbb{N}_{>0}$ be the denominator of λ_1, let $\lambda_2 := \max(\mathrm{slope}(\mathbf{N}_2))$, and let $\varsigma_1 := \max(\frac{\lambda_1}{2}, \lambda_2)$. By assumption $\lambda_1 > \varsigma_1$. Choose a positive integer multiple r_1 of r_0 and a positive integer s_1 such that $r_1 < s_1$, $s_1\varsigma_1 \in \mathbb{N}$ and $s_1\varsigma_1 < r_1\lambda_1$. Let $\mathfrak{g}_{\mathbb{Z}_p}$ be a $\mathbb{Z}_p$-lattice in the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathbf{N}_\mathbb{Q}))$ of $\mathrm{Aut}(\mathbf{N}_\mathbb{Q})$ which contains a $\mathbb{Q}_p$-basis of $\mathrm{Lie}(\mathrm{Aut}(\mathbf{N}_\mathbb{Q}))$, and $\exp_{\mathrm{Aut}(\mathbf{N}_\mathbb{Q})}(\mathfrak{g}_{\mathbb{Z}_p}) \subseteq \mathrm{Aut}(\mathbf{N})$.

Step 1. Reduction to the case when the following conditions hold.

- We may assume that κ is an algebraically closed field.
- We may assume that there exists a positive integer r_0 such that $\mathrm{Ker}(\mathrm{Fr}_Z^{r_0}) = Z[p^{r_0\lambda_1}]$, where $\mathrm{Fr}_Z^{r_0} : Z \to Z^{(p^{r_0})}$ is the r_0-th iterate of the relative Frobenius of Z.

Step 2. We know from 3.8 that there exist constants $n_0, c_0 \in \mathbb{N}$ such that the restriction to $\pi^{-1}\big(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_1\rfloor - c_0}]\big)$ of the Tate-linear automorphism $\exp_{\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})}(p^n v)$ of $\mathrm{TL}(\mathbf{N})$ is equal to the translation by a formal morphism

$$\mathfrak{z}_n[v] : \pi^{-1}\big(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_1\rfloor - c_0}]\big) \longrightarrow Z,$$

for all $n \geq n_0$ and all $v \in \mathfrak{g}_{\mathbb{Z}_p}$, i.e.

$$\exp_{\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})}(p^n v)(x) = \mathfrak{z}_n[v](x) * x$$

for all functorial points x of $\pi^{-1}\big(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_1\rfloor - c_0}]\big)$.

Analyze the maps $\mathfrak{z}_n[v]$ using the first order term in the Baker–Campbell–Hausdorff formula to show the following compatibility property of these maps: There exist constants $n_1, c_1 \in \mathbb{N}$, $n_1 \geq n_0$, such that $\lfloor n/\varsigma_1\rfloor - c_1 \leq \lfloor n/\lambda_2\rfloor - c_0$, and

$$\big(\,\mathfrak{z}_{n+1}\,[v] - [p]_Z \circ \mathfrak{z}_n[v]\big)\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{\lfloor n/\varsigma_1\rfloor - c_1}]} = 0$$

for all $n \geq n_1$ and all $v \in \mathfrak{g}_{\mathbb{Z}_p}$. We will use the following slightly weaker form of this compatibility property.

There exists a constant m_1 such that

$$\big(\,\mathfrak{z}_{(m+1)r_1\lambda_1}\,[v] - [p^{r_1\lambda_1}]_Z \circ \mathfrak{z}_{mr_1\lambda_1}[v]\big)\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]} = 0 \tag{6.1}$$

for all $m \geq m_1$ and all $v \in \mathfrak{g}_{\mathbb{Z}_p}$.

Step 3. Let R_Z and R_T be the affine coordinate rings of Z and $T = \mathrm{TL}(\mathbf{N})$ respectively. Out of the family $\big(\mathfrak{z}_{mr_1\lambda_1}\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]}\big)_{m\geq m_0}$ of maps, which satisfies the compatibility relation (6.1), one produces a tempered virtual formal morphism $\tilde{\mathfrak{z}}[v] : T \dashrightarrow Z$ which interpolates the maps $\mathfrak{z}_{mr_1\lambda_1}\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]}$, such that

$$([p^{mr_1\lambda_1}]_Z \circ \tilde{\mathfrak{z}}[v]) * \mathrm{id}_T$$

is in some sense an "asymptotic expansion" of the automorphism $\exp_{\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})}(p^{mr_1\lambda_1}v)$ up to the first order, for $m \gg 0$.

