A bisection of the Artin conductor

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Version 2.1, September 2, 2003

Abstract Let K be a local field such that the residue field κ is perfect and $\mathrm{char}(\kappa) = p > 0$. For each finite quotient Γ of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ we define a class function bA_Γ on Γ with values in a cyclotomic extension over $\mathbb{Q}_p$. The sum of bA_Γ and its complex conjugate $\overline{\mathrm{bA}_\Gamma}$ is equal to the Artin character for Γ. Let A be an abelian variety over K, and let L/K be a finite Galois extension of K such that A has semistable reduction over $\mathcal{O}_L$. In this situation one has the base change conductor $c(A, K)$ of A, a numerical invariant which measures the difference between the Néron model of A and the Néron model of A_L. In the case when K is the completion at a finite place v of a number field F, and the abelian variety $A_{/K}$ comes from an abelian variety $A_{/F}$ over F, then the base change conductor $c(A, K)$ is the contribution of the place v to the loss of Faltings height of $A_{/F}$, after a finite base field extension E/F such that $A_{/E}$ has semistable reduction over $\mathcal{O}_E$. Assume that the formal completion along the zero section of the Néron model of A_L is a formal torus $\underline{G}$ over $\mathcal{O}_L$. Then the base change conductor $c(A, K)$ is equal to the pairing of $\mathrm{bA}_{\mathrm{Gal}(L/K)}$ with the character of the linear representation of $\mathrm{Gal}(L/K)$ on the character group of the formal torus $\underline{G}$.

§1. Introduction

A longer title, which would summarizes the main result of this article, is *The loss of Faltings height due to stabilization is measured by a bisection of the Artin conductor.* This sentence is elaborated in the next three paragraphs.

Let $\mathcal{O} = \mathcal{O}_K$ be a complete discrete valuation ring with fraction field K and maximal ideal $\mathfrak{p}$. We assume, for simplicity, that the residue field κ of $\mathcal{O}$ is algebraically closed of characteristic $p > 0$. Let A be an abelian variety over K. The base change conductor $c(A, K)$ is a non-negative rational number which measures the failure for A to have semistable reduction over $\mathcal{O}$; see 2.4 for the definition of $c(A, K)$. If K is the completion of the maximal unramified extension of the completion F_v at a finite place $v|p$ of a number field F, and A comes from an abelian variety A_F over F, then $\frac{\log(\mathrm{N}_v)}{[F_v:\mathbb{Q}_p]}c(A, F_v)$ is the contribution of the place v to the difference between the (unstable) Faltings height defined in [F], of A and A_E, where E is any finite extension of F such that A_E has semistable reduction over $\mathcal{O}_E$.

In this paper we provide a formula for the base change conductor $c(A, K)$ when A has *potentially ordinary reduction*, meaning that there exists a finite Galois extension L of K such that the neutral component of the closed fiber of the Néron model $\underline{A_L}^{\mathrm{NR}}$ of A_L over $\mathcal{O}_L$ is an extension of an ordinary abelian variety by a torus. The hypothesis on A implies that the formal completion $\underline{A_L}^{\mathrm{NR}\wedge}$ of $\underline{A_L}^{\mathrm{NR}}$ along the zero section is a formal torus over $\mathcal{O}_L$. There is a natural linear action of the Galois group $\mathrm{Gal}(L/K)$ on the character group of the formal torus $\underline{A_L}^{\mathrm{NR}\wedge}$; the character of this representation is a $\mathbb{Q}_p$-valued class function χ on

[1]partially supported by grant DMS 9800609 and DMS01-00441 from the National Science Foundation

$\mathrm{Gal}(L/K)$. Theorem 7.6 says that $c(A, K)$ is equal to the pairing of χ with a class function $\mathrm{bA}_{\mathrm{Gal}(L/K)}$ on $\mathrm{Gal}(L/K)$.

The definition of this class function $\mathrm{bA}_{\mathrm{Gal}(L/K)}$ can be found in 3.2.1, where a class function bA_Γ on Γ is defined for each finite quotient Γ of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Unlike the Artin character $\mathrm{Ar}_{\mathrm{Gal}(L/K)}$, which is $\mathbb{Q}$-valued, the function $\mathrm{bA}_{\mathrm{Gal}(L/K)}$ takes values in L if $\mathrm{char}(K) = 0$, in the fraction field of $W(\kappa)$ if $\mathrm{char}(K) = p$. Here $W(\kappa)$ is the ring of p-adic Witt vectors with entries in κ. Moreover bA_Γ is a $\mathbb{Q}$-linear combination of ramified characters of Γ with positive coefficients: its pairing with an irreducible character ρ of Γ is a non-negative rational number, equal to zero if and only if ρ is unramified. We would like to think of bA_Γ as a "bisection" of the Artin character Ar_Γ of Γ, because the sum of bA_Γ and its "complex conjugate" $\overline{\mathrm{bA}_\Gamma}$ is equal to Ar_Γ. The values of the Artin character are rational numbers (in fact integers), while the values of bA_Γ are algebraic numbers in finite extensions of $\mathbb{Q}_p$. Otherwise the behavior of the functions bA_Γ for passing to quotient groups and restricting to subgroups is similar to that of the Artin character; see Propositions 3.4 and Prop. 3.5.3.

This paper is a sequel to [CYdS] and [Ch]. In [CYdS] a congruence statement was proved for Néron models of tori over local fields with possibly different characteristics. The main application of this congruence result is a formula for the base change conductor $c(T, K)$ of T, proved independently by E. de Shalit: $c(T, K)$ is equal to one-half of the Artin conductor of the Galois representation on the character group of T. The congruence statement is generalized to abelian varieties in [Ch]. But unlike the case of tori, the base change conductor for abelian varieties may change under an isogeny; see [Ch, 6.10] for counterexamples. On the other hand, [Ch, Thm. 6.8] says that $c(A, K)$ remains unchanged under a K-isogeny invariant if $\mathrm{char}(K) = 0$ and A is potentially ordinary. In view of this result, an optimist may hope that there is a "formula" for $c(A, K)$, for abelian varieties A with potentially ordinary reduction, at least when $\mathrm{char}(K) = 0$. Theorem 7.6 provides such a formula; it also shows that the base change conductor for abelian varieties over a local field M with potentially ordinary reduction is invariant under M-isogeny, if the residue field κ_M of $\mathcal{O}_M$ is perfect.

The starting point of this paper is the following observation: Suppose that A is an abelian variety A over K which has ordinary reduction over $\mathcal{O}_L$ for a finite Galois L/K, i.e. the neutral component of the closed fiber of the Néron model $\underline{A_L}^{\mathrm{NR}}$ of A_L is an extension of an ordinary abelian variety by a torus. Then the formal completion along a suitable subvariety of the closed fiber of the Néron model $\underline{A}^{\mathrm{NR}}$ of A is in some sense a formal Néron model of a rigid analytic subgroup G over K of the rigid analytic space A^{an} attached to A, characterized by the property that $G \times_{\mathrm{Spm}\,K} \mathrm{Spm}\,L$ is equal to the rigid analytic space attached to (or, the generic fiber of) the formal torus $\underline{A_L}^{\mathrm{NR}\wedge}$. The rigid analytic group G above belongs to a class of rigid groups, which we call *concordant rigid groups*. After passing to a finite extension M/K, a concordant rigid group G_1 over K is equal to the generic fiber of a formal torus over $\mathcal{O}_M$; see 4.3 for a precise definition. Each concordant rigid group G has a character group $X^*(G)$, which is a free $\mathbb{Z}_p$-modules of finite rank, with a linear action by the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, operating via a finite quotient. Every $\mathbb{Z}_p$-representation of

$\mathrm{Gal}(K^{\mathrm{sep}}/K)$ with the above property is isomorphic to the character group of a concordant rigid group over K, and every concordant rigid group G is determined by the Galois module $X^*(G)$. The definition of the base change conductor can be extended to concordant rigid groups, using a suitable notion of formal Néron models; see 5.2, 5.9, and 6.1.4.

Passing from an abelian variety A over K with potentially ordinary reduction, to the concordant rigid group G attached to A, is akin to a localization process, under which the base change conductor remains the same. Similarly for tori over K. Moreover every concordant rigid group over K can be obtained as a direct summand of the concordant rigid group attached to a torus over K. The readers may find it attractive to regard the category of concordant rigid groups as the result of this localization (or, p-adic completion) process applied to the category of tori over local fields. More precisely, starts with the abelian category $\mathcal{C}$ of tori over K. Define $\mathcal{C}_{\mathbb{Z}_p}$ to be the abelian category with the same objects as $\mathcal{C}$, with $\mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}_p}}(T_1, T_2) = \mathrm{Hom}_{\mathcal{C}}(T_1, T_2) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, for tori T_1, T_2 over K. Finally, pass to the Karoubian envelope of $\mathcal{C}_{\mathbb{Z}_p}$, whose objects consists of images of idempotents, and with morphisms defined in the standard way, to get an abelian category $\mathcal{C}^{\dagger}_{\mathbb{Z}_p}$. This abelian category $\mathcal{C}^{\dagger}_{\mathbb{Z}_p}$ is isomorphic to the category of concordant rigid groups over K. From this point of view, many results in this paper are localized versions of the corresponding results for tori; they include the existence of Néron models and the isogeny invariance of the base change conductor.

A major technical point of this paper is finding a suitable notion of *formal Néron models* for rigid spaces which would allow us to extend the definition of the base change conductor to concordant rigid groups. In [BS] the authors gave a definition of Néron models for rigid spaces and proved a general existence theorem of Néron models. Unfortunately since the concordant rigid groups are not quasi-compact, the Bosch-Schlöter construction cannot be applied to produce Néron models for them. The notion of formal Néron models adopted in this paper is different, in that we allow formal schemes which are not necessarily π-adic. No general existence theorem of formal Néron models for smooth rigid spaces is proved in this paper. Instead the existence of formal Néron models of concordant rigid groups is deduced from that for tori. Also left unresolved is the question as to when a formally smooth affine formal scheme over $\mathcal{O}$ is the formal Néron model of its generic fiber, see 5.3.4 (i). For the question on the compatibility between the Bosch-Schlöter definition of formal Néron models and the notion adopted in this article, see 5.3.4 (ii).

There is an explicit formula for the base change conductor in the category of concordant rigid groups: For any concordant rigid group over G over K, $c(G, K)$ is equal to the pairing of $\mathrm{bA}_{\mathrm{Gal}(L/K)}$ with the character of the Galois representation $X^*(G)$ for a sufficiently large finite Galois extension L/K; see Thm. 7.5. The proof requires an explicit calculation of the base change morphism for an induced torus. This formula also provides an explanation of the curious factor "$\frac{1}{2}$" in the formula $c(T, K) = \frac{1}{2}\,\mathrm{Ar}_K(X^*(G))$ for the base change conductor of a torus T over K: Since the character of $X^*(T)$ is $\mathbb{Q}$-valued, its pairing with bA_Γ and with its "complex conjugate" $\overline{\mathrm{bA}_\Gamma}$, which is also equal to $\mathrm{Ar}_\Gamma - \mathrm{bA}_\Gamma$, are equal, for every finite quotient Γ of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. In some sense the only point of this paper, besides the technique of localization described above, is the observation that the base change conductor of semiabelian varieties with potentially ordinary reductions defines a bisection of the Artin conductor, which can be expressed explicitly using a class function of the local Galois group, just like the case of the Artin character.

The following is a sketch of the proof of Thm. 7.5, under the assumption that κ is algebraically closed. (Otherwise one has to allow unramified twists in the definition of concordant rigid groups, and a descent argument for the maximal unramified extension K^{sh}/K of K becomes necessary.)

(a) We formulate a notion of *formal Néron models* for smooth rigid spaces in Definition 5.2. One shows that every concordant rigid group G over K is a direct summand of the concordant rigid group attached to a torus T over K. So the formal Néron model of a G can be obtained as a direct summand of a suitable completion of the Néron model of T.

(b) The existence of formal Néron models for concordant rigid groups allows us to define the base change conductor $c(G, K)$ for a concordant rigid group G over K. If G comes from a torus T over K, or an abelian variety A over K with potentially ordinary reduction, the base change conductor of G is equal to that of T or A.

(c) From the isogeny invariance of $c(T, K)$ for tori T, one deduces that the base change conductor of a concordant rigid group G over K is invariant under K-isogeny.

(d) Artin's theorem on characters of a finite group reduces the computation of the $c(G, K)$ to the case when G is the Weil restriction from M to K, of a concordant rigid group G_1 over M with a splitting field L which is tamely ramified over M. The last case is settled by a direct computation, which finishes the proof of Thm. 7.5.

As pointed out by the referee, the main result of this article, a formula for the base change conductor of an abelian variety A over a local field K with potentially ordinary reduction, is a statement about schemes, but the proof involves rigid analytic geometry. A natural question is, where in the proof are the properties of rigid objects used in an essential way? The author does not have a truly satisfactory answer, other than the feeble attempt below.

There is "only" one class of semiabelian varieties over a local field K whose base change conductor has been explicitly computed, namely the Weil restriction $\mathrm{R}_{L/K}(G)$ of a semiabelian variety, such that G has semistable reduction over the ring of integers of a tamely ramified extension of L. These example are not enough to determine the base change conductor, when one works in the category of semiabelian varieties over K. What one gains from using rigid analytic geometry is that there are more objects, other than abelian varieties, or more generally, semiabelian varieties, to work with. Moreover, there are more morphisms between the rigid objects which behave nicely with respect to the base change conductor, than there are between abelian varieties. The available examples of concordant rigid groups whose base change conductor have been computed are still the "induced ones", i.e. those of the form $\mathrm{R}_{L/K}(G)$, such that the base extension G_M of G to a tamely ramified finite extension M of L is the generic fiber of a split formal torus over $\mathcal{O}_M$. But these induced concordant rigid groups generate, with $\mathbb{Q}$-coefficients, the K-group of concordant rigid groups over the local field K. So we can squeeze out, from a meager set of examples, a formula for the base change conductor for all concordant rigid groups over K. Consequently we get a formula for the base change conductor for any semiabelian variety G over K with potentially ordinary reduction, since the base change conductor of G is determined by that of the attached concordant rigid group.

The contents of this paper are outlined as follows. In §2 we collect some notation and definitions. The class functions bA_Γ are defined and studied in §3. The definition of concordant rigid groups is introduced in §4. In §5 we formulate the notion of formal Néron models and prove that each concordant rigid group has a formal Néron model. The proof is based on a statement in commutative algebra which relies on the main result of [Sw]; see 5.3. In §6 we prove the isogeny invariance of the base change conductor for concordant rigid groups. The formula for the base change conductor is proved in §7.

It is a pleasure to thank J.-K. Yu whose idea on congruence of Néron models for tori made this work possible, and to G. Prasad who brought the question on the base change conductor for tori to my attention. I thank S. Bosch for helpful discussions on Néron models and descent for rigid analytic spaces. I would like to thank the referee, for a very careful reading, whose many comments and suggestions improve the exposition considerably. I would also like to thank T. Chinburg and S. Shatz for their interest and encouragement.

§2. Notation and Definitions

(2.1) In this paper $\mathcal{O} = \mathcal{O}_K$ denotes a henselian discrete valuation ring. Let $\mathfrak{p} = \mathfrak{p}_K$ be the maximal ideal of $\mathcal{O}$, and let $\pi = \pi_K$ be a generator of $\mathfrak{p}$. Let K be the fraction field of $\mathcal{O}$, and let $\kappa = \mathcal{O}/\mathfrak{p}$ be the residue field of O. The strict henselization of $\mathcal{O}$ and the completion of $\mathcal{O}$ will be denoted by $\mathcal{O}^{\mathrm{sh}}$ and $\widehat{\mathcal{O}}$ respectively; their field of fractions is denoted by K^{sh} and $\widehat{K}$ respectively. The residue field of $\mathcal{O}^{\mathrm{sh}}$ is κ^{sep}, the separable closure of κ.

For any finite Galois extension field L of K, denote by $\mathcal{O}_L$ the integral closure of K in L, and denote by $\mathrm{Gal}(L/K)$ the Galois group of L/K.

(2.1.1) In §4 and §5 the discrete valuation ring $\mathcal{O}$ is assumed to be complete. In §6 $\mathcal{O}$ is complete and its residue field κ is assumed to have characteristic $p > 0$. In §7 $\mathcal{O}$ is complete, κ is perfect and $\mathrm{char}(\kappa) = p > 0$.

(2.2) We review some basic facts about local fields, see [S1, chap. IV] for more information. Let K be a local field, that is K is the field of fractions of a complete discrete valuation ring $\mathcal{O}$. We assume for simplicity that the residue field $\kappa = \mathcal{O}/\mathfrak{p}$ is perfect; otherwise we will have to restrict our discussions to Galois extension M of K, such that the residue field extension

κ_M/κ_K is separable.

(2.2.1) For any finite extension M of K, let

- π_M be a uniformizing element in the maximal ideal $\mathfrak{p}_M$ of the ring of integers $\mathcal{O}_M$ in M,
- κ_M be the residue field of $\mathcal{O}_M$,
- $f(M/K) = [\kappa_M : \kappa]$ be the degree of the residue field extension,
- $e(M/K)$ be the ramification index of M/K.

(2.2.2) Let L be a finite Galois extension of K, and denote by $\Gamma = \mathrm{Gal}(L/K)$ the Galois group for L/K. The Galois group Γ has a finite decreasing filtration

$$\Gamma = \Gamma_{-1} \supseteq \Gamma_0 \supseteq \Gamma_1 \supseteq \cdots$$

by normal subgroups, called the *lower-numbering filtration*. The subgroup Γ_0 is the inertia subgroup of Γ. The subgroup Γ_1 is the wild ramification subgroup of Γ; Γ_1 is a finite p-group if the residue field has positive characteristic p, trivial if $\mathrm{char}(\kappa) = 0$. The quotient group Γ_0/Γ_1 is a cyclic group, whose order $n = n(L/K)$ is prime to p if $p > 0$. Sylow's theorem tells us that Γ_0 is a semi-direct product of Γ_1 with a cyclic group of order n.

For an integer $i \geq 0$, an element $s \in \Gamma$ is in Γ_i if and only if s operates trivially on $\mathcal{O}_L/\mathfrak{p}_L^{i+1}$. Let i_Γ be the $\mathbb{N}$-valued function on $\Gamma - \{1_\Gamma\}$ such that $i_\Gamma(s) \geq j+1$ if and only if $s \in \Gamma_j$, for any integer $j \geq -1$ and for any element $s \in \Gamma, s \neq 1_\Gamma$.

(2.2.3) The kernel of the map

$$\theta_0 : \Gamma_0 \to \kappa_L^\times, \quad s \mapsto s(\pi_L)/\pi_L \pmod{\mathfrak{p}_L} \in \kappa_L$$

is equal to Γ_1, so θ_0 induces an isomorphism

$$\overline{\theta_0} : \Gamma_0/\Gamma_1 \xrightarrow{\sim} \mu_n(\kappa_L)\,, \quad n = \mathrm{Card}(\Gamma_0/\Gamma_1)\,.$$

(2.2.4) If $\mathrm{char}(K) = 0$, let

$$\omega = \omega_{L/K} : \Gamma_0/\Gamma_1 \xrightarrow{\sim} \mu_n(L)$$

be the composition of $\overline{\theta_0}$ with the inverse of the canonical isomorphism $\mu_n(L) \xrightarrow{\sim} \mu_n(\kappa_L)$. If $\mathrm{char}(K) = p > 0$, let

$$\omega = \omega_{L/K} : \Gamma_0/\Gamma_1 \xrightarrow{\sim} \mu_n(W(\kappa_L))$$

be the composition of $\overline{\theta_0}$ with the inverse of the canonical isomorphism $\mu_n(W(\kappa_L)) \xrightarrow{\sim} \mu_n(\kappa_L)$, where $W(\kappa_L)$ is the ring of all p-adic Witt vectors for κ_L, a discrete valuation ring whose maximal ideal is generated by p.

(2.2.5) Denote by $F = F(L/K)$ the copy of the cyclotomic field $\mathbb{Q}(\mu_n)$, in L if $\mathrm{char}(K) = 0$, in the fraction field of $W(\kappa_L)$ if $\mathrm{char}(K) = p > 0$, so that we have $\omega_{L/K} : \Gamma_0/\Gamma_1 \xrightarrow{\sim} \mu_n(F_{L/K})$ in both cases. Let $\overline{F}$ be the algebraic closure of F, contained in the algebraic closure $\overline{K}$ if $\mathrm{char}(K) = 0$, otherwise contained in the algebraic closure of the fraction field of $W(\kappa^{\mathrm{sep}})$ if $\mathrm{char}(K) = p > 0$.

(2.3) Every abelian variety over K has a Néron model $\underline{A}^{\mathrm{NR}}$; it is a smooth group scheme of finite type over $\mathcal{O}$ with A as its generic fiber, characterized by the following universal property: For every smooth scheme S over $\mathcal{O}$, the canonical map

$$\underline{A}^{\mathrm{NR}}(S) \to A(S \times_{\mathrm{Spec}\mathcal{O}} \mathrm{Spec}K)$$

is a bijection. More generally every semi-abelian variety G over K has a Néron model $\underline{G}^{\mathrm{NR}}$, which is a smooth group scheme locally of finite type over $\mathcal{O}$, characterized by the same universal property. See [BLR] for more information on Néron models.

(2.4) Let G be a semiabelian variety over K. We recall the definition of the *base change conductor* $c(G) = c(G,K)$ of G in [CYdS], [Ch]: Choose a finite separable extension L of K such that $G_L = G \times_{\mathrm{Spec}K} \mathrm{Spec}L$ has semistable reduction over $\mathcal{O}_L$. Let

$$\mathrm{can} : \underline{G}^{\mathrm{NR}} \times_{\mathrm{Spec}\mathcal{O}} \mathrm{Spec}\mathcal{O}_L \to \underline{G_L}^{\mathrm{NR}}$$

be the homomorphism over $\mathcal{O}_L$ extending id_{G_L}. Then

$$c(G,K) := \frac{1}{e(L/K)} \mathrm{length}_{\mathcal{O}_L} \left(\mathrm{Lie}(\underline{G_L}^{\mathrm{NR}}) / \mathrm{can}_*(\mathrm{Lie}(\underline{G}^{\mathrm{NR}}) \otimes_{\mathcal{O}} \mathcal{O}_L) \right) .$$

This definition is independent of the choice of L.

