Families of ordinary abelian varieties: canonical coordinates, p-adic monodromy, Tate-linear subvarieties and Hecke orbits

Ching-Li Chai[1]

version July 30, 2003

Contents

1 Introduction **1**

2 Canonical coordinates **6**

3 Local p-adic monodromy **16**

4 Global p-adic monodromy **19**

5 Tate-linear subvarieties **23**

6 Connection to the Hecke orbit problem **28**

7 Some Conjectures **39**

8 Tate-linear subvarieties of Hilbert modular varieties **42**

§1. Introduction

Throughout this paper p denotes a fixed prime number. An abelian variety A over a field k of characteristic p is said to be *ordinary* if any one of the following equivalent conditions hold.

- $A[p](k^{\mathrm{alg}}) \cong (\mathbb{Z}/p\mathbb{Z})^{\dim(A)}$, where $A[p] = \mathrm{Ker}\,([p]_A : A \to A)$ denotes the subgroup scheme of p-torsion points of A.
- The Barsotti-Tate group $A[p^\infty]$ attached to A is an extension of an étale Barsotti-Tate group by a multiplicative Barsotti-Tate group.
- The abelian variety A (or its Barsotti-Tate group $A[p^\infty]$) has only two slopes, 0 and 1, both with multiplicity $\dim(A)$.
- The formal completion $\hat{A}$ of A along its origin is a formal torus.

[1]partially supported by grants DMS95-02186, DMS98-00609 and DMS01-00441 from the National Science Foundation

Being ordinary is an open condition on the base: For any abelian scheme $A \to S$ over a base scheme $S \to \mathrm{Spec}(\mathbb{F}_p)$, the subset of S consisting of points $s \in S$ such that the fiber A_s is ordinary is a (possibly empty) open subset of S. A general belief, reinforced by experience, is that among abelian varieties in characteristic p, the ordinary ones tend to have properties closer to those for abelian varieties in characteristic 0.

The deformation theory for ordinary abelian varieties is very elegant. Let A_0 be an ordinary abelian variety over a perfect field $k \supseteq \mathbb{F}_p$. Then the universal deformation space $\mathrm{Def}(A_0)$ has a natural structure as a formal torus over the Witt vectors $W(k)$, according to a famous theorem of Serre and Tate, announced in the 1964 Woods Hole Summer School. The zero section of the Serre-Tate formal torus over $W(k)$ corresponds to a deformation of A_0 over $W(k)$, which is the p-adic completion of an abelian scheme $\widetilde{A_0}$ over $W(k)$, known as the *canonical lifting* of A_0. Every endomorphism of A_0 over k lifts uniquely to an endomorphism of $\widetilde{A_0}$ over $W(k)$.

(1.1) This article was motivated by the Hecke orbit problem for reduction of Shimura varieties. We shall restrict our discussions to PEL-type modular varieties. Such a modular variety $\mathcal{M}$ classifies abelian varieties with prescribed symmetries, coming from polarization and endomorphisms; see [19] and also (6.7.1). On a PEL-type modular variety $\mathcal{M}$ over $\overline{\mathbb{F}_p}$ there is a large family of symmetries. These symmetries come from a reductive linear algebraic group G over $\mathbb{Q}$, attached to the PEL-data for $\mathcal{M}$, in the following fashion. There is an infinite étale Galois covering $\widetilde{\mathcal{M}} \to \mathcal{M}$, whose Galois group is a compact open subgroup $K^{(p)}$ of the restricted product

$$G(\mathbb{A}_f^{(p)}) = \prod_{\ell \neq p}{}' G(\mathbb{Q}_\ell)\,,$$

and the action $K^{(p)}$ on $\widetilde{\mathcal{M}}$ extends to a continuous action of $G(\mathbb{A}_f^{(p)})$ on $\widetilde{\mathcal{M}}$. Descending from $\widetilde{\mathcal{M}}$ to $\mathcal{M}$, elements of the non-compact group $G(\mathbb{A}_f^{(p)})$ induce algebraic correspondences on $\mathcal{M}$, known as *Hecke correspondences*. Notice that the $\mathbb{Q}_p$-points of G did not enter into our description of the Hecke symmetries above, since elements of $G(\mathbb{Q}_p)$ in general do not give étale correspondences on $\mathcal{M}$.

The general Hecke orbit problem seeks to determine the Zariski closure of the countable subset of $\mathcal{M}(\overline{\mathbb{F}_p})$, consisting of all points in $\mathcal{M}(\overline{\mathbb{F}_p})$ which belong to the image of a given point $x_0 \in \mathcal{M}(\overline{\mathbb{F}_p})$ under a Hecke correspondence. When x_0 is ordinary, i.e. the abelian variety underlying the point x_0 of $\mathcal{M}$ is an ordinary abelian variety, it seems very likely that the Hecke orbit of x_0 is dense in $\mathcal{M}$ for the Zariski topology. We refer to this expectation as *the ordinary case of the Hecke orbit problem*; see 7.1. This expectation has been verified in [5] when $\mathcal{M} = \mathcal{A}_g$, the moduli space of of g-dimensional principally polarized abelian varieties. The argument in [5] also settles the problem for modular varieties of PEL-type C. In the case of PEL-type A or D, the method in [5] does not establish the ordinary case of the Hecke orbit problem, which remains an open question. Still the method employed in [5] can be extended and sharpened. This is the task we undertake in the present article.

(1.2) The main results in this article, all dealing with families of ordinary abelian varieties, can be divided into four threads,

(a) global version of Serre-Tate coordinates,

(b) p-adic monodromy,

(c) Tate-linear subvarieties in $\mathcal{A}_{g/\overline{\mathbb{F}_p}}$,

(d) local rigidity for formal tori,

to be described below.

(a) Our global version of the Serre-Tate coordinates, in its simplest form, boils down to standard Kummer theory. In general, given an abelian scheme $A \to S$ with ordinary fibers, where S is an scheme such that p is locally in $\mathcal{O}_S$, the canonical coordinates of $A \to S$ is a homomorphism $q(A/S)$, from a smooth $\mathbb{Z}_p$-sheaf of constant rank on $S_{\text{ét}}$ of the form $\mathrm{T}_p(A[p^\infty]) \otimes X^*(A[p^\infty]^{\text{mult}})$, the tensor product of the p-adic Tate module of the maximal étale quotient of $A[p^\infty]$ with the character group of the multiplicative part of $A[p^\infty]$, to a sheaf $\nu_{p^\infty,S}$ on $S_{\text{ét}}$. The sheaf $\nu_{p^\infty,S}$ is defined as the projective limit, of

$$\mathrm{Coker}\left([p^n] : \mathbb{G}_{\mathrm{m},S_{\text{ét}}} \to \mathbb{G}_{\mathrm{m},S_{\text{ét}}}\right) ,$$

and may be thought of as a sheaf of some sort of generalized functions on S. If $g = \dim(A/S)$, then the rank of the smooth $\mathbb{Z}_p$-sheaf $\mathrm{T}_p(A[p^\infty]) \otimes X^*(A[p^\infty]^{\text{mult}})$ is equal to g^2, so that the canonical coordinates $q(A/S)$ give g^2 "local coordinates" for A/S, with values in the sheaf of generalized functions ν_{p^∞}. The global Serre-Tate coordinates of an abelian scheme $A \to S$ depends only on the Barsotti-Tate group $A[p^\infty]$.

In our write-up in §2, we formulate the global canonical coordinates for Barsotti-Tate groups or truncated Barsotti-Tate groups $G \to S$, such that p is (topologically) locally nilpotent in $\mathcal{O}_S$, and $G \to S$ is *ordinary*, in the sense that it is an extension of an étale (truncated) Barsotti-Tate group by an multiplicative one.

There is also a geometric version of the canonical coordinates, due to Mochizuchi, which I learned from de Jong. Here is a formulation for the ordinary locus $\mathcal{A}_g^{\text{ord}}$ of $\mathcal{A}_g$ in characteristic p. Denote by $\mathcal{A}_g^{\text{ord}} \to \mathrm{Spec}(W(\overline{\mathbb{F}_p}))$ the complement in $\mathcal{A}_g \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}(W(\overline{\mathbb{F}_p}))$ of the non-ordinary locus in $\mathcal{A}_g \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}(\overline{\mathbb{F}_p})$. Define a formal scheme C by

$$C := \left(\mathcal{A}_g^{\text{ord}} \times_{\mathrm{Spec}(W(\overline{\mathbb{F}_p}))} \mathcal{A}_g^{\text{ord}}\right)^{/\Delta} ,$$

the formal completion of $\mathcal{A}_g^{\text{ord}} \times_{\mathrm{Spec}(W(\overline{\mathbb{F}_p}))} \mathcal{A}_G^{\text{ord}}$ along the diagonally embedded ordinary locus

$$\Delta := \Delta_{\mathcal{A}_g^{\text{ord}}} : \mathcal{A}_g^{\text{ord}} \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathcal{A}_g^{\text{ord}}$$

of the closed fiber. The first projection $\mathrm{pr}_1 : C \to \mathcal{A}_g^{\text{ord}}$, a formally smooth morphism, can be thought of as the family of formal completions at varying points of $\mathcal{A}_g^{\text{ord}}$. The geometric version of the Serre-Tate canonical coordinates states that the morphism $\mathrm{pr}_1 : C \to \mathcal{A}_g^{\mathrm{ord}}$ has a natural structure as a formal torus over $\mathcal{A}_g^{\mathrm{ord}}$. When one restrict this formal torus to the fiber over a closed point s of S, one recovers the standard point-wise version of the Serre-Tate formal torus for the principally polarized abelian variety (A_s, λ_s).

(b) In §3 and §4, we relate the p-adic monodromy of an ordinary Barsotti-Tate group $G \to S$ to its global canonical coordinates. Here we explain the results when $G \to S$ is the Barsotti-Tate group attached to an abelian scheme $A \to S$ with ordinary fibers, where S is a normal integral scheme of finite type over $\overline{\mathbb{F}_p}$. to the canonical coordinates of $A \to S$. The p-adic monodromy group of $A \to S$ is defined using the formalism of Tannakian categories. Choose a closed point s of S. The Barsotti-Tate group $A[p^\infty]$ of A_s splits uniquely as a direct product of its multiplicative part and its maximal étale quotient. Using s as a base point, we can describe the p-adic monodromy group of $A \to S$, as a subgroup of $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$ whose elements have the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, with $a, b, c, d \in \mathrm{M}_g(\mathbb{Q}_p)$. More precisely, Thm. 4.4 says that this monodromy group $\mathrm{MG}(A_{/S})$ is a semi-direct product of its unipotent radical U and a Levi subgroup L. The unipotent radical U consisting of all elements of $\mathrm{MG}(A_{/S})$ of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, and it corresponds to the minimal formal subtorus of the Serre-Tate formal torus $\mathcal{A}_g^{/s}$ which contains the canonical coordinates of the family $A_{/S}$. The Levi subgroup L consists of elements of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ in $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$, and corresponds to the "naive" p-adic monodromy group of the smooth $\mathbb{Z}_p$-sheaves $X_*(A[p^\infty]^{\mathrm{mult}})$ and $\mathrm{T}_p(A[p^\infty]^{\text{ét}}$ over $S_{\text{ét}}$. The proof of 4.4 uses results in [8].

In a nutshell, the canonical coordinates and the naive p-adic monodromy are the two ingredients of the p-adic monodromy group for $A_{/S}$. The compatibility of the two aspects is partly reflected in the group-theoretic condition that U is stable under the adjoint action of L, as subgroups of $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$. In particular, if one knows that the naive p-adic monodromy group is large, often one can obtain some upper bound on the number of "linear" relations among the canonical coordinates of $A \to S$.

(c) By definition, a Tate-linear subvariety of $\mathcal{A}_g^{\mathrm{ord}}{}_{/\mathrm{Spec}(\overline{\mathbb{F}_p})}$ is a closed subvariety Z of the ordinary locus $\mathcal{A}_g^{\mathrm{ord}}{}_{/\mathrm{Spec}(\overline{\mathbb{F}_p})}$ of $\mathcal{A}_g$ over $\overline{\mathbb{F}_p}$. such that the formal completion $Z^{/z}$ at every closed point z is a formal subtorus of the Serre-Tate formal torus $\mathcal{A}_g^{/z}$ over $\overline{\mathbb{F}_p}$. This "pointwise" condition satisfies a form of "analytic continuation": If one is willing to ignore the possibility that Z may have self-intersection, then one would be able to say that, being Tate-linear at one point $z_0 \in Z$ implies that Z is Tate-linear in an open neighborhood of Z; see 5.3.

The only known examples of Tate-linear subvarieties are the intersection of $\mathcal{A}_g^{\mathrm{ord}}{}_{/\overline{\mathbb{F}_p}}$ with the reduction of (Hecke-translates of) Shimura subvarieties of $\mathcal{A}_g$. See the discussions below.

(d) Most subvarieties of $\mathcal{A}_g^{\mathrm{ord}}$ over $\overline{\mathbb{F}_p}$ are, of course, not Tate-linear. But the Zariski closure in $\mathcal{M}^{\mathrm{ord}} \subset \mathcal{A}_g^{\mathrm{ord}}{}_{/\overline{\mathbb{F}_p}}$ of the prime-to-p Hecke orbit of any ordinary point of a modular variety $\mathcal{M}$ of PEL-type over $\overline{\mathbb{F}_p}$ turns out to be Tate-linear. The proof of this facts depends on a local rigidity result for irreducible closed subsets of formal tori over $\overline{\mathbb{F}_p}$. Here we describe a prototypical special case of this local rigidity result: Let T be a formal torus over $\overline{\mathbb{F}_p}$, and let W be a formally smooth closed formal subvariety of T. If W is stable under the action of $[1+p^{n_0}]_T$ for some integer $n_0 \geq 2$, then W is a formal subtorus of T. In Thm. 6.6, the base field is allowed to be any algebraically closed field, the formal smoothness condition on W is weakened to W being reduced and irreducible, and the invariance condition on W is generalized to allow more general linear action on the character group of the formal torus T.

(1.3) Tate-linear subvarieties in $\mathcal{A}_g^{\mathrm{ord}}{}_{/\overline{\mathbb{F}_p}}$ have another very special property, that they can be linearly lifted to characteristic 0. More precisely, every Tate-linear subvariety Z over $\overline{\mathbb{F}_p}$ can be lifted, over $W(\overline{\mathbb{F}_p})$, to a formal subscheme Z_∞ of the p-adic completion of $\mathcal{A}_g^{\mathrm{ord}}{}_{/W(\overline{\mathbb{F}_p})}$, such that Z_∞ is formally smooth over $W(k)$, and the formal completion of Z_∞ at any closed point z of Z is a formal subtorus of the Serre-Tate formal torus over $W(\overline{\mathbb{F}_p})$. This Tate-linear lift Z_∞ of Z is clearly unique. According to a result of Moonen in [23], [24], if Z_∞ is algebraic, in the sense that there exists a closed subscheme $\widetilde{Z}$ of $\mathcal{A}_g^{\mathrm{ord}}{}_{/W(\overline{\mathbb{F}_p})}$ whose p-adic completion coincides with Z_∞, then Z is the reduction of a Shimura subvariety, meaning that the generic fiber of $\widetilde{Z}$ is a translate of a Shimura subvariety of $\mathcal{A}_g^{\mathrm{ord}}{}_{/B(\overline{\mathbb{F}_p})}$, where $B(k)$ is the fraction field of $W(k)$.

It seems very likely that every Tate-linear subvariety of $\mathcal{A}_g^{\mathrm{ord}}$ is the reduction of a Shimura subvariety. We state this as a conjecture in 7.2. This is perhaps the major open question about Tate-linear subvarieties.

Having ventured into the precarious zone of making conjectures, we formulate some more in §7. Conj. 7.4 is a semi-simplicity statement. It says that the naive p-adic monodromy group for a family $A \to S$ of ordinary abelian varieties over $\overline{\mathbb{F}_p}$ should be semisimple if the base scheme S is of finite type over $\overline{\mathbb{F}_p}$. Moreover the p-adic monodromy group $\mathrm{MG}(A/S)$ determines a reductive group G_p over $\mathbb{Q}_p$ which contains $\mathrm{MG}(A/S)$ as a Siegel parabolic subgroup. Conj. 7.6 and Conj. 7.7 combined amounts to an analogue of the Mumford-Tate conjecture for $A_{/S}$. They assert that, once a closed base point $s \in S$ is chosen, there should be a reductive group G over $\mathbb{Q}$, of Hodge type with respect to the canonical lifting of the fiber A_s over s and an embedding $W(\kappa(s) \hookrightarrow \mathbb{C}$, such that G gives rise to the reductive group G_p over $\mathbb{Q}_p$, and also the ℓ-adic monodromy group G_ℓ of $A_{/S}$ for any prime $\ell \neq p$. The point here is that it is possible to define a candidate for the motivic Galois group for $A_{/S}$, for instance the smallest reductive group over $\mathbb{Q}$ of Hodge type which contains G_p.

The relations between the above conjectures are:

- Conj. 7.2, on Tate-linear subvarieties, implies Conj. 7.1, the ordinary case of the Hecke orbit conjecture.

- Conj. 7.6, an analog of the Mumford-Tate conjecture, implies both the semisimplicity conjecture 7.4 and the conjecture 7.2 on Tate-linear subvarieties.

It seems that if one can show the semisimplicity conjecture 7.4, or at least the semisimplicity of the naive p-adic monodromy group, then one is close to proving the rest of the conjectures.

(1.4) When a conjecture is stated, it often means that the author is unable to find a proof despite the best effort; this paper is no exception. As a feeble substitute for a proof, we offer some evidence for the conjectures. We prove in 8.6, using the main theorem of de Jong in [10], that every Tate-linear subvariety of the ordinary locus of a Hilbert modular variety, attached to a totally real number field, is the reduction of a Shimura subvariety. Our argument can be used to establish the other conjectures in the special case of subvarieties of a Hilbert modular variety, although we do not provide the details in this paper. The proof of 8.6 also works for the case of "fake Hilbert modular varieties", i.e. modular varieties attached to a quaternion algebra over (a product of) totally real number fields, such that the underlying reductive group is an inner twist of (a Weil restriction of scalars of) $GL(2)$. Combining Thm. 6.6 and Thm. 8.6, one deduces immediately the main results of [5]. However, a "Hilbert-free" proof of the main theorem of [5], which would offer a road map to the ordinary case of the Hecke orbit conjecture, seems to require a completely new idea.

Here are some information on the logical dependence of various parts of this article. Either of sections 6 and 8, which contain the local rigidity theorem 6.6 and the result 8.6 on Tate-linear subvarieties of Hilbert modular varieties respectively, can be read by itself. Sections 3 is also logically independent of the rest of this article. However sections 4 and 5 depend on section 2. Readers who prefer theorems to conjectures are encouraged to read §6 and §8 first.

(1.5) This paper has gone through a long incubation period since the summer of 1995. So far the author is still unable to fulfill the original goal, namely, to prove the ordinary case of the Hecke orbit conjecture for modular varieties of PEL-type. My excuse for publishing the relic of a failed attempt is that the methods developed for their original purpose may be useful for other problems. For instance the the proofs of 6.6 and 8.6 have been used by Hida to study the μ-invariant of Katz's p-adic L-functions attached to CM-fields.

It is a pleasure to acknowledge discussions with J. de Jong, H. Hida, F. Oort, F. Pop, J. Tilouine and E. Urban on topics related to this paper. I would also like to thank Harvard University and the National Center of Theoretical Sciences in Hsinchu, where some early versions of this paper were written, for their hospitality.

§2. Canonical coordinates

(2.1) Let S be a scheme over $\mathbb{Z}_{(p)}$. Let S_{zar} (resp. $S_{\text{ét}}$, resp. S_{fppf}) be the small Zariski site (resp. étale site, resp. fppf site) of S. For any natural number $n \geq 0$, consider the $[p^n]$-th power map $[p^n]_{\mathbb{G}_{\mathrm{m}}} : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ of sheaves over S_τ, where $\tau =$ zar, ét or fppf. The map $[p^n]_{\mathbb{G}_{\mathrm{m}}}$ is surjective for $\tau =$ fppf, but may not be surjective if $\tau =$ zar or ét.

(2.1.1) Definition For $\tau =$ ét or zar, define sheaves ν_{p^n,S_τ}, $\tilde{\mu}_{p^n,S_\tau}$, ξ_{p^n,S_τ}, $n \geq 0$, and ν_{p^∞,S_τ} on the site S_τ by

$$\begin{aligned}
\nu_{p^n,S_\tau} &= \mathrm{Coker}\left([p^n]_{\mathbb{G}_{\mathrm{m},S_\tau}} : \mathbb{G}_{\mathrm{m},S_\tau} \to \mathbb{G}_{\mathrm{m},S_\tau}\right) , \\
\tilde{\mu}_{p^n,S_\tau} &= \mathrm{Ker}\left([p^n]_{\mathbb{G}_{\mathrm{m},S_\tau}} : \mathbb{G}_{\mathrm{m},S_\tau} \to \mathbb{G}_{\mathrm{m},S_\tau}\right) , \\
\xi_{p^n,S_\tau} &= \mathrm{Image}\left([p^n]_{\mathbb{G}_{\mathrm{m},S_\tau}} : \mathbb{G}_{\mathrm{m},S_\tau} \to \mathbb{G}_{\mathrm{m},S_\tau}\right) , \\
\nu_{p^\infty,S_\tau} &= \varprojlim_n \nu_{p^n,S_\tau} = \text{ the projective limit of } \left(\nu_{p^n,S_\tau}\right)_n .
\end{aligned}$$

Since the étale site is often used in applications, we abbreviate $\nu_{p^n,S_{\text{ét}}}$ to $\nu_{p^n,S}$, $\tilde{\mu}_{p^n,S_{\text{ét}}}$ to $\tilde{\mu}_{p^n,S}$, $\xi_{p^n,S_{\text{ét}}}$ to $\xi_{p^n,S}$, and ν_{p^∞,S_τ} to $\nu_{p^\infty,S}$.

Customarily, μ_{p^n} denotes the finite locally free group scheme $\mathrm{Ker}([p^n] : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}})) = \mathrm{Spec}\left(\mathbb{Z}[t,t^{-1}]/(t^{p^n}-1)\right)$, where the comultiplication sends t to the element

$$t \otimes t \in \left(\mathbb{Z}[t,t^{-1}]/(t^{p^n}-1)\right) \otimes_{\mathbb{Z}} \left(\mathbb{Z}[t,t^{-1}]/(t^{p^n}-1)\right) .$$

We shall also identify μ_{p^n} with the sheaf represented by it on the fppf site, so that $\mu_{p^n,S}$ is just another notation for the sheaf $\tilde{\mu}_{p^n,S_{\mathrm{fppf}}}$.

(2.1.2) Remark (i) Let R be an Artinian local ring whose residue field κ is perfect of characteristic $p > 0$. Then $\nu_{p^\infty,\mathrm{Spec}\,R}$ is represented by the formal completion of $\mathbb{G}_{\mathrm{m}/\mathrm{Spec}\,R}$ along its unit section.

(ii) We are mainly interested in sheaves of the form $\nu_{p^n}^{\oplus r}$, and also their twists by an unramified Galois representation $\pi_1(S) \to \mathrm{GL}_r(\mathbb{Z}/p^r\mathbb{Z})$. The sheaves $\tilde{\mu}_{p^n}, \xi_{p^n}$ are introduced in order to study the sheaves ν_{p^n} on $S_{\text{ét}}$ via the exact sequences

$$1 \to \tilde{\mu}_{p^n,S} \to \mathbb{G}_{\mathrm{m}S} \to \xi_{p^n,S} \to 1 \quad \text{and} \quad 1 \to \xi_{p^n,S} \to \mathbb{G}_{\mathrm{m}S} \to \nu_{p^n,S} \to 1.$$

of sheaves of abelian groups on S_{fppf}.