The actual meaning of a tempered virtual formal morphism $\tilde{\mathfrak{z}}[v] : T \dashrightarrow Z$ is a continuous κ-linear ring homomorphism $\tilde{\mathfrak{z}}[v]^*$ from R_Z to a tempered perfection

$$\tilde{\mathfrak{z}}[v]^* : R_Z \to (R_T, \mathfrak{m}_T)^{\mathrm{perf},\flat}_{A,b;d}$$

for suitable parameters (A, b, d). The idea is to define $\tilde{\mathfrak{z}}[v]^*(f)$ for every formal function f on Z by

$$\tilde{\mathfrak{z}}[v]^*(f) = \lim_{m\to\infty} \big(\,\mathfrak{z}_{mr_1\lambda_1}\,[v]\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]}\big)^*([p^{mr_1\lambda_1}]_Z^*)^{-1}(f).$$

The compatibility relation (6.1) guarantees that the above limit exists as a tempered virtual function, i.e. an element of $(R_T, \mathfrak{m}_T)^{\mathrm{perf},\flat}_{A,b;d}$ for some parameters (A, b, d).

Step 4. Apply the method of hypocotyl elongation in tempered perfections, 5.4 and 5.6, to conclude that W is stable under the translation by the schematic image of $\tilde{\mathfrak{z}}[v] : W \dashrightarrow Z$.

This assertion means that

$$\Big(\big((q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W\big) \circ \mu^*\Big)(f) = 0 \tag{6.2}$$

for all f in the ideal I_W of R_T which defines W and all $v \in \mathfrak{g}_{\mathbb{Z}_p}$. Here

- $q_W : R_T \twoheadrightarrow R_W$ is the natural surjective ring homomorphism from R_T to the affine coordinate ring R_W of the closed subscheme W of T,
- $q_W^\flat : \big(R_T\big)^{\mathrm{perf},\flat}_{A,b;d} \to \big(R_W\big)^{\mathrm{perf},\flat}_{A,b;d}$ is the continuous ring homomorphism between tempered perfections induced by q_W, and
- $((q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W) \circ \mu^*$ is the composition

$$R_T \xrightarrow{\mu^*} R_Z \hat{\otimes}_\kappa R_T \xrightarrow{(q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W} \big(R_W\big)^{\mathrm{perf},\flat}_{A,b;d} \hat{\otimes}_\kappa R_W .$$

We adopt a good coordinate system on Z, under which the ring endomorphism $[p^{mr_1\lambda_1}]_Z^*$ of R_Z becomes "raising all coordinates of Z to the $p^{r_1 m}$-th power". The property of $\tilde{\mathfrak{z}}[v]$ that $\exp_{\mathrm{Aut}(\mathbf{N}_\mathbb{Q})}(p^{mr_1\lambda_1}v)$ is well approximated by $([p^{mr_1\lambda_1}]_Z \circ \tilde{\mathfrak{z}}[v]) * \mathrm{id}_T$ guarantees that the congruences required by the hypocotyl elongation method are satisfied.

Step 5. Proof of 4.2 (a)–(b).

Note that the statement 4.2 (a) that $(W \cap Z)_{\mathrm{red}}$ is a p-divisible subgroup Z' of Z follows from orbital rigidity of p-divisible formal groups. The conclusion of step 4 implies that $(W \cap Z)_{\mathrm{red}}$ is stable under translation by the sum, over all elements $v \in \mathfrak{g}_{\mathbb{Z}_p}$, of the schematic images of $\tilde{\mathfrak{z}}[v]\big|_W : W \dashrightarrow Z$. It is clear that the restriction of $\tilde{\mathfrak{z}}[v]$ to Z is the endomorphism of Z induced by v. It follows quickly from the assumption that G operates strongly nontrivially on $\mathrm{TL}(\mathbf{N})$ that W is stable under translation by Z'.