§3. A bisection of the Artin conductor

(3.1) Let k be an algebraically closed field of characteristic 0, which we fix through (3.1.1)–(3.1.3). For each positive integer n, let μ_n be the group of n-th roots of unity in k.

(3.1.1) Definition For each positive integer n, define a k-valued function $\mathrm{bA}_n = \mathrm{bA}_{\mu_n}$ on μ_n by

$$\mathrm{bA}_n(\zeta) = \mathrm{bA}_{\mu_n}(\zeta) = \sum_{i=1}^{n-1} \frac{i}{n}\, \zeta^i \qquad \forall \zeta \in \mu_n$$

(3.1.2) Lemma (i) *The function* bA_{μ_n} *is given explicitly by the following formula*

$$\mathrm{bA}_n(\zeta) = \begin{cases} \frac{1}{\zeta - 1} & \zeta \in \mu_n, \quad \zeta \neq 1 \\ \frac{n-1}{2} & \zeta = 1 \end{cases}$$

(ii) *The function* bA_n *satisfies*

$$\sum_{\zeta\in\mu_n} \mathrm{bA}_n(\zeta) = 0\,.$$

(iii) *The sum of the function bA_n and its "complex conjugate" is given by*

$$\mathrm{bA}_n(\zeta) + \mathrm{bA}_n(\zeta^{-1}) = \begin{cases} -1 & \text{if } \zeta \neq 1 \\ n-1 & \text{if } \zeta = 1 \end{cases}$$

Here "complex conjugation" is taken according to the standard convention in the character theory of finite groups, see 3.1.3 (i) below.

(iv) *For every positive integer d and every element $\xi \in \mu_n$, we have*

$$\frac{1}{d}\sum_{\substack{\zeta\in\mu_{nd}\\ \zeta^d=\xi}} \mathrm{bA}_{nd}(\zeta) = \mathrm{bA}_n(\xi)$$

Proof. The formula (i) is easily checked: multiply $\mathrm{bA}_n(\zeta)$ with $\zeta - 1$ and simplify. One can also evaluate $z\frac{d}{dz}\left(\frac{z^n-1}{z-1}\right)$ at $z = \zeta$ for $1 \neq \zeta \in \mu_n$.

The formula (ii) follows from the identity

$$\sum_{\zeta\in\mu_n} \zeta^i = 0$$

for $i = 1, \ldots, n-1$. Another proof can be obtained by applying the residue theorem to the meromorphic differential

$$\frac{dz}{z(z-1)(z^n-1)}\,.$$

The statement (iii) is straightforward. The formula (iv) can be proved by a direct computation. Choose an element $\zeta_1 \in \mu_{nd}$ such that $\zeta_1^d = \xi$. Then

$$\frac{1}{d}\sum_{\substack{\zeta\in\mu_{nd}\\ \zeta^d=\xi}} \mathrm{bA}_{nd}(\zeta) = \frac{1}{nd}\sum_{i=1}^{nd-1} i\zeta_1^i \cdot \frac{1}{d}\sum_{\zeta\in\mu_d} \zeta^i = \frac{1}{n}\sum_{j=1}^{n-1} j\,\xi^j = \mathrm{bA}_n(\xi)\,.$$

For another proof, consider the meromorphic differential

$$\alpha(z) = \frac{dz}{z(z-1)(z^d-\xi)}\,.$$

We may and do assume that $\xi \neq 1$, since the case $\xi = 1$ follows from (ii). The residue at $z = \zeta$, with $\zeta^d = \xi$, is equal to $\frac{1}{d(\zeta-1)\xi}$. The residue at $z = 0$ is ξ^{-1}, while the residue at $z = 1$ is $(1-\xi)^{-1}$. The formula (iv) now follows from the residue theorem. ∎

(3.1.3) Remark (i) For any finite group Γ, denote by $\mathrm{R}(\Gamma)$ the integral character ring, consisting of all $\mathbb{Z}$-linear combinations of characters of finite dimensional linear representations of Γ over k. There is a $\mathbb{Q}$-valued positive definite $\mathbb{Q}$-bilinear symmetric form on the group $\mathrm{R}(\Gamma)_{\mathbb{Q}} := \mathrm{R}(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ of rational virtual characters of the finite group Γ, given by

$$(f_1, f_2) \mapsto (f_1 \,|\, f_2) := \mathrm{Card}(\Gamma)^{-1} \sum_{x \in \Gamma} f_1(x)\, f_2(x^{-1})\,.$$

There is also a $\mathbb{Q}$-linear involution on $\mathrm{R}(\Gamma)_{\mathbb{Q}}$, called the *complex conjugation*, defined by

$$f \;\mapsto \overline{f}; \quad \overline{f}(x) = f(x^{-1}) \;\; \forall x \in \Gamma\,.$$

The function bA_n is an element of $\mathrm{R}(\mu_n)_{\mathbb{Q}} = \mathrm{R}(\mu_n) \otimes_{\mathbb{Z}} \mathbb{Q}$. In fact, the irreducible characters of μ_n are naturally parametrized by $\mathbb{Z}/n\mathbb{Z}$, given by $\chi_i(\zeta) = \zeta^i$, for all $\zeta \in \mu_n$ and all $i \in \mathbb{Z}/n\mathbb{Z}$. The definition of bA_n visibly says that

$$\mathrm{bA}_n = \frac{1}{n}(\chi_1 + \cdots + \chi_{n-1})\,.$$

Lemma 3.1.2 (ii) says that $(\mathrm{bA}_n \,|\, \mathbf{1}_{\mu_n})$, the inner product of $\mathrm{bA}_n \in \mathrm{R}(\mu_n)_{\mathbb{Q}}$ and the trivial character $\mathbf{1}_{\mu_n}$ of μ_n, is equal to 0. Similarly, Lemma 3.1.2 (iii) says that the sum of the rational virtual character bA_n and its complex conjugate $\overline{\mathrm{bA}_n}$ is equal to $\mathbf{u}_{\mu_n} = \mathbf{r}_{\mu_n} - \mathbf{1}_{\mu_n}$, the augmentation character for the group μ_n. Here $\mathbf{r}_{\mu_n}$ denotes the character of the regular representation of μ_n.

(ii) Let $\widehat{\mathbb{Z}}(1)$ be the projective limit of the finite groups μ_n's, where the transition maps are

$$[d] : \mu_{dn} \twoheadrightarrow \mu_n; \quad [d] : \zeta \mapsto \zeta^d \;\; \forall \zeta \in \mu_{nd}\,.$$

Lemma 3.1.2 (iv) says that $([d] : \mu_{dn} \twoheadrightarrow \mu_n)_*(\mathrm{bA}_{nd})$, the push-forward of the rational virtual character bA_{nd} on μ_{nd} to μ_n via the surjection $[d] : \mu_{dn} \twoheadrightarrow \mu_n$, is equal to bA_n. The system of functions $(\mathrm{bA}_{\mu_n})_{n \in \mathbb{Z}_{>0}}$, compatible with respect to the push-forward maps $[d]_*$, defines a linear functional on the space of locally constant functions on the profinite group $\widehat{\mathbb{Z}}(1)$, which we denote by bA. With complete disregard to continuity, we will refer to linear functionals on the space of locally constant functions on a profinite group G as *distributions* on G. Taking the complex conjugation, we get the formula

$$([d] : \mu_{dn} \twoheadrightarrow \mu_n)_*(\overline{\mathrm{bA}_{nd}}) = \overline{\mathrm{bA}_n} \qquad \forall\, n, d,$$

and also a distribution $\overline{\mathrm{bA}}$ on $\widehat{\mathbb{Z}}(1)$ conjugate to bA. The sum $\mathrm{bA} + \overline{\mathrm{bA}}$ is the distribution on $\widehat{\mathbb{Z}}(1)$ defined by the augmentation characters $(\mathbf{u}_{\mu_n})_{n>0}$.

For any finite dimensional linear representations ρ over k of the profinite group $\widehat{\mathbb{Z}}(1)$ with finite image, the pairing between the distribution bA and the character of ρ is given by

$$\mathrm{bA}(\rho) := (\rho_n \,|\, \mathrm{bA}_{\mu_n})_{\mu_n}, \quad n \gg 0\,.$$

Here n is sufficiently large such that ρ factors as a composition $\rho_n \circ \mathrm{pr}_n$, where $\mathrm{pr}_n : \widehat{\mathbb{Z}}(1) \twoheadrightarrow \mu_n$ is the canonical projection and ρ_n is a linear representation of μ_n. Similarly we have

$$\overline{\mathrm{bA}}(\rho) := (\rho_n \,|\, \overline{\mathrm{bA}}_{\mu_n})_{\mu_n}, \quad n \gg 0\,.$$

If ρ is a non-trivial irreducible representation, then both $\mathrm{bA}(\rho)$ and $\overline{\mathrm{bA}}(\rho)$ are positive, in the sense that they are positive rational numbers in k.

(iii) The notation "bA" is an acronym for a "*b*isection of the *A*rtin conductor". The reason for this will become clear later in this section. In the present situation we are dealing only with the group $\widehat{\mathbb{Z}}(1)$, which is related to tamely ramified extensions of local fields as follows. Let K be the field of fractions of a complete discrete valuation ring $\mathcal{O}$ with an algebraically closed residue field κ of characteristic p, where p is either a prime number or 0. The Galois group $\mathrm{Gal}(K^{\mathrm{tame}}/K)$ of the maximal tamely ramified extension K^{tame} of K is canonically identified with $\widehat{\mathbb{Z}}^{(p)}(1)$, the maximal prime-to-$p$ quotient of $\widehat{\mathbb{Z}}(1)$; see (2.2).(We use the convention that $\widehat{\mathbb{Z}}^{(p)}(1) = \widehat{\mathbb{Z}}(1)$ if $p = 0$.) If ρ is a finite dimensional representation of $\mathrm{Gal}(K^{\mathrm{tame}}/K) = \widehat{\mathbb{Z}}^{(p)}(1)$ over k with finite image, then $\mathrm{bA}(\rho) + \overline{\mathrm{bA}}(\rho)$ is equal to the Artin conductor of the Galois representation ρ. The "bisection" here is a partition of the distribution on $\mathrm{Gal}(K^{\mathrm{tame}}/K)$, defined by the Artin character for tamely ramified finite dimensional extensions of K, into a conjugate (but unequal) pair, bA and $\overline{\mathrm{bA}}$.

(3.2) For the rest of this section we assume that the residue field $\kappa = \mathcal{O}/\mathfrak{p}$ is perfect as in 2.2, and we follow the notation there. Let $\mathrm{bA}_{\Gamma_0/\Gamma_1}$ be the F-valued function on Γ_0/Γ_1 defined in 3.1.1, using the canonical isomorphism $\omega_{L/K} : \Gamma_0/\Gamma_1 \to \mu_n(F)$. In other words,

$$\begin{aligned} \mathrm{bA}_{\Gamma_0/\Gamma_1}(\bar{s}) &= \tfrac{1}{\omega(\bar{s})-1} \qquad \text{if } \bar{s} \neq 1_{\Gamma_0/\Gamma_1} \\ \mathrm{bA}_{\Gamma_0/\Gamma_1}(1_{\Gamma_0/\Gamma_1}) &= \tfrac{n-1}{2} \end{aligned}$$

Let $\overline{\mathrm{bA}}_{\Gamma_0/\Gamma_1}(\bar{s}) = \mathrm{bA}_{\Gamma_0/\Gamma_1}(\bar{s}^{-1})$ for every $\bar{s} \in \Gamma_0/\Gamma_1$.

The goal of this section is to define a rational virtual character $\mathrm{bA}_\Gamma \in \mathrm{R}(\Gamma)_{\mathbb{Q}}$ for every finite Galois extension L/K of local fields, which extends the definition of the function bA_{μ_n} in (3.1.1). Here $\mathrm{R}(\Gamma)_{\mathbb{Q}} = \mathrm{R}(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathrm{R}(\Gamma)$ is the ring of $\overline{F}$-valued virtue characters of Γ. This construction must satisfy the compatibility property that for each normal subgroup N of Γ, the push forward $(\Gamma \twoheadrightarrow \Gamma/N)_*\,(\mathrm{bA}_\Gamma)$ of the virtual character bA_Γ of Γ to the quotient group Γ/N is equal to $\mathrm{bA}_{\Gamma/N}$.

Suppose that $\mathrm{char}(\kappa) = 0$. Then $\Gamma_1 = \{1_\Gamma\}$. Let K_0 be the maximal unramified subextension in L/K, that is the fixed field of Γ_0 in L. Since Γ_0 is a cyclic group, canonically isomorphic to $\mu_n(\kappa_L)$ with $n = \mathrm{Card}(\Gamma_0)$, bA_{Γ_0} is already defined in (3.1.1). The natural definition for bA_Γ is

$$\mathrm{bA}_\Gamma := \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{bA}_{\Gamma_0})\,,$$

the virtual character on Γ induced by bA_{Γ_0}. It is not difficult to verify the desired compatibility condition in this case. More work is needed when $\mathrm{char}(\kappa) = p > 0$; our general definition of bA_Γ below is uniform for the two cases.

(3.2.1) Definition Notation as above. Define an F-valued function bA_{Γ_0} on Γ_0 by

$$\mathrm{bA}_{\Gamma_0}(s) = \begin{cases} \frac{1}{\omega(s)-1} & \text{if } s \in \Gamma_0, s \notin \Gamma_1 \\ -\frac{1}{2}\, i_\Gamma(s) & \text{if } s \in \Gamma_1, s \neq 1_{\Gamma_0} \\ \frac{1}{2}\sum_{t\in\Gamma_0-\{1_\Gamma\}} i_\Gamma(t) & s = 1_{\Gamma_0} \end{cases}$$

It is easy to see that bA_{Γ_0} is an F-valued class function on Γ_0. Define

$$\mathrm{bA}_\Gamma = \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{bA}_{\Gamma_0})\,,$$

the F-valued class function on Γ induced from bA_{Γ_0}.

Denote by $\overline{\mathrm{bA}}_{\Gamma_0}$ (resp. $\overline{\mathrm{bA}}_\Gamma$) the F-valued functions on Γ_0(resp. Γ), given by

$$\overline{\mathrm{bA}}_{\Gamma_0}(s) = \mathrm{bA}_{\Gamma_0}(s^{-1}) \ \ \forall s \in \Gamma_0, \quad \overline{\mathrm{bA}}_\Gamma(s) = \mathrm{bA}_\Gamma(s^{-1}) \ \ \forall s \in \Gamma\,.$$

(3.2.2) Lemma (i) *The F-valued class function bA_Γ on Γ is given by the following formula.*

$$\mathrm{bA}_\Gamma(s) = \begin{cases} 0 & \text{if } s \notin \Gamma_0 \\ \sum_{\bar t\in\Gamma/\Gamma_0} \frac{1}{\omega(tst^{-1})-1} & \text{if } s \in \Gamma_0, s \notin \Gamma_1 \\ -\frac{f(L/K)}{2}\, i_\Gamma(s) & \text{if } s \in \Gamma_1, s \neq 1_\Gamma \\ \frac{f(L/K)}{2}\sum_{t\in\Gamma-\{1_\Gamma\}} i_\Gamma(t) \\ = \frac{f(L/K)}{2}\mathrm{ord}_L(\mathcal{D}_{L/K}) = \frac{1}{2}\mathrm{ord}_K(\mathrm{disc}(L/K)) & \text{if } s = 1_\Gamma \end{cases}$$

In the above formula, t denotes an element of Γ which maps to $\bar t \in \Gamma/\Gamma_0$.

(ii) *As for the class function $\overline{\mathrm{bA}}_\Gamma$ on Γ we have $\overline{\mathrm{bA}}_\Gamma = \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\overline{\mathrm{bA}_{\Gamma_0}})$, and*

$$\overline{\mathrm{bA}}_\Gamma(s) = \begin{cases} 0 & \text{if } s \notin \Gamma_0 \\ \sum_{\bar t\in\Gamma/\Gamma_0} \frac{1}{\omega(ts^{-1}t^{-1})-1} & \text{if } s \in \Gamma_0, s \notin \Gamma_1 \\ -\frac{f(L/K)}{2}\, i_\Gamma(s) & \text{if } s \in \Gamma_1, s \neq 1_\Gamma \\ \frac{f(L/K)}{2}\sum_{t\in\Gamma-\{1_\Gamma\}} i_\Gamma(t) \\ = \frac{f(L/K)}{2}\mathrm{ord}_L(\mathcal{D}_{L/K}) = \frac{1}{2}\mathrm{ord}_K(\mathrm{disc}(L/K)) & \text{if } s = 1_\Gamma \end{cases}$$

(iii) *We have $\mathrm{bA}_\Gamma + \overline{\mathrm{bA}}_\Gamma = \mathrm{Ar}_\Gamma$, the Artin character for Γ.*

Proof. The proof of (i), (ii) is easy, hence omitted. Before proving (iii), recall that the Artin character Ar_Γ is given by

$$\mathrm{Ar}_\Gamma(s) = \begin{cases} -f(L/K)\cdot i_\Gamma(s) & \text{if } s \neq 1_\Gamma \\ f(L/K)\cdot \sum_{1_\Gamma\neq s\in\Gamma} i_\Gamma(s) & \end{cases} .$$

To prove (iii), we only need to check the equality for elements $s\in\Gamma_0, s\notin\Gamma_1$; this follows from Lemma 3.1.2 (iii). ∎

(3.3) Proposition *Let* $\mathrm{Inf}^{\Gamma_0}_{\Gamma_0/\Gamma_1}(\mathrm{bA}_{\Gamma_0/\Gamma_1})$ *be the inflation of the rational virtual character* $\mathrm{bA}_{\Gamma_0/\Gamma_1}$ *to* Γ_0 *via the quotient map* $\Gamma_0\to\Gamma_0/\Gamma_1$. *Then we have*

$$\mathrm{bA}_\Gamma = \mathrm{Ind}^\Gamma_{\Gamma_0}\circ\mathrm{Inf}^{\Gamma_0}_{\Gamma_0/\Gamma_1}(\mathrm{bA}_{\Gamma_0/\Gamma_1}) + \frac{[\Gamma_0:\Gamma_1]+1}{2[\Gamma_0:\Gamma_1]}\,\mathrm{Ind}^\Gamma_{\Gamma_1}(\mathbf{u}_{\Gamma_1}) + \frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{[\Gamma_0:\Gamma_j]}\,\mathrm{Ind}^\Gamma_{\Gamma_j}(\mathbf{u}_{\Gamma_j})\,.$$

Similarly we have

$$\overline{bA}_\Gamma = \mathrm{Ind}^\Gamma_{\Gamma_0}\circ\mathrm{Inf}^{\Gamma_0}_{\Gamma_0/\Gamma_1}(\overline{\mathrm{bA}}_{\Gamma_0/\Gamma_1}) + \frac{[\Gamma_0:\Gamma_1]+1}{2[\Gamma_0:\Gamma_1]}\,\mathrm{Ind}^\Gamma_{\Gamma_1}(\mathbf{u}_{\Gamma_1}) + \frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{[\Gamma_0:\Gamma_j]}\,\mathrm{Ind}^\Gamma_{\Gamma_j}(\mathbf{u}_{\Gamma_j})\,.$$

Proof. It is only necessary to prove the first formula. We may and do assume that $\Gamma=\Gamma_0$. Recall that $n=[\Gamma_0:\Gamma_1]$. For each $j\geq 1$, we have

$$\mathrm{Ind}^{\Gamma_0}_{\Gamma_j}(\mathbf{u}_{\Gamma_j})(s) = \begin{cases} 0 & \text{if } s\notin\Gamma_j \\ -[\Gamma_0:\Gamma_j] & \text{if } s\in\Gamma_j, s\neq 1_{\Gamma_0} \\ \mathrm{Card}(\Gamma_0)-[\Gamma_0:\Gamma_j] & \text{if } s=1_{\Gamma_0} \end{cases}$$

Suppose that $s\in\Gamma_0, s\notin\Gamma_1$. Then $\mathrm{Ind}^{\Gamma_0}_{\Gamma_j}(\mathbf{u}_{\Gamma_j})(s)=0$ for all $j\geq 1$, and the right hand side of the formula becomes

$$\mathrm{Inf}^{\Gamma_0}_{\Gamma_0/\Gamma_1}(\mathrm{bA}_{\Gamma_0/\Gamma_1})(s) = \frac{1}{\omega(s)-1} = \mathrm{bA}_{\Gamma_0}(s)\,.$$

Suppose that $s\in\Gamma_1, s\neq 1_{\Gamma_0}$. Then the right hand side of the formula is

$$\frac{n-1}{2}+\frac{n+1}{2n}(-n)-\frac{1}{2}\sum_{\substack{s\in\Gamma_j\\ j\geq 2}} 1 = -\frac{i_{\Gamma_0}(s)}{2} = \mathrm{bA}_{\Gamma_0}(s)\,.$$

For $s=1_{\Gamma_0}$, the right hand side of the formula is equal to

$$\begin{aligned} &\tfrac{[\Gamma_0:\Gamma_1]-1}{2}+\tfrac{[\Gamma_0:\Gamma_1]}{2}(\mathrm{Card}(\Gamma_1)-1)+\tfrac{1}{2}\textstyle\sum_{j\geq 1}(\mathrm{Card}(\Gamma_j)-1) \\ &= \tfrac{1}{2}\textstyle\sum_{j\geq 0}(\mathrm{Card}(\Gamma_j)-1) = \tfrac{1}{2}\sum_{s\in\Gamma_0} i_{\Gamma_0}(s) = \mathrm{bA}_{\Gamma_0}(s)\,. \end{aligned}$$