(2.1.3) Lemma *Let $\pi : S_{\mathrm{fppf}} \to S_{\text{ét}}$ and $\pi' : S_{\mathrm{fppf}} \to S_{\mathrm{zar}}$ be the projection morphism between the respective sites. Then $\mathrm{R}^1\pi_*\mu_{p^n} = \nu_{p^n,S_{\text{ét}}}$, $\mathrm{R}^1\pi'_*\mu_{p^n} = \nu_{p^n,S_{\mathrm{zar}}}$.*

Proof. Both statements follow easily from the exact sequences in 2.1.2 and Hilbert's Theorem 90. ∎

In 2.1.4–2.1.6, we collect some injectivity properties of the presheaf $\mathrm{Spec}\,R \rightsquigarrow R^\times/(R^\times)^{p^n}$.

(2.1.4) Lemma *Let A be a normal integral domain, and let K be the fraction field of A. Then for every integer $n \geq 1$, the map*

$$A^\times/(A^\times)^{p^n} \to K^\times/(K^\times)^{p^n}$$

induced by the inclusion $A^\times \hookrightarrow K^\times$ is injective.

PROOF. A simple induction show that it suffices to verify the case $n = 1$. Suppose that an element $a \in A^\times$ is equal to b^p for some $b \in K^\times$. From the normality of A we deduce that $b \in A$, and $b \in A^\times$ since $b^p \in A^\times$. ∎

Remark As the argument shows, the statement of 2.1.4 holds for every commutative ring A such that $(K^\times)^p \cap A^\times = (A^\times)^p$.

(2.1.5) Lemma *Let $h : A \to B$ be a homomorphism of commutative rings over $\mathbb{F}_p$. Assume that h is étale and faithfully flat, and A is reduced. Then the map*

$$A^\times/(A^\times)^{p^n} \to B^\times/(B^\times)^{p^n}$$

induced by h is injective.

PROOF. Suppose that $a \in A^\times$, $b \in B^\times$, and $b^p = h(a)$. We must show that $b \in h(A)$. In the tensor product $B \otimes_A B$, we have $(b \otimes 1 - 1 \otimes b)^p = a \otimes 1 - 1 \otimes a = 0$. Since B is étale over A, the tensor product $B \otimes_A B$ is also étale over the reduced ring A. So $B \otimes_A B$ is reduced, and $b \otimes 1 = 1 \otimes b$. The assumption that B is faithfully flat over A implies that $b \in h(A)$ by descent. ∎

(2.1.6) Lemma *Let A be an excellent commutative ring over $\mathbb{F}_p$. Let I be an ideal of A such that $1 + I \subset A^\times$. Let B be the I-adic completion of A. Assume that A is reduced and excellent. Then the map*

$$A^\times/(A^\times)^{p^n} \to B^\times/(B^\times)^{p^n}$$

induced by the inclusion $A \to B$ is injective, for any integer $n \geq 1$.

PROOF. We assert that the faithfully flat morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is regular because A is excellent. It suffices to check that $\operatorname{Spec}(B_\mathfrak{m}) \to \operatorname{Spec}(A_\mathfrak{m})$ is regular for each maximal ideal $\mathfrak{m}$ of A, where $A_\mathfrak{m}$ is the localization of A at $\mathfrak{m}$, and $B_\mathfrak{m} = B \otimes_A A_\mathfrak{m}$. The maximal ideal $\mathfrak{m}$ contains I, because $1 + I \subset A^\times$. Let C be the $\mathfrak{m}$-adic completion of A. We have morphisms $\operatorname{Spec} C \to \operatorname{Spec}(B_\mathfrak{m}) \to \operatorname{Spec}(A_\mathfrak{m})$, where $\operatorname{Spec} C \to \operatorname{Spec}(B_\mathfrak{m})$ is faithfully flat and the composition $\operatorname{Spec} C \to \operatorname{Spec}(A_\mathfrak{m})$ is regular. Hence $\operatorname{Spec}(B_\mathfrak{m}) \to \operatorname{Spec}(A_\mathfrak{m})$ is a regular morphism by [16, IV 6.8.3] or [20, 33.B].

Since $\operatorname{Spec} B \to \operatorname{Spec} A$ is regular, so $\operatorname{Spec}(B \otimes_A B) \to \operatorname{Spec} A$ is also regular. This implies that $B \otimes_A B$ is reduced. The argument of Lemma 2.1.5 can now be applied to the present case. ∎

(2.1.7) Lemma *Assume that S is reduced and $p = 0$ in $\mathcal{O}_S$. Let $\bar{\pi} : S_{\text{ét}} \to S_{\text{zar}}$ be the projection from the small étale to the small Zariski site of S. Then*

(i) *The sheaf $\xi_{p^n,S}$ is isomorphic to $\mathbb{G}_\mathrm{m}$, and $\mathrm{R}^1\bar{\pi}_*\,\xi_{p^n,S} = 0$.*

(ii) *There is a natural isomorphism $\bar{\pi}_*(\nu_{p^n,S}) = \nu_{p^n,S_{\text{zar}}}$.*

Proof. The hypotheses implies that the sheaf $\tilde{\mu}_{p^n}$ is trivial. So ξ_{p^n} is isomorphic to $\mathbb{G}_{\mathrm{m}}$, and $\mathrm{R}^1\bar{\pi}_*\,\xi_{p^n,S} = 0$ by Hilbert's theorem 90. The second statement follows from the long exact sequence attached to the short exact sequence in 2.1.2 (iii). ∎

(2.2) Proposition *Let p be a prime number. Let A be a noetherian commutative ring over $\mathbb{Z}_{(p)}$. and let I be an ideal of A which contains the image of p such that A/I is reduced. Let $S = \operatorname{Spec} A$, $S_0 = \operatorname{Spec}(A/I)$, and let $i : S_0 \hookrightarrow S$ be the inclusion map. Suppose that $n_0 \geq 0$ is a natural number such that $p^{n_0-j}\, I^{p^j} = 0$ for every $j = 0, \ldots, n_0$. Let $n \geq n_0$ be an integer. Then the following statements hold.*

(i) *The sheaf $\tilde{\mu}_{p^n,S}$ is equal to*

$$\operatorname{Ker}\left(\mathbb{G}_{\mathrm{m},S_{\text{ét}}} \to i_{\text{ét}*}(\mathbb{G}_{\mathrm{m},S_{0\text{ét}}})\right) .$$

as a subsheaf of $\mathbb{G}_{\mathrm{m}}$ on $S_{\text{ét}}$.

(ii) *The sheaf ξ_{p^n} is isomorphic to $i_{\text{ét}*}(\mathbb{G}_{\mathrm{m},S_{0\text{ét}}})$.*

(iii) *Denote by $\mathrm{ev}_{\mathbb{G}_{\mathrm{m}},S_0} : \mathbb{G}_{\mathrm{m},S_{\text{ét}}} \to i_{\text{ét}*}\left(\mathbb{G}_{\mathrm{m},S_{0\text{ét}}}\right)$ the map of sheaves on $S_{\text{ét}}$ given by "evaluating at S_0"; it sends each element $u \in R^\times$ to the image of u in $(R/IR)^\times$, for every commutative algebra R which is étale over A. Denote by $\mathrm{ev}_{\nu_{p^n},S_0} : \nu_{p^n,S} \to i_{\text{ét}*}\left(\nu_{p^n,S_0}\right)$ the map "evaluating at S_0" induced by $\mathrm{ev}_{\mathbb{G}_{\mathrm{m}},S_0}$. Then the natural map*

$$\operatorname{Ker}\left(\mathrm{ev}_{\mathbb{G}_{\mathrm{m}},S_0}\right) \to \operatorname{Ker}\left(\mathrm{ev}_{\nu_{p^n},S_0}\right)$$

is an isomorphism. In other words, we have a functorial isomorphism

$$\operatorname{Ker}\left(\mathrm{ev}_{\nu_{p^n},S_0}\right)(\operatorname{Spec} R) \xleftarrow{\sim} 1 + IR \ \subset\ R^\times$$

for every commutative étale A-algebra R.

(iv) *Let $S_j = \operatorname{Spec}(R/I^{j+1})$ for $j \in \mathbb{N}$. The sheaf*

$$\tilde{\mu}_{p^n,S} = \operatorname{Ker}(\mathrm{ev}_{\mathbb{G}_{\mathrm{m}},S_0} : \mathbb{G}_{\mathrm{m}S_{\text{ét}}} \to i_{\text{ét}*}(\mathbb{G}_{\mathrm{m}},S_{0\text{ét}}))$$

has a finite separated exhaustive filtration by subsheaves

$$\mathrm{Fil}^j := \operatorname{Ker}(\mathrm{ev}_{\mathbb{G}_{\mathrm{m}},S_0} : \mathbb{G}_{\mathrm{m}S_{\text{ét}}} \to i_{\text{ét}*}(\mathbb{G}_{\mathrm{m}},S_{j\text{ét}}))\,.$$

For each $j \in \mathbb{N}$ we have a natural isomorphism

$$\mathrm{Fil}^j(\operatorname{Spec} R) \xleftarrow{\sim} 1 + I^{j+1}R \subset R^\times$$

for for every commutative étale A-algebra R, compatible with the inclusions $\mathrm{Fil}^{j+1} \subset \mathrm{Fil}^j$.

(v) *Let* $\bar\pi : S_{\text{ét}} \to S_{\text{zar}}$ *be the projection from the small étale to the small Zariski site of* S. *Then* $\mathrm{R}^1\bar\pi_*\,\tilde\mu_{p^n,S} = 0$, $\mathrm{R}^1\bar\pi_*\,\xi_{p^n,S} = 0$, *and* $\bar\pi_*(\nu_{p^n,S}) = \nu_{p^n,S_{\text{zar}}}$.

PROOF. Suppose that R is an étale algebra over A. Then R/IR is reduced because it is étale over A/I. So any element of $\tilde\mu_{p^n}(\operatorname{Spec} R)$ has the form $1+y$ for some $y \in IR$. In other words the inclusion $\tilde\mu_{p^n,S} \subseteq \operatorname{Ker}\left(\mathbb{G}_{\mathrm{m},S_{\text{ét}}} \to i_{\text{ét}*}(\mathbb{G}_{\mathrm{m},S_{0\text{ét}}})\right)$ holds. To verify the inclusion in the other direction, it suffices to show that for every étale A-algebra R and every element $z \in IR$, we have $(1+z)^{p^n} = 1$. A simple calculation shows that for every integer $1 \le a \le p^n$, we have $\operatorname{ord}_p\left(\frac{p^n!}{(p^n-a)!\,a!}\right) = \operatorname{ord}_p\left(\frac{p^n!}{(p^n-p^b)!\,p^b!}\right) = n-b$ if $p^b \parallel a$. Therefore the assumption $p^{n_0-j}\,I^{p^j} = 0$ for every $j = 0,\dots,n_0$ implies that $\left(\frac{p^n!}{(p^n-a)!\,a!}\right) z^a = 0$ for every $a = 1,\dots,p^n$ and every $z \in IR$. i.e. $(1+z)^{p^n} = 1$. We have proved the statement (i).

The statement (ii) follows immediately from (i); we describe an isomorphism α from $i_{\text{ét}*}(\mathbb{G}_{\mathrm{m},S_{0\text{ét}}})$ to ξ_{p^n} below. Suppose that $\operatorname{Spec} R_0$ is étale over S_0, and f is a unit of R_0. There exists an étale A-algebra R such that $R/IR \cong R_0$. Moreover f lifts to a unit $\tilde f$ in R. Then $\tilde f^{p^n}$ is a section of ξ_{p^n} over $\operatorname{Spec} R$, and it does not depend on the choice of the lifting $\tilde f$ of f. The isomorphism α is the map which sends f to $\tilde f^{p^n}$.

Let $\mathrm{ev}_{\xi_{p^n},S_0} : \xi_{p^n,S} \to i_{\text{ét}*}(\xi_{p^n,S_0})$ be the evaluation map induced by $\mathrm{ev}_{\mathbb{G}_{\mathrm{m}},S_0}$. This map $\mathrm{ev}_{\xi_{p^n},S_0}$ is an isomorphism by (ii). So we have a map of exact sequences of sheaves on $S_{\text{ét}}$ given by "evaluating at S_0", from the short exact sequence $1 \to \xi_{p^n,S} \to \mathbb{G}_{\mathrm{m}S} \to \nu_{p^n,S} \to 1$ to the short exact sequence $i_{\text{ét}*}(1 \to \xi_{p^n,S_0} \to \mathbb{G}_{\mathrm{m}S_0} \to \nu_{p^n,S_0} \to 1)$, the push-forward to $S_{\text{ét}}$ of a similar exact sequence of sheaves on $S_{0\text{ét}}$. The statement (iii) follows from the snake lemma.

The statement (iv) follows easily from (iii). Each associated graded sheaf $\mathrm{Fil}^j/\mathrm{Fil}^{j+1}$ for the filtration in (iv) is isomorphic to the sheaf of abelian groups on $S_{\text{ét}}$ attached to the quasi-coherent $\mathcal{O}_S$-module $I^{j+1}\mathcal{O}_S/I^{j+1}\mathcal{O}_S$. Hence $\mathrm{R}^1\bar\pi_*(\mathrm{Fil}^j/\mathrm{Fil}^{j+1}) = 0$ for all j, and $\mathrm{R}^1\bar\pi_*\tilde\mu_{p^n,S} = 0$ by dévissage. The vanishing of $\mathrm{R}^1\bar\pi_*\xi_{p^n,S}$ is immediate from (ii). The statement (v) follows from the long exact sequences attached to the two short exact sequences in 2.1.2 (iii) and Hilbert's theorem 90. ∎

(2.3) We establish some notation about Barsotti-Tate groups. Let S be either scheme over $\mathbb{Z}_{(p)}$ such that p is locally nilpotent in $\mathcal{O}_S$, or a noetherian formal scheme such that p is topologically nilpotent.

(2.3.1) We abbreviate "G is a Barsotti-Tate group over S" to "G is a BT-group over S. Similarly "G is a truncated Barsotti-Tate group of level-n" is shortened to "G is a BT_n-group over S". We identify a truncated Barsotti-Tate group of level-n over S with the corresponding sheaf of $(\mathbb{Z}/p^n\mathbb{Z})$-modules on S_{fppf}.

(2.3.2) Let E_n be an étale BT_n-group over scheme S. Denote by $\mathrm{T}_p(E_n)$ the restriction of E_n to the small étale site $S_{\text{ét}}$ of S, so that $\mathrm{T}_p(E_n) = \underline{\mathrm{Hom}}_{\mathrm{BT}_n,S_{\text{ét}}}(\mathbb{Z}/p^n\mathbb{Z}, E_n)$. The sheaf $\mathrm{T}_p(E_n)$ is a sheaf of free $(\mathbb{Z}/p^n\mathbb{Z})$-modules of finite rank on $S_{\text{ét}}$. Let $\mathrm{T}_p(E_n)^\vee := \underline{\mathrm{Hom}}_{S_{\text{ét}}}(\mathrm{T}_p(E_n), \mathbb{Z}/p^n\mathbb{Z})$.

Suppose that E is an étale BT-group over S. Write $E = \varinjlim_n E_n$, where each E_n is an étale BT_n-group over S. Denote by $\mathrm{T}_p(E)$ the projective limit of $\mathrm{T}_p(E_n)$, so that $\mathrm{T}_p(E) = \underline{\mathrm{Hom}}_{\mathrm{BT},\, S_{\text{ét}}}(\mathbb{Q}_p/\mathbb{Z}_p, E)$. The sheaf $\mathrm{T}_p(E)$ is a sheaf of free $\mathbb{Z}_p$-modules of finite rank on $S_{\text{ét}}$. Let $\mathrm{T}_p(E)^\vee := \underline{\mathrm{Hom}}_{S_{\text{ét}}}(\mathrm{T}_p(E), \mathbb{Z}_p) = \underline{\mathrm{Hom}}_{S_{\text{ét}}}(E, \mathbb{Q}_p/\mathbb{Z}_p)$.

(2.3.3) Let T_n be a multiplicative BT_n-group over S. Denote by $X_*(T_n)$ (resp. $X^*(T_n)$) the sheaf $\underline{\mathrm{Hom}}_{\mathrm{BT}_n,\, S_{\text{ét}}}(\mu_{p^n}, T_n)$ (resp. $\underline{\mathrm{Hom}}_{\mathrm{BT}_n,\, S_{\text{ét}}}(T_n, \mu_{p^n})$.) Both $X_*(T_n)$ and $X^*(T_n)$ are sheaves of free $(\mathbb{Z}/p^n\mathbb{Z})$-modules of finite rank on $S_{\text{ét}}$, and they are $(\mathbb{Z}/p^n\mathbb{Z})$-dual to each other.

Let $T = \varinjlim_n T_n$ be a multiplicative BT-group over S, where each T_n is a BT_n-group over S. Then we denote by $X_*(T)$ (resp. $X^*(T)$) the projective limit of $X_*(T_n)$ (resp. $X^*(T_n)$.) The sheaf $X_*(T)$ is isomorphic to $\underline{\mathrm{Hom}}_{\mathrm{BT},\, S_{\text{ét}}}(\mu_{p^\infty}, T)$, while $X^*(T)$ is isomorphic to $\underline{\mathrm{Hom}}_{\mathrm{BT},\, S_{\text{ét}}}(T, \mu_{p^\infty})$.

(2.4) Proposition *Let E_n be an étale BT_n-group over a scheme S, and let T_n be a multiplicative BT_n-group over S. Denote by $\pi : S_{\mathrm{fppf}} \to S_{\text{ét}}$ the projection from the small fppf site of S to the small étale site of S. Then we have canonical isomorphisms*

(i) $\underline{\mathrm{Ext}}^1_{\mathbb{Z}/p^n\mathbb{Z},\, S_{\mathrm{fppf}}}(E_n, T_n) = 0$,

(ii) $\mathrm{H}^1(S_{\mathrm{fppf}}, \underline{\mathrm{Hom}}_{\mathbb{Z}/p^n\mathbb{Z},\, S_{\mathrm{fppf}}}(E_n, T_n)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathbb{Z}/p^n\mathbb{Z},\, S_{\mathrm{fppf}}}(E_n, T_n)$,

(iii) $\underline{\mathrm{Hom}}_{\mathbb{Z}/p^n\mathbb{Z},\, S_{\mathrm{fppf}}}(E_n, T_n) \xleftarrow{\sim} \mathrm{T}_p(E_n)^\vee \otimes_{\mathbb{Z}/p^n\mathbb{Z}} X_*(T_n) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mu_{p^n, S_{\mathrm{fppf}}}$,

(iv) $\mathrm{R}^1\pi_*\underline{\mathrm{Hom}}_{\mathbb{Z}/p^n\mathbb{Z},\, S_{\mathrm{fppf}}}(E_n, T_n) \xleftarrow{\sim} \mathrm{T}_p(E_n)^\vee \otimes_{\mathbb{Z}/p^n\mathbb{Z}} X_*(T_n) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \nu_{p^n, S_{\text{ét}}}$.

Proof. The statement (i) holds because E_n is locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$; (ii) follows from (i). The statement (iii) is immediate from the definitions. The isomorphism in (iv) comes from the isomorphism in (iii) and the isomorphism $\nu_{p^n} \xrightarrow{\sim} \mathrm{R}^1\pi_*\mu_{p^n}$ in Kummer theory. ∎

(2.4.1) Corollary *Let E_n be an étale BT_n-group over a scheme S, and let T_n be a multiplicative BT_n over S. Denote by $\underline{\mathrm{Ext}}(E_n, T_n)$ the sheafification of the presheaf*

$$\Big(U_{/S} \rightsquigarrow \mathrm{Ext}_{\mathrm{BT}_n, U_{\mathrm{fppf}}}(E_n, T_n) \cong \mathrm{Ext}^1_{\mathbb{Z}/p^n\mathbb{Z}, U_{\mathrm{fppf}}}(E_n, T_n)\Big)_{U \text{ étale over } S}$$

on the small étale site $S_{\text{ét}}$ of S. In other words, $\underline{\mathrm{Ext}}(E_n, T_n) = \mathrm{R}^1\pi_*\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}/p^n\mathbb{Z}, S_{\mathrm{fppf}}}(E_n, T_n)$.

(i) *We have*

$$\underline{\mathrm{Ext}}(E_n, T_n) \cong \mathrm{R}^1\pi_*\underline{\mathrm{Hom}}_{\mathbb{Z}/p^n\mathbb{Z},\, S_{\mathrm{fppf}}}(E_n, T_n) \cong \mathrm{T}_p(E_n)^\vee \otimes_{\mathbb{Z}/p^n\mathbb{Z}} X_*(T_n) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \nu_{p^n, S}.$$

(ii) *Assume that S satisfies one of the following two conditions: either*

– *$p = 0$ in $\mathcal{O}_S$ and S is reduced, or*

– *$S = \operatorname{Spec} R$ is affine and there is an ideal $I \subset R$ which contains the image of p in R, such that R/I is reduced, and $p^{n-j} I^{p^j}$ for $j = 0, \ldots, n$.*

Then

$$\mathrm{Ext}_{\mathrm{BT}_n,\, S_{\mathrm{fppf}}}(E_n, T_n) = \Gamma(S_{\text{ét}}, \underline{\mathrm{Ext}}_{S_{\text{ét}}}(E_n, T_n)) = \mathrm{Hom}_{S_{\text{ét}}}\left(\mathrm{T}_p(E_n) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} X^*(T_n),\ \nu_{p^n,S}\right).$$

Proof. The statement (i) is a restatement of 2.4 (ii), (iv); it remains to prove (ii). According to the Leray spectral sequence

$$\mathrm{H}^j\left(S_{\text{ét}}, \mathrm{R}^i\pi_*\underline{\mathrm{Hom}}_{\mathbb{Z}/p^n\mathbb{Z}, S_{\mathrm{fppf}}}(E_n, T_n)\right) \implies \mathrm{Ext}^1_{\mathbb{Z}/p^n\mathbb{Z},\, S_{\mathrm{fppf}}}(E_n, T_n) \cong \mathrm{Ext}_{\mathrm{BT}_n,\, S_{\mathrm{fppf}}}(E_n, T_n),$$

we only have to show that

$$\mathrm{H}^1\left(S_{\text{ét}}, \pi_*\left(\mathrm{T}_p(E_n)^\vee \otimes_{\mathbb{Z}/p^n\mathbb{Z}} X_*(T_n) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \nu_{p^n,S}\right)\right) = 0$$

under either of the two conditions. In the first case when $p = 0$ in $\mathcal{O}_S$, we have

$$\pi_*\left(\mathrm{T}_p(E_n)^\vee \otimes_{\mathbb{Z}/p^n\mathbb{Z}} X_*(T_n) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mu_{p^n,S_{\mathrm{fppf}}}\right) = 0.$$

In the second case, the proof of Prop. 2.2 (i) tells us that $\pi_*\left(\mu_{p^n,S_{\mathrm{fppf}}}\right)$ is isomorphic to $\mathrm{Ker}(\mathbb{G}_{\mathrm{m}S_{\text{ét}}} \to i_{\text{ét}*}(\mathbb{G}_{\mathrm{m}S_{0\text{ét}}}))$, where $i_0 : S_0 = \operatorname{Spec}(R/I) \hookrightarrow \operatorname{Spec} R = S$ denotes the closed embedding of $\operatorname{Spec}(R/I)$ in $\operatorname{Spec} R$. Hence the sheaf of abelian groups $\pi_*\left(\mu_{p^n,S_{\mathrm{fppf}}}\right)$ on $S_{\text{ét}}$ is a successive extension of sheaves attached to quasi-coherent $\mathcal{O}_{S_0}$-modules as in 2.2 (iii); the same is true for the sheaf $\pi_*\left(\mathrm{T}_p(E_n)^\vee \otimes_{\mathbb{Z}/p^n\mathbb{Z}} X_*(T_n) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mu_{p^n,S_{\mathrm{fppf}}}\right)$. The statement (ii) follows from the assumption that S is affine, because $\mathrm{H}^j(S_{\text{ét}}, \mathcal{F}_{\text{ét}}) = \mathrm{H}^j(S_{0\mathrm{zar}}, \mathcal{F}) = 0$ for each sheaf of abelian groups $\mathcal{F}_{\text{ét}}$ on $S_{\text{ét}}$ attached to a quasi-coherent $\mathcal{O}_{S_0}$-module $\mathcal{F}$, $j \geq 1$. ∎

(2.5) Definition (i) Let S be a scheme over $\mathbb{Z}_{(p)}$ such that p is locally nilpotent in $\mathcal{O}_S$. Let G_n be a BT_n-group over S. Denote by G_n^{mult} the maximal multiplicative BT_n-subgroup of G_n, and $G_n^{\text{ét}}$ the maximal étale quotient of G_n. Assume that G_n is *ordinary*, in the sense that the complex $1 \to G_n^{\mathrm{mult}} \to G_n \to G_n^{\text{ét}} \to 1$ of BT_n-groups over S is exact. Then G_n is an extension of of $G_n^{\text{ét}}$ by G_n^{mult}, giving rise to an element $e(G_n) \in \mathrm{Ext}_{S_{\mathrm{fppf}}}(G_n^{\text{ét}}, G_n^{\mathrm{mult}})$. Define

$$q_n(G_n) \ \in\ \mathrm{Hom}_{S_{\text{ét}}}\left(\mathrm{T}_p(G_n^{\text{ét}}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} X^*(G_n^{\mathrm{mult}}),\ \nu_{p^n,S}\right)$$

to be the image of $e(G_n)$ in

$$\Gamma\left(S_{\text{ét}}, \underline{\mathrm{Ext}}(G_n^{\text{ét}}, G_n^{\mathrm{mult}})\right) = \mathrm{Hom}_{S_{\text{ét}}}\left(\mathrm{T}_p(G_n^{\text{ét}}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} X^*(G_n^{\mathrm{mult}}),\ \nu_{p^n,S}\right).$$

We say that $q_n(G_n)$ is the *canonical coordinates* of the ordinary BT_n-group G_n.