Step 6. It remains to prove the assertion 4.2 (c) that the restriction $\bar{\pi}\big|_{W_1}$ to W_1 of the formal morphism $\bar{\pi} : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ is purely inseparable. Clearly we may and do assume that $(W \cap Z)_{\mathrm{red}}$ is equal to the singleton consisting of the base point of Z. We need to show that W is purely inseparable over $\mathrm{TL}(\mathbf{N}_2)$.

We use the property of the virtual tempered formal morphism $\tilde{\mathfrak{z}}[v]$ that it behaves nicely with respect to the Z-torsor structure of $\mathrm{TL}(\mathbf{N})$:

$$\tilde{\mathfrak{z}}[v](z * x) = v(z) * \tilde{\mathfrak{z}}[v](x) \quad \forall \text{ functorial points } (z, x) \text{ of } Z \times \mathrm{TL}(\mathbf{N}). \tag{6.3}$$

Strictly speaking, the property (6.3) should be interpreted as a statement that a suitable diagram of ring homomorphisms commute. We will gloss over this technical point here and describe the argument in geometric terms. The idea is that, if W is not purely inseparable over $\mathrm{TL}(\mathbf{N}_2)$, then we can find two $\kappa[[t]]$-valued points ξ_1, ξ_2 of W, such that they differ by a non-zero $k[[t]]$-point δ of Z, say $\xi_1 = \delta * \xi_2$. Then (6.3) implies that $v(\delta)$ is a $\kappa[[t]]$-point of $W \cap Z$ for all $v \in \mathfrak{g}_{\mathbb{Z}_p}$. Since G operates strongly non-trivially on $\mathrm{TL}(\mathbf{N})$, we conclude that $W \cap Z$ contains a non-zero $\kappa[[t]]$-valued point. This contradicts the assumption that $(W \cap Z)_{\mathrm{red}} = (0)$. We have finished the sketch of the proof of 4.2, and refer to [**?**] for details.

7. Open questions

Question 7.1. Clarify the relation between *cascades* and Tate-linear formal varieties.

Remark. Many Tate-linear formal varieties, including the sustained deformation spaces $\mathcal{D}ef\,(X)_{\rm sus}$ of p-divisible groups X over fields of characteristic p, have natural cascade structures in the sense of [?]. It is likely that additional constraints need to be imposed on the Weil pairings of the intermediate biextensions of a given cascade constituted from p-divisible formal groups over a field κ of characteristic p, in order to force the cascade to have a Tate-linear structure over κ.

7.2. Let X be a p-divisible group over an algebraically closed field κ of characteristic p. Let $\mathcal{D}ef(X)$ be the *equi-characeristic* p deformation space of X. Let $\mathcal{W} = \mathcal{W}(X)$ be the largest reduced closed subscheme of $\mathcal{D}ef(X)$ such that every $\kappa[[t]]$-point of $\mathcal{W}$ corresponds to a p-divisible group over $\kappa[[t]]$ with constant Newton polygon. Denote by R the affine coordinate ring of $\mathcal{W}$, an augmented complete noetherian local κ-algebra with maximal ideal $\mathfrak{m}$. Let $\mathfrak{X}$ be the universal p-divisible group over $\mathcal{W}$. For every pair (m,n) of positive integer, let $\mathcal{W}_m = \mathrm{Spf}(R/\mathfrak{m}^{m+1})$, and let

$$\mathscr{E}sm(\mathfrak{X})_{m,n} := \mathscr{I}som_{\mathcal{W}_m\times\mathcal{W}_m}\big(\mathrm{pr}_2^*\mathfrak{X}[p^n]|_{\mathcal{W}_m}, \mathrm{pr}_1^*\mathfrak{X}[p^n]|_{\mathcal{W}_m}\big),$$

the Isom scheme over $\mathcal{W}_m\times\mathcal{W}_m$ between the pull-backs of $\mathfrak{X}[p^n]$ of the two projections maps $\mathrm{pr}_1,\mathrm{pr}_2 : \mathcal{W}_m\times\mathcal{W}_m\to\mathcal{W}_m$. Let $r_{m,n;m,n+i} : \mathscr{E}sm(\mathfrak{X})_{m,n+i}\to\mathscr{E}sm(\mathfrak{X})_{m,n}$ be the natural restriction map for every $i\in\mathbb{N}$. Let