We have verified that the two sides of the formula are equal in all cases. ∎

(3.3.1) Corollary *Let $\chi = \chi_\rho$ be the character of an (absolutely) irreducible linear representation ρ of Γ over $\overline{F}$. Then*

$$(\mathrm{bA}_\Gamma \,|\, \chi) = \begin{cases} \frac{1}{2}\,(\mathrm{Ar}_\Gamma \,|\, \chi) & \text{if } \rho \text{ is not trivial on } \Gamma_1 \\ \\ \big(\mathrm{bA}_{\Gamma_0/\Gamma_1} \,|\, \chi_{\overline{\rho_0}}\big) & \text{if } \rho|_{\Gamma_0} \text{ factors through a representation } \overline{\rho_0} \text{ of } \Gamma_0/\Gamma_1 \end{cases}$$

$$\big(\overline{bA}_\Gamma \,|\, \chi\big) = \begin{cases} \frac{1}{2}\,(\mathrm{Ar}_\Gamma \,|\, \chi) & \text{if } \rho \text{ is not trivial on } \Gamma_1 \\ \\ \big(\overline{bA}_{\Gamma_0/\Gamma_1} \,|\, \chi_{\overline{\rho_0}}\big) & \text{if } \rho|_{\Gamma_0} \text{ factors through a representation } \overline{\rho_0} \text{ of } \Gamma_0/\Gamma_1 \end{cases}$$

Proof. This Corollary follows from the fact that

$$(\mathrm{Inf}^{\Gamma}_{\Gamma_j}(\mathbf{u}_{\Gamma_j}) \,|\, \rho) = \begin{cases} 0 & \text{if } \rho \text{ is trivial on } \Gamma_j \\ \dim(\rho) & \text{if } \rho \text{ is not trivial on } \Gamma_j \end{cases}$$

and the definition of the Artin character. ∎

(3.3.2) Corollary (i) *The F-valued class functions bA_Γ and $\overline{\mathrm{bA}}_\Gamma$ are elements of the group $\mathrm{R}(\Gamma)_\mathbb{Q} := \mathrm{R}(\Gamma) \otimes_\mathbb{Z} \mathbb{Q}$, where $\mathrm{R}(\Gamma)$ denotes the Grothendieck group of all finite-dimensional representations of Γ over the algebraic closure $\overline{F}$ of F. Moreover $\mathrm{bA}_\Gamma \in \frac{1}{n'} \cdot \mathrm{R}(\Gamma)$, where $n' = \mathrm{lcm}(n, 2)$.*

(ii) *We have*

$$(\mathrm{bA}_\Gamma \,|\, \mathbf{1}_\Gamma) = 0, \quad \big(\overline{\mathrm{bA}}_\Gamma \,|\, \mathbf{1}_\Gamma\big) = 0\,.$$

(iii) *The pairing of bA_Γ (resp. $\overline{\mathrm{bA}_\Gamma}$) with the character χ_ρ of any finite dimensional representation ρ over $\overline{F}$ is a non-negative rational number whose denominator divides n', where $n' = \mathrm{lcm}(n, 2)$ as in* (i). *Moreover $(\mathrm{bA}_\Gamma \,|\, \chi_\rho) = 0$ if and only if ρ is unramified; same for $\overline{\mathrm{bA}}_\Gamma$.*

Proof. In statement (i), the first sentence is immediate from Prop. 3.3. To show that $\mathrm{bA}_\Gamma \in \frac{1}{n'}$, it suffices to check that $(\mathrm{bA}_\Gamma|\chi) \in \frac{1}{n'}\mathbb{Z}$ for every absolutely irreducible character χ of Γ. That is a consequence of 3.3.1 when χ is not tamely ramified, by the integrality of the Artin conductor. When χ is tamely ramified, the denominator of $(\mathrm{bA}_\Gamma|\chi) \in \frac{1}{n}\mathbb{Z}$ by the last displayed formula in 3.1.3 (i).

The statement (ii) is clear from Prop. 3.3. The first sentence of (iii) is a reformulation of the last sentence of (i). To verify the last sentence of (iii), notice that $(\mathrm{bA}_\Gamma \,|\, \chi_\rho)$ is a non-negative rational number by 3.3.1, so it suffices to check it for absolutely irreducible characters, which is easy to see from the formula in 3.3.1. ∎

(3.3.3) Corollary *Let χ_ρ be an effective $\overline{F}$-valued character of $\Gamma = \Gamma_{L/K}$ as in 3.3.2* (iii). *Then the following estimates of $(\mathrm{bA}_{\Gamma_{L/K}}|\chi_\rho)$ hold.*

(i) *With $n = n_{L/K}$ as before, we have*

$$\frac{1}{2}\left(\mathrm{Sw}_\Gamma|\chi_\rho\right) + \min\left(\frac{1}{n}, \frac{n-1}{n}\right) \cdot \left(\mathrm{Ar}_\Gamma - \mathrm{Sw}_\Gamma|\chi_\rho\right)$$
$$\leq \ (\mathrm{bA}_\Gamma|\chi_\rho) \ \leq \ \frac{1}{2}\left(\mathrm{Sw}_\Gamma|\chi_\rho\right) + \max\left(\frac{1}{2}, \frac{n-1}{n}\right) \cdot \left(\mathrm{Ar}_\Gamma - \mathrm{Sw}_\Gamma|\chi_\rho\right)\,,$$

where Sw_Γ *is the Swan character of* Γ*.*

(ii) (weaker form, with coefficients independent of L/K)

$$\frac{1}{2}\left(\mathrm{Sw}_\Gamma|\chi_\rho\right) \leq (\mathrm{bA}_\Gamma|\chi_\rho) \leq (\mathrm{Ar}_\Gamma|\chi_\rho) - \frac{1}{2}\left(\mathrm{Sw}_\Gamma|\chi_\rho\right)$$

If either of the above inequalities is an equality, then ρ is unramified and all three terms in the displayed inequality above are equal to zero.

(iii) (a weak lower bound)

$$(\mathrm{bA}_\Gamma|\chi_\rho) \geq \min\left(\frac{1}{2}, \frac{1}{n}\right) \cdot \left(\deg(\chi_\rho) - (\chi_\rho|\mathbf{1}_\Gamma)\right)$$

Proof. Clearly (ii) follows from (i). To show (i), we may and do assume that ρ is absolutely irreducible. If ρ is tamely ramified, then the Swan conductor of ρ is 0, and the displayed inequality follows from the last displayed formula for bA_n in 3.1.3 (i). If ρ is not tamely ramified, then the displayed inequality reduces to the fact that the Swan conductor of ρ is smaller than the Artin conductor of ρ, since $\min(\frac{1}{n}, \frac{n-1}{n}) \leq \frac{1}{2}$ and $\max(\frac{1}{2}, \frac{n-1}{n}) \geq \frac{1}{2}$.

To show inequality (iii), we may and do assume that ρ is absolutely irreducible and non-trivial. If ρ is not tamely ramified, then $(\mathrm{bA}_\Gamma|\chi_\rho) = \frac{1}{2}(\mathrm{Ar}_\Gamma|\chi_\rho) \geq \frac{1}{2}\deg(\chi_\rho)$. If ρ is tamely ramified, then $n > 1$ since ρ is assumed to be non-trivial, so $(\mathrm{bA}_\Gamma|\chi_\rho) \geq \frac{1}{n}\deg(\chi_\rho)$ by 3.1.3 (i). In both cases the inequality (iii) holds. ∎

(3.4) Proposition *Let N be a normal subgroup of Γ, and let $\alpha : \Gamma \twoheadrightarrow \Gamma/N$ be the canonical surjection. Let $\alpha_*(\mathrm{bA}_\Gamma)$ be the push-forward of the virtual character bA_Γ to N, defined by*

$$\alpha_*(\mathrm{bA}_\Gamma)(\bar{t}) = \frac{1}{\mathrm{Card}(N)} \sum_{s\in\Gamma, s\mapsto t} \mathrm{bA}_\Gamma(s) \qquad \forall\ t \in \Gamma/N\,.$$

Then

$$\alpha_*(\mathrm{bA}_\Gamma) = \mathrm{bA}_{\Gamma/N}\,.$$

Similarly we have

$$\alpha_*(\overline{\mathrm{bA}}_\Gamma) = \overline{\mathrm{bA}}_{\Gamma/N}\,.$$

PROOF. It suffices to prove that $\alpha_*(\mathrm{bA}_\Gamma) = \mathrm{bA}_{\Gamma/N}$. Let $\iota : \Gamma_0 \hookrightarrow \Gamma$ be the inclusion of Γ_0 in Γ. Let $N_0 = N \cap \Gamma_0$, and let $\tau : \Gamma_0/N_0 \to \Gamma/N$ be the injection induced by the inclusion $\Gamma_0 \hookrightarrow \Gamma$. Let $\beta : \Gamma_0 \twoheadrightarrow \Gamma_0/N_0$ be the canonical surjection. From $\alpha \circ \iota = \tau \circ \beta$, we deduce that

$$\alpha_*(\mathrm{bA}_\Gamma) = \alpha_*(\iota_*(\mathrm{bA}_{\Gamma_0})) = \tau_*(\beta_*(\mathrm{bA}_{\Gamma_0})\,.$$

Hence it suffices to prove that $\beta_*(\mathrm{bA}_{\Gamma_0}) = \mathrm{bA}_{\Gamma_0/N_0}$. We will verify this equality by comparing the values of both sides at each element $t \in \Gamma_0/N_0$, that is

$$(\dagger) \qquad \beta_*(\mathrm{bA}_{\Gamma_0})(t) = \mathrm{bA}_{\Gamma_0/N_0}(t) \quad \forall t \in \Gamma_0/N_0\,.$$

Denote by K_0 the subfield of L attached to Γ_0; K_0/K is the maximal unramified subextension of L/K. Let M_0 be the subfield of L attached to $N_0 = N \cap \Gamma_0$; we have $\mathrm{Gal}(M_0/K_0) = \Gamma_0/N_0$.

Assume first that $t \notin (\Gamma_0/N_0)_1 = \Gamma_1 N_0/N_0$. Let $s \in \Gamma_0$ be an element of Γ_0 lying over t. Then $s \cdot n \notin \Gamma_1$ for every $n \in N_0$, and $\mathrm{bA}_{\Gamma_0}(sn) = \frac{1}{\omega_{\Gamma_0}(sn)-1} = \mathrm{bA}_{\Gamma_0/\Gamma_1}(\overline{sn})$, where $\overline{sn}$ denotes the image of sn in Γ_0/Γ_1. On the other hand $\mathrm{bA}_{\Gamma_0/N_0}(t) = \frac{1}{\omega_{\Gamma_0/N_0}(t)-1} = \mathrm{bA}_{\Gamma_0/N_0\Gamma_1}(\tilde{t})$, where $\tilde{t}$ is the image of t in $\Gamma_0/N_0\Gamma_1$, which is also the image of sn in $\Gamma_0/N_0\Gamma_1$ for every $n \in N_0$. The equality $(\dagger)$ when $t \notin (\Gamma_0/N_0)_1$ follows from 3.1.2 (iv).

Suppose that t is an element of $(\Gamma_0/N_0)_1 = \Gamma_1 \cdot N_0/N_0$ and $t \neq 1_{\Gamma_0/N_0}$. Choose a set of representatives $\{1 = n_1, n_2, \ldots, n_{[N_0:N_1]}\}$ for N_0/N_1 in N_0. Let s be an element of Γ_1 which maps to t; so $s \notin N_1$. In this case the equation $(\dagger)$ we need to check reads

$$\sum_{j=1}^{[N_0:N_1]} \sum_{n \in N_1} \mathrm{bA}_{\Gamma_0}(s \cdot n \cdot n_j) = -\frac{1}{2}\,\mathrm{Card}(N_0) \cdot i_{\Gamma_0/N_0}(t)\,.$$

Let j be any natural number, $2 \le j \le [N_0 : N_1]$. We know that $n_j \notin \Gamma_1$ and hence $s \cdot n \cdot n_j \notin \Gamma_1$, for each $n \in N_1$, and

$$\mathrm{bA}_{\Gamma_0}(s \cdot n \cdot n_j) = \frac{1}{\omega(n_j) - a} = \mathrm{bA}_{N_0/N_1}(\bar{n}_j)\,.$$

So we get

$$\begin{aligned}
&\textstyle\sum_{j=2}^{[N_0:N_1]} \sum_{n \in N_1} \mathrm{bA}_{\Gamma_0}(s \cdot n \cdot n_j) \\
&= \mathrm{Card}(N_1) \cdot \textstyle\sum_{\substack{\bar{x} \in N_0/N_1 \\ \bar{x} \neq 1}} \mathrm{bA}_{N_0/N_1}(\bar{x}) \\
&= -\tfrac{1}{2}\,\mathrm{Card}(N_1) \cdot ([N_0 : N_1] - 1) && \text{by 3.1.2 (iv)} \\
&= -\tfrac{1}{2} \textstyle\sum_{j=2}^{[N_0:N_1]} \sum_{n \in N_1} i_{\Gamma_0}(s \cdot n \cdot n_j) && \text{since } i_{\Gamma_0}(s \cdot n \cdot n_j) = 1 \ \ \forall j \ge 2\,.
\end{aligned}$$

On the other hand we have $\mathrm{bA}_{\Gamma_0}(s \cdot n) = -\frac{1}{2}\, i_{\Gamma_0}(s \cdot n)$ for each $n \in N_1$, since $s \cdot n \in \Gamma_1$ and $s \cdot n \neq 1_{\Gamma_1}$. Therefore

$$\sum_{j=1}^{[N_0:N_1]} \sum_{n \in N_1} \mathrm{bA}_{\Gamma_0}(s \cdot n \cdot n_j) = -\frac{1}{2} \sum_{j=1}^{[N_0:N_1]} i_{\Gamma_0}(s \cdot n \cdot n_j) = -\frac{1}{2}\,\mathrm{Card}(N_0) \cdot i_{\Gamma_0/N_0}(t)$$

by [S1, IV, §1, Prop. 3]. We have checked the desired equality $(\dagger)$ when $t \in (\Gamma_0/N_0)_1$, $t \neq 1_{\Gamma_0/N_0}$.

It remains to verify the equality (†) when $t = 1_{\Gamma_0/N_0}$. Since $\sum_{t\in\Gamma_0/N_0} i_{\Gamma_0/N_0}(t) = 0$ and $\sum_{s\in\Gamma_0} i_{\Gamma_0}(s) = 0$, and we have checked the equality (†) for all other elements of Γ_0/N_0, the equality (†) for $t = 1_{\Gamma_0/N_0}$ follows. One can also check this equality directly: Let M_0 be the fixed field of N_0 in L, and let K_0 be the fixed field of Γ_0 in L. Because

$$\mathrm{bA}_{\Gamma_0}(n) = \mathrm{bA}_{N_0}(n) \quad \forall 1 \neq n \in N_0\,,$$

we get

$$\begin{aligned}&\textstyle\sum_{n\in N_0} \mathrm{bA}_{\Gamma_0}(n) = \sum_{n\in N_0} \mathrm{bA}_{N_0}(n) + \frac{1}{2}\mathrm{ord}_{K_0}(\mathrm{disc}(L/K_0)) - \frac{1}{2}\mathrm{ord}_{M_0}(\mathrm{disc}(L/M_0))\\ &= \tfrac{1}{2}[L:M_0]\cdot \mathrm{ord}_{M_0}(\mathrm{disc}(M_0/K_0)) = [L:M_0]\cdot \mathrm{bA}_{\Gamma_0/N_0}(1_{\Gamma_0/N_0})\,. \qquad \blacksquare\end{aligned}$$

(3.4.1) Remark Proposition 3.4 shows that the system of virtual characters

$$\big(\mathrm{bA}_{\mathrm{Gal}(L/K)}\big)_{L,\ L/K \text{ finite Galois}}$$

defines an $\overline{F}$-valued distribution bA_K on the space of all locally constant $\overline{F}$-valued functions on the profinite group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. For a locally constant function ϕ on $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ with values in $\overline{F}$ which is the pull-back of a function $\phi_{L/K}$ on a finite Galois extension L/K, we have $\mathrm{bA}_K(\phi) := (\mathrm{bA}_{\mathrm{Gal}(L/K)}\,|\,\phi_{L/K})$, independent of the choice of L. Similarly we have an $\overline{F}$-valued distribution $\overline{\mathrm{bA}}_K$ on $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ defined by the system $\big(\overline{\mathrm{bA}}_{\mathrm{Gal}(L/K)}\big)_{L,\ L/K \text{ finite Galois}}$.

(3.5) To determine the pairing of bA_Γ with an induced character, we have to compute the restriction of bA_Γ to a subgroup.

(3.5.1) Lemma *Let H_0 be a subgroup of Γ_0 and let M be the subfield of L fixed by H_0. Then the restriction $\mathrm{res}^{\Gamma_0}_{H_0}(\mathrm{bA}_{\Gamma_0})$ of the virtual character bA_{Γ_0} to H_0 is given by*

$$\mathrm{res}^{\Gamma_0}_{H_0}(\mathrm{bA}_{\Gamma_0}) = \mathrm{bA}_{H_0} + \frac{1}{2}\,\mathrm{ord}_{M_0}(\mathrm{disc}(M_0/K_0))\cdot \mathbf{r}_{H_0}\,.$$

Similarly

$$\mathrm{res}^{\Gamma_0}_{H_0}(\overline{\mathrm{bA}}_{\Gamma_0}) = \overline{\mathrm{bA}}_{H_0} + \frac{1}{2}\,\mathrm{ord}_{M_0}(\mathrm{disc}(M_0/K_0))\cdot \mathbf{r}_{H_0}\,.$$

Proof. We only need to verify the first formula. To do so, observe that the value at any non-trivial element $h \in H_0$ of both sides of the equality are visibly equal. The same argument as in the last paragraph of the proof of Prop. 3.4 shows the equality for the trivial element 1_{H_0}. ∎

(3.5.2) We need some notations for the next proposition. For any subgroup S of a group Γ, any element $g \in \Gamma$ and any linear representation $\rho : S \to \mathrm{Aut}(V)$ of the group S, denote by ${}^x\rho$ the linear representation of the subgroup xSx^{-1} on V, given by ${}^x\rho(y) = \rho(x^{-1}y\,x)$ for every $y \in xSx^{-1}$.

Let L/K be a finite Galois extension of a local field K with Galois group Γ as before. The group Γ/Γ_0 operates on Γ_0/Γ_1 by conjugation; this action is equal to the restriction of the natural action of Γ/Γ_0 on κ_L, post-composed with the inverse of $\overline{\theta_0} : \Gamma_0/\Gamma_1 \xrightarrow{\sim} \mu_n(\kappa_L)$. Since group of automorphisms $\mathrm{Aut}(F) = \mathrm{Aut}_{\text{field}}(F(L/K))$ of the cyclotomic field F is equal to $\mathrm{Aut}_{\text{group}}(\mu_n(F))$, the Γ/Γ_0 on $\mu_n(\kappa_L) \xleftarrow{\sim} \mu_n(F)$ gives homomorphism

$$\lambda : \Gamma \twoheadrightarrow \Gamma/\Gamma_0 \to \mathrm{Aut}_{\text{field}}(F)\,.$$

For any F-valued function ϕ on a subset T of Γ and for any element $s \in \Gamma$, denote by $\lambda(s) \cdot \phi$ the function on T such that $(\lambda(s) \cdot \phi)(t) = \lambda(s)(\phi(t))$ for each $t \in T$. More generally, for any $\bar{F}$-valued function ϕ on a subset T of Γ and any automorphism σ of $\bar{F}$, denote by $\sigma \cdot \phi$ the function on T such that $(\sigma \cdot \phi)(t) = \sigma(\phi(t))$ for all $t \in T$.