(ii) Let S be as in (i). Let G be a BT-group over S. Denote by G_{mult} (resp. $G_{\text{ét}}$) the maximal multiplicative BT-subgroup (resp. the maximal étale BT-quotient group) of G. Assume that G is *ordinary*, in the sense that the complex $1 \to G^{\text{mult}} \to G \to G^{\text{ét}} \to 1$ of *BT*-groups over S is exact. This assumption means that we have compatible short exact sequences of BT_n-groups $1 \to G_n^{\text{mult}} \to G_n \to G_n^{\text{ét}} \to 1$ for $n \geq 1$. Define

$$\begin{aligned} q(G) &= \lim_{n\to\infty} q_n(G_n) \\ &\in \varprojlim_n \mathrm{Hom}_{S_{\text{ét}}}\left(\mathrm{T}_p(G_n^{\text{ét}}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} X^*(G_n^{\text{mult}}),\ \nu_{p^n,S}\right) \\ &= \mathrm{Hom}_{S_{\text{ét}}}\left(\mathrm{T}_p(G^{\text{ét}}) \otimes_{\mathbb{Z}_p} X^*(G^{\text{mult}}),\ \nu_{p^\infty,S}\right) . \end{aligned}$$

We call $q(G)$ the *canonical coordinates* of the ordinary BT-group G.

(ii)' Let S be a noetherian formal scheme over $\mathbb{Z}_p$ such that p is locally topologically nilpotent. Let G be a BT-group over S. The same formula in (ii) above makes sense in the present situation and defines an element

$$q(G) \in \mathrm{Hom}_{S_{\text{ét}}}\left(\mathrm{T}_p(G^{\text{ét}}) \otimes_{\mathbb{Z}_p} X^*(G^{\text{mult}}),\ \nu_{p^\infty,S}\right)$$

again called the *canonical coordinates* of G.

(2.5.1) Remark Suppose that R is an Artinian local ring whose residue field κ is algebraically closed of characteristic $p > 0$. Let $S = \operatorname{Spec} R$, and let $G := A[p^\infty]$ be the BT-group attached to an abelian scheme A over S whose closed fiber is an ordinary abelian variety. Then both $\mathrm{T}_p(G^{\text{ét}})$ and $X^*(G^{\text{mult}})$ are constant on $S_{\text{ét}}$, $\nu_{p^\infty,S}$ is represented by the formal completion $\widehat{\mathbb{G}_{\mathrm{m}/S}}$ of $\mathbb{G}_{\mathrm{m}/S}$ along the unit section, and $q(G)$ coincides with the classical definition of the Serre-Tate coordinates of G. See [18] for an exposition of the Serre-Tate coordinates.

One can regard local sections of ν_{p^∞} as some sort of "generalized functions" on S, so the use of the word "coordinates" can be partially justified from this point of view.

(2.6) Proposition *Let S be a scheme such that p is locally nilpotent in $\mathcal{O}_S$.*

(i) *Let G_n, H_n be two ordinary BT_n-groups over S. Let $\alpha : G_n \to H_n$ be a homomorphism of BT-groups over S. Denote by $\alpha_{\text{mult}} : G_n^{\text{mult}} \to H_n^{\text{mult}}$ (resp. $\alpha_{\text{ét}} : G_n^{\text{ét}} \to H_n^{\text{ét}}$) the homomorphism induced by α between the étale (resp. multiplicative) part of the two BT_n-groups. Then we have*

$$q_n(G;\, a \otimes \alpha_{\text{mult}}^*(\lambda)) = q_n(H;\, \alpha_{\text{ét}*}(a) \otimes \lambda)$$

for any local section a of $\mathrm{T}_p(G_n^{\text{ét}})$ and any local section λ of $X^*(H_n^{\text{mult}})$.

(ii) *Let G, H be two ordinary BT-groups over S. Let $\alpha : G \to H$ be a homomorphism of BT-groups over S. Denote by $\alpha_{\mathrm{mult}} : G^{\mathrm{mult}} \to H^{\mathrm{mult}}$ (resp. $\alpha_{\text{ét}} : G^{\text{ét}} \to H^{\text{ét}}$) the homomorphism induced by α between the étale (resp. multiplicative) part of the two BT-groups. Then we have*

$$q(G;\, a \otimes \alpha_{\mathrm{mult}}^{*}(\lambda)) = q(H;\, \alpha_{\text{ét}*}(a) \otimes \lambda)$$

for any local section a of $\mathrm{T}_p(G^{\text{ét}})$ and any local section λ of $X^*(H^{\mathrm{mult}})$.

Proof. The statement (ii) follows from (i). To prove (i), we may and do assume that S is strictly local. Consequently G^{mult} and $G^{\text{ét}}$ are constant. So it suffices to check (i) when $G^{\mathrm{mult}} \cong \mu_{p^n}$ and $G^{\text{ét}} \cong \mathbb{Z}/p^n\mathbb{Z}$. In this case the statement (ii) follow from the functoriality in Kummer theory. ∎

(2.7) Proposition *Let S be a scheme such that p is locally nilpotent in $\mathcal{O}_S$.*

(i) *Let G_n be an ordinary BT_n-group over S. Let G_n^t be the Cartier dual of G_n. The canonical pairing $G_n \times G_n^t \to \mu_{p^n}$ induces isomorphisms*

$$\mathrm{T}_p(G_n^{\text{ét}}) \cong X^*(G_n^{t,\mathrm{mult}}), \qquad \mathrm{T}_p(G_n^{t,\text{ét}}) \cong X^*(G_n^{\mathrm{mult}}).$$

With the above isomorphisms, one can regard the canonical coordinates of G_n and G_n^t as

$$\begin{aligned} q_n(G_n) &\in \ \mathrm{Hom}_S(\mathrm{T}_p(G_n^{\text{ét}}) \otimes \mathrm{T}_p(G_n^{t,\text{ét}}), \nu_{p^n,S}), \\ q_n(G_n^t) &\in \ \mathrm{Hom}_S(\mathrm{T}_p(G_n^{t,\text{ét}}) \otimes \mathrm{T}_p(G_n^{\text{ét}}), \nu_{p^n,S}). \end{aligned}$$

Then

$$q_n(G_n;\, a \otimes b) = q_n(G_n^t;\, b \otimes a)^{-1} \qquad \text{for all } a \in \mathrm{T}_p(G_n^{\text{ét}}), b \in \mathrm{T}_p(G_n^{t,\text{ét}}),$$

where $q_n(G_n^t;\, b \otimes a)^{-1}$ denotes the inverse of $q_n(G_n^t;\, b \otimes a)$ according to the group structure of $\nu_{p^n,S}$.

(ii) *Let G be an ordinary BT-group over S. Let G^t be the Serre dual of G. The canonical pairing $G \times G^t \to \mu_{p^\infty}$ induces isomorphisms*

$$\mathrm{T}_p(G^{\text{ét}}) \cong X^*(G^{t,\mathrm{mult}}), \qquad \mathrm{T}_p(G^{t,\text{ét}}) \cong X^*(G^{\mathrm{mult}}).$$

With the above isomorphisms, one can regard the canonical coordinates of G and G^t as

$$\begin{aligned} q(G) &\in \ \mathrm{Hom}_S(\mathrm{T}_p(G^{\text{ét}}) \otimes \mathrm{T}_p(G^{t,\text{ét}}), \nu_{p^\infty,S}), \\ q(G^t) &\in \ \mathrm{Hom}_S(\mathrm{T}_p(G^{t,\text{ét}}) \otimes \mathrm{T}_p(G^{\text{ét}}), \nu_{p^\infty,S}). \end{aligned}$$

Then

$$q(G;\, a \otimes b) = q(G^t;\, b \otimes a)^{-1} \quad \text{for all } a \in \mathrm{T}_p(G^{\text{ét}}), b \in \mathrm{T}_p(G^{t,\text{ét}}).$$

Proof. It suffices to prove the statement (i) about BT_n-groups, since (ii) follows from (i) by taking the limit. To prove (i), by étale descent it suffices to prove it when both $\mathrm{T}_p(G^{\text{ét}})$ and $X^*(G^{\text{mult}})$ are constant, and $S = \operatorname{Spec} R$ for a local ring R. Assume this is the case. Then from the definition of the canonical coordinates one sees that it suffices to prove (ii) in the special case when $G^{\text{mult}} = \mu_{p^n}$ and $G^{\text{ét}} = \mathbb{Z}/p^n\mathbb{Z}$. Assume now that we are in this special case. The extension $1 \to \mu_{p^n} \to G_n \to \mathbb{Z}/p^n\mathbb{Z} \to 1$ comes from an element $\bar{f} \in R^\times/(R^\times)^{p^n}$. Let $f \in R^\times$ be a representative of $\bar{f}$. Then G_n can be described as follows:

$$G_n(U) = \left\{ (g,a) \in \Gamma(U, \mathcal{O}_U^\times \times \mathbb{Z}_U) \,|\, g^{p^n} = f^a \right\} / \langle (f, p^n) \rangle$$

for every S-scheme U. See [8, 4.3.1]. Similarly the element $\bar{f}^{-1} \in R^\times/(R^\times)^{p^n}$ gives rise to a BT_n-group H_n such that

$$H_n(U) = \left\{ (h,b) \in \Gamma(U, \mathcal{O}_U^\times \times \mathbb{Z}_U) \,|\, h^{p^n} = f^{-b} \right\} / \langle (f^{-1}, p^n) \rangle$$

for every S-scheme U. Define a pairing $G_n \times H_n \to \mu_{p^n}$ by

$$(g,a) \times (h,b) \;\mapsto\; g^b \cdot h^a$$

for every $(g,a) \in G_n(U)$ and $(h,b) \in H_n(U)$ as above. It is routine to check that this pairing is well-defined, and realizes H_n as the Cartier dual of G_n. Both $\mathrm{T}_p(G_n^{\text{ét}})$ and $\mathrm{T}_p(H_n^{\text{ét}})$ are canonically isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. From the definition of the canonical coordinates we have $q_n(G_n; \bar{1} \otimes \bar{1}) = \bar{f}$, and $q_n(G_n; \bar{1} \otimes \bar{1}) = \bar{f}^{-1}$. We have verified (ii) under the assumptions specified at the beginning of the proof. ∎

(2.7.1) Remark When $p = 0$ in $\mathcal{O}_S$, one can also verify the special case above using the explicit description of the Dieudonné crystal of G_n given in [8, 4.3.2]: With the notation in *loc. cit.*, we have $\mathbb{D}(G_n) = M(d\,\log(\tilde{f}), \log(\sigma(\tilde{f})\tilde{f}^{-p}))$; $\mathbb{D}(G_n)$ has an A_n-basis v, $\tilde{v}$ such that

$$\begin{array}{rrll} \nabla\, v = & 0 & \nabla\, \tilde{v} & = v \otimes d\log(\tilde{f}) \\ \mathrm{F}\, v^\sigma = & v & \mathrm{F}\, \tilde{v}^\sigma & = \log(\sigma(\tilde{f})\tilde{f}^{-p})) \cdot v + p \cdot \tilde{v} \\ \mathrm{V}\, v = & p \cdot v^\sigma & \mathrm{V}\, \tilde{v} & = -\log(\sigma(\tilde{f})\tilde{f}^{-p})) \cdot v^\sigma + \tilde{v}^\sigma \end{array}$$

Similarly $\mathbb{D}(H_n) = M(d\,\log(\tilde{f}^{-1}), \log(\sigma(\tilde{f}^{-1})\tilde{f}^{p}))$; $\mathbb{D}(H_n)$ has an A_n-basis w, $\tilde{w}$ such that

$$\begin{array}{rrll} \nabla\, w = & 0 & \nabla\, \tilde{w} & = -\, w \otimes d\log(\tilde{f}) \\ \mathrm{F}\, w^\sigma = & w & \mathrm{F}\, \tilde{w}^\sigma & = -\log(\sigma(\tilde{f})\tilde{f}^{-p})) \cdot w + p \cdot \tilde{w} \\ \mathrm{V}\, w = & p \cdot w^\sigma & \mathrm{V}\, \tilde{w} & = \log(\sigma(\tilde{f})\tilde{f}^{-p})) \cdot w^\sigma + \tilde{w}^\sigma \end{array}$$

We define a pairing between

$$\langle\ ,\ \rangle \;:\; \mathbb{D}(G_n) \times \mathbb{D}(H_n) \;\longrightarrow\; \mathbb{D}(\mu_{p^n}) \;=\; A_n$$

by

$$\langle v, w\rangle = 0, \quad \langle v, \tilde{w}\rangle = 1, \quad \langle \tilde{v}, w\rangle = 1, \quad \langle \tilde{v}, \tilde{w}\rangle = 0.$$

One verifies that this pairing is well-defined, horizontal, and is compatible with F and V. Moreover the pairing $\langle\ ,\ \rangle$ identifies $\mathbb{D}(H_n)$ as $\mathbb{D}(G_n)^\vee(-1)$, so H_n is the Cartier dual of G_n.

(2.8) Proposition *Let S be a scheme such that p is locally nilpotent in $\mathcal{O}_S$. Let $S_0 = S \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$. Let A be an abelian variety over S such that $A_0 := A \times_S S_0$ is ordinary. We identify $X^*(A[p^\infty]^{\mathrm{mult}})$ with $\mathrm{T}_p(A^t[p^\infty]^{\text{ét}})$ using the duality pairing*

$$\langle\ ,\ \rangle_A : A[p^\infty] \times A^t[p^\infty] \to \mathbb{G}_{\mathrm{m}}[p^\infty]$$

attached to A. So $q(A)$, the canonical coordinates of $A[p^\infty]$, can be identified with an element of $\operatorname{Hom}_S\left(\mathrm{T}_p(A[p^\infty]^{\text{ét}}) \otimes \mathrm{T}_p(A^t[p^\infty]^{\text{ét}}),\ \nu_{p^\infty,S}\right)$. In the same fashion we regard $q(A^t)$ as an element of $\operatorname{Hom}_S\left(\mathrm{T}_p(A^t[p^\infty]^{\text{ét}}) \otimes \mathrm{T}_p\left((A^t)^t[p^\infty]^{\text{ét}}\right),\ \nu_{p^\infty,S}\right)$. Let $j_A : A \xrightarrow{\sim} (A^t)^t$ be the canonical isomorphism from the abelian variety A to its double dual; it induces an isomorphism from $\mathrm{T}_p\left(A[p^\infty]^{\text{ét}}\right)$ to $\mathrm{T}_p\left((A^t)^t[p^\infty]^{\text{ét}}\right)$. With the above notation, we have

$$q(A;\, a \otimes b) = q(A^t;\, b \otimes j_A(a))$$

for all $a \in \mathrm{T}_p(A[p^\infty]^{\text{ét}})$ and all $b \in \mathrm{T}_p(A^t[p^\infty]^{\text{ét}})$.

Proof. We have the following relation between the duality pairing $\langle\ ,\ \rangle_A : A[p^\infty] \times A^t[p^\infty] \to \mathbb{G}_{\mathrm{m}}[p^\infty]$ for A and also the duality pairing $\langle\ ,\ \rangle_{A^t} : A^t[p^\infty] \times (A^t)^t[p^\infty] \to \mathbb{G}_{\mathrm{m}}[p^\infty]$ for A^t :

$$\langle x,\, y\rangle_A = \langle y,\, j_A(x)\rangle_{A^t}^{-1}$$

for every $x \in A[p^\infty](U)$, every $y \in A^t[p^\infty](U)$, and every scheme U over S. We refer the readers to the discussion in § 5.1 of [3] for the signs in the duality pairing for abelian varieties. Prop. 2.8 follows from the displayed formula above and 2.7 (ii). ∎

§3. Local p-adic monodromy

(3.1) We set up notation for this section. Let k be an algebraically closed field of characteristic p, and let R is a complete Noetherian normal integral domain over k. Let $S = \operatorname{Spec} R$. Let $\mathfrak{m}$ be the maximal ideal of R.

(3.1.1) Let G be an ordinary BT-group over S, which sits in a short exact sequence

$$0 \to T \to G \to E \to 0$$

of BT-groups over S, where T multiplicative and E is étale. Let $s_0 \in S$ be the closed point of S. Denote by G_0 (resp. T_0, resp. E_0) the fiber of G (resp. T, resp. E) over s_0. The sheaf of free $\mathbb{Z}_p$-modules $\mathrm{T}_p(E)$ (resp. $X^*(T)$) on $S_{\text{ét}}$ is constant; its fiber over s_0 is $\mathrm{T}_p(E_0)$ (resp. $X^*(T_0)$.) Let $U_{\mathbb{Z}_p} = U(G_0)_{\mathbb{Z}_p} := \mathrm{T}_p(E_0)$, the p-adic Tate module of E_0. Let $V_{\mathbb{Z}_p} = V(G_0)_{\mathbb{Z}_p} := X_*(T_0)$, the cocharacter group of T_0. Let $U_{\mathbb{Q}_p} := U_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathrm{T}_p(E_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $V_{\mathbb{Q}_p} := V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = X_*(T_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Denote by $U_{\mathbb{Z}_p}^\vee$ and $V_{\mathbb{Z}_p}^\vee$ (resp. $U_{\mathbb{Q}_p}^\vee$ and $V_{\mathbb{Q}_p}^\vee$) the $\mathbb{Z}_p$-dual of $U_{\mathbb{Z}_p}$ and $V_{\mathbb{Z}_p}$ (resp. the $\mathbb{Q}_p$-dual of $U_{\mathbb{Q}_p}$ and $V_{\mathbb{Q}_p}$.)

(3.1.2) The canonical coordinates for G can be regarded as a homomorphism

$$q(G) : U_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p}^{\vee} = \mathrm{T}_p(E_0) \otimes_{\mathbb{Z}_p} X^*(T_0) \longrightarrow 1 + \mathfrak{m} \subset R^{\times} .$$

If G is the BT-group attached to an ordinary abelian scheme A over S, then $q(G)$ coincides with the Serre-Tate coordinates of A.

(3.1.3) Let $N(G)^{\perp} := \mathrm{Ker}(q(G)) \subseteq U_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p}^{\vee}$. Since the group $1 + \mathfrak{m} \subset R^{\times}$ is torsion free, $(U_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p}^{\vee})/N(G)^{\perp}$ is a free $\mathbb{Z}_p$-module of finite rank. Define a $\mathbb{Z}_p$-direct factor $N(G)$ of $U_{\mathbb{Z}_p}^{\vee} \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p} = \mathrm{Hom}_{\mathbb{Z}_p}(U_{\mathbb{Z}_p}, V_{\mathbb{Z}_p})$ by

$$N(G) := \{\, n \in U_{\mathbb{Z}_p}^{\vee} \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p} \mid \langle n, a\rangle = 0 \ \ \forall a \in N(G)^{\perp} \,\} .$$

(3.1.4) Remark The canonical coordinates $q(G)$ of G defines a morphism

$$f \,:\, \mathrm{Spf}\ R \longrightarrow U_{\mathbb{Z}_p}^{\vee} \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}_{\mathrm{m}}}$$

from Spf R to a formal torus $U_{\mathbb{Z}_p}^{\vee} \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}_{\mathrm{m}}}$ over k. The $\mathbb{Z}_p$-direct factor $N(G)$ of $U_{\mathbb{Z}_p}^{\vee} \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p}$ is characterized by the property that $N(G) \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}_{\mathrm{m}}}$ is the smallest formal subtorus of $U_{\mathbb{Z}_p}^{\vee} \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}_{\mathrm{m}}}$ which contains the schematic image of f.

(3.2) Let $W = W(k)$ be the ring of p-adic Witt vectors with components in k. Let $K = B(k)$ be the fraction field of $W(k)$. Let F-Isoc(S/K) be the $\otimes$-category over $\mathbb{Q}_p$ consisting of convergent F-isocrystals on S/K; see [2], [7]. We often omit the adjective "convergent" and simply refer to objects in F-Isoc(S/K) as F-isocrystals on S, in view of Dwork's lemma.

(3.2.1) Let $\mathbb{D}(G)$ be the contravariant Dieudonné crystal attached to G as defined in [3]. Let $\mathbb{T} = \mathbb{T}(G)$ be the the $\otimes$-subcategory of F-Isoc(S/K) generated by the image $\mathbb{D}(G) \otimes K$ of $\mathbb{D}(G)$ in F-Isoc(S/K).

(3.2.2) The closed point s_0 of S gives rise to a fiber functor ω from $\mathbb{T}(G)$ to Vec_K, the $\otimes$-category of finite dimensional K-vector spaces; it sends each F-isocrystal $\mathcal{M}$ to the fiber $\mathcal{M}_{s_0}$ of $\mathcal{M}$ over s_0.

There is an increasing $\mathbb{Z}$-filtration on $\mathbb{T}(G)$ in the following sense. For every object $\mathcal{M}$ in $\mathbb{T}$, there is a functorial filtration of finite length

$$\cdots \subseteq \mathcal{M}_{i-1} \subseteq \mathcal{M}_i \subseteq \mathcal{M}_{i+1} \subseteq \cdots$$

by subobjects of $\mathcal{M}$, such that $\mathcal{M}_i = 0$ for $i \ll 0$ and $\mathcal{M}_i = \mathcal{M}$ for $i \gg 0$, and each subquotient $\mathcal{M}_i/\mathcal{M}_{i-1}$ has the form $\mathcal{F}(-i)$ for some unit-root F-isocrystal $\mathcal{F}$ on S. Here $\mathcal{F}(-i)$ denotes the $(-i)$-th Tate-twist of $\mathcal{F}$: the isocrystal underlying $\mathcal{F}(-i)$ is the same as that of $\mathcal{F}$, while $\Phi_{\mathcal{F}(-i)} = p^i \cdot \Phi_{\mathcal{F}}$. We call this filtration the *slope filtration* on $\mathbb{T}(G)$, and refer to [12] for the existence of the slope filtration. The slope filtration is compatible with the structure of $\otimes$-category of $\mathbb{T}$ in the usual way. For $\mathcal{M} = \mathbb{D}(G) \otimes K$, we have $\mathcal{M}_{-1} = (0)$, $\mathcal{M}_1 = \mathcal{M}$, $\mathcal{M}/\mathcal{M}_0 = V^{\vee}(-1) \otimes_{\mathbb{Z}_p} \mathcal{O}_{S/K}$, $\mathcal{M}_0 = U^{\vee} \otimes_{\mathbb{Z}_p} \mathcal{O}_{S/K}$.