$$\mathscr{E}sm^{\rm st}(\mathcal{X})_{m,n} := \mathrm{Image}\big(\mathscr{E}sm(\mathfrak{X})_{m,n+N}\to\mathscr{E}sm(\mathfrak{X})_{m,n}\big),\quad N\gg 0,$$

be the schematic image of $r_{m,n;m,n+N}$ for sufficiently large N, a scheme of finite type over $\mathcal{W}_m\times\mathcal{W}_m$. Let

$$\mathcal{R}(\mathfrak{X})_{m,n} := \mathrm{Image}\big(\mathscr{E}sm(\mathfrak{X})_{m,n}\to\mathcal{W}_m\times\mathcal{W}_m\big)$$

be the schematic image of the map $\mathscr{E}sm(\mathfrak{X})_{m,n}\to\mathcal{W}_m\times\mathcal{W}_m$. Since each $\mathcal{R}(\mathfrak{X})_{m,n}$ is an Artinian scheme, $\mathcal{R}(\mathfrak{X})_{m,n}$ stabilizes as $n\to\infty$. Let

$$\mathcal{R}(\mathfrak{X})_m = \mathcal{R}(\mathfrak{X})_{m,n}\quad\text{for } n\gg 0.$$

Let S_m be the affine coordinate ring of $\mathcal{R}(\mathfrak{X})_m$, a quotient of $(R/\mathfrak{m}^{m+1})\hat{\otimes}_\kappa(R/\mathfrak{m}^{m+1})$. We have natural κ-linear homomorphisms

$$q_{m,m+1} : S_{m+1}\to S_m,$$

corresponding to the natural morphisms

$$j_{m+1,m} : \mathcal{R}(\mathfrak{X})_m\to\mathcal{R}(\mathfrak{X})_{m+1}.$$

Let $S := \varprojlim_m S_m$, and let $\mathcal{R}(\mathfrak{X}) := \mathrm{Spf}(S)$. Let

$$f_X : \mathcal{R}(\mathfrak{X})\to\mathcal{W}(X)$$

be the morphism induced by the first projection $\mathrm{pr}_1 : \mathcal{W}(X)\times\mathcal{W}(X)\to\mathcal{W}(X)$

Similarly, for a polarized p-divisible group (Y,λ) over k, we have a formal morphism

$$f_{Y,\lambda} : \mathcal{R}(\mathfrak{Y},\lambda) \to \mathcal{W}(Y,\lambda)$$

defined in an analogous way, where $\mathcal{W}(Y,\lambda)$ is the largest reduced closed subscheme of $\mathcal{D}ef(Y,\lambda)$ corresponding to deformations of (Y,λ) with constant Newton polygon.

Question. Are the formal morphisms f_X and $f_{Y,\lambda}$ *flat*? Do they have other regularity properties, if any?

Remark. (a) Question 7.2 stems from an attempt to analyze the equivalence relation "lying on the same central leaf" in a given Newton polygon stratum of a Siegel modular variety in characteristic p. A good understanding of the morphism $\mathcal{R}(\mathfrak{Y},\lambda) \to \mathcal{W}(Y,\lambda)$ may lead to a notion of "families of Tate-linear formal varieties" involving a notion of "families of Tate unipotent groups".

(b) It seems plausible that for "generic" X and (Y,λ), the morphisms f_X and $f_{Y,\lambda}$ are flat.

Question 7.3. Develop a theory of families of Tate-linear formal varieties, i.e. Tate-linear formal schemes over general base schemes. Does a version of 2.11 hold in such a theory?

Remark. It is tempting to define the notion of a Tate unipotent group $\mathbf{N}$ over a general base scheme S exactly as in 2.1, without the restriction that S is the spectrum of a field of characteristic p, then define $\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ to be the Tate-linear formal scheme over S attached to $\mathbf{N}$. However such a definition may be too restrictive. It may be better to impose weaker conditions. A starting point can be the following: one requires that $\mathbf{N}$ is a sheaf of nilpotent groups on the big fpqc site of S, and there exists a p-divisible group X over S with the following properties.