(3.5.3) Proposition *Let H be a subgroup of the Galois group Γ of a finite Galois extension L/K as above. Let M be the subfield of L attached to H, and let $H_0 = H \cap \Gamma_0$ be the inertial subgroup of H. Let $\{s_1, \ldots, s_{f(M/K)}\}$ be a set of representatives for the double coset $H\backslash\Gamma/\Gamma_0$ in Γ. Then the restriction of the virtual character bA_Γ to H is given by*

$$\begin{aligned}
\mathrm{res}^\Gamma_H(\mathrm{bA}_\Gamma) &= \textstyle\sum_{j=1}^{f(M/K)} \mathrm{Ind}^H_{H_0}\left({}^{s_j}\mathrm{bA}_{s_j^{-1}\cdot H_0\cdot s_j}\right) + \frac{1}{2}\,\mathrm{ord}_K(\mathrm{disc}(M/K)) \cdot \mathbf{r}_H \\
&= \textstyle\sum_{j=1}^{f(M/K)} \mathrm{Ind}^H_{H_0}\left(\lambda(s_j^{-1}) \cdot \mathrm{bA}_{H_0}\right) + \frac{1}{2}\,\mathrm{ord}_K(\mathrm{disc}(M/K)) \cdot \mathbf{r}_H \\
&= \textstyle\sum_{j=1}^{f(M/K)} \lambda(s_j^{-1}) \cdot \mathrm{bA}_H + \frac{1}{2}\,\mathrm{ord}_K(\mathrm{disc}(M/K)) \cdot \mathbf{r}_H\,.
\end{aligned}$$

Proof. The last equality is immediate from the definition of induced characters. The first equality follows from Mackey's theorem and Lemma 3.5.1: The restriction $\mathrm{res}^\Gamma_H(\mathrm{bA}_\Gamma)$ of bA_Γ to H is equal to

$$\textstyle\sum_{j=1}^{f(M/K)} \mathrm{Ind}^H_{H_0}\left({}^{s_j}\left(\mathrm{res}^{\Gamma_0}_{s_j^{-1}\cdot H_0\cdot s_j}(\mathrm{bA}_{\Gamma_0})\right)\right) \qquad \text{by Mackey's theorem}$$

$$= \textstyle\sum_{j=1}^{f(M/K)} \mathrm{Ind}^H_{H_0}\left({}^{s_j}\mathrm{bA}_{s_j^{-1}\cdot H_0\cdot s_j}\right) + \frac{1}{2}\,\mathrm{ord}_K(\mathrm{disc}(M/K)) \cdot \mathbf{r}_H \qquad \text{by Lemma 3.5.1}$$

The second equality in the Proposition is a consequence of the following formula

$$(\dagger\dagger) \qquad {}^{s_j}\mathrm{bA}_{s_j^{-1}\cdot H_0\cdot s_j} = \lambda(s_j^{-1}) \cdot \mathrm{bA}_{H_0}\,.$$

To prove this, we examine the values of both sides of $(\dagger\dagger)$ at an element $h \in H_0$. Clearly we have equality if $h \in H_1$, since $i_{H_0}(h) = i_{s_j^{-1}\cdot H_0\cdot s_j}(h) = i_{\Gamma_0}(h) \in \mathbb{Z}$. Suppose that $h \in H_0, h \notin H_1$. Then

$$\begin{aligned}
\left({}^{s_j}\mathrm{bA}_{s_j^{-1}\cdot H_0\cdot s_j}\right)(h) &= \mathrm{bA}_{s_j^{-1}\cdot H_0\cdot s_j}(s_j^{-1} h s_j) &&= \tfrac{1}{\omega(s_j^{-1} h s_j))-1} \\
= \lambda(s_j^{-1})\left(\tfrac{1}{\omega(h)-1}\right) &= \lambda(s_j^{-1})\,(\mathrm{bA}_{H_0}(h)) &&= (\lambda(s_j^{-1}) \cdot \mathrm{bA}_{H_0})(h)
\end{aligned}$$

We have proved $(\dagger\dagger)$ and finished the proof of the Proposition. ∎

(3.5.4) Corollary *Let H be a subgroup of Γ and let M be the subfield of L corresponding to H. Suppose that ψ is a class function on H with values in $\overline{F}$. Let $\{s_1, \ldots s_{f(M/K)}\}$ be a set of representatives for $H\backslash\Gamma/\Gamma_0$ in Γ as in Prop. 3.5.3. For each $j = 1, \ldots, f(M/K)$, let σ_j be an automorphism of the field $\overline{F}$ which extends the action of $\lambda(s_j)$ on F. Then*

$$\begin{aligned}(\mathrm{bA}_\Gamma \,|\, \mathrm{Ind}_H^\Gamma(\psi)) &= \textstyle\sum_{j=1}^{f(M/K)} \left(\lambda(s_j^{-1}) \cdot \mathrm{bA}_{H_0} \,|\, \psi\right) + \frac{1}{2}\,\mathrm{ord}_M(\mathrm{disc}(M/K)) \cdot \psi(1_H) \\ &= \textstyle\sum_{j=1}^{f(M/K)} \sigma_j^{-1}\,(\mathrm{bA}_{H_0} \,|\, \sigma_j \cdot \psi) + \frac{1}{2}\,\mathrm{ord}_M(\mathrm{disc}(M/K)) \cdot \psi(1_H)\end{aligned}$$

$$\begin{aligned}(\overline{\mathrm{bA}_\Gamma} \,|\, \mathrm{Ind}_H^G(\psi)) &= \textstyle\sum_{j=1}^{f(M/K)} \left(\lambda(s_j^{-1}) \cdot \overline{\mathrm{bA}}_{H_0} \,|\, \psi\right) + \frac{1}{2}\,\mathrm{ord}_M(\mathrm{disc}(M/K)) \cdot \psi(1_H) \\ &= \textstyle\sum_{j=1}^{f(M/K)} \sigma_j^{-1}\left(\overline{\mathrm{bA}}_{H_0} \,|\, \sigma_j \cdot \psi\right) + \frac{1}{2}\,\mathrm{ord}_M(\mathrm{disc}(M/K)) \cdot \psi(1_H)\end{aligned}$$

In particular if $\psi \in \mathrm{R}(H)_{\mathbb{Q}}$ and $\sigma_j \cdot \psi = \psi$ for each $j = 1, \ldots, f(M/K)$, then

$$\begin{aligned}(\mathrm{bA}_\Gamma \,|\, \mathrm{Ind}_H^\Gamma(\psi)) &= \;\; f(M/K) \cdot (\mathrm{bA}_H \,|\, \psi) + \tfrac{1}{2}\,\mathrm{ord}_M(\mathrm{disc}(M/K)) \cdot \psi(1_H) \\ (\overline{\mathrm{bA}}_\Gamma \,|\, \mathrm{Ind}_H^\Gamma(\psi)) &= \;\; f(M/K) \cdot (\overline{\mathrm{bA}}_H \,|\, \psi) + \tfrac{1}{2}\,\mathrm{ord}_M(\mathrm{disc}(M/K)) \cdot \psi(1_H)\,.\end{aligned}$$

The condition that $\sigma_j \cdot \psi = \psi$ for each j is satisfied if ψ takes values in $\mathrm{frac}(W(\kappa)) \supseteq \mathbb{Q}_p$.

§4. A class of rigid groups

In this section, we assume that $\mathcal{O}$ is a complete discrete valuation ring.

(4.1) We recall some basic facts about rigid analytic geometry and establish notation along the way.

(4.1.1) For any topological ring A and any natural number $d \geq 0$, denote by $A\{x_1, \ldots, x_d\}$ the ring consisting of all formal power series of the form

$$f(x_1, \ldots, x_d) = \sum_{n_1,\ldots,n_d \geq 0} a_{n_1,\ldots,n_d}\, x_1^{n_1} \cdots x_d^{n_d} = \sum_{\underline{n} \in \mathbb{N}^d} a_{\underline{n}}\, \underline{x}^{\underline{n}}, \quad a_{\underline{n}} \in A \;\; \forall\, \underline{n} \in \mathbb{N}^d$$

such that $a_{\underline{n}} = a_{n_1,\ldots,n_d} \to 0$ as $|\underline{n}| := n_1 + \ldots + n_d \to \infty$. The ring of *strictly convergent* power series $\mathcal{O}\{x_1, \ldots, x_d\}$ over $\mathcal{O}$, described above, is the π-adic completion of the polynomial algebra $\mathcal{O}[x_1, \ldots, x_d]$. The algebra $K\{x_1, \ldots, x_d\}$ is equal to $\mathcal{O}\{x_1, \ldots, x_d\} \otimes_{\mathcal{O}} K$. The affinoid space $\mathrm{Spm}\, K\{x_1, \ldots, x_d\}$, whose underlying set is the set of all maximal ideals of $K\{x_1, \ldots, x_d\}$, is often referred to as the d-dimensional closed unit ball. A *Tate algebra*, or *affinioid algebra* over K is a quotient of $K\{x_1, \ldots, x_d\}$ for some $d \geq 0$. Let R be a quotient of $\mathcal{O}\{x_1, \ldots, x_d\}$. Then $R \otimes_{\mathcal{O}} K$ is a Tate algebra over K, and $\mathrm{Spm}(R \otimes_{\mathcal{O}} K)$ is the *generic fiber* of Spf R. The functor which sends Spf R to $\mathrm{Spm}(R \otimes_{\mathcal{O}} K)$ is compatible with localization, and allows one to associate to each π-adic formal scheme $\mathfrak{X}$ locally of finite type over $\mathcal{O}$ a rigid analytic space $\mathfrak{X}_K$ locally of finite type over K, called the *generic fiber* of $\mathfrak{X}$. Raynaud's original approach to rigid analytic spaces over K as the generic fiber of π-adic formal schemes locally of finite type over $\mathcal{O}$ was further developed in [BL].

(4.1.2) The *generic fiber* of the formal $\mathcal{O}$-scheme $\mathrm{Spf}\,\mathcal{O}[[\mathrm{x}_1,\ldots,\mathrm{x}_\mathrm{d}]]$ is the d-dimensional open unit ball, defined as the inductive limit

$$\lim_{n\to\infty} \mathrm{Spm}\left(K\{x_1,\ldots,x_d,x_{1,n},\ldots,x_{d,n}\}/(x_1^n-\pi x_{1,n},\ldots,x_d^n-\pi x_{d,n})\right),$$

where the transition maps are induced by the algebra homomorphisms

$$\begin{array}{c} K\{x_1,\ldots,x_d,x_{1,n+1},\ldots,x_{d,n+1}\}/(x_1^{n+1}-\pi x_{1,n+1},\ldots,x_d^{n+1}-\pi x_{d,n+1}) \\ \longrightarrow\ K\{x_1,\ldots,x_d,x_{1,n},\ldots,x_{d,n}\}/(x_1^n-\pi x_{1,n},\ldots,x_d^n-\pi x_{d,n}) \\ x_{i,n+1}\ \mapsto\ x_i\cdot x_{i,n}\quad \text{for } i=1,\ldots,d\ . \end{array}$$

Similarly one can attach to each adic (but not necessarily π-adic) formal scheme $\mathfrak{X}$ locally of finite type over $\mathcal{O}$ a rigid analytic space $\mathfrak{X}_K$ over K, called the *generic fiber* of $\mathfrak{X}$. See [Berth, §0.2.6], and also [dJ, §7] and [RZ, pp. 229–234], for more information about the generic fiber of a formal scheme over $\mathcal{O}$.

(4.1.3) The global sections of the structure sheaf of the open unit ball $B=B^d_-(0;1)$ consists of all formal power series $\sum_{\underline{n}} a_{\underline{n}}\,\underline{x}^{\underline{n}}$ with coefficients $a_{\underline{n}}\in K$ such that $\lim_{|\underline{n}|\to\infty}|a_{\underline{n}}|\,\epsilon^{|\underline{n}|}=0$ for all $0<\epsilon<1$. The formal power series ring $\mathcal{O}[[x_1,\ldots x_d]]$ can be recovered from the rigid analytic structure of the open ball B: it is equal to the set of functions $f(\underline{x})\in\Gamma(B,\mathcal{O}_B)$ such that $|f(a)|\le 1$ for all $a=(a_1,\ldots,a_d)$ with $a_i\in K^{\mathrm{alg}}$, $|a_i|\le 1$, for $i=1,\ldots,d$. In other words, the sup norm of $f(\underline{x})$ on $B^d_-(0;1)$ is ≤ 1.

(4.2) We establish notation about formal tori and their character groups. Assume that $\mathrm{char}(\kappa)=p>0$.

(4.2.1) A *split formal torus* over $\mathcal{O}$ is a formal group scheme over $\mathcal{O}$ which is isomorphic to the formal completion along the unit section of the special fiber of a split $\mathcal{O}$-torus $\mathrm{Spec}\,\mathcal{O}[x_1^{\pm1},\ldots,x_d^{\pm1}]$, or equivalantly, the formal completion along the unit section of $\mathrm{Spec}\,\mathcal{O}[x_1^{\pm1},\ldots,x_d^{\pm1}]$, since $\mathcal{O}$ is complete.

(4.2.2) A *formal torus* over $\mathcal{O}$ is the smooth formal group scheme $\mathfrak{T}$ over $\mathcal{O}$ attached to a Barsotti-Tate group $T=(T_n)_{n\in\mathbb{N}}$ over $\mathcal{O}$ of multiplicative type. For each $n\ge 1$ the truncated Barsotti-Tate group at level n is equal to the kernel of $[p^n]$ on $\mathfrak{T}$. We refer to [Me] and [Il] for information on Barsotti-Tate groups. The generic fiber of a d-dimensional formal $\mathcal{O}$-torus $\mathfrak{T}$ is isomorphic to a d-dimensional open unit ball because $\mathfrak{T}$ is smooth, as recalled in 4.1.2.

(4.2.3) Let $\mathfrak{T}$ be a formal torus over $\mathcal{O}$ attached to a Barsotti Tate group $(T_n)_{n\in\mathbb{Z}}$ over $\mathcal{O}$. The character group $X^*(\mathfrak{T})$ (resp. cocharacter group $X_*(\mathfrak{T})$) of $\mathfrak{T}$ is the étale sheaf $\underline{\mathrm{Hom}}((T_n),(\mu_n))$ (resp. $\underline{\mathrm{Hom}}((\mu_n),(T_n))$) over $\mathcal{O}$. We will identify $X^*(\mathfrak{T})$ (resp. $X_*(\mathfrak{T})$) with its geometric generic fiber, together with the natural action by $\mathrm{Gal}(K^{\mathrm{sh}}/K)=\mathrm{Gal}(\kappa^{\mathrm{sep}}/\kappa)$. Both $X^*(\mathfrak{T})$ and $X_*(\mathfrak{T})$ are free $\mathbb{Z}_p$-modules of rank $\dim(\mathfrak{T})$, and they are naturally $\mathbb{Z}_p$-dual to each other. The last assertion follows from the corresponding duality statement for finite locally free group schemes of multiplicative type; see [SGA3, Exposé X].

(4.3) Definition Let $\mathcal{O}$ be a complete discrete valuation ring whose residue field κ has characteristic $p > 0$.

(i) A *concordant rigid group* G over K is a rigid analytic group over K with the property that there exists a finite Galois extension L/K and a formal torus $\mathfrak{T}_{\mathcal{O}_L}$ over $\mathcal{O}_L$ such that the base extension $G \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, L$ of G to L is isomorphic, as a rigid analytic group, to the generic fiber $\mathfrak{T}_{\mathcal{O}_L, L}$ of $\mathfrak{T}_{\mathcal{O}_L}$. Such a field L above is called a *stabilizing field* of G.

(ii) A G is *strongly concordant rigid group* G over K is a rigid analytic group over K with the property that there exists a finite Galois extension L/K and a split formal torus $\mathfrak{T}_{\mathcal{O}_L}$ over $\mathcal{O}_L$ such that the base extension $G \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, L$ of G to L is isomorphic, as a rigid analytic group, to the generic fiber $\mathfrak{T}_{\mathcal{O}_L, L}$ of $\mathfrak{T}_{\mathcal{O}_L}$. A field L above is called a *splitting field* of G.

(4.3.1) Remark (i) The rigid analytic space T^{an} attached to a torus T over K is *not* a concordant rigid group, instead there is a rigid open subgroup T inside T^{an} which is a strongly concordant rigid group.

(ii) When $\mathrm{char}(\kappa) = p > 0$, formal tori over $\mathcal{O}$ have rigidity properties similar to those for tori over $\mathcal{O}$. On the other hand if $\mathrm{char}(\kappa) = 0$, any connected formal group which is flat and topologically of finite type over $\mathcal{O}$ is isomorphic to a formal additive group. In particular we do not have a good notion of "the character group of a concordant rigid group" if $\mathrm{char}(\kappa) = 0$. Therefore we required that $\mathrm{char}(\kappa) = p$ in Definition 4.3, even though the conditions makes sense when $\mathrm{char}(\kappa) = 0$ as well.

(iii) Let L be a stabilizing field of a concordant rigid group G as in 4.3 (i). Then there is a natural action of the Galois group $\mathrm{Gal}(\kappa_L^{\mathrm{sep}}/\kappa_L)$ of the residue field κ_L of $\mathcal{O}_L$ on the character group $X^*(\mathfrak{T}_{\mathcal{O}_L})$ of the formal torus $\mathfrak{T}_{\mathcal{O}_L}$. The rigid group G over K is concordant if and only if the Galois action above factors through the Galois group of a finite extension of κ_L.

(iv) If the residue field of $\mathcal{O}_K$ is separably closed, then every concordant rigid group over K is strongly concordant.

(4.4) Definition Assume that $\mathrm{char}(\kappa) = p > 0$. Let T be a concordant rigid group over K, and let L/K be a finite Galois extension such that $T_L := T \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, L$ is the generic fiber of a formal torus $\mathfrak{T}_{\mathcal{O}_L}$ over $\mathcal{O}_L$. Then the *character group* $X^*(T)$ of T is defined to be the character group $X^*(\mathfrak{T}_{\mathcal{O}_L})$ of the formal torus $\mathfrak{T}_{\mathcal{O}_L}$, together with the natural linear action on $X^*(\mathfrak{T}_{\mathcal{O}_L})$ by $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. This action by $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ extends the $\mathrm{Gal}(L^{\mathrm{sh}}/L)$-action on the character group $X^*(\mathfrak{T}_{\mathcal{O}_L})$ of the formal torus $\mathfrak{T}_{\mathcal{O}_L}$.

(4.4.1) Remark (i) Let $\mathfrak{T}$ be a split formal torus over $\mathcal{O}$. Let $\mathfrak{T}_K$ be the generic fiber of $\mathfrak{T}$. Then $X^*(\mathfrak{T})$ is naturally isomorphic to the group of all K-homomorphisms from the rigid group $\mathfrak{T}_K$ to $(\widehat{\mathbb{G}_m})_K$, where $(\widehat{\mathbb{G}_m})_K$ denotes the formal completion of $\mathbb{G}_m$ along the zero section of its closed fiber, as in 4.2.1. Each homomorphism $f : \mathfrak{T}_K \to (\widehat{\mathbb{G}_m})_K$ can be regarded as a rigid analytic function on $\mathfrak{T}_K$ whose sup norm is at most one, hence defines a map from $\mathfrak{T}$ to $\widehat{\mathbb{G}_m}$ according to 4.1.3. Similarly $X_*(\mathfrak{T})$ is naturally isomorphic to the group of all K-homomorphisms from $(\widehat{\mathbb{G}_m})_K$ to $\mathfrak{T}_K$.

To check the above assertions, we may and do assume that $\mathfrak{T}$ is $\widehat{\mathbb{G}_{\mathrm{m}}}$ over $\mathcal{O}_L$. We must show that every endomorphism of $(\widehat{\mathbb{G}_m})_L$ is a $\mathbb{Z}_p$-power of the identity endomorphism. We use the standard coordinate on $\widehat{\mathbb{G}_{\mathrm{m}}}$, and represent a given endomorphism as a rigid analytic function $f(x) = \sum_{n\geq 0} a_n\, x^n$, with $a_n \in \mathcal{O}_L$ for every $n \geq 0$, on $(\widehat{\mathbb{G}_m})_L$, as in 4.1.3, such that $(1 + f(x))^i = 1 + f((1 + x)^i - 1)$ for all $i \in \mathbb{Z}$. The functional equation above implies that $f(x)$ comes from a formal function on $\widehat{\mathbb{G}_{\mathrm{m}}}$, so the given endomorphism of $\widehat{\mathbb{G}_{\mathrm{m}}}_L$ comes from an element of $\widehat{\mathbb{G}_{\mathrm{m}}})\mathcal{O}_L \xleftarrow{\sim} \mathbb{Z}_p$.

(ii) Suppose that $\mathfrak{T}$ is a formal torus over $\mathcal{O}$. Let $M = \widehat{K^{\mathrm{sh}}}$. Then one can identify $X^*(\mathfrak{T})$ with the group of all M-homomorphisms from $\mathfrak{T}_K \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, M$ to $(\widehat{\mathbb{G}_{\mathrm{m}}})_M$ as in (i) above.

(iii) The action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ in 4.4 comes from (a limit of) finite étale descent of schemes. See the proof of 4.4.2 below.

(iv) In the context of 4.4, the character group $X^*(T)$ of T can be identified with the set of all rigid homomorphisms from $\mathfrak{T}_K \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, \widehat{L^{\mathrm{sh}}}$ to $(\widehat{\mathbb{G}_{\mathrm{m}}})_{\widehat{L^{\mathrm{sh}}}}$.

(4.4.2) Lemma *Notation as above.*

(i) *The action of the Galois group* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ *on the character group of a concordant rigid group T is trivial on a subgroup of finite index of the inertia subgroup* $\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{sh}}) \subset \mathrm{Gal}(K^{\mathrm{sep}}/K)$.

(ii) *Conversely, for any free $\mathbb{Z}_p$-module M of finite rank and any homomorphism ρ :* $\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}_{\mathbb{Z}_p}(M)$ *such that* $\rho(\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{sh}}))$ *is finite, there exists a concordant rigid group T over K such that $X^*(T)$ is isomorphic to M as* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$*-modules.*

(iii) *The rigid group T in (ii) above is strongly concordant if and only if the Galois representation ρ has finite image. Moreover the concordant rigid group T is determined by (M, ρ) up to (non-unique) isomorphisms.*

Proof. First we make explicit the action of the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $X^*(T)$. Let L be a finite Galois extension of K such that $G \times_{\mathrm{Spm}\, K} \mathrm{Spm}, L$ is the generic fiber of a formal torus $\mathfrak{T}$ over L. Then for every finite Galois extension L_1 of K containing L and every element $\sigma \in \mathrm{Gal}(L_1/K)$, we have a natural σ-linear automorphism $\beta_{L_1,\sigma}$ of $G \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, L_1$, which in turn comes from a σ-linear automorphism $\alpha_{L_1,\sigma}$ of $\mathfrak{T} \times_{\mathrm{Spf}\ \mathcal{O}_L} \mathrm{Spf}\ \mathcal{O}_{L_1}$, by 4.1.3 and the argument in 4.4.1 (i). Restricting $\alpha_{L_1,\sigma}$ to the p^n-torsion subgroup of the formal torus $\mathfrak{T} \times_{\mathrm{Spf}\ \mathcal{O}_L} \mathrm{Spf}\ \mathcal{O}_{L_1}$, for a fixed n and sufficiently large L_1, the family of automorphisms $\{\alpha_{L_1,\sigma} \,|\, \sigma \in \mathrm{Gal}(L_1/K)\}$ gives an action of $\mathrm{Gal}(L_1/K)$ on the character group of the p^n-torsion subgroup of the formal torus $\mathfrak{T}$. These Galois representations form a compatible family as n varies, and we get the desired action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $X^*(T) = X^*(\mathfrak{T})$ after passing to the limit. By construction, the image of the inertia group of the above Galois representation is trivial. We have shown the statement (i). The first part of (iii) follows from the description of the Galois action on $\mathfrak{T}$ and standard finite Galois descent. We refer to SGA 1 Exposé VIII and [BLR, Chap. 6] for finite Galois descent. The second part of (iii) is a consequence of the reconstruction of a concordant rigid group G from the Galois action on its character group, described below.