For each object $\mathcal{M}$ in $\mathbb{T} = \mathbb{T}(G)$, the fiber $\mathcal{M}_{s_0}$ of $\mathcal{M}$ at s_0 can be considered as a finite dimensional K-vector space with a p-semilinear action Φ. The hypothesis that G is ordinary implies that the slopes of $\mathcal{M}_{s_0}$ in the Dieudonné-Manin classification are integers. For each $i \in \mathbb{Z}$, denote by $(\mathcal{M}_{s_0})^{p^{-i}\Phi}$ the $\mathbb{Q}_p$-subspace of $\mathcal{M}_{s_0}$ consisting of all elements $x \in \mathcal{M}_{s_0}$ such that $\Phi(x^\sigma) = p^i \cdot x$. The Dieudonné-Manin classification tells us that the subspaces $(\mathcal{M}_{s_0})^{p^{-i}\Phi}$ of $\mathcal{M}_{s_0}$ are linearly independent over both $\mathbb{Q}_p$ and K, and they generate $\mathcal{M}_{s_0}$ over K. It is easy to see that the formation of the $(\mathcal{M}_{s_0})^{p^{-i}\Phi}$'s is exact in $\mathcal{M}$ and compatible with the following operations: $\otimes$, $\oplus$, $\underline{\mathrm{Hom}}$ (the internal Hom), and $\vee$ (the dual). So we obtain another fiber functor on $\mathbb{T}$ with values in the category of $\mathbb{Q}_p$-vector spaces:

(3.2.3) Definition Let $\omega_{\mathbb{Q}_p}$ be the fiber functor from $\mathbb{T}$ to the category of finite dimensional graded $\mathbb{Q}_p$-vector spaces given by

$$\omega_{\mathbb{Q}_p}(\mathcal{M}) := \oplus_i \, (\mathcal{M}_{s_0})^{p^{-i}\Phi}$$

Note that we have a canonical isomorphism

$$\omega_{\mathbb{Q}_p}(\mathcal{M}) = \oplus_i \, (\mathcal{M})_{s_0}^{p^{-i}\Phi} \xrightarrow{\sim} \oplus_i \, (\mathcal{M}_{i,s_0}/\mathcal{M}_{i-1,s_0})^{p^{-i}\Phi} \, .$$

As an example, $(\mathbb{D}(G) \otimes K)_0$ is a unit-root F-crystal, $(\mathbb{D}(G) \otimes K)_1 = \mathbb{D}(G) \otimes K$, and $\omega_{\mathbb{Q}_p}(\mathbb{D}(G) \otimes K) = U_{\mathbb{Q}_p}^\vee \oplus V_{\mathbb{Q}_p}^\vee(-1)$.

The dual of $\mathbb{D}(G)$ is canonically isomorphic to the covariant crystal attached to G in [21], and $\omega_{\mathbb{Q}_p}((\mathbb{D}(G) \otimes K)^\vee) = V_{\mathbb{Q}_p}(1) \oplus U_{\mathbb{Q}_p}$.

(3.2.4) We have seen that $\mathbb{T}(G)$ is a neutral Tannakian category over $\mathbb{Q}_p$. The general formalism of Tannakian categories tells us that $\mathbb{T}(G)$ is equivalent to the category of finite-dimensional $\mathbb{Q}_p$-linear representations of the algebraic group $\mathrm{Aut}^{\otimes}_{\mathbb{T}(G)}(\omega_{\mathbb{Q}_p})$. In particular $(\mathbb{D}(G) \otimes K)^\vee$ corresponds to a representation

$$\rho_G : \mathrm{Aut}^{\otimes}_{\mathbb{T}(G)}(\omega_{\mathbb{Q}_p}) \subset \mathrm{GL}(\omega_{\mathbb{Q}_p}((\mathbb{D}(G) \otimes K)^\vee)) = \mathrm{GL}(V(1)_{\mathbb{Q}_p} \oplus U_{\mathbb{Q}_p}) \, .$$

This homomorphism ρ_G is an embedding by construction. Moreover the subspace $V(1)_{\mathbb{Q}_p}$ of $\omega_{\mathbb{Q}_p}((\mathbb{D}(G) \otimes K)^\vee)$ is stable under ρ_G. It is easy to see that the image of $\mathrm{Aut}^{\otimes}_{\mathbb{T}(G)}(\omega_{\mathbb{Q}_p})$ in $\mathrm{GL}(U_{\mathbb{Q}_p})$ is trivial, while the image of $\mathrm{Aut}^{\otimes}_{\mathbb{T}(G)}(\omega_{\mathbb{Q}_p})$ in $\mathrm{GL}(V(1)_{\mathbb{Q}_p})$ is equal to $\mathbb{G}_{\mathrm{m}} \cdot \mathrm{Id}_{V(1)_{\mathbb{Q}_p}}$. So $\mathrm{Aut}^{\otimes}_{\mathbb{T}(G)}(\omega_{\mathbb{Q}_p})$ is a semidirect product of $\mathbb{G}_{\mathrm{m}}$ with its unipotent radical, which is commutative and naturally identified with a $\mathbb{Q}_p$-subspace of $\mathrm{Hom}_{\mathbb{Q}_p}(U_{\mathbb{Q}_p}, V(1)_{\mathbb{Q}_p})$.

(3.3) Theorem *Notation as above. Then*

$$\mathrm{Aut}^{\otimes}_{\mathbb{T}(G)}(\omega_{\mathbb{Q}_p}) \xrightarrow[\sim]{\rho_G} \left\{ \begin{pmatrix} \lambda \cdot \mathrm{Id}_{V(1)_{\mathbb{Q}_p}} & B \\ 0 & \mathrm{Id}_{U_{\mathbb{Q}_p}} \end{pmatrix} \in \mathrm{GL}(V(1)_{\mathbb{Q}_p} \oplus U_{\mathbb{Q}_p}) \;\middle|\; \begin{array}{c} \lambda \in \mathbb{Q}_p^\times \\ B \in N(G)(1)_{\mathbb{Q}_p} \end{array} \right\}$$

Proof. We only need to show that the kernel of the natural surjection $\mathrm{Aut}^{\otimes}_{\mathbb{T}(G)}(\omega_{\mathbb{Q}_p}) \to \mathbb{G}_{\mathrm{m}}$, which sends an element $\begin{pmatrix} \lambda \cdot \mathrm{Id}_{V(1)_{\mathbb{Q}_p}} & B \\ 0 & \mathrm{Id}_{U_{\mathbb{Q}_p}} \end{pmatrix}$ of $\mathrm{Aut}^{\otimes}_{\mathbb{T}(G)}(\omega)$ to λ, consists of all elements of the form $\begin{pmatrix} \mathrm{Id}_{V(1)_{\mathbb{Q}_p}} & B \\ 0 & \mathrm{Id}_{U_{\mathbb{Q}_p}} \end{pmatrix}$ with $B \in N(G)(1)_{\mathbb{Q}_p}$. Each element $\alpha \in U_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} V^{\vee}_{\mathbb{Z}_p}$ defines a homomorphism

$$\begin{aligned} h_\alpha \ : \ & \left\{ \begin{pmatrix} \lambda \cdot \mathrm{Id}_{V(1)_{\mathbb{Q}_p}} & B \\ 0 & \mathrm{Id}_{U_{\mathbb{Q}_p}} \end{pmatrix} \in \mathrm{GL}(V(1)_{\mathbb{Q}_p} \oplus U_{\mathbb{Q}_p}) \,\middle|\, \lambda \in \mathbb{Q}_p^{\times},\ B \in N(G)(1)_{\mathbb{Q}_p} \right\} \\ & \longrightarrow \left\{ \begin{pmatrix} \lambda \cdot \mathrm{Id}_{\mathbb{Q}(1)} & b \\ 0 & \mathrm{Id}_{\mathbb{Q}_p} \end{pmatrix} \in \mathrm{GL}(\mathbb{Q}(1) \oplus \mathbb{Q}_p) \,\middle|\, \lambda \in \mathbb{Q}_p^{\times},\ b \in \mathbb{Q}(1)_{\mathbb{Q}_p} \right\} \end{aligned}$$

which sends $\begin{pmatrix} \lambda \cdot \mathrm{Id}_{V(1)_{\mathbb{Q}_p}} & B \\ 0 & \mathrm{Id}_{U_{\mathbb{Q}_p}} \end{pmatrix}$ to $\begin{pmatrix} \lambda \cdot \mathrm{Id}_{\mathbb{Q}(1)} & \langle \alpha, B \rangle \\ 0 & \mathrm{Id}_{\mathbb{Q}_p} \end{pmatrix}$, where

$$\langle \ , \ \rangle \ : \ (U_{\mathbb{Q}_p} \otimes V^{\vee}_{\mathbb{Q}_p}) \times (U^{\vee}_{\mathbb{Q}_p} \otimes V(1)_{\mathbb{Q}_p}) \ \longrightarrow \ \mathbb{Q}(1)$$

denotes the natural pairing. What we need to show is that $h_\alpha \circ \rho_G(\mathrm{Aut}^{\otimes}_{\mathbb{T}(G)}(\omega))$ is equal to $\mathbb{G}_{\mathrm{m}} \cdot \mathrm{Id}_{\mathbb{Q}_p(1)} \times \mathrm{Id}_{\mathbb{Q}_p}$ if and only if $\alpha \in N(G)^{\perp}_{\mathbb{Z}_p}$. The homomorphism $h_\alpha \circ \rho_G$ corresponds to the F-isocrystal attached to the extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\mathbb{G}_{\mathrm{m}}[p^\infty]$ given by the element $\langle \alpha, q(G) \rangle \in \nu_{p^\infty}(R)$. So it suffices to show that an element of $\nu_{p^\infty}(R)$ is trivial if and only if the F-isocrysal attached to the Diedonné crystal of the corresponding extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\mathbb{G}_{\mathrm{m}}[p^\infty]$ is isomorphic to the direct sum of $\mathbb{D}(\mathbb{G}_{\mathrm{m}}[p^\infty]) \otimes K$ with $\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p) \otimes K$. This can be checked directly using the formula in [8, 4.3.2], or by the fact that the crystalline Dieudonné functor on the category of p-divisible groups over S is fully faithful, see [8]. ∎

§4. Global p-adic monodromy

(4.1) In this section k is an algebraically closed field of characteristic $p > 0$, and S is an irreducible normal scheme of finite type over k. Let G be a ordinary BT over S, so we have a short exact sequence $1 \to T \to G \to E \to 1$ of BT-groups over S, where T is multiplicative and E is étale. Let $U := \mathrm{T}_p(E)$, $V := X_*(T)$; each is a smooth sheaf of free $\mathbb{Z}_p$-modules of finite rank on $S_{\text{ét}}$.

(4.1.1) For each geometric point $\bar{s}$ of S. Denote by $G_{\bar{s}}$ (resp. $T_{\bar{s}}$, resp. $E_{\bar{s}}$) the fiber of G (resp. T, resp. E) over $\bar{s}$. Denote by $U_{\bar{s}}$ the fiber of U over $\bar{s}$, and let $U_{\bar{s},\mathbb{Q}_p} = U_{\bar{s}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$; similarly for other smooth sheaves of $\mathbb{Z}_p$-modules on $S_{\text{ét}}$.

(4.2) The canonical coordinates of G is a homomorphism

$$q(G) \ : U \otimes_{\mathbb{Z}_p} V^{\vee} = \mathrm{T}_p(E) \otimes_{\mathbb{Z}_p} X^*(T) \ \longrightarrow \ \nu_{p^\infty, S}$$

of sheaves on $S_{\text{ét}}$. Let

$$N(G)^{\perp} := \operatorname{Ker}(q(G)) \subseteq U \otimes V^{\vee}$$

(4.2.1) Proposition (i) *The sheaf $N(G)^{\perp}$ is a smooth subsheaf of $U \otimes V^{\vee}$, and $U \otimes V^{\vee}/N(G)^{\perp}$ is a torsion-free smooth sheaf of free $\mathbb{Z}_p$-modules on $S_{\text{ét}}$.*

(ii) *Let $N(G) \subseteq U^{\vee} \otimes_{\mathbb{Z}_p} V$ be the orthogonal complement of $N(G)^{\perp}$. Then $N(G)$ is a smooth sheaf of free $\mathbb{Z}_p$-modules on $S_{\text{ét}}$.*

Proof. We only have to prove (i), since (ii) follows from (i). Lemma 2.1.4 and Lemma 2.1.5 imply that $N(G)^{\vee}$ is a smooth sheaf of $\mathbb{Z}_p$-modules on $S_{\text{ét}}$. To show that $U \otimes V^{\vee}/N(G)^{\perp}$ is torsion-free, it suffices to show that $\nu_{p^{\infty},S}(\mathcal{O}^{\text{sh}}_{S,x})$ is torsion free for every closed point $x \in S$, where $\mathcal{O}^{\text{sh}}_{S,x}$ denotes the strict henselisation of $\mathcal{O}_{S,x}$. By Lemma 2.1.6, it suffices to show that $\nu_{p^{\infty},S}(\mathcal{O}^{\wedge}_{S,x})$ is torsion free, where $\mathcal{O}^{\wedge}_{S,x}$ is the completion of $\mathcal{O}_{S,x}$. We know that $\nu_{p^{\infty},S}(\mathcal{O}^{\wedge}_{S,x})$ is isomorphic to the subgroup $1 + \mathfrak{m}_x\mathcal{O}^{\wedge}_{S,x}$ of $(\mathcal{O}^{\wedge}_{S,x})^{\times}$. For any $a \in \mathfrak{m}_x\mathcal{O}^{\wedge}_{S,x}$, if $(1+a)^{p^m} = 1$ for some $m \in \mathbb{N}$, then $1 + a = 1$ because $\mathcal{O}^{\wedge}_{S,x}$ is an integral domain. ∎

(4.3) Let $W = W(k)$ be the ring of p-adic Witt vectors with components in k. Let $K = K(k)$ be the fraction field of $W(k)$. Denote by F-Isoc(S/K) the $\otimes$-category of convergent F-isocrystals over S/K as in §3. The contravariant Diedonné crystal $\mathbb{D}(G)$ attached to G gives rise to an object $\mathbb{D}(G) \otimes K$ in F-Isoc(S/K). Let $\mathbb{T}(G)$ be the $\otimes$-subcategory generated by $\mathbb{D}(G) \otimes K$. As in §3, the $\otimes$-category $\mathbb{T}(G)$ has a functorial slope filtration, compatible with the $\otimes$-structure. The construction/definition of the fiber functor in 3.2.3 can be generalized to the present situation as well. In particular, every object $\mathcal{M}$ in $\mathbb{T}(G)$ has a unique finite increasing filtration such that each associate graded piece $\mathcal{M}_i/\mathcal{M}_{i-1}$ is a the $(-i)^{\text{th}}$ Tate twist of a unit-root isocrystal.

(4.3.1) Definition Let $\bar{s}$ be a geometric point of S. Define a fiber functor $\omega_{\bar{s},\mathbb{Q}_p}$ from $\mathbb{T}(G)$ to the $\otimes$-category of finite dimensional graded $\mathbb{Q}_p$-vector spaces by

$$\omega_{\bar{s},\mathbb{Q}_p}(\mathcal{M}) := \oplus_i \, (\mathcal{M}_{\bar{s}})^{p^{-i}\Phi}$$

for each object $\mathcal{M}$ in $\mathbb{T}(G)$. Notice that there is a canonical isomorphism

$$\omega_{\bar{s},\mathbb{Q}_p}(\mathcal{M}) \xrightarrow{\sim} \oplus_i \, (\mathcal{M}_{i,s_0}/\mathcal{M}_{i-1,\bar{s}})^{p^{-i}\Phi} \,.$$

(4.3.2) Notation as above. The object $(\mathbb{D}(G)\otimes K)^{\vee}$ of $\mathbb{T}(G)$ corresponds to a homomorphism

$$\rho_{G,\bar{s}} : \operatorname{Aut}_{\mathbb{T}(G)}(\omega_{\bar{s},\mathbb{Q}_p}) \longrightarrow \operatorname{GL}(V(1)_{\bar{s},\mathbb{Q}_p} \oplus U_{\bar{s},\mathbb{Q}_p}) \,,$$

which is an embedding. We will identify the monodromy group $\operatorname{Aut}_{\mathbb{T}(G)}(\omega_{\bar{s},\mathbb{Q}_p})$ with its image under $\rho_{G,\bar{s}}$.

(4.3.3) Let $\mathbb{T}'(G)$ be the $\otimes$-subcategory of the $\mathbb{T}(G)$ generated by the direct sum

$$(\mathbb{D}(G^{\text{mult}}) \otimes K) \oplus (\mathbb{D}(G^{\text{ét}}) \otimes K)\,.$$

The generator $(\mathbb{D}(G^{\text{mult}}) \otimes K) \oplus (\mathbb{D}(G^{\text{ét}}) \otimes K)$ of $\mathbb{T}'(G)$ corresponds to an embedding

$$\rho'_{G,\bar{s}} : \mathrm{Aut}_{\mathbb{T}'(G)}(\omega_{\bar{s},\mathbb{Q}_p}) \hookrightarrow \mathrm{GL}(V(1)_{\bar{s},\mathbb{Q}_p}) \times \mathrm{GL}(U_{\bar{s},\mathbb{Q}_p})\,,$$

and we identify $\mathrm{Aut}_{\mathbb{T}'(G)}(\omega_{\bar{s},\mathbb{Q}_p})$ with its image in $\mathrm{GL}(V(1)_{\bar{s},\mathbb{Q}_p}) \times \mathrm{GL}(U_{\bar{s},\mathbb{Q}_p})$ under $\rho'_{G,\bar{s}}$.

(4.3.4) Let $\mathbb{T}''(G)$ be the $\otimes$-subcategory in the $\otimes$-category of smooth $\mathbb{Q}_p$-sheaves on $S_{\text{ét}}$ generated by $V_{\mathbb{Q}_p} \oplus U_{\mathbb{Q}_p}$. For each geometric point $\bar{s}$ of S, the functor

$$\bar{s}^* : \mathbb{T}''(G) \;\to \mathrm{Vec}_{\mathbb{Q}_p}$$

defined by "pull-back by $\bar{s}$" is a fiber functor of $\mathbb{T}'(G)$ with values in the category of finite dimensional $\mathbb{Q}_p$-vector spaces. The object $V_{\mathbb{Q}_p} \oplus U_{\mathbb{Q}_p}$ of $\mathbb{T}'(G)$ defines an embedding

$$\rho''_{G,\bar{s}} : \mathrm{Aut}_{\mathbb{T}''(G)}(\bar{s}^*) \hookrightarrow \mathrm{GL}(V_{\bar{s},\mathbb{Q}_p}) \times \mathrm{GL}(U_{\bar{s},\mathbb{Q}_p})\,,$$

and we identify $\mathrm{Aut}_{\mathbb{T}''(G)}(\bar{s}^*)$ with its image in $\mathrm{GL}(V_{\bar{s},\mathbb{Q}_p}) \times \mathrm{GL}(U_{\bar{s},\mathbb{Q}_p})$ under ρ''_G.

(4.3.5) We have canonical isomorphisms

$$\mathrm{GL}(\omega_{\bar{s},\mathbb{Q}_p}(G^{\text{mult}})) \cong \mathrm{GL}(V_{\bar{s},\mathbb{Q}_p})\,, \qquad \mathrm{GL}(\omega_{\bar{s},\mathbb{Q}_p}(G^{\text{ét}})) \cong \mathrm{GL}(U_{\bar{s},\mathbb{Q}_p})\,.$$

The second isomorphism comes from $\omega_{\bar{s},\mathbb{Q}_p}(\mathbb{D}(G^{\text{ét}})^\vee) = U_{\bar{s},\mathbb{Q}_p}$. The first isomorphism comes from $\omega_{\bar{s},\mathbb{Q}_p}(\mathbb{D}(G^{\text{mult}})^\vee) = V(1)_{\bar{s},\mathbb{Q}_p}$, using the standard notation on Tate twists.

(4.3.6) Lemma *Notation as above. We have*

$$\rho'_{G,\bar{s}}(\mathrm{Aut}_{\mathbb{T}'(G)}(\bar{s}^*)) = \{(\lambda \cdot A, D) \in \mathrm{GL}(V(1)_{\bar{s},\mathbb{Q}_p}) \times \mathrm{GL}(U_{\bar{s},\mathbb{Q}_p}) \,|\, (A, D) \in \rho''_{G,\bar{s}}(\mathrm{Aut}_{\mathbb{T}''(G)}(\bar{s}^*))\}$$

(4.4) Theorem *Notation as above. Let $\bar{s}$ be a geometric point of S. Then we have a canonical isomorphism*

$$\mathrm{Aut}_{\mathbb{T}(G)}(\omega_{\bar{s},\mathbb{Q}_p}) \xrightarrow{\sim} \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \;\middle|\; \begin{array}{c} B \in N(G)(1)_{\bar{s}} \\ (A, D) \in \mathrm{Aut}_{\mathbb{T}'(G)}(\bar{s}^*) \subseteq \mathrm{GL}(V(1)_{\bar{s},\mathbb{Q}_p}) \times \mathrm{GL}(U_{\bar{s},\mathbb{Q}_p}) \end{array} \right\}$$

induced by ρ'_G.

Proof. By Theorem 3.3, the image of ρ'_G clearly contains the group described at the right-hand-side of the displayed formula in the statement; denote that $\mathbb{Q}$-group by $\Gamma\mathrm{al}_{\bar{s}}$. So we obtain a $\otimes$-functor $\psi : \mathcal{T}(G) \;\longrightarrow \mathrm{Rep}(\Gamma\mathrm{al}_{\bar{s}})$. We want to show that ψ is an equivalence of categories. It suffices to prove this assertion in the case when $\bar{s}$ is a geometric generic point $\bar{\eta}$ of S;

The group $\Gamma\mathrm{al}_{\bar\eta}$ is a $\mathbb{Q}_p$-algebraic subgroup of $\mathrm{GL}(V(1)_{\bar\eta,\mathbb{Q}_p} \oplus U_{\bar\eta,\mathbb{Q}_p})$. Let $\Gamma\mathrm{al}_{\bar\eta,\mathbb{Z}_p}$ be the intersection of $\Gamma\mathrm{al}_{\bar\eta}$ with $\mathrm{GL}(V(1)_{\bar\eta,\mathbb{Z}_p} \oplus U_{\bar\eta,\mathbb{Z}_p}$; it is a $\mathbb{Z}_p$-model of $\Gamma\mathrm{al}_{\bar\eta}$). Let $\Gamma\mathrm{al}'_{\bar\eta}$ be the image of $\Gamma\mathrm{al}_{\bar\eta}$ in $GL(V(1)_{\bar\eta,\mathbb{Q}_p}) \times \mathrm{GL}(U_{\bar\eta,\mathbb{Q}_p})$; up to a factor $\mathbb{G}_\mathrm{m}$ it is equal to the $\mathbb{Q}_p$-Zariski closure of the image of $\mathrm{Gal}(\bar\eta/\eta)$ in $GL(V(1)_{\bar\eta,\mathbb{Q}_p}) \times \mathrm{GL}(U_{\bar\eta,\mathbb{Q}_p})$. Let $\Gamma\mathrm{al}'_{\bar\eta,\mathbb{Z}_p}$ be the image of $\Gamma\mathrm{al}_{\bar\eta,\mathbb{Z}_p}$ in $GL(V(1)_{\bar\eta,\mathbb{Z}_p}) \times \mathrm{GL}(U_{\bar\eta,\mathbb{Z}_p})$. So $\Gamma\mathrm{al}_{\bar\eta,\mathbb{Z}_p}$ is a semi-direct product of $\Gamma\mathrm{al}'_{\bar\eta,\mathbb{Z}_p}$ with $N(G)(1)_{\bar\eta}$.