(a) The Lie $\mathbb{Q}_p$-algebra $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ of the Mal'cev completion $\mathbf{N}_{\mathbb{Q}}$ is isomorphic to the $\mathbb{Q}_p$-Tate module $\mathbf{V}_p(X)$ attached to a X.
(b) $\mathbf{T}_p(X)$ is a sheaf of Lie $\mathbb{Z}_p$-submodules of $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$.
(c) The exponential of the sheaf of Lie $\mathbb{Z}_p$-subalgebras $\mathbf{T}_p(X)$ of $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ is an fqpc sheaf of nilpotent subgroups of $\mathbf{N}_{\mathbb{Q}}$ isogenous to $\mathbf{N}$.

Question 7.4. Let Z be a reduced irreducible closed subscheme of a central leaf $\mathcal{C}$ in a Siegel modular variety over $\overline{\mathbb{F}_p}$. Suppose that the formal completion $Z^{/x_0}$ of Z at an $\overline{\mathbb{F}_p}$-point $x_0 \in Z(\overline{\mathbb{F}_p})$ is a Tate-linear formal subvariety of $\mathcal{C}^{/x_0}$. Show that the formal completion $Z^{/x}$ is a Tate-linear formal subvariety of $\mathcal{C}^{/x}$ for every $x \in Z(\overline{\mathbb{F}_p})$.

Remark. The question 7.4 asks for a "principle of analytic continuation" of Tate linearity. One may need to pass to the normalization of Z to sidestep the potential issue that $Z^{/x}$ might have self-intersection.

7.5. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Let $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ be the ring of integers of $\overline{\mathbb{Q}}_p$. Let $S_{\overline{\mathbb{Q}}}$ be a Shimura subvariety of a Siegel modular variety $\mathscr{A}_{g,d,n/\overline{\mathbb{Q}}}$ over $\overline{\mathbb{Q}}$. Let S^{Zar} be the Zariski closure of $S_{\overline{\mathbb{Q}}}$ in $\mathscr{A}_{g,d,n/\mathcal{O}_{\overline{\mathbb{Q}_p}}}$, and let $S_{\overline{\mathbb{F}_p}}$ be the closed fiber of S^{Zar}, which is a closed subscheme of the closed fiber $\mathscr{A}_{g,d,n/\overline{\mathbb{F}_p}}$ of $\mathscr{A}_{g,d,n/\mathcal{O}_{\overline{\mathbb{Q}_p}}}$.

Question. Suppose that $S_{\overline{\mathbb{F}}_p}$ is contained in the Zariski closure of a central leaf $\mathcal{C}$ in $\mathscr{A}_{g,d,n/\overline{\mathbb{F}}_p}$. Let $x_0 \in (S_{\overline{\mathbb{F}}_p} \cap \mathcal{C})(\overline{\mathbb{F}}_p)$ be an $\overline{\mathbb{F}}_p$-point of $S_{\overline{\mathbb{F}}_p} \cap \mathcal{C}$. Show that the formal completion $(S_{\overline{\mathbb{F}}_p} \cap \mathcal{C})^{/x_0}$ of $S_{\overline{\mathbb{F}}_p} \cap \mathcal{C}$ at x_0 is a Tate-linear formal subvariety of $\mathcal{C}^{/x_0}$.

Remark. The desired conclusion in 7.5 is known when p is large relative to g. So this question is about small primes, including primes ramified with respect to the input data for the Shimura variety $S_{\overline{\mathbb{Q}}}$.

Question 7.6. Let $\mathcal{C}$ be a central leaf in a Siegel modular variety over $\overline{\mathbb{F}}_p$. Let $x_0 \in \mathcal{C}(\overline{\mathbb{F}}_p)$ be an $\overline{\mathbb{F}}_p$-point of $\mathcal{C}$. Determine which Tate-linear formal subvarieties of $\mathcal{C}^{/x_0}$ are of the form $(S_{\overline{\mathbb{F}}_p} \cap \mathcal{C})^{/x_0}$ for some Shimura subvariety S of $\mathscr{A}_{g,d,n/\overline{\mathbb{Q}}}$ as in 7.5.

Question 7.7. (a) Determine whether every Tate-linear formal variety over a field of characteristic p admits a lifting to characteristic 0 (in the sense of an answer to 7.3).