Suppose we are given a free $\mathbb{Z}_p$-module M of finite rank, and an action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on M such that the restriction of the Galois action to the inertia subgroup $I_K = \mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{ur}})$ factors through a finite quotient $\mathrm{Gal}(L'/K^{\mathrm{ur}})$. We have to construct a concordant group G over K whose character group coincides with the given Galois representation M,. Let L be a finite Galois subextension of K in K^{sep} such that $L' \subseteq L \cdot K^{\mathrm{sep}}$. For each $n \in \mathbb{N}$, consider the action of $\mathrm{Gal}(K^{\mathrm{sep}}/L)$ on M/p^nM, which factors through the maximal unramified extension L^{ur} of L. So M/p^nM is the character group of a finite local free group scheme $\mathfrak{T}_n$ over $\mathcal{O}_L$ of multiplicative type. The finite group schemes $\mathfrak{T}_n$ over $\mathcal{O}_L$ form an inductive system, where the transition map $\mathfrak{T}_n \to \mathfrak{T}_{n+1}$ is defined by the canonical projection $M/p^{n+1}M \to M/p^nM$. The inductive limit $\mathfrak{T} := \varinjlim_n \mathfrak{T}_n$ is a formal torus over $\mathcal{O}_L$, whose character group is canonically isomorphic to M, and the action of $\mathrm{Gal}(K^{\mathrm{sep}}/L)$ on $X^*(\mathfrak{T})$ coincides with the restriction to $\mathrm{Gal}(K^{\mathrm{sep}}/L)$ on $X^*(\mathfrak{T})$ of the given Galois representation on M.

Consider the generic fiber $\mathfrak{T}_L$ of $\mathfrak{T}$. The action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $X^*(\mathfrak{T})$ defines a semi-linear action of the finite Galois group $\mathrm{Gal}(L/K)$ on the formal torus $\mathfrak{T}$ over $\mathcal{O}_L$. Passing to the generic fiber, we get a semi-linear action of $\mathrm{Gal}(L/K)$ on $\mathfrak{T}_L$, or equivalently, a descent data on the split concordant rigid group $\mathfrak{T}_L$ over L, with respect to the finite Galois extension L/K. By finite Galois descent of for rigid analytic spaces, discussed in 4.5 below, $\mathfrak{T}_L$ descends to a rigid analytic group G over K, which is concordant by construction. Notice that the criterion for effectivity in 4.5.3 is satisfied, because $\mathfrak{T}_L$ is the direct limit of an increasing sequence of affinoid spaces. Clearly the cocharacter group of G coincides with the given Galois representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on M. ∎

(4.5) There are several places in this article where finite Galois descent for rigid spaces is required. Unfortunately this topic does not seem to have been systematically documented in the literature on rigid analytic spaces, for instance [BGR], nor in the closely related topics on non-archimedean analytic spaces and adic spaces, treated in [Berk] and [H] respectively. The discussion below owe much to S. Bosch.

(4.5.1) Lemma *Let k_1/k be a finite Galois extension of complete discrete valuation rings. Let $\Gamma = \mathrm{Gal}(k_1/k)$ be the Galois group of k_1/k. Let A_1 be an affinoid algebra over k_1. Assume that we have an action of Γ on A_1 by ring automorphisms, extending the tautological action of Γ on k_1. Let $A = A_1^\Gamma$ be the subring of Γ-invariants in A_1. Then A is a Tate algebra, and the natural homomorphism $h : k_1 \otimes_k A \to A_1$ is an isomorphism.*

Proof. That h is an isomorphism follows from standard finite Galois descent. It remains to show that A is an affinoid algebra over k.

Let $x_1, \ldots, x_m$ be a system of affinoid generators of A_1, and let $\alpha : k_1\{X_1, \ldots, X_m\} \twoheadrightarrow A_1$ be the k_1-linear surjection from the ring of strictly convergent power series $k_1\{X_1, \ldots, X_m\}$ to A_1 such that $\alpha(X_i) = x_i$ for $i = 1, \ldots, m$. Let $\sigma_1, \ldots, \sigma_n$ be the elements of Γ. For each x_i, let $y_{i,1}, \ldots, y_{i,n}$ be the elementary symmetric polynomials in $\sigma_1(x_i), \ldots, \sigma_n(x_i)$. Clearly $y_{i,j} \in A$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$

Let $B = k\{Y_{i,j}\}$ be the strictly convergent power series ring over k in the variables $Y_{i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Clearly $B_1 := B \otimes_k$ is isomorphic to the strictly convergent power series ring $k_1 \cong k_1\{Y_{i,j}\}$ over k_1. Consider the homomorphism $\beta : B \to A$ which sends $Y_{i,j}$ to $y_{i,j}$. By [BGR, 6.3.2/2], $\iota \circ \beta \otimes_k k_1 : B \otimes_k k_1 \to A_1$ is finite, where $\iota : A \hookrightarrow A$ is the inclusion map. Hence $\beta \otimes_k k_1 : B \otimes_k k_1 to A \otimes_k k_1$ is finite. Therefore $\beta : B \to A$ is finite. We have proved that A is an affinoid algebra over k. ∎

(4.5.2) With Lemma 4.5.1 at hand, we have at our disposal the standard formalism of descent as in SGA 1 Exposé VIII and [BLR, Chap. 6], and the usual properties for descent of morphisms hold. As for effectivity of descent, we only present a simple one in 4.5.3.

(4.5.3) Lemma *Let k_1/k be a finite Galois extension of complete discrete valuation rings. Let $\Gamma = \mathrm{Gal}(k_1/k)$ be the Galois group of k_1/k. Let X be a rigid analytic space over k_1. Suppose that we have a semi-linear action of Γ on X, giving rise to a descent datum on X with respect to $k \to k_1$. Assume that every Γ-orbit on X is contained in an affinoid open subspace of X. Then the descent datum is effective. In other words, there exists, up to unique isomorphism, a rigid analytic space Y over k, and an isomorphism $Y \times_{\mathrm{Spm}\, k} \mathrm{Spm}\, k_1 \xrightarrow{X}$, compatible with the given semi-linear action of Γ on X.*

Proof. The standard argument in the case of schemes, as found in [BGR, p. 141], works in the present situation. ∎

§5. Néron models for rigid concordant groups

In this section K denotes the fraction field of a complete discrete valuation ring $\mathcal{O}$. The purpose of this section is to formulate a definition of formal Néron models for concordant rigid groups and establish their existence.

(5.1) Bosch and Schlöter have developed a theory of Néron models for smooth rigid analytic varieties in [BS]. Let K be the fraction field of a complete discrete valuation ring $\mathcal{O}$. According to their definition, a π-adic formal Néron model of a smooth rigid space X over K is a smooth π-adic formal scheme $\mathfrak{U}$ locally of finite type over $\mathcal{O}$ whose generic fiber $\mathfrak{U}_K$ is an open rigid subspace of X, such that the following universal property is satisfied:

(∗) If $\mathfrak{Z}$ is a smooth π-adic formal scheme of finite type over $\mathcal{O}$, and $f_K : \mathfrak{Z}_K \to X$ is a rigid K-morphism, then there exists a unique $\mathcal{O}$-morphism $f_{\mathcal{O}} : \mathfrak{Z} \to \mathfrak{U}$ of π-adic formal schemes which induces f_K on the generic fibers.

(5.1.1) Bosch and Schlöter proved a general existence theorem of π-adic formal Néron models. It says that the necessary and sufficient condition for a smooth rigid K-group X to have a quasi-compact π-adic formal Néron model is that the set of K^{sh}-valued points $X(K^{\mathrm{sh}})$ of X is bounded; i.e. it is contained in a quasi-compact rigid subspace of X. They also clarified the relation between the Néron model for a smooth K-group G and the π-adic formal Néron model for the rigid analytic K-group G^{an} attached to G: Suppose that G is of finite type over K with Néron model $\underline{G}^{\mathrm{NR}}$, and assume either that $\underline{G}^{\mathrm{NR}}$ is quasi-compact or that G is commutative, then the π-adic completion along the special fiber of $\underline{G}^{\mathrm{NR}}$ is a π-adic formal Néron model of G^{an}.

(5.1.2) Since the concordant rigid groups are not quasi-compact, it will be too restrictive to use only the π-adic smooth formal schemes which are locally of finite type over $\mathcal{O}$. To illustrate this point, let $\mathfrak{T}$ be a split formal torus over $\mathcal{O}$, and let T be the generic fiber of $\mathfrak{T}$. A natural candidate for the formal Néron model for T is the split formal torus $\mathfrak{T}$ itself, which is certainly not π-adic unless $\dim(T) = 0$.

(5.2) Definition Let X be a smooth rigid analytic space over K. A *formal Néron model* of X is a locally noetherian formal scheme $\mathfrak{X}$ over $\mathcal{O}$ which is formally smooth, separated and locally of finite type over Spf $\mathcal{O}$, together with an isomorphism from the generic fiber $\mathfrak{X}_K$ to an open rigid subspace of X, such that the following universal property is satisfied:

(†) For every noetherian adic ring R over $\mathcal{O}$ which is formally smooth and topologically of finite type over $\mathcal{O}$, and for every K-morphism $f_K : (\mathrm{Spf}\ R)_K \to X$ of rigid spaces, there exists a unique $\mathcal{O}$-morphism $f_{\mathcal{O}} : \mathrm{Spf}\ R \to \mathfrak{X}$ of formal schemes which induces f_K on the generic fibers.

Remark It is immediate from the universal property (†) that any two formal Néron models of a smooth rigid analytic space X are isomorphic up to unique isomorphism. But different rigid analytic spaces may have the same formal Néron models. For instance $\mathfrak{X}$ is also the Néron model of its generic fiber $\mathfrak{X}_K$.

Notation The formal Néron model of a smooth rigid space X over K is denoted by $\underline{X}^{\mathrm{fm\,NR}}$, if it exists.

(5.2.1) Remark By definition, the formal scheme $\mathfrak{X}$ in 5.2 are covered by affine open formal subschemes of the form Spf (R, I), where R is a noetherian adic $\mathcal{O}$-algebra which is formally smooth and topologically of finite type over $\mathcal{O}$. More concretely, R is an $\mathcal{O}$-algebra and I is an ideal of R such that

(a) The ring R is noetherian, and is complete and separated with respect to the I-adic topology.

(b) The ideal $h^{-1}(I)$ of $\mathcal{O}$ contains a power of the maximal ideal $\mathfrak{p}$ of $\mathcal{O}$, where $h : \mathcal{O} \to R$ is the structural map of R as an $\mathcal{O}$-algebra. In other words h is a continuous homomorphism for the π-adic topology on $\mathcal{O}$ and the I-adic topology on R.

(c) The continuous homomorphism $h : \mathcal{O} \to R$ is formally smooth.

(d) The ring R/I is finitely generated over $\mathcal{O}$.

Adic $\mathcal{O}$-algebras (R, I) satisfying the above properties are exactly those used as "test objects" in the condition (†) of 5.2 above.

(5.2.2) Examples of noetherian adic rings formally smooth and topologically of finite type over $\mathcal{O}$.

(1) The π-adic completion of a smooth $\mathcal{O}$-algebra of finite type.

(2) A formal power series ring $A[[y_1, \ldots, y_n]]$, where A is a smooth $\mathcal{O}$-algebra of finite type.

(3) The J-adic completion of a smooth $\mathcal{O}$-algebra A of finite type over $\mathcal{O}$, where J is an ideal of A containing π such that A/J is smooth over κ.

(4) The I-adic completion of a smooth $\mathcal{O}$-algebra A of finite type, where I is an ideal of A containing π. Of course this class contains all the above examples.

(5.2.3) Remark For the purpose of this article it will be enough to use the following variant of Definition 5.2: Require that the formal scheme $\mathfrak{X}$ is covered by open affines Spf R where R belongs to Example (3) above, and in the condition (†) use only adic rings R in Example (3) above. The reason is that every concordant rigid group has a Néron model, and the Néron model is covered by affine opens of the form Spf R where R is an adic ring in Example (3) above.

Using the variant notion of formal Néron model would simplify some part of the exposition of this paper involving commutative algebra. However the definition adopted in 5.2 seems more appealing.

(5.3) Proposition *Let $h : (\mathcal{O}, \mathfrak{p}) \to (A, I)$ be a formally smooth continuous homomorphism of noetherian adic rings such that A is topologically finitely generated over $\mathcal{O}$. Then the $\mathcal{O}$-algebra A is isomorphic to a filtered inductive limit of smooth $\mathcal{O}$-algebras.*

(5.3.1) Lemma *Let $h : (\mathcal{O}, \mathfrak{p}) \to (A, I)$ be a continuous homomorphism as in Proposition 5.3.*

(i) *There exists an adic ring (B, J) which is the J_1-adic completion of a polynomial algebra $\mathcal{O}[x_1, \ldots, x_n]$ for an ideal J_1 containing π, and a surjection $\beta : B \to A$, such that the inverse image of an ideal of definition of A is an ideal of definition of B.*

(ii) *Let (B, J) be an adic ring as in* (i) *above. Then there exists an $\mathcal{O}$-algebra homomorphism $i : A \to B$ such that $\beta \circ i = \mathrm{id}_R$.*

(iii) *Let (B, J) be as above. Then there exists a projective A-module M of finite rank and an isomorphism*

$$B \xrightarrow{\sim} A[[M]] = \bigoplus_{m \geq 0} \mathrm{Sym}^m_A(M)$$

Proof of Lemma 5.3.1. Let $a_1, \ldots, a_n$ be a finite set of topological generators of A. Let P be the polynomial algebra $\mathcal{O}[x_1, \ldots, x_n]$ over $\mathcal{O}$, and let $\beta_1 : P \to A$ be the homomorphism such that $\beta_1(x_i) = a_i$ for $i = 1, \ldots, n$. Let $J_1 = \beta_1^{-1}(I)$ be the inverse image of I under β_1, and let (B, J) be the J_1-adic completion of (P, J_1). Then β_1 induces a continuous homomorphism from (B, J) to (A, I), which is surjective by assumption. This proves (i). Statement (ii) follows from the formal smoothness of A. The statement (iii) follows from EGA 0_{IV} 19.5.3 and EGA 0_{IV} 19.5.4, because B and A are formally smooth over $\mathcal{O}$. The reader may want to consult (a part of) [dJ, Lemma 1.3.3] for some similar statements and their proofs. ∎

Proof of Proposition 5.3. According to the main theorem of [Sw], every regular homomorphism of commutative rings $a : S \to R$ is a filtered inductive limit of smooth homomorphism of rings $a_\alpha : S \to R_\alpha$. So it suffices to prove that $h : \mathcal{O} \to A$ is regular, i.e. all fibers of h are geometrically regular. This should be "well-known", but we cannot find a ready reference. So a proof is provided here; it reduces the regularity of h to the geometric regularity of formal fibers of a polynomial ring over $\mathcal{O}$.

We recall two basic properties of regular homomorphisms. Suppose that $\phi : R_1 \to R_2$ and $\psi : R_2 \to R_3$ are homomorphism of commutative rings.

(a) (transitivity) If ϕ and ψ are both regular, then so is $\psi \circ \phi$.

(b) (descent) If $\psi \circ \phi$ is regular and ψ is faithfully flat, then ϕ is regular.

See [EGA, IV 6.8.3] or [Ma, 33.B] for a proof of these facts.

Let $i : A \to B$ and $\beta : B \to A$ be as in Lemma 5.3.1. Since $i : A \to B$ is faithfully flat, it suffices to prove that B is regular over $\mathcal{O}$ by (b) above. Denote by C the polynomial ring $\mathcal{O}[x_1, \ldots, x_n]$; C is regular over $\mathcal{O}$ because it is smooth over $\mathcal{O}$. So we only have to show that B is regular over C. Since each local ring of a fiber of $\mathrm{Spec}\, B \to \mathrm{Spec}\, C$ is a localization of a fiber of $\mathrm{Spec}\, B_{\mathfrak{m}} \to \mathrm{Spec}\, C_{\mathfrak{m} \cap C}$ for some maximal ideal $\mathfrak{m}$ of B, it suffices to check that for every maximal ideal $\mathfrak{m}$ of B, $B_{\mathfrak{m}}$ is regular over $C_{\mathfrak{m}_1}$, where $\mathfrak{m}_1 = \mathfrak{m} \cap C$. Since B is the J_1-adic completion of C, the maximal ideal $\mathfrak{m}$ contains $J_1 B$. So $\mathfrak{m}_1$ is a maximal ideal of C containing J_1, and $\mathfrak{m} = \mathfrak{m}_1 \cdot B$. By definition, the $\mathfrak{m}$-adic completion $B_{\mathfrak{m}}^{\wedge}$ of $B_{\mathfrak{m}}$ is canonically isomorphic to the $\mathfrak{m}_1$-adic completion $C_{\mathfrak{m}_1}^{\wedge}$ of $C_{\mathfrak{m}_1}$. Since $B_{\mathfrak{m}}^{\wedge}$ is faithfully flat over $B_{\mathfrak{m}}$, to prove that $B_{\mathfrak{m}}$ is regular over $C_{\mathfrak{m}_1}$, it suffices to prove that $B_{\mathfrak{m}}^{\wedge} = C_{\mathfrak{m}_1}^{\wedge}$ is regular over $C_{\mathfrak{m}_1}$. This holds because if a noetherian ring R has the property that all of its formal fibers are regular, the same is true for every finitely generated R-algebra; see [EGA, IV 7.4.4] or [Ma, 33.G]. (Alternatively, the ring $\mathcal{O}$ is excellent because it is a complete noetherian local ring. Hence C is excellent because it is finitely generated over $\mathcal{O}$. See [EGA, IV 7.8.3 (ii), (iii)], or [Ma, 34.A].) ∎

(5.3.2) Proposition *Let X be a smooth K-scheme of finite type, which has a Néron model $\underline{X}^{\mathrm{NR}}$ over $\mathcal{O}$. Let Z be a reduced closed subscheme of $\underline{X}^{\mathrm{NR}} \times_{\operatorname{Spec}\mathcal{O}} \operatorname{Spec}\kappa$, and let $\mathfrak{X} = (\underline{X}^{\mathrm{NR}})^{/Z}$ be the formal completion of $\underline{X}^{\mathrm{NR}}$ along Z. Then $\mathfrak{X}$ is a formal Néron model of $\mathfrak{X}_K$.*

Proof. This Propostion, as well as Prop. 5.4.1, are surely known to the experts, but the author does not know a reference. Let R be a noetherian adic ring which is formally smooth and topologically of finite type over $\mathcal{O}$. Suppose that $f_K : (\operatorname{Spf}\ R)_K \to \mathfrak{X}_K$ is a K-morphism of rigid analytic spaces. We have to show that f_K extends to a morphism $f : \operatorname{Spf}\ R \to \mathfrak{X}$ of formal schemes. By Prop. 5.3, R is a filtered inductive limit of smooth $\mathcal{O}$ algebras R_α. Post-composing f_K with the composition of natural morphism $\mathfrak{X}_K \to X^{\mathrm{an}} \to X$ of ringed spaces, one obtains a K-morphism $g_{_K} : \operatorname{Spec}(\varinjlim_\alpha R_\alpha \otimes K) \to X$ of ringed spaces, which is necessarily a K-morphism of schemes over K. By the universal property of Néron models, $g_{_K}$ extends uniquely to a morphism $g : \operatorname{Spec} R \to \underline{X}^{\mathrm{NR}}$. Let I be the largest ideal of definition of the adic ring R. Since the map $\mathfrak{X} \to \underline{X}^{\mathrm{NR}}$ is a monomorphism in the category of ringed spaces, the uniqueness in the previous sentence implies the uniqueness of the extension $f : \operatorname{Spf}\ R \to \mathfrak{X}$ of f_K. We claim that $g(\operatorname{Spec}(R/I)) \subseteq Z$. This claim implies that g factors as a composition $\operatorname{Spf}\ R \to \mathfrak{X} \to \underline{X}^{\mathrm{NR}}$ of morphisms of ringed spaces. The assertion that $g(\operatorname{Spec}(R/I)) \subseteq Z$ also implies that the map $\operatorname{Spf}\ R \to \mathfrak{X}$ is compatible with the adic topologies of its source and target, giving the existence of the required map $f : \operatorname{Spf}\ R \to \mathfrak{X}$.

It remains to prove the claim. Since I is equal to the intersection of all maximal ideals containing I, it suffices to prove that $g(x_0) \in Z$ for every closed point x_0 of $\operatorname{Spec}(R/I)$. Since R is formally smooth over $\mathcal{O}$, x_0 is the image of the closed point of an $\mathcal{O}$-morphism $x : \operatorname{Spf}\ \mathcal{O}_M \to \operatorname{Spf}\ R$ for some finite unramified extension M of K. Under f_K, the point x of $(\operatorname{Spf}\ R)_K$ is mapped to a point of $\mathfrak{X}$, therefore the composition of morphisms of ringed spaces $g \circ x : \operatorname{Spf}\ \mathcal{O}_M \to \operatorname{Spf}\ R \to \underline{X}^{\mathrm{NR}}$ factors through $\mathfrak{X} \to \underline{X}^{\mathrm{NR}}$. This implies that $g(x_0) \in Z$. The claim has been proved. ∎

(5.3.3) Corollary *Let $R = \mathcal{O}[[x_1, \ldots, x_n]]$, the formal power series ring over $\mathcal{O}$, endowed with the topology defined by the maximal ideal of R. Let $B = (\operatorname{Spf}\ R)_K$ be the n-dimensional rigid open unit ball over $\mathcal{O}$. Then $\operatorname{Spf}(R)$ is the formal Néron model of B. In particular, every formal torus $\mathfrak{T}$ over $\mathcal{O}$ is the formal Néron model of the concordant rigid group $\mathfrak{T}_K$.*

PROOF. Let G be an n-dimensional semiabelian variety over K. Then R is isomorphic to the formal completion of $\underline{G}^{\mathrm{NR}}$ at the zero section of its closed fiber. ∎

(5.3.4) Remark (i) Let R be a complete I-adic algebra over $\mathcal{O}$, where I is an ideal of R which contains a power of the maximal ideal of $\mathcal{O}$. Assume that R is formally smooth over $\mathcal{O}$. The author does not know, under the above conditions, whether Spf R is the formal Néron model of its generic fiber $(\mathrm{Spf}\ R)_K$.