(4.4.1) Lemma *Let $\mathcal{H}$ be a tensor construction from $\mathbb{D}(G)$, $\mathbb{D}(G)^\vee$ and the Tate twists. and let $H := \psi(\mathcal{H})$, a representation of $\Gamma\mathrm{al}_{\bar\eta}$ on $\omega_{\bar\eta,\mathbb{Q}_p}(\mathcal{H})$. Suppose that M is a $\mathbb{Q}_p$-subspace of H which is stable under $\Gamma\mathrm{al}_{\bar\eta}$. Then there exists a unique subobject $\mathcal{M}$ of $\mathcal{H}$ such that $\psi(\mathcal{M}) = M$.*

Proof. The uniqueness assertion follows trivially from the fact that ψ is exact and faithful. Let n be a positive integer. Choose a finite étale covering ${}^nS$ of S to trivialize both $U_{\mathbb{Z}_p}/p^nU_{\mathbb{Z}_p}$ and $V_{\mathbb{Z}_p}/p^nV_{\mathbb{Z}_p}$. Then we cover ${}^nS$ with affine opens ${}^nT_j$ such that $q_n(G[p^n]_{/{}^nT_j})$ comes from a homomorphism

$$q_{n,j} \,:\, \left((U_{\mathbb{Z}_p}/p^nU_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} (V_{\mathbb{Z}_p}/p^nV_{\mathbb{Z}_p})^\vee\right)({}^nT_j) \longrightarrow \Gamma({}^nT_j, \mathcal{O}^\times_{{}^nT_j}) \Big/ (\Gamma({}^nT_j, \mathcal{O}^\times_{{}^nT_j}))^{p^n} \ .$$

Over each ${}^nT_j$ we can use the explicit formula for the crystal attached to $G_{{}^nT_j/W_n}$ in [8, 4.3.2].

By definition, the isocrystal $\mathcal{H}$ over S comes from a crystal $\mathcal{H}_\infty$ on S/W. Moreover a suitable Tate twist $\mathcal{H}_\infty(a)$ of $\mathcal{H}_\infty$ is an F-crystal on S/W. Apply the tensor construction used to obtain H from $V(1)_{\bar\eta,\mathbb{Q}_p} \oplus U_{\bar\eta,\mathbb{Q}_p}$, to $V(1)_{\bar\eta,\mathbb{Z}_p} \oplus U_{\bar\eta,\mathbb{Z}_p}$, we obtain a $\mathbb{Z}_p$-lattice $H_{\mathbb{Z}_p}$ in H. Let $M_{\mathbb{Z}_p} = M \cap H_{\mathbb{Z}_p}$. Clearly $M_{\mathbb{Z}_p}$ is stable under the natural action of $\Gamma\mathrm{al}_{\bar\eta,\mathbb{Z}_p}$.

For each n, let $H_n := H_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} (\mathbb{Z}/p^n\mathbb{Z})$ and $M_n := M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} (\mathbb{Z}/p^n\mathbb{Z})$. The group $\Gamma\mathrm{al}_{\bar\eta,\mathbb{Z}_p}$ operates naturally on H_n and M_n. Over each ${}^nT_j$, the Dieudonné crystal $\mathbb{D}(G)^\vee$ can be described explicitly in terms of $N(G)$, as a differential equation on

$$(V(1)_{\bar\eta,\mathbb{Z}_p} \oplus U_{\bar\eta,\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} (\mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathcal{O}_{\widetilde{{}^nT_j}} \ ,$$

where $\widetilde{{}^nT_j}$ is a $W_n(k)$-lifting of ${}^nT_j$; see [8, 4.3.2]. Using the tensor construction involved, one obtains a differential equation on $H_n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathcal{O}_{\widetilde{{}^nT_j}}$ with nilpotent connection; denote the corresponding crystal on ${}^nT_j/W_n(k)$ by $\mathcal{H}_{n,j}$. Moreover $M_n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathcal{O}_{\widetilde{{}^nT_j}}$ is stable under the connection; denote the corresponding crystal on ${}^nT_j/W_n(k)$ by $\mathcal{M}_{n,j}$. The $\Gamma\mathrm{al}_{\bar\eta,Zp}$-invariance of M_n implies that the crystals $\mathcal{M}_{n,j}$ descends to a crystal $\mathcal{M}_n$ on $S/W_n(k)$. It is not difficult to see that $\mathcal{M}_n$'s are compatible as n varies, giving a crystal $\mathcal{M}_\infty$ on $S/W(k)$. Let $\mathcal{M}$ be the isocrystal attached to $\mathcal{M}_\infty$. It is easy to see from the construction of $\mathcal{M}_\infty$ that $\mathcal{M}$ is a sub-isocrystal of $\mathcal{H}$, and $\mathcal{M}$ inherits a Frobenius structure from that of $\mathcal{H}$. Moreover $\psi(\mathcal{M}) = M$. This proves the existence assertion. ∎

(4.4.2) Lemma *Let $\mathcal{H}$ be a tensor construction from $\mathbb{D}(G)$, $\mathbb{D}(G)^\vee$ and the Tate twists. and let $H := \psi(\mathcal{H})$, a representation of $\Gamma\mathrm{al}_{\bar\eta}$ on $\omega_{\bar\eta,\mathbb{Q}_p}(\mathcal{H})$. Let $M_1 \subset M_2$ be $\mathbb{Q}_p$-subspaces of H which is stable under $\Gamma\mathrm{al}_{\bar\eta}$, and let $M = M_2/M_1$.*

(i) *Then there exists a subquotient $\mathcal{M}$ of $\mathcal{H}$ such that $\psi(\mathcal{M}) = M$.*

(ii) *If v is an element of M which is fixed by $\Gamma\mathrm{al}_{\bar{\eta}}$, then there exists a horizontal global section $\tilde{v}$ of $\mathcal{M}$ fixed by the Frobenius, such that $\psi(\tilde{v}) = v$.*

PROOF. We use the notation in the proof of Lemma 4.4.1. Apply the construction in 4.4.1 to M_i, $i = 1, 2$, we obtain $M_{1,n} \subset M_{2,n} \subset H_n$, and crystals $\mathcal{M}_{i,n}$ on $S/W_n(k)$. As n varies, the crystals $\mathcal{M}_{i,n}$ gives crystals $\mathcal{M}_{i,\infty} \subset \mathcal{M}_{i,\infty} \subset \mathcal{H}_\infty$ on $S/W(k)$. Let $\mathcal{M}_i$ be the isocrystal attached to $\mathcal{M}_\infty$, with Frobenius structure induced by that of $\mathcal{H}$. Then we have $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{H}$ as F-isocrystals. Let $\mathcal{M} = \mathcal{M}_2/\mathcal{M}_1$. Clearly $\psi(\mathcal{M}) = M$. This proves (i)

Multiplying the $\Gamma\mathrm{al}_{\bar{\eta}}$-fixed vector v by a suitable power of p, we may and do assume that v is the image of an element $w \in M_{2,\mathbb{Z}_p}$. Let $v_{2,n}$ be the image of w in $M_{2,n}/M_{1,n}$. Using [8, 4.3.2] and the construction of $M_{i,n}$, $i = 1, 2$, it is easy to see that $v_{2,n}$ gives a section of $\mathcal{M}_{2,n}/\mathcal{M}_{1,n}$ over $S/W_n(k)$. The sections $v_{2,n}$ are compatible as n varies, giving a section $\tilde{v}$ of $\mathcal{M}$ over $S/W(k)$. Going through the same argument again and consider the Frobenius structure, one sees that the horizontal section $\tilde{v}$ is fixed by the Frobenius: $\Phi(F^*(\tilde{v})) = \tilde{(v)}$. This proves (ii) ∎

END OF THE PROOF OF PROP. 4.4. Lemma 4.4.2 implies that ψ is fully faithful, and moreover for every object $\mathcal{W}$ of $\mathbb{T}(G)$ and every subobject M of $\psi(\mathcal{W})$, there exists a subobject $\mathcal{M}$ of $\mathcal{W}$ such that $\psi(\mathcal{M}) = M$. Indeed Lemma 4.4.2 shows that this is so for subquotient of tensor constructions using $\mathbb{D}(G), \mathbb{D}(G)^\vee$ and Tate twists. Therefore it hold for every object of $\mathbb{T}(G)$. ∎

(4.4.3) Remark If one prefers, in the proof of Theorem 4.4 one may assume that S is smooth over k. Indeed by de Jong's theorem on alteration in [9], there exists a smooth k-scheme $\widetilde{S}$ and a proper surjective morphism $f : \widetilde{S} \to S$ which is finite étale over a Zariski dense open subscheme of S. One then applies the theorem [25, 4.6] of Ogus, which says that the category of convergent isocrystals satisfy descent for proper surjective morphisms of k-schemes of finite type.

§5. Tate-linear subvarieties

(5.1) Definition Let k be an algebraically closed field of characteristic p. Let $g \geq 1$ be a positive integer. We fix an auxiliary positive integer Nsuch that $(N, p) = 1$, $N \geq 3$. Denote by $\mathcal{A}_{g,1,N_{/k}}$ the moduli space of g-dimensional principally polarized abelian varieties over k with level-N-structure; often we will simply write $\mathcal{A}_{g_{/k}}$, suppressing other subscripts. Suppose that Z is an irreducible closed subvariety of $\mathcal{A}^{\mathrm{ord}}_{g,1,N_{/k}}$ over k, where $\mathcal{A}^{\mathrm{ord}}_{g,1,N_{/k}}$ denotes the ordinary locus in $\mathcal{A}_{g,1,N_{/k}}$.

(i) Let z be a closed point of Z. We say that Z is *Tate-linear at z* if the formal completion of Z at z is a formal subtorus of the Serre-Tate formal torus $\mathcal{A}^{/z}_{g,1,N_{/k}}$.

(ii) We say that Z is *Tate-linear* if it is Tate-linear at every closed point of Z.

(iii) Denote by $f : Y \to Z$ the normalization of Z. We say that Z is *weakly Tate-linear* if for every closed point y of Y, the morphism $Y^{/y} \to \mathcal{A}_{g,1,N}^{/f(y)}{}_{/k}$ induced by f is an isomorphism from $Y^{/y}$ to a formal subtorus of the Serre-Tate formal torus $\mathcal{A}_{g,1,N}^{/f(y)}{}_{/k}$.

(5.1.1) Remark (1) Clearly if Z is Tate-linear at a closed point z_0, then Z is smooth over k at z_0. Hence if Z is Tate-linear, then it is smooth over k. Singularities are allowed for weakly Tate-linear subvarieties; they "come from self-intersection". A weakly Tate-linear subvariety Z of $\mathcal{A}_{g,1,N}^{\mathrm{ord}}{}_{/k}$ is Tate-linear at all closed points of the smooth locus of Z.

(2) Although we defined Tate-linearity only in the equicharacteristic p situation, the definition makes sense for closed subschemes of $\mathcal{A}_{g,1,N}{}^{\mathrm{ord}}_{/W(k)}$ over $W(k)$, where $\mathcal{A}_{g,1,N}{}^{\mathrm{ord}}_{/W(k)}$ denotes the complement of the non-ordinary locus of the closed fiber in the moduli space $\mathcal{A}_{g,1,N}{}_{/W(k)}$ over $W(k)$. One can also weaken the requirement so that each formal completion is the translation of a formal subtorus of the Serre-Tate torus by a torsion point, over a finite flat extension of $W(k)$. The characteristic 0 fiber of a Tate-linear subvariety is a subvariety of $\mathcal{A}_{g,1,N}{}_{/B(k)}$ of Hodge type according to a theorem of Moonen [22], where $B(k)$ is the fraction field of $W(k)$. See 5.4.2 for more discussions.

(5.2) Proposition *Let Z be an irreducible closed subscheme of $\mathcal{A}_g^{\mathrm{ord}}{}_{/k}$. Denote by $q(A/Z)$ the canonical coordinates of the ordinary BT-group $A[p^\infty] \to Z$ over Z; it is a homomorphism from the sheaf* $\mathrm{T}_p(A[p^\infty]_Z^{\text{ét}}) \otimes X^*(A[p^\infty]_Z^{\mathrm{mult}})$ *on $Z_{\text{ét}}$ to $\nu_{p^\infty,Z}$. Denote by $N(A/Z)^\perp$ the kernel of $q(A/Z)$; it is a subsheaf of* $\mathrm{T}_p(A[p^\infty]_Z^{\text{ét}}) \otimes_{\mathbb{Z}_p} X^*(A[p^\infty]_Z^{\mathrm{mult}})$*. Let $N(A/Z)$ be the orthogonal complement of $N(A/Z)^\perp$, a smooth $\mathbb{Z}_p$-sheaf on $Z_{\text{ét}}$.*

(i) *For any closed point z of Z, Z is Tate-linear at z if and only if* $\mathrm{rank}(N(A/Z)_z) = \dim(Z)$*.*

(ii) *The subvariety Z is Tate-linear if and only if $N(A/Z)$ is a smooth cotorsion-free sheaf of $\mathbb{Z}_p$-submodules of* $\mathrm{T}_p(A[p^\infty]_Z^{\text{ét}})^\vee \otimes X_*(A[p^\infty]_Z^{\mathrm{mult}})$ *such that* $\mathrm{rank}(N(A/Z)) = \dim(Z)$*.*

Proof. Let z be a closed point of Z. Then the smallest formal subtorus of the Serre-Tate formal torus $\mathcal{A}_g^{/z}$ containing the formal completion $Z^{/z}$ of Z at z is canonically isomorphic to $N(G)_z \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}_{\mathrm{m}}}$. Hence Z is Tate-linear at z if and only if $\mathrm{rank}(N(A/Z)_z) = \dim(Z)$; (i) is proved.

If Z is Tate-linear, then Z is smooth, hence $N(A/Z)^\perp$ is a smooth cotorsion-free sheaf of $\mathbb{Z}_p$-submodules by 4.2.1. The rest of (ii) are immediate from (i). ∎

(5.3) Proposition *Let Z be a closed irreducible subscheme of $\mathcal{A}_{g,1,N}^{\mathrm{ord}}{}_{/k}$ over an algebraically closed field k of characteristic p. Suppose Z is Tate-linear at a closed point z of Z. Then Z is a weakly Tate-linear subvariety of $\mathcal{A}_g^{\mathrm{o}}$.*

PROOF. We may and do assume that k is algebraically closed. Suppose that Z is Tate-linear at a closed point z. Consider

$$\mathrm{T}_p(A_z[p^\infty]^{\text{ét}}) \otimes X^*(A_z[p^\infty]^{\text{mult}}) \xrightarrow{q(A/Z)_z} \nu_{p^\infty}(\mathcal{O}_{Z,z}) \xrightarrow{\beta} \nu_{p^\infty}(\mathcal{O}^\wedge_{Z,z}),$$

where $\mathcal{O}^\wedge_{Z,z}$ denotes the completion of the local ring $\mathcal{O}_{Z,z}$, and β is the natural map from $\nu_{p^\infty}(\mathcal{O}_{Z,z})$ to $\nu_{p^\infty}(\mathcal{O}^\wedge_{Z,z}) = 1 + \mathcal{O}^\wedge_{Z,z}$. The assumption implies that $\mathrm{Coimage}(\beta \circ q(A/Z)_z)$ is a free $\mathbb{Z}_p$-module of rank $\dim(Z)$. Let $f : Y \to Z$ be the normalization of Z. From 2.1.4–2.1.6, we deduce that $N(A/Y)^\perp$ is a cotorsion-free smooth sheaf of $\mathbb{Z}_p$-submodules of $\mathrm{T}_p(A[p^\infty]^{\text{ét}}) \otimes X^*(A[p^\infty]^{\text{mult}}_Y)$. Therefore the rank of $N(A/Y)$ can be evaluated at the point z of Y. We conclude that the rank of $N(A/Y)$ is equal to $\dim(Z) = \dim(Y)$. At each closed point y of Y, $q(A/Y)_y$ gives the Serre-Tate coordinates when one passes to the formal completion of Y at y. The fact that the rank of $N(A/Y)$ is equal to $\dim(Y)$ shows that the natural map $Y^{/y} \to \mathcal{A}^{/f(y)}_{g,1,N}$ factors through the formal subtorus of $\mathcal{A}^{/f(y)}_{g,1,N}$ whose cocharacter group is $N(A/Y)_y$. Therefore the schematic image of $Y^{/y}$ in $\mathcal{A}^{/f(y)}_{g,1,N}$ is equal to the formal subtorus with cocharacter group $N(A/Y)_y$. Let z_1 be a closed point of Z and let $\{y_1, \ldots, y_b\}$ be the points of Y above z_1. The above consideration tells us that the reduced formal scheme $Z^{/z_1}$ is the schematic union of a finite number of formal subtori of the same dimension. Hence the formal completion of the normalization Y of Z along $\{y_1, \ldots, y_b\}$ is equal to the disjoint union of these formal subtori. This proves that Z is a weakly Tate-linear subvariety. ∎

(5.3.1) Remark It seems natural to expect that the adjective "weakly" can be eliminated from the statement of 5.3. In other words, it is plausible that every weakly Tate-linear subvariety of $\mathcal{A}^{\text{ord}}_{g,1,N/k}$ is Tate-linear, assuming that $N \geq 3$, $(N,p) = 1$.

(5.4) Proposition *Consider the moduli scheme $\mathcal{A}^{\text{ord}}_{g,1,n}$ of ordinary principally polarized abelian varieties over over $\mathbb{Z}_{(p)}$, and write it as $\mathcal{A}^{\circ}_g$ for short. Let $\Delta_{\mathcal{A}^\circ_g}$ be the diagonally embedded $\mathcal{A}^{\text{ord}}_{g,1,N}$ in $(\mathcal{A}^\circ_g)^2 = \mathcal{A}^{\text{ord}}_{g,1,n} \times_{\operatorname{Spec}\mathbb{Z}_{(p)}} \mathcal{A}^{\text{ord}}_{g,1,n}$. Denote by $(\mathcal{A}^\circ_g)^{2,\wedge}$ the formal completion of $(\mathcal{A}^\circ_g)^2$ along $\Delta_{\mathcal{A}^\circ_g}$. Then $(\mathcal{A}^\circ_g)^{2,\wedge}$, considered as a formal scheme over $\mathcal{A}^\circ_g$ via the first projection, has a natural structure as a formal torus over $\mathcal{A}^\circ_g$. Restricting this formal torus to a closed point gives the usual Serre-Tate coordinates at the point. More precisely,*

$$(\mathcal{A}^\circ_g)^{2,\wedge} \cong \underline{\mathrm{Hom}}_{\mathbb{Z}_p}\left(\mathrm{T}_p(A[p^\infty]^{\text{ét}}), A^\wedge\right).$$

Here A denotes the universal abelian scheme over $\mathcal{A}^\circ_g$, $\mathrm{T}_p(\mathbb{A}[p^\infty]^{\text{ét}})$ denotes the Tate module attached to the maximal étale quotient of the BT-group attached to A, and $A^\wedge$ denotes the formal completion of A, a formal torus over $\mathcal{A}^\circ_g$. The character group of the formal torus $(\mathcal{A}^\circ_g)^(2,\wedge)$ is $\mathrm{S}^2(\mathrm{T}_p(A[p^\infty]^{\text{ét}}))$, the second symmetric product over $\mathbb{Z}_p$ of the free $\mathbb{Z}_p$-module $\mathrm{T}_p(A[p^\infty]^{\text{ét}})$.

Proof. All we have to do is to construct a morphism from the formal torus

$$\underline{\mathrm{Hom}}_{\mathbb{Z}_p}\left(\mathrm{T}_p(A[p^\infty]^{\text{ét}}), A^\wedge\right)$$

over $\mathcal{A}_g^{\mathrm{o}}$ to $(\mathcal{A}_g^{\mathrm{o}})^{2,\wedge}$, and verify that it is an isomorphism. The "push-out" construction used in the proof of [18, Theorem 2.1] gives us a Barsotti-Tate group over $\underline{\mathrm{Hom}}_{\mathbb{Z}_p}\left(\mathrm{T}_p(A[p^\infty]^{\text{ét}}), A^\wedge\right)$. By the Serre-Tate theorem (see [18, Theorem 1.2.1]) we get an abelian scheme over the scheme $\underline{\mathrm{Hom}}_{\mathbb{Z}_p}\left(\mathrm{T}_p(A[p^\infty]^{\text{ét}}), A^\wedge\right)$. This gives the arrow

$$\alpha' : \underline{\mathrm{Hom}}_{\mathbb{Z}_p}\left(\mathrm{T}_p(A[p^\infty]^{\text{ét}}), A^\wedge\right) \to (\mathcal{A}_g^{\mathrm{o}})^{2,\wedge}$$

we need. To see that this is isomorphism, one first observes that α induces an isomorphism when one passes to the formal completion at a closed point of $\mathcal{A}_g^{\mathrm{o}}$. Since α is a morphism over $\mathcal{A}_g^{\mathrm{o}}$, α is an isomorphism. It is also possible to construct the inverse arrow of α in a step-by-step fashion using standard arguments in deformation theory. This is left to the reader since the argument involved does not provide any extra insight. ∎

(5.4.1) Remark (i) Prop. 5.4, as a generalization of the Serre-Tate moduli space, was already known to S. Mochizuki in 1994, according to J. de Jong.

(ii) One can use Prop. 5.4 to give another proof of Prop. 5.3, at least in the case when Z is normal. We leave this point to the interested reader.

(iii) Prop. 5.4 can also be formulated for an arbitrary ordinary principally polarized abelian scheme $A_{/S}$ of relative dimension g with level-N-structure over $\mathbb{Z}_{(p)}$: The ordinary abelian scheme $A_{/S}$ gives rise to a classifying map $f_A : S \to \mathcal{A}_g^{\mathrm{o}}$. The graph Γ_f of f_A is a closed subscheme of $S \times_{\operatorname{Spec}\mathbb{Z}_{(p)}} \mathcal{A}_g^{\mathrm{o}}$. Then the completion of $S \times_{\operatorname{Spec}\mathbb{Z}_{(p)}} \mathcal{A}_g^{\mathrm{o}}$ along Γ_f has a natural structure of a formal torus. Of course it is just the pull-back of the formal torus in Proposition 5.4.

(5.4.2) Remark Although we formulated proposition 5.3 only over a field of characteristic p, the same proof is valid in the mixed characteristic case as well. Much stronger results are available in the mixed characteristic case: A theorem in [22], published in [23], [24], says that if X is a closed subscheme of $\mathcal{A}_{g,1,N}$ faithfully flat over $W(k)$, and there exists a closed point x_0 of the fiber of X over k, such that the formal completion $X^{/x_0}$ of X at x_0, as a formal subscheme of the Serre-Tate formal torus $\mathcal{A}_{g,1,N}^{/x_0}$ over $W(k)$, is equal to the translation by a torsion point of a formal subtorus, then the characteristic 0 fiber of X is a subvariety of Hodge type in $\mathcal{A}_{g,1,N}$.

(5.5) Proposition *Let Z be an irreducible closed subvariety of $\mathcal{A}^{\mathrm{ord}}_{g,1,N_{/k}}$ over k. Assume that Z is a Tate-linear subvariety. Denote by ${\mathcal{A}^{\mathrm{ord}}_{g,1,N}}^{\wedge}_{/W(k)}$ the p-adic completion of $\mathcal{A}^{\mathrm{ord}}_{g,1,N_{/W(k)}} \to$ $\operatorname{Spec} W(k)$. Then there exists a unique closed formal subscheme $\tilde{Z}$ of ${\mathcal{A}^{\mathrm{ord}}_{g,1,N}}^{\wedge}_{/W(k)}$, which is formally smooth over $W(k)$, and such that $\tilde{Z} \times_{\operatorname{Spf} W(k)} \operatorname{Spec} k = Z$, and the formal completion of $\tilde{Z}$ at every closed point z of Z is a formal subtorus of the Serre-Tate formal torus $(\mathcal{A}_{g,1,N_{/W(k)}})^{/z}$ over $W(k)$.*

Before giving the proof of 5.5, it will be useful to record a lemma relating the standard deformation theory to the canonical coordinates. Maximal generality is not attempted.