(b) Generalize the notion of complex multiplication tdo Tate-linear formal varieties over a field of characteristic p.

(c) Let T be a Tate-linear formal variety over a finite field κ of characteristic p, with sufficiently many complex multiplication in the sense of an answer to (b) above. Does T admit a "quasi-canonical lifting" to characteristic 0 in a suitable sense?

(d) Let $\mathcal{O}$ be the ring of integers of a finite extension field of $\mathbb{Q}_p$. Is there a good p-adic Hodge theory for Tate-linear formal varieties over $\mathrm{Spec}(\mathcal{O})$?

7.8. Let $\mathcal{O}$ be the ring of integer of a finite extension field K of $\mathbb{Q}_p$. Let $\kappa = \mathcal{O}/\mathfrak{m}$ be the residue field of $\mathcal{O}$. Let X, Y be p-divisible groups over $\mathcal{O}$ whose closed fibers X_κ, Y_κ are isoclinic of slopes μ_X, μ_Y respectively, and $\mu_X < \mu_Y$. For every positive integer n, let $\mathscr{H}_n = \mathscr{H}om_{\mathrm{Spec}(\mathcal{O})}(X[p^n], Y[p^n])$ be the Hom scheme over $\mathrm{Spec}(\mathcal{O})$ from $X[p^n]$ to $Y[p^n]$, and let $\pi_{n,n+1} : \mathscr{H}_{n+1} \to \mathscr{H}_n$ be the homomorphism over $\mathcal{O}$ such that

$$(Y[p^n] \hookrightarrow Y[p^{n+1}]) \circ \pi_{n,n+1}(h) = h \circ (X[p^n] \hookrightarrow X[p^{n+1}])$$

for every functorial point h of $\mathscr{H}_{n+1}$. Let H_n be the Zariski closure in $\mathscr{H}_n$ of the generic fiber of $\mathscr{H}_n$. Clearly H_n is a commutative finite locally free group scheme over $\mathcal{O}$, because $\mathscr{H}_n$ is proper over $\mathcal{O}$.

Question. Is the homomorphism $\pi_{n,n+1}|_{H_{n+1}} : H_{n+1} \to H_n$ induced by $\pi_{n,n+1}$ flat (hence faithfully flat) over $\mathcal{O}$, for every positive integer n?

Remark. Question 7.8 is a test of a naive approach to the general question on lifting Tate-linear formal varieties from characteristic p to characteristic 0. It seems likely that the answer is negative in general. If so, one would like to find some conditions on the p-divisible groups X and Y over $\mathcal{O}$ under which the answer to 7.8 is affirmative.

7.9. There are many obvious questions about tempered perfections. We mention two here. Let κ be a perfect field of characteristic p. For any augmented complete noetherian local domain $(R, \mathfrak{m})$ over κ, denote by $(R, \mathfrak{m})^{\mathrm{tmp\,perf}}$ the ring consisting of all tempered virtual functions on $\mathrm{Spf}(R)$.

(i) Investigate the spaces consisting of all continuous valuations of topological κ-algebras $(R,\mathfrak{m})^{\mathrm{tmp\,perf}}$ (respectively $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$), endowed with suitable topologies. Describe them explicitly in the case when R is the formal power series ring over κ in 2 variables.

(ii) Develop a geometric theory of tempered perfections of not-necessarily-local formal schemes, modeled on the usual theory of formal schemes.

Remark. (a) Perhaps the strongest motivation here is that tempered perfections might be useful in questions unrelated to Tate-linear structures.

(b) A subsequent quest after (ii) is to develop a theory of crystals over tempered perfections of formal schemes, such as tempered perfections of $(\mathcal{C}\times\mathcal{C})^{/\Delta_{\mathcal{C}}}$, where $\mathcal{C}$ is a central leaf in a Siegel modular variety $\mathscr{A}_{g,d,n\overline{\mathbb{F}}_p}$, and $(\mathcal{C}\times\mathcal{C})^{/\Delta_{\mathcal{C}}}$ is the formal completion of $\mathcal{C}\times\mathcal{C}$ along its diagonal subscheme $\Delta_{\mathcal{C}}$.

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