(ii) Let $\mathfrak{U}$ be a π-adic formal scheme of finite type over $\mathcal{O}$. Let X be a rigid analytic space, such that $\mathfrak{U}_K$ is an open rigid subspace of X. Assume that $\mathfrak{U}$ is a formal Néron model of X in the sense of Bosch and Schlöter. Then it is natural to ask, whether $\mathfrak{U}$ is a formal Néron model of X in the sense of 5.2. The author does not know the answer either.

(5.4) Let X_K be a smooth K-scheme which has a Néron model $\underline{X}^{\mathrm{NR}}$ over $\mathcal{O}$. Let L be a finite Galois extension of K, and assume that $X_L := X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ has a Néron model $\underline{X_L}^{\mathrm{NR}}$ over $\mathcal{O}_L$. Let $\mathrm{can}_{L,K} : \underline{X}^{\mathrm{NR}} \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}\, \mathcal{O}_L \to \underline{X_L}^{\mathrm{NR}}$ be the natural base change map. Let W be a reduced closed subscheme of $\underline{X_L}^{\mathrm{NR}} \times_{\mathrm{Spec}\, \mathcal{O}_L} \mathrm{Spec}\, \kappa_L$, and assume that W is stable under the natural action of $\mathrm{Gal}(L/K)$. Let $Z_1 := \mathrm{can}_{L,K}^{-1}(W)$ with reduced structure. By descent fpqc descent for closed subschemes, explained in SGA 1 Exposé VIII and [BLR, Chap. 6], there exists a unique reduced closed subscheme Z of $\underline{X}^{\mathrm{NR}} \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}\, \kappa$ such that $(Z \times_{\mathrm{Spec}\, \kappa} \mathrm{Spec}\, \kappa_L)_{\mathrm{red}}$ is equal to Z_1. Let $\mathfrak{X} = (\underline{X}^{\mathrm{NR}})^{/Z}$ be the formal completion of $\underline{X}^{\mathrm{NR}}$ along Z. Denote by $\mathfrak{Y}$ the formal completion of $\underline{X_L}^{\mathrm{NR}}$ along W. It is well-known that the rigid space X_L^{an} attached to X_L is naturally isomorphic to $X^{\mathrm{an}} \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, L$, the base change to $\mathrm{Spm}\, L$ of the rigid K-space X^{an} attached to the smooth K-scheme X. Let $\mathfrak{Y}_L$ be the generic fiber of $\mathfrak{Y}$, so that $\mathfrak{Y}_L$ is a rigid analytic subspace of X_L^{an}, and $\mathfrak{Y}_L$ is stable under the natural action of $\mathrm{Gal}(L/K)$. By Galois descent for rigid analytic spaces, there exists a unique rigid subspace Y of X^{an} such that $\mathfrak{Y}_L = Y \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, L$.

(5.4.1) Proposition *Notation as above. Then the formal scheme $\mathfrak{X}$ over* Spf $\mathcal{O}$ *is a formal Néron model of the rigid analytic K-space Y.*

PROOF. Let (R, I) be a noetherian adic ring which is formally smooth and topologically of finite type over $\mathcal{O}$. We may and do assume that $\mathrm{rad}(I) = I$. Suppose that $f_K : (\mathrm{Spf}\ R)_K \to Y$ is a K-morphism of rigid analytic spaces. We have to show that f_K extends uniquely to a morphism $f : \mathrm{Spf}\ R \to \mathfrak{X}$ of formal $\mathcal{O}$-schemes. Let $i : Y \hookrightarrow X^{\mathrm{an}}$ be the inclusion map. The same argument in 5.3.2 shows that $i \circ f_K : (\mathrm{Spf}\ R)_K \to X^{\mathrm{an}}$ extends uniquely to a morphism $g : \mathrm{Spec}\, R \to \underline{X}^{\mathrm{NR}}$, and the composition of $g \times_{\mathrm{Spec}\mathcal{O}} \mathrm{Spec}\, \mathcal{O}_L : \mathrm{Spec}\, R \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}\, \mathcal{O}_L \to \underline{X}^{\mathrm{NR}} \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}\, \mathcal{O}_L$ with $\mathrm{can}_{L,K} : \underline{X}^{\mathrm{NR}} \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}\mathcal{O}_L \to \underline{X_L}^{\mathrm{NR}}$ is equal to the unique extension of the composition $\mathrm{Spf}(R \otimes_{\mathcal{O}} \mathcal{O}_L)_L \xrightarrow{f_K \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, L} \mathfrak{Y}_L \hookrightarrow (\underline{X_L}^{\mathrm{NR}})^{\mathrm{an}}$. Moreover the morphism $\mathrm{can}_{L,K}$ maps $(\mathrm{Spec}(R/I) \otimes_{\mathcal{O}} \mathcal{O}_L)_{\mathrm{red}}$ into W, by the proof of the claim at the end of the proof of Prop. 5.3.2. Therefore $g(\mathrm{Spec}(R/I) \subseteq Z$. Passing to the completions, we obtain the required extension $f : \mathrm{Spf}\ R \to \mathfrak{Y}$ of f_K. ∎

(5.5) Assume that $\mathrm{char}(\kappa) = p > 0$. Let A be a potentially ordinary abelian variety over K. Let L be a finite Galois extension K such that $A_L := A \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ has semi-stable reduction over $\mathcal{O}_L$, and the neutral component of the closed fiber of the Néron model $\underline{A_L}^{\mathrm{NR}}$ of A_L is an extension of an ordinary abelian variety by a torus.

Let Z_1 be the inverse image of the zero section of the closed fiber of $\underline{A_L}^{\mathrm{NR}}$ under $\mathrm{can}_{K,L} : \underline{A}^{\mathrm{NR}} \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}\, \mathcal{O}_L \to \underline{A_L}^{\mathrm{NR}}$. Denote by Z the reduced closed subscheme of $\underline{A}^{\mathrm{NR}}$ such that $(Z \times_{\mathrm{Spec}\kappa} \mathrm{Spec}\kappa_L)_{\mathrm{red}}$ is equal to $(Z_1)_{\mathrm{red}}$. Let $(\underline{A_L}^{\mathrm{NR}})^{\wedge}$ be the formal completion of $\underline{A_L}^{\mathrm{NR}}$ along the special fiber of the zero section of $\underline{A_L}^{\mathrm{NR}}$. Since $\mathcal{O}$ is complete, $(\underline{A_L}^{\mathrm{NR}})^{\wedge}$ is canonically isomorphic to the formal completion of $\underline{A_L}^{\mathrm{NR}}$ along the zero section of $\underline{A_L}^{\mathrm{NR}}$. Denote by Y_L the generic fiber of $(\underline{A_L}^{\mathrm{NR}})^{\wedge}$. Denote by $\mathfrak{Y}$ the formal completion of $\underline{A}^{\mathrm{NR}}$ along Z.

(5.5.1) Corollary *Notation as in 5.5 above. Then Y_L descends to a concordant rigid subgroup Y of A^{an}, and $\mathfrak{Y}$ is a formal Néron model of Y.*

PROOF. The fact that Y_L descends to a rigid open subgroup of A^{an} follows from Galois descent for rigid analytic spaces, as explained in 4.5. The criterion for effectivity in 4.5.3 applies, because Y_L is the union of an increasing sequence of affinoid open subspaces. That $\mathfrak{Y}$ is a formal Néron model is a special case of Prop. 5.4.1. It remains to show that Y is a concordant rigid group over K. Our hypotheses imply that $(\underline{A_L}^{\mathrm{NR}})^{\wedge}$ is a formal torus over $\mathcal{O}_L$. Hence Y is a concordant rigid group. ∎

Remark (i) In the situation of 5.5.1, write the neutral component of the closed fiber of $\underline{A_L}^{\mathrm{NR}}$ as an extension of an abelian variety B_{κ_L} by a torus H_{κ_L}. If one does not assume that B_{κ_L} is ordinary, then the statement of Cor. 5.5.1 has to be modified: The formal $\mathcal{O}$-scheme $\mathfrak{X}$ is still a formal Néron model of Y, but Y is no longer a concordant rigid group. The formal completion $(\underline{A_L}^{\mathrm{NR}})^{\wedge}$ of $\underline{A_L}^{\mathrm{NR}}$ along the zero section of its special fiber contains a maximal formal subtorus $\mathfrak{T}$ over $\mathcal{O}_L$, and the quotient of $(\underline{A_L}^{\mathrm{NR}})^{\wedge}$ by $\mathfrak{T}$ is the smooth formal group attached to a Barsotti-Tate group $G = (G_n)_{n\in\mathbb{N}}$ over $\mathcal{O}_L$ such that neither 0 nor 1 is a slope of the closed fiber G_{κ_L} of G. The generic fiber of $\mathfrak{T}$ descends to a concordant rigid group T over K, and T is a subgroup of the rigid group Y.

(ii) In general, the concordant rigid group Y over K attached to the abelian variety A in 5.5.1 is not strongly concordant. For instance, if the residue field κ of $\mathcal{O}_K$ is finite, and the closed fiber of the Néron model $\underline{A_L}^{\mathrm{NR}}$ has a non-trivial abelian part, then Y is not strongly concordant.

(5.5.2) Corollary *Let T be a torus over K. Let L/K be a finite Galois splitting field of T, and let $\mathfrak{T}_{\mathcal{O}_L}$ be the split formal torus over $\mathcal{O}_L$ with character group $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let G be the rigid K-subgroup of T^{an} such that $G \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, L$ is equal to the generic fiber of $\mathfrak{T}_{\mathcal{O}_L}$. Let $\mathrm{can}_{L,K} : \underline{T}^{\mathrm{NR}} \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}\, \mathcal{O}_L \to \underline{T_L}^{\mathrm{NR}}$ be the base change morphism between the Néron models of T and T_L. Let Z_1 be the inverse image under $\mathrm{can}_{L,K}$ of the zero section of the closed fiber of $\underline{T_L}^{\mathrm{NR}}$. Let Z be the reduced closed subscheme of $\underline{T}^{\mathrm{NR}} \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}\, \kappa$ such that*

$(Z \times_{\mathrm{Spec}\,\kappa} \mathrm{Spec}\,\kappa_L)_{\mathrm{red}} = (Z_1)_{\mathrm{red}}$. *Then G is a strongly concordant rigid group, and the formal completion of $\underline{T}^{\mathrm{NR}}$ along Z is a formal Néron model of G.*

PROOF. The proof is similar to that of Cor. 5.5.1 and is omitted. ∎

(5.6) Lemma *Let X_1, X_2 be smooth rigid analytic spaces over K.*

(i) *Suppose that X_1 and X_2 admit formal Néron models $\mathfrak{X}_1, \mathfrak{X}_2$ over* $\mathrm{Spf}\ \mathcal{O}$. *Then the product formal scheme $\mathfrak{X}_1 \times_{\mathrm{Spf}\ \mathcal{O}} \mathfrak{X}_2$ is a formal Néron model of $X_1 \times_{\mathrm{Spm}\,K} X_2$.*

(ii) *Conversely, suppose that X_1 and X_2 are rigid analytic groups, and that the product $X := X_1 \times_{\mathrm{Spm}\,K} X_2$ admits a formal Néron model $\mathfrak{X}$ over $\mathcal{O}$. Then there exist formal Néron models $\mathfrak{X}_1, \mathfrak{X}_2$ for X_1 and X_2 respectively, and $\mathfrak{X}$ is canonically isomorphic to $\mathfrak{X}_1 \times_{\mathrm{Spf}\ \mathcal{O}} \mathfrak{X}_2$.*

PROOF. The statement (i) is immediate from the definition of formal Néron models, because "forming the formal Néron model over $\mathcal{O}$" is the right adjoint to "taking the generic fiber over K".

Suppose X_1, X_2 are rigid analytic groups as in (ii). From the product structure of X we get two commuting endomorphisms h_1, h_2 of the rigid group X such that $h_1^2 = \mathrm{id}_{X_1}$, $h_2^2 = \mathrm{id}_{X_2}$, $h_1 \circ h_2 = h_2 \circ h_1 = 0$, and $h_1 \cdot h_2 = h_2 \cdot h_1 = \mathrm{id}_X$. By (i) and the universal property of formal Néron models, $\mathfrak{X}$ inherits from X the structure of a formal group scheme over $\mathcal{O}$. Moreover each h_i extends to an endomorphism $\mathbf{h}_i$ of $\mathfrak{X}$, and the identities for the idempotents h_i extend to similar identities for $\mathbf{h_i}$, $i = 1, 2$. This decomposition of $\mathrm{id}_{\mathfrak{X}}$ into commuting idempotents $\mathbf{h}_1, \mathbf{h}_2$ defines a decomposition of $\mathfrak{X}$ into a product $\mathfrak{X}_1 \times_{\mathrm{Spf}\ \mathcal{O}} \mathfrak{X}_2$, and one sees that each $\mathfrak{X}_i$ is the formal Néron model of X_i. ∎

(5.7) Proposition *Let X be a smooth rigid analytic space over K. Let $\mathfrak{X}$ be a locally noetherian formally smooth formal scheme locally of finite type over $\mathcal{O}$ such that the generic fiber $\mathfrak{X}_K$ is an open rigid subspace of X. Let $M = \widehat{K^{\mathrm{sh}}}$ be the completion of the maximal unramified extension of K. Then $\mathfrak{X}$ is a formal Néron model of X if and only if $\mathfrak{X} \times_{\mathrm{Spf}\ \mathcal{O}}$* $\mathrm{Spf}\ \mathcal{O}_M$ *is a formal Néron model of $X \times_{\mathrm{Spm}\,K} \mathrm{Spm}\,M$.*

PROOF. Our proof is different from the proof of the similar statement in Thm. 7.2/3 (ii) of [BLR], which uses a criterion for the existence of Néron models. Suppose that $\mathfrak{X}$ is a Néron model for X. We claim that the ring homomorphism $\mathcal{O} \to \mathcal{O}_M$ is regular. This will imply that every adic noetherian ring (R, I) formally smooth and topologically of finite type over $\mathcal{O}_M$ is a filtered inductive limit of smooth $\mathcal{O}_M$-algebras, and it is also a filtered inductive limit of smooth $\mathcal{O}$-algebras. So the universal property (†) for $\mathfrak{X} \times_{\mathrm{Spf}\ \mathcal{O}} \mathrm{Spf}\ \mathcal{O}_M$ follows from the universal property (†) for $\mathfrak{X}$. The claim that the ring homomorphism $\mathcal{O} \to \mathcal{O}_M$ is regular means that κ^{sep} is geometrically regular over κ, and M is geometrically regular over K. Clearly κ^{sep} is smooth and hence geometrically regular over κ. If $\mathrm{char}(K) = 0$, then K is perfect and M is geometrically regular over K. So we may assume that $\mathrm{char}(K) = p > 0$.

By the structure theorem for equal characteristic complete local rings, the ring $\mathcal{O}$ has a coefficient field because K is formally smooth over $\mathbb{F}_p$. In other words we may assume that the homomorphism $\mathcal{O} \to \mathcal{O}_M$ is equal to the natural inclusion $\kappa[[t]] \hookrightarrow \kappa^{\mathrm{sep}}[[t]]$ of power series rings. We know that for any field E containing a field F, E is geometrically regular over F if and only if E is separable over F. So it suffices to show that for every finite purely inseparable extension k of κ and every $n \geq 0$,

$$\kappa^{\mathrm{sep}}((t)) \otimes_{\kappa((t))} k((t^{1/p^n})) \xrightarrow{\sim} k((t^{1/p^n}))\,.$$

This statement holds because both sides are purely inseparable over $\kappa((t))$ of degree $p^n[k:\kappa]$ and the arrow is surjective.

Conversely, suppose that $\mathfrak{X} \times_{\mathrm{Spf}\ \mathcal{O}} (\mathrm{Spf}\ \mathcal{O}_M)$ is a formal Néron model of $X \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, M$. Let R be an I-adically complete noetherian ring formally smooth and topologically of finite type over $\mathcal{O}$. Let $f_K : (\mathrm{Spf}\ R)_K \to X$ be a morphism of rigid spaces over K. Let R' be the $I \otimes \mathcal{O}_M$-adic completion of $R \otimes_{\mathcal{O}} \mathcal{O}_M$. Then R' is formally smooth and topologically of finite type over $\mathcal{O}_M$, and f_K defines a morphism $f_M : (\mathrm{Spf}\ R')_M \to X \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, M$. So f_M extends to a morphism $f_{\mathcal{O}_M} : \mathrm{Spf}\ R' \to \mathfrak{X} \times_{\mathrm{Spf}\ \mathcal{O}} \mathrm{Spf}(\mathcal{O}_M)$. Clearly $f_{\mathcal{O}_M}$ is compatible with the natural action of $\mathrm{Gal}(M/K) = \mathrm{Gal}(\kappa^{\mathrm{sep}}/\kappa)$ on $\mathrm{Spf}(\mathcal{O}_M)$ and $\mathfrak{X} \times_{\mathrm{Spf}\ \mathcal{O}} \mathrm{Spf}(\mathcal{O}_M)$. In other words, the map $f_{\mathcal{O}_M}$ is compatible with the descent data on both the source and the target. Applying étale descent for morphisms, we see that $f_{\mathcal{O}_M}$ comes from a morphism $f : \mathrm{Spf}\ \mathcal{O} \to \mathfrak{X}$ of formal schemes. ∎

(5.8) Lemma *Let Γ be a finite group, and let V be a finite dimensional linear representation of Γ over $\mathbb{Q}_p$.*

(a) *There exists a $\mathbb{Q}$-rational finite dimensional linear representation W of Γ and a Γ-equivariant isomorphism between V and a Γ-invariant direct summand of $W \otimes_{\mathbb{Q}} \mathbb{Q}_p$.*

(b) *Let W be as in* (a) *above, and let $\alpha : V \oplus V' \xrightarrow{\sim} W \otimes_{\mathbb{Q}} \mathbb{Q}_p$ be an isomorphism of $\mathbb{Q}_p[\Gamma]$-modules. Suppose that $V_{\mathbb{Z}_p}$ (resp. $V'_{\mathbb{Z}_p}$) is a $\mathbb{Z}_p$-lattice in V (resp. V') which is stable under Γ. Then there exists a Γ-invariant $\mathbb{Z}$-lattice $W_{\mathbb{Z}}$ in W such that α induces an isomorphism $V_{\mathbb{Z}_p} \oplus V'_{\mathbb{Z}_p} \xrightarrow{\sim} W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.*

(c) *Let W, α be as in* (b). *Suppose that $V_{\mathbb{Z}_p,1} \subseteq V_{\mathbb{Z}_p,2}$ are two $\mathbb{Z}_p$-lattices of V stable under Γ, and $V'_{\mathbb{Z}_p}$ is a Γ-invariant $\mathbb{Z}_p$-lattice of V'. Then there exists Γ-invariant $\mathbb{Z}$-lattices $W_{\mathbb{Z},1} \subseteq W_{\mathbb{Z},2}$ of W such that $\alpha(V_{\mathbb{Z}_p,i} \oplus V'_{\mathbb{Z}_p}) = W_{\mathbb{Z},i} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for $i = 1, 2$.*

PROOF. The statement (a) is easy: One may assume that V is an irreducible $\mathbb{Q}_p$-module. In this case it suffices to take $W = \mathbb{Q}[\Gamma]$.

Suppose that $W, V_{\mathbb{Z}_p}, V'_{\mathbb{Z}_p}$ and α are as in (b). We identify V with $\alpha(V) \subset W \otimes_{\mathbb{Q}} \mathbb{Q}_p$, similarly for V'. Let $v_1, \ldots, v_a$ (resp. $v'_1, \ldots, v'_b$) be a set of generators for the $\mathbb{Z}_p[\Gamma]$-module $V_{\mathbb{Z}_p}$ (resp. $V'_{\mathbb{Z}_p}$.) Pick $w_1, \ldots, w_a, w'_1, \ldots, w'_b \in W$ such that $v_i - w_i \in pV_{\mathbb{Z}_p} \oplus pV'_{\mathbb{Z}_p}$ and $v'_j - w'_j \in pV_{\mathbb{Z}_p} \oplus pV'_{\mathbb{Z}_p}$ for all $i = 1, \ldots, a$ and all $j = 1, \ldots, b$. Let $W_{\mathbb{Z}}$ be the $\mathbb{Z}[\Gamma]$-submodule of W generated by $w_1, \ldots, w_a, w'_1, \ldots, w_b$. Then $W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p = V_{\mathbb{Z}_p} \oplus V'_{\mathbb{Z}_p}$ by Nakayama's lemma. This proves (b). The argument for (b) also proves (c). ∎

(5.9) Theorem *Suppose that* $\mathrm{char}(\kappa) = p > 0$*. Let* G *be a strongly concordant rigid group over* K*. Then* G *admits a formal Néron model over* $\mathcal{O}$*.*

Proof. The idea of the proof is that every strongly concordant rigid group G over a local field K is, up to an unramified twist, a direct summand of the concordant rigid group attached to a torus T over K. This allows us to obtain the formal Néron model of G from the Néron model for T.