(5.5.1) Lemma *Let $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$ be affine schemes. Let $\iota : S \hookrightarrow S'$ be a nilpotent immersion, defined by an ideal $I \subset R'$. Let $S_0 \hookrightarrow S'$ be closed immersion defined by an ideal $K \subset R'$. Let G be a Barsotti-Tate group over S, and let $G_0 = G \times_S S_0$. Assume that p is nilpotent in $R_0 := R'/K$, and $I \cdot K = (0)$. Then*

(i) *There exists a BT-group $G' \to S'$ which extends $G \to S$.*

(ii) *The only S'-automorphism of $G'_{/S'}$ whose restriction to $G_{/S}$ is $\mathrm{Id}_{G_{/S}}$, is the identity automorphism of $G'_{/S}$.*

(iii) *The set of all liftings $G' \to S'$ of $G \to S$ over S' has a natural structure as a torsor for $\mathrm{t}_{G_0} \otimes_{R_0} \mathrm{t}_{G_o^t} \otimes_{R_0} I$.*

Lemma 5.5.1 is a theorem of Grothendieck; see [15], [21] and [17, Thm. 4.4].

(5.5.2) Lemma *Notation as in 5.5.1. Suppose that*

- *We have $K = pR'$, $I = p^n R'$ for a positive integer $n \geq 1$.*
- *The commutative ring $R_0 := R/pR$ is normal and excellent.*
- *The smooth $\mathbb{F}_p$-sheaves $X_*(G_0^{\mathrm{mult}}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and $\mathrm{T}_p(G_0^{\text{ét}}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ on S_0 are constant.*

Then the following statements hold.

(i) *There are natural isomorphisms*

$$\mathrm{t}_{G_0} \cong (X_*(G_0^{\mathrm{mult}}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \otimes_{\mathbb{F}_p} R_0, \quad \mathrm{t}_{G_0^t} \cong (\mathrm{T}_p(G_0^{\text{ét}})^\vee \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \otimes_{\mathbb{F}_p} R_0 .$$

(ii) *The kernel of the canonical map*

$$X_*(G_0^{\mathrm{mult}}) \otimes_{\mathbb{Z}_p} \mathrm{T}_p(G_0^{\text{ét}})^\vee \otimes_{\mathbb{Z}_p} \nu_{p^\infty,S'} \longrightarrow X_*(G_0^{\mathrm{mult}}) \otimes_{\mathbb{Z}_p} \mathrm{T}_p(G_0^{\text{ét}})^\vee \otimes_{\mathbb{Z}_p} \nu_{p^\infty,S}$$

is naturally isomorphic to the sheaf on $(S_0)_{\text{ét}}$ attached to the R_0-module $\mathrm{t}_{G_0} \otimes_{R_0} \mathrm{t}_{G_0^t} \otimes_{R_0} I$.

(iii) *Let G_1, G_2 be two liftings over S' of the BT-group $G \to S$. Let α be the element of $\mathrm{t}_{G_0} \otimes_{R_0} \mathrm{t}_{G_0^t} \otimes_{R_0} I$ such that $[G_1] + \alpha = [G_2]$ according to the torsor structure on the set of all S'-liftings of the BT-group $G \to S$. Let β be the element of the kernel of the map in the displayed formula in* (ii) *above, which corresponds to α under the canonical isomorphism in* (ii). *Then the canonical coordinates $q(G_1 \to S')$ and $q(G_2 \to S')$ for $G_1 \to S'$ and $G_1 \to S'$ are related by $q(G_2 \to S') = q(G_1 \to S') + \beta$.*

Proof. The statement (i) is immediate from the hypotheses. The statement (ii) follows from (i) and Prop. 2.2 (iii). To prove (iii), it suffices to verify it after making a base change to the completion of a closed point of the base scheme S', by 2.1.6 and 2.2. But then the statement follows from known properties of the standard Serre-Tate coordinates. ∎

Proof of Prop. 5.5. A Tate-linear lifting of Z over a truncated Witt ring $W_n(k)$ is unique if it exists. Therefore it suffices to show that, for every positive integer n, there exists a Tate-linear lift Z_n of Z over $W_n(k)$. In other words, there exists a closed subscheme of $\mathcal{A}_g^{\mathrm{ord}}{}_{/W_n(k)}$ which is formal smooth over $W_n(k)$, and such that the formal completion $Z_n^{/z}$ at any closed point z of Z is a formal subtorus of the Serre-Tate formal torus $\mathcal{A}_{g/W_n(k)}^{/z}$.

By the uniqueness of Tate-linear lifting, we can localize in the étale topology. First, we may and do assume that Z is affine; write $Z = \mathrm{Spec}(R_0)$. Passing to a finite étale cover, we may and do assume that the smooth $\mathbb{F}_p$-sheaves $(X_*(A[p^\infty]^{\mathrm{mult}})\otimes_{\mathbb{Z}_p}\mathbb{F}_p$ and $\mathrm{T}_p(A[p^\infty]^{\text{ét}})\otimes_{\mathbb{Z}_p}\mathbb{F}_p$ on Z are constant We shall show, by induction on n, that there exists a Tate-linear lift Z_n of Z over $W_n(k)$, for every positive integer n.

Assume, by induction, that Z_n exists. Since Z_n is affine and smooth over $W_n(k)$, there exists an affine scheme $S_{n+1} = \mathrm{Spec}(R_n)$, which is smooth over $W_{n+1}(k)$, and such that $S_{n+1} \times_{\mathrm{Spec}(W_{n+1}(k))} \mathrm{Spec}(W_n(k)) \cong Z_n$. Since $A_{g,1,N} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}(W_{n+1}(k))$ is smooth over $W_{n+1}(k)$, there exists a morphism $f_{n+1} : S_{n+1} \to A_{g,1,N}^{\mathrm{ord}} \times_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \mathrm{Spec}(W_{n+1}(k))$ which extends the closed immersion $Z_n \hookrightarrow A_{g,1,N}^{\mathrm{ord}} \times_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \mathrm{Spec}(W_n(k))$. Notice that such a morphism f_{n+1} is necessarily a closed immersion. By Lemma 5.5.2, we can adjust f_{n+1} by a suitable element of $\mathrm{t}_{A/Z} \otimes_{R_0} \mathrm{t}_{A^t/Z} \otimes_{R_0} (p^n R_n)$ so that the coimage of the canonical coordinates

$$q(A/S_{n+1}) : \mathrm{T}_p(A[p^\infty]^{\text{ét}}) \otimes_{\mathbb{Z}_p} X_*(A[p^\infty]^{\mathrm{mult}})^\vee \to \nu_{p^\infty, S_{n+1}}$$

of the ordinary scheme $A \to S_{n+1}$ induced by the adjusted morphism f_{n+1} is a smooth $\mathbb{Z}_p$-sheaf, whose rank is equal to $\dim(Z)$. Then the subscheme

$$Z_{n+1} := f_{n+1}(S_{n+1}) \subseteq \mathcal{A}_{g,1,N}^{\mathrm{ord}}{}_{/W_{n+1}(k)}$$

is a Tate-linear lifting in $\mathcal{A}_{g,1,N}^{\mathrm{ord}}{}_{/W_{n+1}(k)}$ of Z over $W_{n+1}(k)$. We have finished the induction step. ∎

§6. Connection to the Hecke orbit problem

In this section we first consider the a local version of the ordinary Hecke orbit problem; see 6.5 for the set-up, and Theorem 6.6 for the solution. Then we apply Theorem 6.6 to relate the Hecke orbit problem to Tate-linear subvarieties.

(6.1) Proposition *Let k be an algebraically closed field. Let R be a topologically finitely generated complete local domain over k. In other words, R is isomorphic to a quotient $k[[x_1, \ldots, x_n]]/P$, where P is a prime ideal of the power series ring $k[[x_1, \ldots, x_n]]$. Then there exists an injective local homomorphism $\iota : R \hookrightarrow k[[y_1, \ldots, y_d]]$ of complete local k-algebras, where $d = \dim(R)$.*

Proof. Denote by $f : X \to \mathrm{Spec}\, R$ the normalization of the blowing-up of the closed point s_0 of $S := \mathrm{Spec}\, R$. Let $D = (f^{-1}(s_0))_{\mathrm{red}}$ be the exceptional divisor with reduced structure;

it is a scheme of finite type over k. The maximal points of D are contained in the regular locus X_{reg} of X, hence there exists a dense open subscheme $U \subset D$ such that $U \subset X_{\mathrm{reg}}$. Pick a closed point x_0 in U. Then the completion $\mathcal{O}^{\wedge}_{X,x_0}$ of the local ring $\mathcal{O}_{X,x_0}$ is isomorphic to $k[[y_1, \ldots, y_d]]$, and the natural map $R \to \mathcal{O}^{\wedge}_{X,x_0}$ is an injection. ∎

(6.1.1) Remark (i) Prop. 6.1 can be regarded as a very weak version of desingularization. In fact let R be the completion of the local ring at a closed point x of an algebraic variety X over k, and let $f : Y \to X$ be a generically finite morphism of algebraic varieties such that there exists a closed point $y \in Y$ above x and Y is smooth at y. Then the natural map from $R := \mathcal{O}^{\wedge}_{X,x} \to \mathcal{O}_{Y,y}$ gives the desired inclusion.

(ii) It is also possible to prove Prop. 6.1 using Néron's desingularization: One first produces an injective homomorphism $k[[t]] \to R$ which is "generically smooth" in a suitable sense, and a finite extension $k[[t]] \to k[[x]]$ such that there exists a $k[[t]]$-algebra homomorphism $e : R \to k[[x]]$. Then one uses Néron's desingularization procedure to smoothen $R \otimes_{k[[t]]} k[[x]]$ along the section e. This proof is more complicated than the one given above though. The author would like to acknowledge discussion with F. Pop on alternative proofs of Prop. 6.1.

(6.2) Proposition *Let k be a field of characteristic $p > 0$. Let $q = p^r$ be a positive power of p, $r \in \mathbb{N}_{>0}$. Let $F(x_1, \ldots, x_m) \in k[x_1, \ldots, x_m]$ be a polynomial with coefficients in k. Suppose that we are given elements $c_1, \ldots, c_m$ in k and a natural number $n_0 \in \mathbb{N}$ such that $F(c_1^{q^n}, \ldots, c_m^{q^n}) = 0$ in k for all $n \geq n_0$, $n \in \mathbb{N}$. Then $F(c_1^{q^n}, \ldots, c_m^{q^n}) = 0$ for all $n \in \mathbb{N}$; in particular $F(c_1, \ldots, c_m) = 0$.*

PROOF. P We may and do assume that k is perfect. Let $\sigma : k \to k$ be the automorphism of k such that $\sigma(y) = y^{q^{-1}}$ for all $y \in k$. For each $n \in \mathbb{N}$ and each polynomial $f(\mathbf{x}) = \sum_{I \in \mathbb{N}^m} a_I \, \mathbf{x}^I \in k[\mathbf{x}]$, denote by $\sigma^n(f(\mathbf{x}))$ the result of applying σ^n to the coefficients of $f(\mathbf{x})$; i.e. $\sigma^n(f(\mathbf{x})) := \sum_{I \in \mathbb{N}^m} \sigma^n(a_I) \, \mathbf{x}^I \in k[\mathbf{x}]$. Here $\mathbf{x}$ stands for $(x_1, \ldots, x_m)$. The map $f \mapsto \sigma(f)$ is a σ-linear automorphism of the ring $k[\mathbf{x}]$, and it preserves the increasing filtration of $k[\mathbf{x}]$ by degree: For each $a \in \mathbb{N}$, let V_a be the k-subspace of $k[\mathbf{x}]$ consisting of all polynomials in $k[\mathbf{x}]$ of degree at most a. Then $\sigma : f \to \sigma(f)$ is a σ-linear isomorphism from V_a to itself, for each $a \in \mathbb{N}$.

Let I be the ideal in $k[\mathbf{x}]$ generated by all polynomials $\sigma^n(F(\mathbf{x}))$ with $n \geq n_0$. We claim that $\sigma(I) = I$. It is clear that $\sigma(I) \subseteq I$, for $\sigma(I)$ is generated by the polynomials $\sigma^n(F(\mathbf{x}))$, $n \geq n_0 + 1$. On the other hand, for each $a \in \mathbb{N}$, σ induces a σ-linear isomorphism from $I \cap V_a$ to $\sigma(I) \cap V_a$. Therefore $\dim_k(I \cap V_a) = \dim_k(\sigma(I) \cap V_a)$. Since $I \cap V_a \supseteq \sigma(I) \cap V_a$, we deduce that $I \cap V_a = \sigma(I) \cap V_a$, for every $a \in \mathbb{N}$. Hence the k-vector space $I \cap V_a$ is spanned by $\mathbb{F}_q[\mathbf{x}] \cap I \cap V_a$, by descent, for each $a \in \mathbb{N}$. It follows that the ideal $I \subset k[\mathbf{x}]$ is generated by $I \cap \mathbb{F}_q[\mathbf{x}]$. Since $(c_1, \ldots, c_m) \in \mathrm{Spec}(k[x_1, \ldots, x_m]/I)(k)$ and I is defined over $\mathbb{F}_q$, $(\sigma^b(c_1), \ldots, \sigma^b(c_m))$ lies in the zero locus of I for each $b \in \mathbb{N}$. The Proposition follows. ∎

(6.2.1) Corollary *Notation as in 6.2. Let d be the degree of $F(x_1,\ldots,x_m)$. Let V be the set of all homogeneous polynomials in $k[x_1,\ldots,x_m]$ of degree d if $F(x_1,\ldots,x_m)$ is homogeneous; and let V be the set of all polynomials in $k[x_1,\ldots,x_m]$ of degree at most d if $F(x_1,\ldots,x_m)$ is not homogeneous. Assume instead that $F(c_1^{q^n},\ldots,c_m^{q^n})=0$ in k for all $n_0\le n\le n_1$, and $n_1-n_0\ge \dim_k(V)$. Then $F(c_1^{q^n},\ldots,c_m^{q^n})=0$ for all $n\in\mathbb{N}$.*

Proof. For each $a\in\mathbb{N}$, let $W_a=\sum_{n_0\le n\le n_0+a} k\cdot\sigma^n(F(\mathbf{x}))$. Clearly $W_a\subseteq W_{a+1}\subseteq V$ for all $a\in\mathbb{N}$. Suppose that $W_a=W_{a+1}$ for some a, then

$$W_{a+2}=k\langle\sigma^{n_0}(F(\mathbf{x})),\sigma(W_{a+1})\rangle=k\langle\sigma^{n_0}(F(\mathbf{x})),\sigma(W_a)\rangle=W_{a+1}\,.$$

So if $W_a=W_{a+1}$, then $W_a=W_b$ for all $b\ge a$. Since $n_1-n_0\ge\dim(V)$, the ideal I in the proof of 6.2 is generated by $W_{n_1-n_0}$. So the apparently weaker assumption here is actually the same as that in 6.2. ∎

(6.3) Proposition *Let k be a field of characteristic $p>0$. Let $f(\mathbf{u},\mathbf{v})\in k[[\mathbf{u},\mathbf{v}]]$, $\mathbf{u}=(u_1,\ldots,u_a)$, $\mathbf{v}=(v_1,\ldots,v_b)$, be a formal power series in the variables $u_1,\ldots,u_a,v_1,\ldots,v_b$ with coefficients in k. Let $\mathbf{x}=(x_1,\ldots,x_m)$, $\mathbf{y}=(y_1,\ldots,y_m)$ be two new sets of variables. Let $\mathbf{g}(\mathbf{x})=(g_1(\mathbf{x}),\ldots,g_a(\mathbf{x}))$ be an a-tuple of power series without the constant terms: $g_i(\mathbf{x})\in(\mathbf{x})k[[\mathbf{x}]]$ for $i=1,\ldots,a$. Let $\mathbf{h}(\mathbf{y})=(h_1(\mathbf{y}),\ldots,h_b(\mathbf{y}))$, with $h_j(\mathbf{y})\in(\mathbf{y})k[[\mathbf{y}]]$ for $j=1,\ldots,b$. Let $q=p^r$ be a positive power of p. Let $n_0\in\mathbb{N}$ be a natural number. Let $(d_n)_{n\ge n_0}$ be a sequence of natural numbers such that $\lim_{n\to\infty}\frac{q^n}{d_n}=0$. Suppose we are given power series $R_{j,n}(\mathbf{v})\in k[[\mathbf{v}]]$, $j=1,\ldots,b$, $n\ge n_0$, such that $R_{j,n}(\mathbf{v})\equiv 0 \mod (\mathbf{v})^{d_n}$ for all $j=1,\ldots,b$ and all $n\ge n_0$. For each $n\ge n_0$, let $\phi_{j,n}(\mathbf{v})=v_j^{q^n}+R_{j,n}(\mathbf{v})$, $j=1,\ldots,b$. Let $\Phi_n(\mathbf{v})=(\phi_{1,n}(\mathbf{v}),\ldots,\phi_{b,n}(\mathbf{v}))$ for $n\ge n_0$. Assume that*

$$f(\mathbf{g}(\mathbf{x}),\Phi_n(\mathbf{h}(\mathbf{x})))=f\left(g_1(\mathbf{x}),\ldots,g_a(\mathbf{x}),\phi_{1,n}(h(\mathbf{x})),\ldots,\phi_{b,n}(h(\mathbf{x}))\right)=0$$

as an element in $k[[\mathbf{x}]]$, for all $n\ge n_0$. Then $f(g_1(\mathbf{x}),\ldots,g_a(\mathbf{x}),h_1(\mathbf{y}),\ldots,h_b(\mathbf{y}))=0$ in $k[[\mathbf{x},\mathbf{y}]]$.

Proof. Let $\mathbf{t}=(t_{i,J})$ be an infinite set of variables parametrized by indices $(i,J)\in\{1,\ldots,b\}\times(\mathbb{N}^m-\{0\})$. Let

$$H_i(\mathbf{t};y)=\sum_{i,J} t_{i,J}\,\mathbf{y}^J\,,$$

so that if we write $h_i(\mathbf{y})=\sum_{i,J} c_{i,J}\,\mathbf{y}^J$ with all $c_{i,J}\in k$, and let $\mathbf{c}=(c_{i,J})_{i,J}$, then $h_i(\mathbf{y})=H_i(\mathbf{c};\mathbf{y})$ for each $i=1,\ldots,b$. Consider the formal power series

$$f(g_1(\mathbf{x}),\ldots,g_a(\mathbf{x}),H_1(\mathbf{t};\mathbf{y}),\ldots,H_b(\mathbf{t};\mathbf{y}))\,\in\,k[\mathbf{t}][[\mathbf{x},\mathbf{y}]]$$

and write it as

$$f(\mathbf{g}(\mathbf{x}),\mathbf{H}(\mathbf{t};\mathbf{y}))=\sum_{I,J\in\mathbb{N}^m} A_{I,J}(\mathbf{t})\,\mathbf{x}^I\mathbf{y}^J\,,$$

where $\mathbf{H}(\mathbf{t})$ is short for $(H_1(\mathbf{t}), \dots, H_b(\mathbf{t}))$. Our assumption on $\Phi_n(\mathbf{v})$ implies that

$$f(\mathbf{g}(\mathbf{x}), \Phi_n(\mathbf{h}(\mathbf{x}))) \equiv f(\mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{x})^{q^n}) \mod (\mathbf{x})^{d_n} \quad \forall\, n \geq n_0\,.$$

Suppose that $f(\mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{y})) = \sum_{I,J} A_{I,J}(\mathbf{c})\, \mathbf{x}^I \mathbf{y}^J \neq 0$. Define a positive integer M_2 by

$$M_2 := \inf \{\, |J| \,:\, \exists\, I \text{ s.t. } A_{I,J}(\mathbf{c}^{q^n}) \neq 0 \text{ for infinitely many } n \in \mathbb{N} \}\,.$$

Notice that the subset of $\mathbb{Z}_{>0}$ on the right-hand-side of the definition of M_2 is non-empty: There exists multi-indices I_0, J_0 such that $A_{I_0,J_0}(\mathbf{c}) \neq 0$, because $f(\mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{y})) \neq 0$. Hence $A_{I_0,J_0}(\mathbf{c}^{q^n}) \neq 0$ for infinitely many $n \in \mathbb{N}$, by Prop. 6.2. Define $M_1 \in \mathbb{Z}_{>0}$ by

$$M_1 := \inf \{\, |I| \,:\, \exists\, J \text{ with } |J| = M_2 \text{ s.t. } A_{I,J}(\mathbf{c}^{q^n}) \neq 0 \text{ for infinitely many } n \in \mathbb{N} \,\}\,.$$

By Prop. 6.2, if either $|J| < M_2$, or if $|J| = M_2$ and $|I| = M_1$, then $A_{I,J}(\mathbf{c}^{q^n}) = 0$ for all $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} \frac{q^n}{d_n} = 0$, there exists a natural number n_2 such that $q^{n_2} > 2M_1$ and $M_1 + q^n M_2 < d_n$ for all $n \geq n_2$. We have

$$f(\mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{x})^{q^n}) = f(\mathbf{g}(\mathbf{x}), \mathbf{H}(\mathbf{c}^{q^n}; \mathbf{x}^{q^n})) = \sum_{I,J} A_{I,J}(\mathbf{c}^{q^n})\, \mathbf{x}^{I+q^nJ}\,.$$

So we obtain the following congruence

$$\begin{aligned} 0 = f(\mathbf{g}(\mathbf{x}), \Phi_n(\mathbf{h}(\mathbf{x}))) &\equiv f(\mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{x})^{q^n}) && \mod (\mathbf{x})^{d_n} \\ &\equiv \sum_{|I|=M_1, |J|=M_2} A_{I,J}(\mathbf{c}^{q^n})\, \mathbf{x}^{I+q^nJ} && \mod (\mathbf{x})^{M_1+q^nM_2+1} \end{aligned}$$

for all $n \geq n_2$. The leading terms of the above congruence relations give us equalities

$$\sum_{|I|=M_1, |J|=M_2} A_{I,J}(\mathbf{c}^{q^n})\, \mathbf{x}^{I+q^nJ} = 0 \quad \forall\, n \geq n_2 \qquad (*)$$

in the polynomial ring $k[\mathbf{x}]$.

We claim that all terms in left-hand-side of the equality (*) have different multi-degrees. Suppose that two pairs of indices (I_1, J_1), (I_2, J_2) both satisfy $|I_1| = |I_2| = M_1$, $|J_1| = |J_2| = M_2$, and $I_1 + q^nJ_1 = I_2 + q^nJ_2$ for some $n \geq n_2$. Then $I_1 = I_2$ and $J_1 = J_2$ because $q^n > 2M_1$. We have verified the claim. So the equality (*) means that $A_{I,J}(\mathbf{c}^{q^n}) = 0$ if $|I| = M_1$, $|J| = M_2$, and $n \geq n_2$. This contradicts the definition of M_1. ∎

(6.3.1) Remark When $a = b$ and $g_i(\mathbf{x}) = h_i(\mathbf{x})$ for all $i = 1, \dots, a$, we obtain the following special case of Prop. 6.3. Let $T = \mathrm{Spf}\ k[[u_1, \dots, u_a]]$, and let $\Phi_n : T \to T$, $n \geq n_0$, be a family of morphisms which are very close to the Frobenius morphisms Fr_{q^n} as in the statement of Prop. 6.3, where $\mathrm{Fr}_{q^n} : T \to T$ corresponds to the morphism $u_i \mapsto u_i^{q^n}$. Then for any closed formal scheme Z of T, the schematic closure of the union of the graphs of Φ_n, n running over all integers $n \geq n_0$, contains $Z \times Z$.

(6.3.2) Remark Prop. 6.3 is the key technical ingredient to the proof of Thm. 6.6. The case of 6.3 made explicit in 6.3.1 *is* the proof of the case of Thm. 6.6, when the group $G(\mathbb{Z}_p)$ is equal to $\mathbb{Z}_p^\times$, and the action of $\mathbb{Z}_p^\times$ on N is "multiplication with elements of $\mathbb{Z}_p^\times$ according to the $\mathbb{Z}_p$-module structure of N".

(6.4) Lemma *Let G be a connected linear algebraic group over a field F of characteristic 0, let V be a finite dimensional vector space over F, and let $\rho : G \to \mathrm{GL}(V)$ be an F-rational linear representation of G. Let $\mathfrak{g} = \mathrm{Lie}(G)$, and let $d\rho : \mathfrak{g} \to \mathrm{End}(V)$ be the differential of ρ. The following statements are equivalent:*

(i) *The trivial representation $\mathbf{1}_G$ is not a subquotient of (ρ, V).*

(ii) *There exist elements $w_{i,j} \in \mathfrak{g}$, where $i = 1, \ldots, r$, $j = 1, \ldots, n_i$, such that*

$$\sum_{i=1}^{r} d\rho(w_{i,1}) \circ \cdots \circ d\rho(w_{i,n_i}) \in \mathrm{GL}(V)\,.$$

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. Conversely, assume (i). Replacing (ρ, V) by its semi-simplification, we may assume that (ρ, V) is isomorphic to a direct sum $\oplus_{m=1}^{b} (\rho_m, V_m)$ of irreducible representations of G. Each V_m is an irreducible $\mathfrak{g}$-module under $d\rho_m$. By Jacobson's density theorem, for each $m = 1, \ldots, b$, the statement (ii) holds with (ρ, V) replaced by (ρ_m, V_m). An application of Sublemma 6.4.1 below with $r = b$ finishes the proof. ∎

(6.4.1) Sublemma Let K be an infinite field. Let $V_1, \ldots, V_b$ be finite dimensional vector spaces over K, and let $A_1, \ldots, A_r$ be K-liner endomorphisms of $V = \oplus_{m=1}^{b} V_m$ such that $A_i(V_m) \subseteq V_m$ for each $i = 1, \ldots, r$ $m = 1, \ldots, b$. Assume that for each $m = 1, \ldots, b$, there exists an i, $1 \leq i \leq r$, such that $\det(A_i|V_m) \neq 0$. Then there exist elements $\lambda_1, \ldots, \lambda_r$ in K such that $\sum_{i=1}^{r} \lambda_i A_i \in \mathrm{GL}(V)$.

Proof. Let $t_1, \ldots, t_r$ be variables, and consider the polynomial

$$f(t_1, \ldots, t_r) := \det\left(\sum_{i=1}^{r} t_i A_i\right) = \prod_{m=1}^{b} \det\left(\sum_{i=1}^{r} t_i A_i|V_m\right) \in K[t_1, \ldots, t_r]\,.$$

It suffices to show that $f(t_1, \ldots, t_r) \neq 0$: Every rational variety of positive dimension over an infinite field K has at least a K-rational point, and the variety $\mathrm{Spec}(K[t_1, \ldots, t_r, \frac{1}{f(t_1,\ldots,t_r)}])$ is clearly rational over K. For each $m = 1, \ldots, b$, the polynomial

$$f_m(t_1, \ldots, t_r) := \det(\sum_{i=1}^{r} t_i T_i|V_m) \in K[t_1, \ldots, t_r]$$

is not equal to zero by assumption, hence their product $f(t_1, \ldots, t_r)$ is not equal to zero. ∎

(6.5) We set up notation for the main result, Thm. 6.6, of this section. Let k be a field of characteristic $p > 0$. Let T be a formal torus over k with cocharacter group N. In other words N is a free $\mathbb{Z}_p$-module of finite rank, and $T = N \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}_{\mathrm{m}}}$, where $\widehat{\mathbb{G}_{\mathrm{m}}}$ is the completion of $\mathbb{G}_{\mathrm{m}/k}$ along the unit section.

(6.5.1) Let G be a connected linear algebraic group over $\mathbb{Q}_p$. Let $V = N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and let $\rho : G \to \mathrm{GL}(V)$ be a $\mathbb{Q}_p$-rational linear representation of G on V. Let $G(\mathbb{Z}_p)$ be an open subgroup of $G(\mathbb{Q}_p)$ such that the $\mathbb{Z}_p$-lattice $N \subset V$ is stable under $G(\mathbb{Z}_p)$. The group $G(\mathbb{Z}_p)$ operates on the formal torus T via its action on the cocharacter group N of T.

(6.5.2) Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G, and let $d\rho : \mathfrak{g} \to \mathrm{End}(V)$ be the differential of ρ. Let $\mathfrak{g}_{\mathbb{Z}_p}$ be a $\mathbb{Z}_p$-lattice in $\mathfrak{g}$ such that the $\mathbb{Z}_p$-lattice N in V is stable under $d\rho(\mathfrak{g}_{\mathbb{Z}_p})$. For each element $w \in \mathfrak{g}_{\mathbb{Z}_p}$, denote by $\alpha(w)$ the endomorphism of the formal torus T induced by the endomorphism $d\rho(w)$ of N.

(6.6) Theorem *Notation as in 6.5. Assume that the trivial representation $\mathbf{1}_G$ is not a subquotient of (ρ, G). Suppose that Z is a reduced and irreducible closed formal subscheme of T which is closed under the action of an open subgroup U of $G(\mathbb{Z}_p)$. Then there exists a unique $\mathbb{Z}_p$-direct summand N_1 of N such that $Z = N_1 \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}_{\mathrm{m}}}$. Moreover N_1 is stable under the action of U, and $V_1 := N_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a subrepresentation of (ρ, V).*

PROOF. We may and do assume that $k = k^{\mathrm{alg}}$. Notice that once we show that Z is a formal subtorus of T, the last sentence in the statement of 6.6 follows immediately.

Choose an integer $n_0 \geq 2$ such that $\exp_G(p^{n_0} w) \in U$ for every $w \in \mathfrak{g}_{\mathbb{Z}_p}$. The rest of the proof is organized into several steps. Among them the first step is the crucial one; it uses Prop. 6.1 and Prop. 6.3.

Step 1. Denote by $\mu : T \times T \to T$ the group law of the formal torus T. Then for every $w \in \mathfrak{g}_{\mathbb{Z}_p}$, we have

$$\mu \circ (\mathrm{Id} \times \alpha(w))(Z \times Z) \subseteq Z\,.$$

We recall that $\alpha(w)$ is the endomorphism of T induced by $d\rho(w) \in \mathrm{End}_{\mathbb{Z}_p}(N)$.

PROOF OF STEP 1. Write $\widehat{\mathbb{G}_{\mathrm{m}}} = \mathrm{Spf}[[u]]$ with comultiplication $k[[u]] \to k[[u, v]]$ given by $u \mapsto u+v+uv$. Choose a $\mathbb{Z}_p$-basis of N, which gives an isomorphism $T \cong \mathrm{Spf}(k[[u_1, \ldots, u_d]])$, $d = \dim(T) = \mathrm{rank}_{\mathbb{Z}_p}(N)$. The comultiplication map of the coordinate ring of T is

$$\mu^* : k[[u_1, \ldots, u_d]] \to k[[u_1, \ldots, u_d, v_1, \ldots, v_d]]; \quad \mu^* : u_i \mapsto u_i + v_i + u_i v_i \ \ \forall\, i\,.$$

Since the closed formal subscheme $Z \subseteq T$ is assumed to be reduced and irreducible, it corresponds to a prime ideal P of $k[[u_1, \ldots, u_d]]$. By Prop. 6.1, there exists an injective k-algebra homomorphism

$$\iota \,:\, k[[u_1 \ldots, u_d]]/P \hookrightarrow k[[x_1, \ldots, x_m]] \quad m = \dim(Z)\,.$$

Let $g_i(\mathbf{x}) = \iota(u_i)$, $i = 1, \ldots, d$, where $\mathbf{x} = (x_1, \ldots, x_m)$.

For each $w \in \mathfrak{g}_{\mathbb{Z}_p}$ and $n \geq n_0$, we know that $\exp_G(p^n w) \in U$ if $n \geq n_0$, therefore

$$\begin{aligned}\rho(\exp_G(p^n w)) &= \mathrm{Id}_N + d\rho(w) \cdot \sum_{i\geq 1} \frac{p^{in}}{i!}\, d\rho(w)^{i-1} \\ &= \mathrm{Id}_N + d\rho(w)\left(p^n \cdot \mathrm{Id}_N + \frac{p^{2n}}{2!}\, d\rho(w) + \frac{p^{3n}}{3!}\, d\rho(w)^2 + \cdots\right)\end{aligned}$$

Since $n \geq n_0 \geq 2$, we have $\lim_{i\to\infty} \frac{p^{in}}{i!} = 0$ in $\mathbb{Z}_p$ by the following estimate on the p-adic valuation ord_p of $\frac{p^{in}}{i!}$:

$$\mathrm{ord}_p(\frac{p^{in}}{i!}) = in - \sum_{m\geq 1} \lfloor \frac{i}{p^m} \rfloor \geq in - \frac{i}{p-1}\,.$$

We also have $\frac{p^{in}}{i!} \in \mathbb{Z}_p$ for each $i \geq 1$. So

$$E_{p^n w} := \sum_{i\geq 1} \frac{p^{in}}{n!}\, d\rho(w)^{i-1}$$

is an endomorphism of N. Let $\beta_{n,w}$ be the endomorphism of the formal torus T induced by $E_{p^n w}$, and let $\Phi_n(w)$ be the continuous endomorphism of the complete local k-algebra $k[[u_1, \ldots, u_d]]$ corresponding to $\beta_{n,w}$. For each $n \geq n_0$ and for $i = 1, \ldots, d$, let

$$\phi_{i,n}(w) := \Phi_n(w)(u_i) \in k[[u_1, \ldots, u_d]] = k[[\mathbf{u}]]$$

and define $R_{i,n}(\mathbf{u}) \in k[[\mathbf{u}]]$ by

$$\phi_{i,n}(w) = u_i^{p^n} + R_{i,n}(\mathbf{u})\,.$$

From the definition of $E_{p^n w}$, $\beta_{n,w}$ and $\phi_{i,n}(w)$, it is immediate that

$$R_{i,n}(\mathbf{u}) \equiv 0 \mod (\mathbf{u})^{p^{\lfloor 2n - \frac{2}{p-1} \rfloor}}\,.$$

We want to show that, for each element $f_1(\mathbf{u}) \in P$, the image

$$f(\mathbf{u}, \mathbf{v}) := (\mathrm{Id} \times \alpha(w))^* \circ \mu^*(f_1(\mathbf{u}))$$

of $f_1(\mathbf{u})$ in $k[[\mathbf{u}, \mathbf{v}]]$ under the ring homomorphism $(\mathrm{Id} \times \alpha(w))^* \circ \mu^*$ belongs to the ideal

$$\mathrm{pr}_1^*(P) \cdot k[[\mathbf{u}, \mathbf{v}]] + \mathrm{pr}_2^*(P) \cdot k[[\mathbf{u}, \mathbf{v}]]\,.$$

Here $\alpha(w)^* : k[[\mathbf{u}]] \to k[[\mathbf{u}]]$ is the endomorphism of $k[[\mathbf{u}]]$ corresponding to the endomorphism $\alpha(w)$ of T, and $\mathrm{pr}_1^*, \mathrm{pr}_2^* : k[[\mathbf{u}]] \to k[[\mathbf{u}, \mathbf{v}]]$ correspond to the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : T \times T \to T$. Equivalently, we must show that

$$(\iota \hat{\otimes} \iota)(f(\mathbf{u}, \mathbf{v}) = f(g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}), g_1(\mathbf{y}), \ldots, g_m(\mathbf{y}))$$

is equal to 0 in $k[[\mathbf{x},\mathbf{y}]]$. Since the automorphism of T induced by $\rho(\exp_G(p^n w))$ is equal to $\mu\circ(\mathrm{Id}\times\alpha(w))\circ(\mathrm{Id}\times\beta_{n,w})$, the assumption that Z is stable under the action of $\exp_G(p^n w)$ translates into

$$f(g_1(\mathbf{x}),\ldots,g_m(\mathbf{x}),\phi_{1,n}(w)(\mathbf{g}(\mathbf{x})),\ldots,\phi_{1,n}(w)(\mathbf{g}(\mathbf{x})))=0\ \ \forall\, n\geq n_0\,,$$

where $\mathbf{g}(\mathbf{x})$ is short for $(g_1(\mathbf{x}),\ldots,g_m(\mathbf{x}))$. Applying Prop. 6.3 to the present situation, we conclude that $f(\mathbf{g}(\mathbf{x}),\mathbf{g}(\mathbf{y}))=0$. This finishes the proof of Step 1.

Step 2. Let $(w_{i,j})$, $i=1,\ldots,r$, $j=1,\ldots,n_i$ be a finite family of elements in $\mathfrak{g}_{\mathbb{Z}_p}$. Consider the following homomorphism

$$\begin{array}{rcl} s:\overbrace{T\times\cdots\times T}^{(\mathrm{r+1})-\mathrm{times}} & \longrightarrow & T \\ (x_0,x_1,\ldots,x_r) & \longmapsto & x_0+\sum_{i=1}^{r}\alpha(w_{i,1})\circ\cdots\alpha(w_{i,n_i})(x_i) \end{array}$$

of formal tori. Then $s(Z\times Z\times\cdots\times Z)\subseteq Z$. In particular we have $\sigma(Z\times Z)\subseteq Z$, where

$$\sigma:T\times T\to T$$

is the homomorphism of formal tori defined by

$$\sigma\,:\,(x,y)\,\mapsto\,x+\sum_{i=1}^{a}\alpha(w_{i,1})\circ\cdots\alpha(w_{i,n_i})(y)\,.$$

Proof of Step 2. One sees from Step 1 that $\alpha(w)(Z)\subseteq Z$ for every $w\in\mathfrak{g}_{\mathbb{Z}_p}$. The assertion in Step 2 now follows by an easy induction.

Step 3. The closed formal scheme $Z\subseteq T$ is closed under the group law. In other words, $\mu(Z\times Z)\subseteq Z$.

Proof of Step 3. According to Lemma 6.4, one can find $w_{i,j}\in\mathfrak{g}_{\mathbb{Z}_p}$, $i=1,\ldots,r$, $j=1,\ldots,n_i$, such that the element

$$A:=\sum_{i=1}^{r}d\rho(w_{i,1})\circ\cdots\circ d\rho(w_{i,n_i})$$

is an injective endomorphism of the $\mathbb{Z}_p$-module N. Let $\alpha:T\to T$ be the endomorphism of T induced by A. Then $\mathrm{Id}\times\alpha:Z\times Z\to Z\times Z$ is a dominant morphism. By Step 2, $\mu\circ(\mathrm{Id}\times\alpha)(Z\times Z)\subseteq Z$. Therefore $\mu(Z\times Z)\subseteq Z$.

Step 4. The closed formal subscheme $Z\subseteq T$ is a formal subtorus of T.

Proof of Step 4 and End of Proof of Theorem 6.6. For every abelian group M and any integer a, denote by $[a] = [a]_M$ the multiplication by a on M. For each integer n, let T_n be the kernel of the endomorphism $[p^n]_T : T \to T$ of T. Step 3 tells us that $Z \cap T_n$ is stable under the group law for each $n \geq 0$. Since $[-1]$ is equal to $[p^n - 1]$ on T_n, $Z \cap T_n$ is a subgroup scheme of T_n for each $n \geq 0$. Taking the limit as $n \to \infty$, we conclude that Z is a formal subgroup scheme of T. Since Z is assumed to be reduced and irreducible, Z is a formal subtorus of T. This finishes the proof of Step 4 and Theorem 6.6. ∎

(6.6.1) Remark (i) In some sense the effect of Prop. 6.1 is to reduce the reduces the proof of Thm. 6.6 to the case when Z is formally smooth over k.

(i) In application to the Hecke orbit problem, often one only needs a weakened version of Thm. 6.6, when the irreducible closed formal subscheme Z is assumed, in addition, to be formally smooth over k.

(iii) The statement of Thm. 6.6 has to be modified if the irreducibility assumption on Z is eliminated. Then one can only conclude that Z is the union of a finite number of formal subtori of T.

(iv) The analogue of 6.6 in the case of mixed characteristics seems quite plausible. Using the same notation as in 6.5, the statement of the analogue is as follows. Let $\tilde{T}$ be the formal torus over $W(k)$ with cocharacter group N. Let $\tilde{Z}$ be an irreducible closed formal subscheme of $\tilde{T}$ which is flat over $W(k)$ and stable under the action of an open subgroup U of $G_{\mathbb{Z}_p}$. Then (it seems quite likely that) Z is a formal subtorus of $\tilde{T}$ over $W(k)$.

With the help of Thm. 6.6, it is not difficult to verify the above conjectural statement if one adds the additional assumption that $\tilde{Z}$ is formally smooth over $W(k)$.

(v) The naive characteristic-zero analogue of 6.6 is *false*. We leave it to the readers to find an example of an irreducible closed subscheme Z of a two-dimensional formal torus T over $\mathbb{C}$ stable under $[n]_T$ for all $0 \neq n \in \mathbb{Z}$ such that Z is not a formal subtorus of T.

(6.6.2) Remark An examination of the proof of Thm. 6.6 reveals that the statement of Thm. 6.6 can be strengthened as follows. In the set-up of 6.6, assume that G is a p-adic analytic Lie group, $\mathfrak{g}$ is the Lie algebra of G, a finite dimensional vector space over $\mathbb{Q}_p$. Let $\rho : G \to GL(V)$ be an analytic linear representation of G such that the trivial $\mathbb{Q}_p$-linear representation $d\rho : \mathfrak{g} \to \mathrm{End}(V)$ of $\mathfrak{g}$ on V does not contain the trivial representation of $\mathfrak{g}$ as a subquotient. The irreducible closed formal subscheme Z of T is assumed to be stable under the action of an open subgroup U of G. Then the conclusion of Thm. 6.6 still holds, by the same proof. Only the statement of Lemma 6.4 requires an obvious modification to the present context, namely the statement of Lemma 6.4 (i) should be changed to: "The trivial representation of the Lie algebra $\mathfrak{g}$ is not a subquotient of $(d\rho, V)$."

(6.7) As an application of Theorem 6.6, we shall show that the ℓ-adic Hecke orbit of an *ordinary point* of a modular variety of PEL-type over $\mathbb{F}$ is a linear. Here the adjective "ordinary" is taken in the naive sense, meaning that the underlying abelian variety is ordinary.

(6.7.1) We recall the definition of PEL-data with good reduction at a prime number p; see [19, §5]. A PEL-data consists of

- a finite-dimensional central simple algebra B over $\mathbb{Q}$, which is unramified above p, (i.e. B is unramified at every place v of the center of B which divides p.)
- a positive definite involution $* = *_B$ of B,
- an order $\mathcal{O}_B \subset B$ which is maximal at p and stable under the involution $*$ of B,
- a non-degenerate alternating $\mathbb{Q}$-bilinear pairing $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{Q}$ on a finitely generated B-module V, such that the B-module structure is hermitian with respect to the involution $*$ and the pairing $\langle\cdot|\cdot\rangle$,
- a $*$-homomorphism $h : \mathbb{C} \to C \otimes_{\mathbb{Q}} \mathbb{R}$, where C is the commutator subalgebra $\mathrm{End}_B(V_{\mathbb{R}})$ of B in $\mathrm{End}_{\mathbb{Q}}(V)$, with the involution $*_C$ induced by $*_B$ and the pairing $\langle\cdot|\cdot\rangle$, such that the real-valued symmetric bilinear form $(v, w) \mapsto \langle v|h(\sqrt{-1})w\rangle$ on V is positive definite,
- a self-dual $\mathcal{O}_B$-lattice Λ_p in $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$,
- a compact open subgroup $K^{(p)}$ of $G(\mathbb{A}_f^{(p)})$, where G is the linear algebraic group over $\mathbb{Q}$ such that its R-valued points for any $\mathbb{Q}$-algebra R are given by

$$G(R) = \left\{x \in (C \otimes_{\mathbb{Q}} R)^\times \,|\, xx^{*_C} \in R^\times\right\} .$$

(6.7.2) Let $\mathcal{M}$ be the PEL-type moduli scheme defined by a given PEL input-data as above, defined over $\mathrm{Spec}\, \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, where E is the Shimura reflex field attached to the PEL-data. Points of $\mathcal{M}$ have the form $(A, \lambda, \iota, \bar\eta)$, where A is an abelian variety, with a separable polarization λ, an $\mathcal{O}_B$-action ι, and a level-structure $\bar\eta$, compatible with the given PEL-data. We refer to [19, §5] for the precise definition. We fix a geometric point $\mathbf{p} : \mathcal{O}_E \to \overline{\mathbb{F}_p}$ of $\mathrm{Spec}\, \mathcal{O}_E$ in characteristic p, and denote by $\mathcal{M}_{/\overline{\mathbb{F}_p}}$ the fiber product $\mathcal{M} \times_{\mathrm{Spec}\, \mathcal{O}_E, \mathbf{p}} \mathrm{Spec}\, \overline{\mathbb{F}_p}$. Abusing terminology, we call $\mathcal{M}_{/\overline{\mathbb{F}_p}}$ a reduction of $\mathcal{M}$ in characteristic p.

(6.7.3) For any prime number $\ell \neq p$, the elements of $G(\mathbb{Q}_\ell)$ gives rise to finite étale algebraic correspondences on $\mathcal{M}_{\overline{\mathbb{F}_p}}$. We call these the ℓ-adic Hecke correspondences on $\mathcal{M}_{/\overline{\mathbb{F}_p}}$. Similarly algebraic correspondences on $\mathcal{M}_{/\overline{\mathbb{F}_p}}$ defined by elements of $G(\mathbb{A}_f^{(p)})$ will be called the prime-to-p Hecke correspondences of $\mathcal{M}_{/\overline{\mathbb{F}_p}}$.

Let x_0 be a point of $\mathcal{M}_{/\overline{\mathbb{F}_p}}(k)$, where k is an algebraically closed field containing $\mathbb{F}_p$. Let ℓ be a prime different from p. The countable subset of $\mathcal{M}_{/\overline{\mathbb{F}_p}}(k)$, consisting of points which are images of x_0 under some ℓ-adic (resp. prime-to-p) Hecke correspondences on $\mathcal{M}_{/\overline{\mathbb{F}_p}}$, is called the ℓ-adic (resp. prime-to-p) Hecke orbit of x_0, denoted by $\mathcal{H}_\ell(x_0)$ (resp. $\mathcal{H}^{(p)}(x_0)$.) Let $Z_\ell(x_0)$ (resp. $Z^{(p)}(x_0)$) be the Zariski closure of $\mathcal{H}_\ell(x_0)$ (resp. $\mathcal{H}^{(p)}(x_0)$) in $\mathcal{M}_{/\overline{\mathbb{F}_p}}$. It is easy to see that $Z_\ell(x_0)$ and $Z^{(p)}(x_0)$ are both smooth at x_0, using with the general fact that $Z_\ell(x_0)$ and $Z^{(p)}(x_0)$ are generically smooth and the group-like property of Hecke correspondences. This statement generalizes the standard fact that every orbit (with the reduced structure) for the action of an algebraic group acting on an algebraic variety is smooth.