Let L be a finite Galois extension of K such that the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on the character group $X^*(G)$ of G factors through $\Gamma = \mathrm{Gal}(L/K)$. By Lemma 5.8, there exists a torus T over K which is split over L and a concordant rigid group G' over K such that $G \times_{\mathrm{Spm}\, K} G'$ is isomorphic to the concordant rigid group over K attached to T as in Cor. 5.5.2. According to Cor. 5.5.2, $G \times_{\mathrm{Spm}\, K} G'$ has a formal Néron model over $\mathcal{O}$. Hence G also has a formal Néron model over $\mathcal{O}$ by Lemma 5.6. ∎

The following Proposition says that 5.10 remains true when the adjective "strongly" is removed from its statement. We will not use this strengthening of 5.10 in this article.

(5.10) Proposition *Suppose that* $\mathrm{char}(\kappa) = p > 0$*. Let* G *be a concordant rigid group over* K*. Then* G *admits a formal Néron model over* $\mathcal{O}$*.*

Proof. Let M be the completion of the maximal unramified extension K^{sh} of K. We know from Thm. 5.10 that $G_M := G \times_{\mathrm{Spm}\, K} \mathrm{Spm}\, M$ admits a formal Néron model $\underline{G_M}^{\mathrm{fm\,NR}}$. We would like to show that $\underline{G_M}^{\mathrm{fm\,NR}}$ can be descended to a formal Néron model of G over $\mathcal{O}_K$. We use the standard theory of fpqc descent for schemes. The standard references are SGA 1 Exposé VIII, and [BLR, Chap. 6], especially 6.1 and 6.2.

By the universal property for $\underline{G_M}^{\mathrm{fm\,NR}}$, there is a natural semi-linear action of the Galois group $\mathrm{Gal}(M/K) = \mathrm{Gal}(\kappa^{\mathrm{sep}}/\kappa)$ on $\underline{G_M}^{\mathrm{fm\,NR}}$. Moreover this action is continuous with respect to the adic topology of the structure sheaf. The action of $\mathrm{Gal}(M/K)$ defines a descent datum for the formal scheme $\underline{G_M}^{\mathrm{fm\,NR}}$ with respect to $\mathrm{Spf}\ \mathcal{O}_M \to \mathrm{Spf}\ \mathcal{O}$. More precisely, let $\mathcal{I}$ be the largest ideal if definition of the noetherian adic formal scheme $\underline{G_M}^{\mathrm{fm\,NR}}$, and let $b \geq 1$ be a positive integer such that $\mathcal{I}$ contains π_M^b. For each $m \geq 1$, the $\mathrm{Gal}(M/K)$-action defines a descent datum of $\underline{\mathrm{Spec}}(\mathcal{O}_{\underline{G_M}^{\mathrm{fm\,NR}}}/\mathcal{I}^m)$ with respect to the infinite étale Galois covering $\mathrm{Spec}(\mathcal{O}_M/\pi_M^j) \to \mathrm{Spec}(\mathcal{O}_K/\pi_K^j)$, if $j \geq bm$. These descent data for varying m and j form an inductive system, indexed by (m, j) such $j \geq mb$. This is the descent datum for $\underline{G_M}^{\mathrm{fm\,NR}}$ referred to above. Here $\underline{\mathrm{Spec}}(?)$ is the notation for the relative spectrum of a sheaf of commutative rings, represented by the question mark, on the topological space underlying a scheme.

It is known that neutral component of the closed fiber of the Néron model of a torus is affine. To verify this, it suffices to check that the closed fiber of the Néron model of a torus does not contain a non-trivial abelian variety as a subquotient, a fact which is readily see from the ℓ-adic representation attached to this torus, for a prime number ℓ-different from p.

For each $m \geq 1$, the scheme $\underline{\mathrm{Spec}}(\mathcal{O}_{\underline{G_M}^{\mathrm{fm\,NR}}}/\mathcal{I}^m)$ is affine. By fpqc descent for affine schemes, each descent datum in the inductive system of descent data . Therefore for each

(m,j) with $j \geq mb$, the descent datum for $\underline{\mathrm{Spec}}(\mathcal{O}_{\underline{G_M}^{\mathrm{fm\,NR}}}/\mathcal{I}^m)$ with respect to the étale map $\mathrm{Spec}(\mathcal{O}_M/\pi_M^j) \to \mathrm{Spec}(\mathcal{O}_K/\pi_K^j)$. The descended schemes form an inductive system, and the inductive limit is a formally smooth formal scheme $\mathfrak{G}$ over $\mathcal{O}$ such that $\mathfrak{G} \times_{\mathrm{Spf}\ \mathcal{O}} \mathrm{Spf}\ \mathcal{O}_M = \underline{G_M}^{\mathrm{fm\,NR}}$. By Prop. 5.7, $\mathfrak{G}$ is a formal Néron model of G.

§6. Isogeny invariance of the base change conductor

In this section $\mathcal{O}$ is a complete discrete valuation ring, such that the residue field κ is separably closed and has characteristic $p > 0$.

(6.1) Let G be a concordant rigid group over K, or equivalently, a strongly concordant rigid group over K. We saw in Thm. 5.9 that G has a formal Néron model $\underline{G}^{\mathrm{fm\,NR}}$.

(6.1.1) For any adic formal scheme $\mathfrak{X}$ over $\mathcal{O}$, with an ideal of definition $\mathcal{I}$, denote by $\Omega^1_{\mathfrak{X}/\mathcal{O}}$ the sheaf of continuous finite differentials on $\mathfrak{X}$. In other words,

$$\Omega^1_{\mathfrak{X}/\mathcal{O}} = \varprojlim_n \Omega^1_{\underline{\mathrm{Spec}}(\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^n)/\mathcal{O}}\,.$$

If $\mathfrak{X}$ is the J-adic completion of a scheme X over $\mathcal{O}$ with respect to an ideal J of $\mathcal{O}_X$, then $\Omega^1_{\mathfrak{X}/\mathcal{O}}$ is the J-adic completion of $\Omega^1_{X/\mathcal{O}}$. If $\mathfrak{X}$ is formally smooth over $\mathcal{O}$, then $\Omega^1_{\mathfrak{X}/\mathcal{O}}$ is a free $\mathcal{O}$-module whose rank is equal to the relative dimension of $\mathfrak{X}$ over $\mathcal{O}$. The last assertion follows from 5.3.1 (iii).

(6.1.2) Let $\mathfrak{G}$ be a smooth adic formal group scheme over $\mathcal{O}$. Denote by $\Omega(\mathfrak{G}/\mathcal{O})$ the pull-back of $\Omega^1_{\mathfrak{G}/\mathcal{O}}$ along the zero section of $\mathfrak{G}$. The Lie algebra $\mathrm{Lie}(\mathfrak{G}) = \mathrm{Lie}(\mathfrak{G}/\mathcal{O})$ of the smooth formal group scheme $\mathfrak{G}$ over $\mathcal{O}$ is defined as the $\mathcal{O}$-dual of $\Omega(\mathfrak{G}/\mathcal{O})$. The Lie algebra $\mathrm{Lie}(\mathfrak{G})$ of the smooth formal group $\mathfrak{G}$ over $\mathcal{O}$ is a free $\mathcal{O}$-module of rank, equal to the relative dimension of $\mathfrak{G}$ over $\mathcal{O}$. If $\mathfrak{G}$ is a formal completion of a smooth group scheme $\underline{G}$ over $\mathcal{O}$, then $\mathrm{Lie}(\mathfrak{G})$ is canonically isomorphic to the Lie algebra $\mathrm{Lie}(\underline{G}/\mathcal{O})$ of $\underline{G}$.

(6.1.3) Let L be a finite Galois extension of K such that $G \times_{\mathrm{Spm}\,K} \mathrm{Spm}\,L$ is the generic fiber of a formal torus over $\mathcal{O}_L$. Denote by $\underline{G_L}^{\mathrm{fm\,NR}}$ the formal Néron model of $G \times_{\mathrm{Spm}\,K} \mathrm{Spm}\,L$. Let

$$\mathrm{can}_{L,K} : \underline{G}^{\mathrm{fm\,NR}} \times_{\mathrm{Spf}\ \mathcal{O}} \mathrm{Spf}\ \mathcal{O}_L \to \underline{G_L}^{\mathrm{fm\,NR}}$$

be the natural base change map between the formal Néron models.

(6.1.4) Definition Notation as above. The *base change conductor* $c(G,K)$ of a concordant rigid group G over K is defined by

$$c(G,K) := \frac{1}{e(L/K)} \mathrm{length}_{\mathcal{O}_L} \left(\mathrm{Lie}(\underline{G_L}^{\mathrm{NR}}) / \mathrm{can}_*(\mathrm{Lie}(\underline{G}^{\mathrm{NR}} \otimes_{\mathcal{O}} \mathcal{O}_L)) \right)\,.$$

This definition is independent of the choice of the Galois extension L/K splitting G. This assertion is easy to verify, using 5.3.3, and the fact that for any two stabilizing fields $L_1 \subset L_2$ of G, the map

$$\mathrm{can}_{L_2,L_1} : \underline{G_{L_1}}^{\mathrm{fm\,NR}} \times_{\mathrm{Spf}\ \mathcal{O}_{L_1}} \mathrm{Spf}\ \mathcal{O}_{L_2} \to \underline{G_{L_2}}^{\mathrm{fm\,NR}}$$

is an isomorphism between split formal groups over $\mathcal{O}_{L_2}$.

(6.1.5) Lemma *Notation as in Lemma 5.5.1. Then $c(Y,K) = c(A,K)$. In other words the base change conductor for an abelian variety A over a local field K can be computed from the concordant rigid group Y over K attached to A.*

Proof. Immediate from the construction of Y in Lemma 5.5.1 and the definition of the base change conductor. ∎

(6.2) Definition A K-homomorphism $\alpha : G_1 \to G_2$ between concordant rigid groups is a *K-isogeny* if α induces an isomorphism $\alpha_* : X_*(G_1) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} X_*(G_2) \otimes_{\mathbb{Z}} \mathbb{Q}$. Equivalently, there exists a K-homomorphism $\beta : G_2 \to G_1$ such that $\beta \circ \alpha = n \cdot \mathrm{id}_{G_1}$ and $\alpha \circ \beta = n \cdot \mathrm{id}_{G_2}$ for some non-zero integer n.

(6.2.1) Lemma *Let A_1, A_2 be potentially ordinary abelian varieties over K, and let Y_1, Y_2 be the concordant rigid groups over K attached to A_1, A_2 as in 5.5.1. If A_1, A_2 are isogenous over K, then Y_1, Y_2 are isogenous over K.*

Proof. Let $\alpha : A_1 \to A_2$ be a K-rational isogeny. Let L be a finite Galois extension of K such that A_1 and A_2 have semistable reduction over $\mathcal{O}_L$. Then α induces a homomorphism from the Néron model $\underline{A_{1L_1}}^{\mathrm{NR}}$ of A_{1/L_1} to the Néron model $\underline{A_{2L_2}}^{\mathrm{NR}}$ of A_{2/L_2} with a quasi-finite kernel, hence also a homomorphism between the formal completion of the Néron models along the zero section of their the special fibers. That last homomorphism is an isogeny between split formal groups over $\mathcal{O}$. ∎

(6.3) Proposition *Assume that the residue field κ of $\mathcal{O}$ is perfect, i.e. it is algebraically closed. Let G_1 and G_2 be K-isogenous concordant rigid groups over K. Then $c(G_1, K) = c(G_2, K)$.*

Proof. Let L be a common splitting field of G_1, G_2 finite Galois over K. Let $\Gamma = \mathrm{Gal}(L/K)$. Fix a K-isogeny $h : G_1 \to G_2$, and identify $X_*(G_1)$ with a $\mathbb{Z}_p[\Gamma]$-submodule of finite index in $X_*(G_2)$ via h. By Lemma 5.8 (c), there exists a finitely generated $\mathbb{Q}_p[\Gamma]$-module V', a Γ-invariant $\mathbb{Z}_p$-lattice $V'_{\mathbb{Z}_p}$ in V', a finitely generated $\mathbb{Q}[\Gamma]$-module W, an isomorphism $\alpha : V \oplus V' \xrightarrow{\sim} W \otimes \mathbb{Q}_p$, and Γ-invariant $\mathbb{Z}$-lattices $W_{\mathbb{Z},1} \subseteq W_{\mathbb{Z},2}$ in W such that $\alpha_*(X_*(G_i) \oplus V'_{\mathbb{Z}_p}) = W_{\mathbb{Z},i} \otimes \mathbb{Z}_p$ for $i = 1, 2$. Let T_i be the K-torus with $X_*(T_i) = W_{\mathbb{Z},i}$, and let G' be the concordant rigid group over K with $X_*(G') = V'_{\mathbb{Z}_p}$. The proofs of Thm. 5.9 and Lemma 5.6 imply that $c(G_i, K) + c(G', K) = c(T_i, K)$ for $i = 1, 2$. By Theorem 12.1 of [CYdS], we know that $c(T_1, K) = c(T_2, K)$. Hence $c(G_1, K) = c(G_2, K)$. ∎

(6.3.1) Corollary *Let $\mathcal{O}$ be as in Prop. 6.3. Suppose that A_1, A_2 are abelian varieties over K with potentially ordinary reduction, and A_1, A_2 are isogenous over K. Then $c(A_1, K) = c(A_2, K)$.*

Proof. Let Y_1, Y_2 be the concordant rigid groups over K attached to A_1, A_2 as in 5.5.1. Then $c(A_i, K) = c(Y_i, K)$ for $i = 1, 2$ by Lemma 6.1.5, and $c(Y_1, K) = c(Y_2, K)$ by Prop. 6.3. Hence $c(A_1, K) = c(A_2, K)$. ∎

(6.3.2) Remark The statement of the main congruence result, Thm. 8.5 of [CYdS], holds for concordant rigid groups. The proof is similar to that of Thm. 6.3: Let G be a concordant rigid group over K, let L be a finite Galois extension of K which splits G, and let $\Gamma = \mathrm{Gal}(L/K)$. Apply Lemma 5.8 (b), with $V_{\mathbb{Z}_p} = X_*(G)$ and $V = X_*(G) \otimes \mathbb{Q}_p$ to obtain $W, W_{\mathbb{Z}}, V', V_{Zp}$ and α. Let T be the torus over K with $X_*(T) = W_{\mathbb{Z}}$. Let h, δ be the invariants for T, K defined in 8.1 and 8.5 of [CYdS]. Suppose that $N \geq 1$ and $m \geq \max(N+\delta+2h, 3h+1)$ be positive integers. Using Lemma 5.6 and the proof of Thm. 5.9, one deduces from Thm. 8.5 of [CYdS] that the congruence class modulo level N, $\underline{G}^{\mathrm{fm\,NR}} \otimes_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spf}(\mathcal{O}/\mathfrak{p}^N)$, of the formal Néron model $\underline{G}^{\mathrm{fm\,NR}}$, is determined by the quadruple $(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}_L/\mathfrak{p}^m\mathcal{O}_L, \Gamma, X_*(G))$ modulo level N. Here we do not assume that the residue field κ is perfect.

§7. A Formula for the base change conductor

In this section the residue field κ is assumed to be perfect and of characteristic $p > 0$.

(7.1) Lemma *Let L/K be a finite Galois extension of K with Galois group Γ. Assume that L/K is totally ramified, i.e. $\kappa_L = \kappa$, or equivalently $\Gamma = \Gamma_{-1} = \Gamma_0$, the inertia group. Let χ be the character of a linear combination of Γ on a finite dimensional $\mathbb{Q}_p$-vector space. Then χ is a $\mathbb{Q}$-linear representations of characters induced from linear representations of cyclic subgroups Γ of order prime to p on finite dimensional $\mathbb{Q}_p$-vector spaces.*

Proof. By Artin's theorem [S2, Thm. 17], χ is a $\mathbb{Q}$-linear combination of characters induced from one-dimensional $\overline{\mathbb{Q}_p}$-valued characters of cyclic subgroups Γ. Averaging over the $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-orbit, one sees that χ is a $\mathbb{Q}$-linear combination of characters induced from linear representations of cyclic subgroups of Γ on finite dimensional $\mathbb{Q}_p$-vector spaces. So we may and do assume that Γ is a cyclic group.

The cyclic group Γ is a direct product of a cyclic p-group Γ_1 and a cyclic group H whose order is prime to p. Since every irreducible $\mathbb{Q}_p$-rational representation is the tensor product of an irreducible $\mathbb{Q}_p$-rational representation of Γ_1 and an irreducible $\mathbb{Q}_p$-rational representation of H, it suffices to verify the assertion for Γ_1 and H separately. For H it is trivial. For Γ_1 the assertion follows from the basic fact that the cyclotomic extension $\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p$ is totally ramified over $\mathbb{Q}_p$ of degree $\phi(p^n)$, for every positive integer n. ∎

(7.2) Let M be a finite separable extension of K. On can define the Weil restriction of scalar functor $\mathbf{R}_{M/K}$, which is a functor from the category of rigid analytic spaces over M, to the category of set-valued contravariant functors on the rigid analytic spaces over K. For any rigid space Y over K, Y-points of the attached functor $\mathbf{R}_{M/K}(Y)$ is equal to the set of all morphisms $Y \times_{\operatorname{Spm} K} \operatorname{Spm} M$ to G over M. There is also a natural candidate rigid space $\mathrm{R}_{M/K}(Y)$, obtained by gluing affinoid opens representing $\mathbf{R}_{M/K}(U)$, where U runs through affinoid open subspces in Y, together with a natural map from $\mathrm{R}_{M/K}(Y)$ to $\mathbf{R}_{M/K}(Y)$. See [Berta] for more information, as well as criteria for representability.

The Weil restriction functor, when applied to concordant rigid groups, attaches to any concordant rigid group G over M, a concordant rigid group $\mathrm{R}_{M/K}(G)$ which represents $\mathbf{R}_{M/K}(G)$. In fact let L a stabilizing field for G, which is a finite Galois extension of M. Then any finite Galois extension of K containing L is a stabilizing field of $\mathrm{R}_{M/K}(G)$. To see the last assertion, first notice that $\mathrm{R}_{M/K}(G)$ represents $\mathbf{R}_{M/K}(G)$, since G is the union of an increasing sequence of affinoid open subspaces. Then one only needs to check that the base extension of $\mathrm{R}_{M/K}(G)$ to L is the rigid group attached to a formal torus over L, which is straightforward, as in the case of tori.

Let G be a concordant rigid group over M as above. Then the cocharacter group of $\mathrm{R}_{M/K}(G)$ is equal to $\mathrm{Ind}_M^K(X_*(G))$, the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module induced from the $\mathrm{Gal}(K^{\mathrm{sep}}/M)$-module $X_*(G)$. This statement is not hard to check, using the equivalence of categories between concordant rigid groups and linear representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on free $\mathbb{Z}_p$-modules of finite rank such that the inertia subgroup has finite image, as in 4.4.2.

(7.2.1) Lemma *Let G be a concordant rigid group over M as above, and let $\underline{G}^{\mathrm{fm\,NR}}$ be the formal Néron model of G over $\mathcal{O}_M$. Let $\mathrm{R}_{M/K}(G)$ be the Weil restriction of scalars of G from M to K as above. Then $\mathrm{R}_{\mathcal{O}_M/\mathcal{O}}(\underline{G}_M^{\mathrm{fm\,NR}})$ is a formal Néron model of $\mathrm{R}_{M/K}(G)$.*

Proof. Suppose that T be a torus over M as in the proof of 5.9, such that G is a direct summand of the concordant rigid group G' attached to T. Let $\underline{T}^{\mathrm{NR}}$ be the Néron model of T over $\mathcal{O}_M$. Then $\mathrm{R}_{\mathcal{O}_M/\mathcal{O}_K}(\underline{T}^{\mathrm{NR}})$ is the Néron model of $\mathrm{R}_{M/K}(T)$ by [BLR, 7.6/6].

The formal Néron model $\underline{G'}^{\mathrm{fm\,NR}}$ of G' over $\mathcal{O}_M$ is a formal completion of $\underline{T}^{\mathrm{NR}}$ by 5.4.1. The concordant rigid group attached to $\mathrm{R}_{M/K}(T)$ is $\mathrm{R}_{M/K}(G')$, and the formal Néron model of $\mathrm{R}_{M/K}(G')$ is a formal completion of $\mathrm{R}_{\mathcal{O}_M/\mathcal{O}}(\underline{T}^{\mathrm{NR}})$ by 5.4.1 and the fact recalled in the previous paragraph. It is easy to check, from the proof of 5.4.1, that the formal Nŕon model of $\mathrm{R}_{M/K}(G')$ is canonically isomorphic to $\mathrm{R}_{\mathcal{O}_M/\mathcal{O}}(\underline{G'}^{\mathrm{fm\,NR}})$.

The formal Néron model of G is the direct summand of $\underline{G'}^{\mathrm{fm\,NR}}$ cut out by the idempotent e of G' whose image is G, by the proof of 5.9. Similarly, the formal Néron model of $\mathrm{R}_{M/K}(G)$ is the direct summand of $\mathrm{R}_{\mathcal{O}_M/\mathcal{O}}(\underline{G'}^{\mathrm{fm\,NR}})$ cut out by the idempotent e. Since the formation of Weil restriction commutes with passing to the direct summand given by an idempotent, the formal Néron model of $\mathrm{R}_{M/K}(G)$ is canonically isomorphic to $\mathrm{R}_{\mathcal{O}_M/\mathcal{O}}(\underline{G}^{\mathrm{fm\,NR}})$. ∎

(7.3) Proposition *Assume that the residue field κ is algebraically closed. Let M be a totally ramified finite separable extension of K. Let G be a concordant rigid group over M. Then*

$$c(\mathrm{R}_{M/K}(G),K) = c(G,M) + \frac{1}{2}\,\mathrm{ord}_K(\mathrm{disc}(M/K))\,\dim(G)\,.$$

PROOF. Let L be a finite Galois splitting field of G over M which is also a Galois extension of K. By the argument for [BLR, 7.6/6], the Weil restriction of $\underline{G}^{\mathrm{fm\,NR}}$ from $\mathcal{O}_M$ to $\mathcal{O}$ is a formal Néron model of $\mathrm{R}_{M/K}(G)$; denote it by $\underline{R}$. Let $\underline{R}^\dagger$ be the split formal torus $X_*(\mathrm{R}_{M/K}(G))\otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}_{\mathrm{m}}}$ over $\mathcal{O}_L$; it is a formal Néron model of the base extension $\mathrm{R}_{M/K}(G)\times_{\mathrm{Spm}\,K}\mathrm{Spm}\,L$. In the above tensor product $X_*(\mathrm{R}_{M/K}(G))\otimes_{\mathbb{Z}_p}\widehat{\mathbb{G}_{\mathrm{m}}}$, $X_*(\mathrm{R}_{M/K}(G))$ is a finite free $\mathbb{Z}_p$-module, so the tensor product is a split formal torus over $\mathcal{O}_L$, with cocharacter group $X_*(\mathrm{R}_{M/K}(G))$. Similarly $X_*(G)\otimes_{\mathbb{Z}_p}\widehat{\mathbb{G}_{\mathrm{m}}}$ is a formal Néron model for G_L.

Let $\mathrm{can}_{L,K} : \underline{R}\otimes_{\mathrm{Spf}\,\mathcal{O}}\mathrm{Spf}\,\mathcal{O}_L \to \underline{R}^\dagger$ be the base change map for $\underline{R}$. Let $\mathcal{L} = \mathrm{Lie}(\underline{G}^{\mathrm{fm\,NR}})$, the Lie algebra of $\underline{G}^{\mathrm{NR}}$. Similarly let $\mathcal{L}^\dagger = \mathrm{Lie}(\underline{G_L}^{\mathrm{NR}})$. The base change morphism $\mathrm{can}_{L,M}$ for $\underline{G}^{\mathrm{NR}}$ induces an $\mathcal{O}_L$-linear map $\lambda := \mathrm{can}_{L,M_*}$ from $\mathcal{L}\otimes_{\mathcal{O}_M}\mathcal{O}_L$ to $\mathcal{L}^\dagger$.

Lemma *The tangent map can_{L,K_*} of the base change homomorphism $\mathrm{can}_{L,K}$ can be explicitly described as follows.*

(i) *The relative Lie algebra $\mathrm{Lie}(\underline{R})$ of $\underline{R}$ can be naturally identified with $\mathcal{L}$, regarded as an $\mathcal{O}$-module instead of an $\mathcal{O}_M$-module.*

(ii) *The relative Lie algebra $\mathrm{Lie}(\underline{R}^\dagger)$ can be identified with $\mathrm{Ind}_{\mathrm{Gal}(L/M)}^{\mathrm{Gal}(L/K)}(\mathcal{L}^\dagger)$, the $\mathcal{O}_L$-module $\mathbb{Z}_p[\mathrm{Gal}(L/K)]\otimes_{\mathbb{Z}_p[\mathrm{Gal}(L/M)]}\mathcal{L}^\dagger$. Here $\mathrm{Gal}(L/M)$ operates semi-linearly on $\mathcal{L}^\dagger$, and $\mathrm{Gal}(L/K)$ operates semi-linearly on $\mathrm{Ind}_{\mathrm{Gal}(L/M)}^{\mathrm{Gal}(L/K)}(\mathcal{L}^\dagger)$.*

(iii) *Under the identifications in (i) and (ii) above, the tangent map can_{L,K_*} of the base change map $\mathrm{can}_{L,K}$, from $\mathcal{O}_L\otimes_{\mathcal{O}}\mathcal{L}$ to $\mathrm{Ind}_{\mathrm{Gal}(L/M)}^{\mathrm{Gal}(L/K)}(\mathcal{L}^\dagger)$, is equal to the $\mathcal{O}_L$-linear map which sends each $v\in\mathcal{L}$ to the element $\sum_{\bar\sigma\in\mathrm{Gal}(L/K)/\mathrm{Gal}(L/M)}\bar\sigma\otimes\lambda(v)$ in $\mathrm{Ind}_{\mathrm{Gal}(L/M)}^{\mathrm{Gal}(L/K)}(\mathcal{L}^\dagger)$.*

PROOF. To check the above assertions, it suffices to verify the analogous statements for tori, as in the proof of 7.2.1. The details for the tori case are left to the tireless reader as an exercise, using [BLR, 7.6/6] and the formula [CYdS, A1.7] for the Lie algebra of Néron models of tori due to E. de Shalit. ∎

We resume the proof of Prop. 7.3. It remains to compute the order of the determinant of the map can_{L,K_*}. Choose an $\mathcal{O}_M$-basis $v_1,\dots,v_{\dim(G)}$ of $\mathcal{L}$, then choose an $\mathcal{O}_M$-basis $w_1,\dots,w_{\dim(G)}$ of $\mathcal{L}^\dagger$ with respect to which the matrix of λ is triangular: $\lambda(v_j) = \sum_{i=1}^{\dim(G)} a_{ij}\,w_i$, and $a_{ij}=0$ if $i>j$. This is possible because $\mathcal{O}_L$ is a discrete valuation ring. Pick an $\mathcal{O}_K$-basis $x_1,\dots,x_{[M:K]}$ of $\mathcal{O}_M$. Then the element $x_s\,v_j$, with $1\le s\le[M:K]$ and $1\le j\le\dim(G)$, form an $\mathcal{O}_K$-basis of $\mathcal{L}$. The elements $\sigma_r\otimes w_i$,with $1\le r\le[M:K]$ and

$1 \le i \le \dim(G)$, form an $\mathcal{O}_L$-basis of $\mathrm{Ind}_{\mathrm{Gal}(L/M)}^{\mathrm{Gal}(L/K)}(\mathcal{L}^\dagger)$. The effect of the linear map can_{L,K_*} on the basis elements $x_s\, v_j$ is

$$\mathrm{can}_{L,K_*} \;:\; x_s\, v_j \;\mapsto\; \sum_{r,i} \sigma_r(x_s)\sigma_r(a_{ij})\,\sigma_r \otimes w_i\,.$$

From the above matrix representation of the tangent of $\mathrm{can}_{L,K}$ one computes its determinant:

$$\mathrm{ord}_K(\det(\mathrm{can}_{L,K_*})) \;=\; \mathrm{ord}_K\left(\det(a_{ij})^{[M:K]}\cdot\det(\sigma_r(x_s))^{\dim(G)}\right)$$
$$= \mathrm{ord}_M(\det(a_{ij})) + \dim(G)\,\mathrm{ord}_K(\det(\sigma_r(x_s))) = c(G,M) + \tfrac{1}{2}\,\mathrm{ord}_K(\mathrm{disc}(M/K))\,\dim(G)\,.$$

∎

Remark For the proof of Thm. 7.5, we will only need Prop. 7.3 in the case when κ is algebraically closed, G has a finite Galois tamely ramified splitting field L/M, and $X^*(G)$ is a direct summand of $\mathbb{Z}_p[\mathrm{Gal}(L/M)]$. In that case the formal Néron model of $\mathrm{R}_{M/K}(G)$ is a direct summand of a completion of $\mathrm{R}_{\mathcal{O}_L/\mathcal{O}}(\mathbb{G}_\mathrm{m})$, by tameness, and the base change conductor can be explicitly computed.

(7.4) Proposition *Assume that the residue field κ is algebraically closed. Let G be a concordant rigid group over K with a stabilizing field L which is tamely ramified finite extension of K. Denote by $\chi_{X^*(G)}$ the character of the representation of $\Gamma_0 := \mathrm{Gal}(LK^{\mathrm{sh}}/K^{\mathrm{sh}}) = \mathrm{Gal}(L\widehat{K^{\mathrm{sh}}}/\widehat{K^{\mathrm{sh}}})$ on the character group $X^*(G)$ of G. Then*

$$c(G,K) = (\mathrm{bA}_{\Gamma_0}|\chi_{X^*(G)})\,,$$

where bA_{Γ_0} is the $F_{L/K}$-valued central function on Γ_0 defined in 3.2.1, and $(\mathrm{bA}_{\Gamma_0}|\chi_{X^*(G)})$ is the pairing of bA_{Γ_0} with the character $\chi_{X^*(G)}$ of Γ_0, defined by the first displayed formula in 3.1.3.

Proof. By Prop. 6.3, we may assume that $X^*(G)\otimes\mathbb{Q}_p$ is an irreducible $\mathbb{Q}_p[\Gamma]$-module. Again by Prop. 6.3, we may assume that $X^*(G)$ is a direct summand of $\mathbb{Q}_p[\Gamma]$, since $n := \mathrm{Card}(\Gamma)$ is prime to p.

Denote by T the concordant rigid group $\mathrm{R}_{L/K}(\widehat{\mathbb{G}_\mathrm{m}}^{\mathrm{an}})$ induced from the one-dimensional split concordant rigid group. By [BLR, 7.6/6], the Weil restriction of $\mathbb{G}_\mathrm{m}$ from $\mathcal{O}_L$ to $\mathcal{O}$ is an open subgroup scheme of the Néron model of $\mathrm{R}_{L/K}(\mathbb{G}_\mathrm{m})$, since the split group scheme $\mathbb{G}_\mathrm{m}$ over $\mathcal{O}_K$ is an open subgroup scheme of the Néron model of its generic fiber. So $\underline{T}^{\mathrm{fm\,NR}}$ is a formal completion of $\mathrm{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_\mathrm{m})$. The proof of Thm. 5.9 tells us that $\underline{G}^{\mathrm{fm\,NR}}$ is a direct summand of $\underline{T}^{\mathrm{fm\,NR}}$; similarly $\underline{G_L}^{\mathrm{fm\,NR}}$ is a direct summand of $\underline{T_L}^{\mathrm{fm\,NR}}$. Moreover these two direct sum decompositions are compatible with the base change map $\mathrm{can}_{L,K}$. Therefore Prop. 7.4 follows from Lemma 7.4.1 below. ∎

(7.4.1) Lemma *Let L be a finite extension of K which is both totally ramified and tamely ramified. Let $n = [L : K]$. Let $\psi = \psi_{L/K} : \Gamma \to L^\times$ be the composition of $\theta_0 : \Gamma \to \mu_n(\kappa_L)$ with the Teichmüller lifting of $\kappa_L^\times$ to $L^\times$, where θ_0 is defined in 2.2.3. The character ψ is equal to $\omega_{L/K}$ if $\mathrm{char}(K) = 0$. Let $\underline{R} = R_{\mathcal{O}_L/\mathcal{O}}(\mathbb{G}_{\mathrm{m}})$ the Néron model (of finite type over $\mathcal{O}$) for $\mathrm{R}_{L/K}(\mathbb{G}_{\mathrm{m}})$. Let $\underline{R}^\dagger$ be the split torus over $\mathcal{O}$ with character group $X^*(\mathrm{R}_{L/K}(\mathbb{G}_{\mathrm{m}}))$. Let $\mathrm{can}_{L,K} : \underline{R} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}_L \to \underline{R}^\dagger$ be the canonical base change map. Then*

$$\mathrm{can}_{L,K}^*(\Omega(\underline{R}^\dagger)(\psi^j)) = \pi_L^j \cdot (\Omega(\underline{R}) \otimes_{\mathcal{O}} \mathcal{O}_L)(\psi^j))$$

for $j = 0, 1, \ldots, n-1$. Here $\Omega(?) = \mathrm{Lie}(?)^\vee$ denotes the functor "dual of the Lie algebra of". For any $\mathcal{O}_L$-submodule M of $\Omega(\mathrm{R}_{L/K}) \otimes_K L$, $M(\psi^j)$ denotes the intersection of M with the ψ^j-eigenspace of $\Omega(\mathrm{R}_{L/K}) \otimes_K L$.

Proof. The Weil restriction $\underline{S} := \mathrm{R}_{\mathcal{O}_L/\mathcal{O}}(\mathbb{G}_a)$ is a ring scheme, and has $\underline{R}$ as its group of units. Likewise $\underline{R}^\dagger$ is the group of units of the ring scheme $\underline{S}^\dagger := X_*(\mathrm{R}_{L/K}(\mathbb{G}_{\mathrm{m}})) \otimes_{\mathbb{Z}} \mathbb{G}_a$ over $\mathcal{O}_L$. Moreover the base change map can extends to a morphism of ring schemes from $\underline{S} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}_L$ to $\underline{S}^\dagger$. So it suffices to verify the statement of 7.4.1 for the cotangent spaces of the ring schemes. By Prop. 5.7, we may and do assume that κ is separably closed.

Attached to each element $\tau \in \Gamma$ is a regular function χ_τ on $\underline{S}$ whose restriction to the K-rational points $\underline{S}(K) = L$ is equal to $\tau : L \to L$. The functions χ_τ gives an $\mathcal{O}_L$-basis of the cotangent space of $\underline{S}^\dagger$, after standard identification. For each $j = 0, \ldots, n-1$, the ψ^j-eigenspace of the cotangent space of $\underline{S}^\dagger$ is the $\mathcal{O}_L$-span of $\sum_{\tau\in\Gamma} \psi^{-j}(\tau)\,\chi_\tau$.

There exists a generator π_L of $\mathfrak{p}_L$ such that $\tau(\pi_L) = \psi(\tau)\,\pi_L$, since κ is separably closed. The elements π_L^i, $i = 0, \ldots, n-1$ form an $\mathcal{O}$-basis of $\mathcal{O}_L$, so every element of $\mathcal{O}_L$ can be written as $\sum_{i=1}^n x_i\,\pi_L^{i-1}$ with all $x_i \in \mathcal{O}$ in a unique way. This choice of basis of $\mathcal{O}_L$ gives an identification of $\underline{S}$ with $\mathrm{Spec}\,\mathcal{O}[x_1, \ldots, x_n]$. With this coordinate system for $\underline{S}$, the function χ_τ goes to $\sum_{i=1}^n \tau(\pi_L^{i-1})\,x_i$ under $\mathrm{can}_{L,K}^*$. A simple computation shows that $\sum_{\tau\in\Gamma} \psi^{-j}(\tau)\,\chi_\tau$ goes to $n\,\pi_L^j\,x_{j+1}$ for each $j = 0, \ldots, n-1$. We are done because n is a unit in $\mathcal{O}$. ∎

Remark The main theorem of [Ed] holds for the formal Néron models of concordant rigid groups, because the proof of Thm. 5.9 allows us to deduce it from the corresponding statement in [Ed]. One can use this result to give another proof of Prop. 7.4.

(7.5) Theorem *Assume that the residue field κ of $\mathcal{O}$ is algebraically closed and $\mathrm{char}(\kappa) = p > 0$. Let G be a concordant rigid group over K. Let L be a finite Galois stabilizing field of G. Let $\Gamma_0 = \mathrm{Gal}(LK^{\mathrm{sh}}/K^{\mathrm{sh}}) = \mathrm{Gal}(\widehat{LK^{\mathrm{sh}}}/\widehat{K^{\mathrm{sh}}})$. Let $\chi_{X^*(G)}$ be the character of the linear representation of Γ_0 on the character group of G. Then $c(G, K) = (\mathrm{bA}_{\Gamma_0}|\chi_{X^*(G)})$.*

Proof. By Prop. 6.3, $c(G, K)$ is an additive function of $X_*(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Prop. 7.3, Cor. 3.5.4 and Lemma 7.1 tell us that it suffices to check the formula in the case when L/K is tamely ramified. Prop. 7.4 finishes the proof. ∎

(7.6) Theorem *Let $\mathcal{O} = \mathcal{O}_K$ be a henselian discrete valuation ring such that the residue field κ of $\mathcal{O}$ is perfect and* $\mathrm{char}(\kappa) = p > 0$*. Let A be an abelian variety over K and let L be a finite Galois extension of K such that the neutral component of the $\underline{A_L}^{\mathrm{NR}}$ is an extension of an ordinary abelian variety by a torus. Let $\Gamma_0 = \mathrm{Gal}(LK^{\mathrm{sh}}/K^{\mathrm{sh}}) = \mathrm{Gal}(L\widehat{K^{\mathrm{sh}}}/\widehat{K^{\mathrm{sh}}})$. Let $X^* := X^*(\underline{A_L}^{\mathrm{fm\,NR}})$ be the character group of the formal completion along the zero section of the Néron model $\underline{A_L}^{\mathrm{NR}}$. Denote by χ_{X^*} the character of the action of Γ_0 on X^*. Then $c(A,K) = (\mathrm{bA}_{\Gamma_0}|\chi_{X^*})$.*

PROOF. Passing to the completion of the maximal unramified extension of K, we may and do assume that the residue field κ of $\mathcal{O}_K$ is algebraically closed. The theorem follows easily from Thm. 7.5 and Lemma 6.1.5. ∎

(7.6.1) Remark (i) The same argument shows that the formula for the base change conductor holds for potentially ordinary abelian varieties over K. Therefore one can view Thm. 7.6 as a generalization of [CYdS, 11.3, 12.1], which says that the base change conductor of a torus over K is equal to one-half of the Artin conductor of the Galois representation on the character group of the torus. Also, Thm. 7.6 generalizes [Ch, 6.8], which states that the base change conductor for abelian varieties over K with potentially ordinary reduction is an isogeny invariant if $\mathcal{O}_K$ has characteristics $(0,p)$. It also answers [Ch, Question 8.6] in the affirmative, eliminating the assumption that K has characteristic 0 from [Ch, 6.8].

(ii) Suppose that $K = \kappa((t))$, such that κ is algebraically closed and $\mathrm{char}(\kappa) = 0$. Let A be an abelian variety over K, and let L be a finite Galois extension over K such that A_L has semistable reduction over $\mathcal{O}_L$. Then $\mathrm{Gal}(L/K)$ operates on the closed fiber of $\underline{A_L}^{\mathrm{NR}}$ by κ-automorphisms, and one obtains a linear representation of $\mathrm{Gal}(L/K)$ on $\mathrm{Lie}(\underline{A_L}^{\mathrm{NR}} \times_{\mathrm{Spec}\,\mathcal{O}_L} \mathrm{Spec}\,\kappa)$. The class function $\mathrm{bA}_{\mathrm{Gal}(L/K)}$ on $\mathrm{Gal}(L/K)$ takes values in the constant field κ of L. In this situation the base change conductor $c(A,K)$ is equal to the pairing of $\mathrm{bA}_{\mathrm{Gal}(L/K)}$ with the character of the representation on the dual of the Lie algebra $\mathrm{Lie}(\underline{A_L}^{\mathrm{NR}} \times_{\mathrm{Spec}\,\mathcal{O}_L} \mathrm{Spec}\,\kappa)$. This assertion is an analogue of Thm. 7.6, and can be proved by a similar but simpler argument. Details are omitted here. One can also prove this assertion using the main result of [Ed], which says that $\underline{A}^{\mathrm{NR}}$ (resp. $\mathrm{Lie}(\underline{A}^{\mathrm{NR}})$) is canonically isomorphic to the fixed point subscheme (resp. subspace) of $\mathrm{Gal}(L/K)$ operating on $\underline{A_L}^{\mathrm{NR}}$ (resp. $\mathrm{Lie}(\underline{A_L}^{\mathrm{NR}})$).)

(7.6.2) Corollary *Notation and assumption as in 7.6.*

(i) *We have the following estimate*

$$\frac{1}{4}\,\mathrm{Sw}(A,K) \le \frac{1}{4}\,\mathrm{Sw}(A,K) + \frac{1}{2}\min\left(\frac{1}{n}, \frac{n_{L/K}-1}{n_{L/K}}\right)(\mathrm{Ar}(A,K) - \mathrm{Sw}(A,K))$$
$$\le\; c(A,K) \;\le\; \frac{1}{4}\,\mathrm{Sw}(A,K) + \frac{1}{2}\,\max\left(\frac{1}{2}, \frac{n_{L/K}-1}{n_{L/K}}\right)(\mathrm{Ar}(A,K) - \mathrm{Sw}(A,K))$$
$$\le \frac{1}{2}\mathrm{Ar}(A,K) - \frac{1}{4}\mathrm{Sw}(A,K)$$

for the base change conductor $c(A,K)$ *of* A.

(ii) *If either* $\frac{1}{4}\,\mathrm{Sw}(A,K) = c(A,K)$ *or* $c(A,K) = \frac{1}{2}\mathrm{Ar}(A,K) - \frac{1}{4}\mathrm{Sw}(A,K)$*, then* A *has good reduction over* $\mathcal{O}_K$*, and all five terms in the inequality in (i) above are equal to zero.*

(iii) *Let* $\mathrm{add}(A,K)$ *be the dimension of the additive part of the neutral component of the closed fiber of the Néron model* $\underline{A}^{\mathrm{NR}}$ *of* A*. Then*

$$c(A,K) \ge \min\left(\frac{1}{2}, \frac{1}{n}\right)\mathrm{add}(A,K)$$

Proof. In the present situation, the Artin (resp. Swan) conductor of χ_{X^*} is equal to one-half of the Artin (resp. Swan) conductor $\mathrm{Ar}(A,K)$ (resp. Sw(A,K)) of A; i.e. one-half of the Artin (resp. Swan) conductor of the ℓ-adic Tate module $\mathrm{V}_\ell(A)$, for any prime number $\ell \neq p$. The statements (i), (ii) follow from Cor. 3.3.3 (i), (ii). The statement (iii) follows from Cor. 3.3.3 (iii), and the fact that $\mathrm{add}(A,K)$ is equal to the dimension of the nontrivial part of the representation of Γ_0 on $X^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. ∎

(7.6.3) Remark (i) It seems natural to ask whether any of the estimates in 7.6.2 (i) and (iii) are still true when A does not have potentially ordinary reduction. Although these estimates do hold in the small number of examples known to the author, including those in [Ch, 6.10.2], there are not enough data to support a prediction one would feel confident about. In this direction it may be worthwhile to point out that, using the results in [Ed], one can show that the estimates in 7.6.2 (i) and (iii) hold when A acquires ordinary reduction over the ring of integers of a tamely ramified extension of the base field K.

(ii) When the residue field κ of $\mathcal{O}_K$ is not perfect, the question on the isogeny invariance of the base change conductor of potentially ordinary semiabelian varieties is completely open, even for tori.

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