Let $x_0 = (A_0, \lambda_0, \iota_0, \bar\eta_0) \in \mathcal{M}_{/\overline{\mathbb{F}_p}}(k)$ be a geometric point of $\mathcal{M}_{/\overline{\mathbb{F}_p}}$ as above. Assume moreover that A_0 is an ordinary abelian variety. By the Serre-Tate theorem, the formal completion $\mathcal{M}^{/x_0}$ of $\mathcal{M}_{/\overline{\mathbb{F}_p}}$ at x_0 has a natural structure as formal torus, naturally isomorphic to

$$\underline{\mathrm{Hom}}^{\mathrm{sym}}_{\mathcal{O}_B\otimes_{\mathbb{Z}_p}\mathbb{Z}_p}(\mathrm{T}_p(A_0[p^\infty]^{\text{ét}}), A_0[p^\infty]^{\mathrm{mult}})\,.$$

(6.8) Proposition *Notation as above. In particular $x_0 = (A_0, \lambda_0, \iota_0, \bar\eta_0)$ is a point of the PEL-type moduli space $\mathcal{M}_{/\overline{\mathbb{F}_p}}$ over $\overline{\mathbb{F}_p}$, and A_0 is is an ordinary abelian variety. The the formal completion $Z^{(p)}(x_0)^{/x_0}$ at x_0 of the Zariski closure of the prime-to-p Hecke orbit of x_0 is a formal subtorus of the Serre-Tate formal torus $\mathcal{M}^{/x_0}_{/\overline{\mathbb{F}_p}}$.*

PROOF. Denote by D the the finite dimensional semi-simple $\mathbb{Q}$-algebra $\mathrm{End}_{\mathcal{O}_B}(A_0)\otimes_{\mathbb{Z}}\mathbb{Q}$, and let $*_D$ be the Rosati involution on D induced by the polarization λ_0. Let H be the reductive algebraic group over $\mathbb{Q}$ whose $\mathbb{Q}$-rational points consist of all elements $h \in D^\times$ such that $h \cdot h^{*_D} \in \mathbb{Q}^\times$. It is easy to see that there is a natural embedding $\xi : H(\mathbb{A}_f^{(p)}) \hookrightarrow G(\mathbb{A}_f^{(p)})$ of $H(\mathbb{A}_f^{(p)}$ as a subgroup of $G(\mathbb{A}_f^{(p)})$. Moreover elements of $H(\mathbb{Q}_p)$ operates naturally as quasi-isogenies on the Barsotti-Tate group $A[p^\infty]$. Since x_0 is defined over some finite field, D is not too small: it contains a commutative semisimple algebra K_1 over $\mathbb{Q}$ such that $[K_1 : K]^2 = \dim_K(C)$, where K is the center of simple algebra B.

Let U_p be the compact open subgroup of $H(\mathbb{Q}_p)$ consisting of all elements of $H(\mathbb{Q}_p)$ which operate naturally as isomorphisms on $A[p^\infty]$. We have a natural action of U_p on the Serre-Tate torus $\mathcal{M}^{/x_0}_{/\overline{\mathbb{F}_p}}$, by "transport of structures".

Consider the subgroup $H(\mathbb{Q}) \cap U_p$ of $H(\mathbb{Q})$, consisting of all elements $h \in H(\mathbb{Q})$ whose image in $H(\mathbb{Q}_p)$ belongs to U_p. The Hecke correspondences given by elements of the image of the composition

$$H(\mathbb{Q}) \cap U_p \hookrightarrow H(\mathbb{A}_f^{(p)}) \xrightarrow{\xi} G(\mathbb{A}_f^{(p)}$$

all have x_0 as a fixed point. Therefore the closed formal subscheme $Z^{(p)}(x_0)$ is stable under the action of the image in U_p of any elements of $H(\mathbb{Q}) \cap U_p$. By the weak approximation theorem, $H(\mathbb{Q}) \cap U_p$ is p-adically dense in U_p. Hence the $Z^{(p)}(x_0)$ is stable under the action of any element of U_p.

We are all set to apply Theorem 6.6. The closed formal subscheme $Z^{(p)}(x_0)$ of the formal torus $\mathcal{M}^{/x_0}_{/\overline{\mathbb{F}_p}}$ is formally smooth, since $Z^{(p)}(x_0)$ is smooth at x_0. The action of U_p on $\mathcal{M}^{/x_0}_{/\overline{\mathbb{F}_p}}$ comes from the natural linear action of U_p on the cocharacter group

$$\mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{T}_p(A_0[p^\infty]^{\text{ét}}), X_*(A_0[p^\infty]^{\text{mult}})\right) .$$

It is easy to see that the above linear representation of U_p, when tensored with $\mathbb{Q}_p$, does not contain the trivial representation as a subquotient. In fact this statement already holds with U_p replaced with its intersection with the group of unitary similitudes attached to the CM-algebra K. So all conditions in 6.6 are met, and Prop. 6.8 follows. ∎

(6.8.1) Remark The argument in 6.8 actually proves a more general statement: Let $W \subset \mathcal{M}_{/\overline{\mathbb{F}_p}}$ be a closed subvariety of $\mathcal{M}_{/\overline{\mathbb{F}_p}}$ stable under all prime-to-p Hecke correspondences. Assume that W contains an ordinary point $x_0 \in \mathcal{M}_{/\overline{\mathbb{F}_p}}(\overline{\mathbb{F}_p})$, and the completion $W^{/x_0}$ of W at x_0 is reduced and irreducible. Then $W^{/x_0}$ is a formal subtorus of the Serre-Tate formal torus $\mathcal{M}^{/x_0}_{/\overline{\mathbb{F}_p}}$.

(6.8.2) Remark It is tempting to try to prove a stronger version of 6.8, with $Z^{(p)}(x_0)^{/x_0}$ replaced by $Z_\ell(x_0)^{/x_0}$, using the strengthened form of Thm. 6.6 in Remark 6.6.2. The biggest obstacle is that the algebraic group H may not have much $\mathbb{Z}[1/\ell]$-points other than those coming from the subgroup $\mathbb{G}_{\mathrm{m}} \subset H$. This difficulty appears, for instance, in the case when D is a CM-field and every place of the maximal totally real subfield F of D above ℓ stays prime in D. For then the group of ℓ-units $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}[1/\ell])^\times$ in $F^\times$ is of finite index in the group of ℓ-units $(\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}[1/\ell])^\times$ in $D\times$.

§7. Some Conjectures

In this section we present several conjectures on families of ordinary abelian varieties in characteristic p. A shorter description of them can be found in 1.3.

(7.1) Conjecture *Given a PEL-input data with good reduction at p as in 6.7.1. Let $\mathcal{M}_{/\overline{\mathbb{F}_p}}$ be a reduction in characteristic p of the modular variety attached to the given PEL-type as in 6.7.2. Let x be a closed point of $\mathcal{M}$ such that the abelian variety A_x is ordinary.*

(i) *The prime-to-p Hecke orbit of x is Zariski dense in $\mathcal{M}_{\overline{\mathbb{F}_p}}$.*

(ii) *Let ℓ be a prime number different from p. Then the ℓ-adic Hecke orbit of x in $\mathcal{M}_{/Fpbar}$ is Zariski dense in $\mathcal{M}_{/Fpbar}$.*

Remark Clearly 7.1(ii) implies 7.1(i).

(7.2) Conjecture *Let k be an algebraically closed field of characteristic $p > 0$. Let $X_0 \subseteq \mathcal{A}_g$ be a Tate-linear subvariety of $\mathcal{A}_g^{\mathrm{ord}}{}_{/}k$ over k. Then X_0 can be lifted to a Tate-linear subvariety X of $\mathcal{A}_g^{\mathrm{ord}}{}_{/W(k)}$ which is smooth over $W(k)$.*

(7.2.1) Remark (i) The statement of 7.2 means that the Tate-linear formal lifting X_∞ of X_0, which is a closed formal subscheme of the p-adic completion of $\mathcal{A}_g^{\rm ord}{}_{/W(k)}$, is equal to the p-adic completion of a closed subscheme X of $\mathcal{A}_g^{\rm ord}{}_{/W(k)}$, necessarily smooth over $W(k)$. By Grothendieck's algebraization theorem, the assertion of 7.2 is equivalent to saying that X_∞ can be extended to a closed formal subscheme of the p-adic completion of one (hence every) toroidal compactification of $\mathcal{A}_g$.

(ii) Suppose that 7.2 holds, and $k = \overline{\mathbb{F}_p}$. Then [22, 5.2] tells us that the generic fiber of $X \times_{\operatorname{Spec} W(k),\tau} \operatorname{Spec}\mathbb{C}$ is a subvariety of $\mathcal{A}_g$ of Hodge type, for every embedding $\tau : W(k) \hookrightarrow \mathbb{C}$. So we can view 7.2 as a (conjectural) characterization, in terms of a simple geometric property in characteristic p, for a subvariety of $\mathcal{A}_g^{\rm ord}$ over k to be the reduction of (a Hecke translate of) a Shimura subvariety of $\mathcal{A}_g$ in characteristic 0.

(7.3) We establish notation for the rest of this section. Following the notation in §4, let k is an algebraically closed field of characteristic $p > 0$, let S be an irreducible normal scheme of finite type over k, and let $\bar{s}$ be a geometric point of S. Let A be a principally polarized abelian scheme over S of relative dimension $g \geq 1$ with ordinary fibers. The Barsotti-Tate group $A[p^\infty] \to S$ attached to $A \to S$ is an example of ordinary Barsotti-Tate group considered in §4. Define smooth $\mathbb{Z}_p$-sheaves U, V on $S_{\text{ét}}$ by $U = \mathrm{T}_p(A[p^\infty]^{\text{ét}})$, $V := X_*(A[p^\infty]^{\rm mult})$ as in 4.1.

(7.3.1) Denote by $\mathrm{GSp}_{\bar{s},\mathbb{Q}_p}$ the group of symplectic similitudes on $\omega_{\bar{s},\mathbb{Q}_p}(\mathbb{D}(A)^\vee) = V(1)_{\bar{s},\mathbb{Q}_p} \oplus U_{\bar{s},\mathbb{Q}_p}$, with respect to the symplectic pairing induced by the principal polarization.

We often write elements of $\mathrm{GSp}_{\bar{s},\mathbb{Q}_p}$ in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ according to the decomposition $\omega_{\bar{s},\mathbb{Q}_p}(\mathbb{D}(A)^\vee) = V(1)_{\bar{s},\mathbb{Q}_p} \oplus U_{\bar{s},\mathbb{Q}_p}$ above. For instance, the entry "B" denotes a element of $\mathrm{Hom}(U_{\bar{s},\mathbb{Q}_p}, V(1)_{\bar{s},\mathbb{Q}_p})$. Let $P_{\bar{s},\mathbb{Q}_p}$ be the parabolic subgroup of $\mathrm{GSp}_{\bar{s},\mathbb{Q}_p}$ consisting of element of $\mathrm{GSp}_{\bar{s},\mathbb{Q}_p}$ of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. Let $L_{\bar{s},\mathbb{Q}_p}$ be the parabolic subgroup of $\mathrm{GSp}_{\bar{s},\mathbb{Q}_p}$, often called the standard Siegel parabolic subgroup, consisting of element of $\mathrm{GSp}_{\bar{s},\mathbb{Q}_p}$ of the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$;

(7.4) Conjecture *There exists a connected reductive subgroup $G_{\bar{s},\mathbb{Q}_p}$ of $\mathrm{GSp}_{\bar{s},\mathbb{Q}_p}$ such that*

(i) *The neutral component $(\mathrm{Aut}_{\mathbb{T}(A[p^\infty])}(\omega_{\bar{s},\mathbb{Q}_p}))^0$ of $\mathrm{Aut}_{\mathbb{T}(A[p^\infty])}(\omega_{\bar{s},\mathbb{Q}_p})$, the p-adic monodromy group of $A_{/S}$, is a parabolic subgroup of $G_{\bar{s},\mathbb{Q}_p}$, and is equal to the neutral component $(G_{\bar{s},\mathbb{Q}_p} \cap P_{\bar{s}})^0$ of $G_{\bar{s}} \cap P_{\bar{s},\mathbb{Q}_p}$.*

(ii) *The neutral component $(G_{\bar{s},\mathbb{Q}_p} \cap L_{\bar{s},\mathbb{Q}_p})^0$ of $G_{\bar{s},\mathbb{Q}_p} \cap L_{\bar{s},\mathbb{Q}_p}$ is a Levi subgroup of the connected p-adic monodromy group $(\mathrm{Aut}_{\mathbb{T}(A[p^\infty])}(\omega_{\bar{s},\mathbb{Q}_p}))^0$.*

(7.4.1) Remark A consequence of 7.4 is that $\mathrm{Aut}_{\mathbb{T}'(A[p^\infty])}(\omega_{\bar{s}^*,\mathbb{Q}_p})$, the monodromy group of the smooth $\mathbb{Q}_p$-sheaf $V(A[p^\infty])(1)_{\mathbb{Q}_p} \oplus U(A[p^\infty])_{\mathbb{Q}_p}$, is a (possibly disconnected) reductive group over $\mathbb{Q}_p$.

(7.5) We set up notation for a conjecture for the family of ordinary abelian varieties $A_{/S}$, analogous to the Mumford-Tate conjecture for abelian varieties over number fields. For each geometric point $\bar{s}$ of S, denote by $\widetilde{A_{\bar{s}}}$ the Serre-Tate canonical lifting of $A_{\bar{s}}$, which is an abelian scheme over $W(\kappa(\bar{s}))$. Let $\tau : W(\kappa(\bar{s})) \hookrightarrow \mathbb{C}$ be an embedding of $W(\kappa(\bar{s}))$ into the field of complex numbers.

(7.5.1) Denote by $\mathrm{H}_{\bar{s},\tau}$ the first Betti cohomology group $\mathrm{H}_1(\tau(\widetilde{A_{\bar{s}}})(\mathbb{C}), \mathbb{Q})$ of the complex points of $\tau(\widetilde{A_{\bar{s}}})$. The Betti cohomology group $\mathrm{H}_{\bar{s},\tau}$ carries a natural Hodge structure. There is a canonical isomorphism

$$\mathrm{H}_{\bar{s},\tau} \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} \omega_{\bar{s},\mathbb{Q}_p}(\mathbb{D}(A)^\vee)$$

because $\omega_{\bar{s},\mathbb{Q}_p}(\mathbb{D}(A)^\vee)$ is canonically isomorphic to the first étale homology group of A.

(7.5.2) Denote by $\mathrm{GSp}_{\bar{s},\mathbb{Q}}$ the group of symplectic similitudes on $\mathrm{H}_{\bar{s},\tau}$ with respect to the symplectic pairing on $\mathrm{H}_{\bar{s},\tau}$, induced by the principal polarization on $A_{\bar{s}}$.

Recall that a connected reductive subgroup G of $\mathrm{GSp}_{\bar{s},\mathbb{Q}}$ over $\mathbb{Q}$ is said to be *of Hodge type* if there is a family of Hodge cycles $\{c_\alpha\}$ in tensor constructions of $\mathrm{H}_{\bar{s},\tau}$ such that G is the largest subgroup of $\mathrm{GSp}_{\bar{s},\mathbb{Q}}$ fixing each c_α. A reductive Lie subalgebra $\mathfrak{g}$ of $\mathrm{Lie}(\mathrm{GSp}_{\bar{s},\mathbb{Q}})$ is said to be *of Hodge type* if $\mathfrak{g}$ is a Hodge substructure of $\mathrm{Lie}(\mathrm{GSp}_{\bar{s},\mathbb{Q}})$. A connected reductive $\mathbb{Q}$-subgroup $\mathrm{GSp}_{\bar{s},\mathbb{Q}}$ is of Hodge type if and only if its Lie algebra is of Hodge type.

(7.6) Conjecture (i) *There exists a reductive subgroup $G_{\bar{s},\tau}$ of $\mathrm{GSp}_{\bar{s},\mathbb{Q}}$ over $\mathbb{Q}$ of Hodge type such that $G_{\bar{s},\tau} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{Q}_p$ satisfies the requirements of 7.4.*

(ii) *There exists a reductive Lie subalgebra $\mathfrak{g}_{\bar{s},\tau}$ over $\mathbb{Q}$ of Hodge type such that*

$$(\mathfrak{g}_{\bar{s},\tau} \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cap \mathrm{Hom}\big(U(A[p^\infty]_{\bar{s}})_{\mathbb{Q}_p}, V(A[p^\infty]_{\bar{s}})(1)_{\mathbb{Q}_p}\big) = N(A[p^\infty])(1)_{\bar{s}}$$

(7.6.1) Remark (i) By Deligne's theorem of absolute Hodge cycles on abelian varieties, the statements in 7.6 are independent of the choice of the complex embedding $\tau : W(\kappa(\bar{s})) \hookrightarrow \mathbb{C}$.

(ii) Clearly, Conj. 7.6 (i) implies Conj. 7.6 (ii).

(iii) Let G_1 be the smallest algebraic subgroup over $\mathbb{Q}$ of $\mathrm{GSp}_{\bar{s},\mathbb{Q}}$ which is of Hodge type and such that $\mathrm{Lie}(G_1) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ contains $N(A[p^\infty])(1)_{\bar{s}}$. Then G_1 is a candidate for 7.6 (i), and *Lie*(G_1) is a candidate for 7.6 (ii).

(7.7) Conjecture *Suppose that Conjecture 7.6 holds. Let ℓ be a prime number, $\ell \neq p$. For every geometric point $\bar{s}$ of S, denote by $\mathrm{Gal}_{\bar{s},\mathbb{Q}_\ell}(A)$ the ℓ-adic monodromy group of A with base point $\bar{s}$, namely the $\mathbb{Q}_\ell$-Zariski closure of the image of $\pi_1(A, \bar{s})$ in $\mathrm{GL}(H_{1,\text{ét}}(A, \mathbb{Q}_\ell))$. Then $G_{\bar{s},\tau} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{Q}_\ell$ is equal to $\mathrm{Gal}_{\bar{s},\mathbb{Q}_\ell}(A)^0$.*

(7.7.1) Remark Conjectures 7.6 and 7.7 combined can be thought of as a generalization of the Mumford-Tate conjecture for ordinary abelian varieties in characteristic p. A reformulation is that the group G_1 defined in 7.6.1 satisfy the statements in 7.6 and 7.7.

(7.8) Here some relations between the conjectures.

(1) Conj. 7.2 implies Conj. 7.1 (i).

(2) Conj. 7.6 (ii) implies Conj. 7.2.

(3) Conj. 7.6 implies Conj. 7.4.

(7.8.1) Remark

(i) Prop. 6.8 is the main ingredient of the implication (1), since it is not hard to check that the only non-empty Shimura subvariety of $\mathcal{M}_{/\overline{\mathbb{F}_p}}$ stable under all ℓ-adic Hecke correspondences is $\mathcal{M}_{/\overline{\mathbb{F}_p}}$.

(ii) The implications (2) and (3) are immediate.

(iii) It seems that Conj. 7.4 (ii), a semisimplicity statement about the naive p-adic monodromy group, offers a possible approach to the conjectures in this section. At the present time, Conj. 7.4 is known to hold only in a few instances. They include Igusa's theorem on the local monodromy at a supersingular elliptic curve, and also its generalization in [6].

(iv) Exploiting the action of the local automorphism group at a *basic* point of $Z_\ell(x_0)$, one can (often) show that $Z_\ell(x_0)$ is equal to $Z^{(p)}(x_0)$. In other words, the Zariski closure of the ℓ-adic Hecke orbit of x_0 in $\mathcal{M}_{/\overline{\mathbb{F}_p}}$ is equal to the Zariski closure of the prime-to-p Hecke orbit of x_0 in $\mathcal{M}_{/\overline{\mathbb{F}_p}}$. Therefore Conj. 7.2(ii) is equivalent to the apparently weaker statement, 7.2(i).

§8. Tate-linear subvarieties of Hilbert modular varieties

In this section we prove a special case of Conj. 7.2: Every Tate-linear subvariety of a Hilbert modular variety comes from a Shimura subvariety. This statement holds for Hilbert modular varieties attached to a product of totally real number fields, that is a product of a finite number of Hilbert modular varieties in the usual sense, each attached to a totally real number field. The proof uses a theorem of de Jong on extending homomorphisms between Barsotti-Tate groups. For simplicity of exposition we restrict to the traditional case of one totally real field.

(8.1) We fix a totally real number field F and a prime number p for the rest of this section.

(8.1.1) Fix an auxiliary integer $N \geq 3$, $(N, p) = 1$. Let $\mathcal{M}(F)_{/\mathbb{Z}} = \mathcal{M}(F)_{N/\mathbb{Z}}$ be the Hilbert modular scheme over $\operatorname{Spec}\mathbb{Z}$ classifying abelian schemes $A \to S$ of relative dimension $[F : \mathbb{Q}]$, with multiplication by $\mathcal{O}_F$ and a principal level-N structure as defined in [13]. For any scheme

T, a T-valued point of $\mathcal{M}(F)_{/\mathbb{Z}}$ is the isomorphism class of a triple (A, ι, η), where $A \to T$ is an abelian scheme over T of relative dimension $[F : \mathbb{Q}]$, $\iota : \mathcal{O}_F \to \mathrm{End}_T(A)$ is a ring homomorphism, and η is a level-N structure, such that the natural map

$$\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_F}(A, A^t) \otimes_{\mathcal{O}_F} A \to A^t$$

is an isomorphism. Let $\mathcal{M}(F) = \mathcal{M}(F)_{/\mathbb{F}_p} := \mathcal{M}(F)_{/\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{F}_p$ be the reduction of $\mathcal{M}(F)_{/\mathbb{Z}}$ modulo p, classifying abelian scheme in characteristic p with real multiplication by $\mathcal{O}_F$ as above.

(8.1.2) Denote by $\mathcal{M}(F)^{\mathrm{ord}}$ the *ordinary* locus of $\mathcal{M}(F)$, i.e. the dense open subscheme of $\mathcal{M}(F)$ whose points correspond to ordinary abelian varieties with multiplication by $\mathcal{O}_F$. We remark that for everu $A \to T$ in $\mathcal{M}(F)^{\mathrm{ord}}(T)$, the relative Lie algebra $\mathrm{Lie}(A/T)$ satisfies the freeness condition in [26]: It is a free $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_T$-module of rank one. It is well-known that $\mathcal{M}(F)^{\mathrm{ord}}$ is smooth of dimension $g = [F : \mathbb{Q}]$. Similarly, for every totally real field E, denote by $\mathcal{M}(E)$ the Hilbert modular scheme over $\mathbb{F}_p$ attached to E, and $\mathcal{M}(E)^{\mathrm{ord}}$ the ordinary locus in $\mathcal{M}(E)$.

(8.1.3) For any point $x \in \mathcal{M}(E)^{\mathrm{ord}}(\overline{\mathbb{F}_p})$, let A_x be the corresponding abelian variety with multiplication by $\mathcal{O}_E$. The Barsotti-Tate group $A_x[p^\infty]$ splits canonically as a direct sum

$$A_x[p^\infty] = \oplus_{\wp|p} A_x[\wp^\infty]\,,$$

where $\wp$ runs through primes ideals $\wp$ of $\mathcal{O}_E$ above p. Each $A_x[\wp^\infty]$ is a Barsotti-Tate group of height $2\,[\mathcal{O}_\wp : \mathbb{Z}_p]$, and sits in the middle of a short exact sequence

$$0 \to A_x[\wp^\infty]^{\mathrm{mult}} \to A_x[\wp^\infty] \to A_x[\wp^\infty]^{\text{ét}} \to 0\,,$$

where $A_x[\wp^\infty]^{\mathrm{mult}}$ and $A_x[\wp^\infty]^{\text{ét}}$ denotes the toric part and the maximal étale quotient of $A_x[\wp^\infty]$ respectively. The p-adic Tate module $\mathrm{T}_p(A_x[\wp^\infty]^{\text{ét}})$ of $A_x[\wp^\infty]^{\text{ét}}$ is a free $\mathcal{O}_\wp$-module of rank one, so is the character group of $A_x[\wp^\infty]^{\mathrm{mult}}$. The formal completion at x of the moduli scheme $\mathcal{M}(F) \times_{\mathrm{Spec}\,\mathbb{F}_p} \mathrm{Spec}\,\overline{\mathbb{F}_p}$ has a canonical product structure

$$(\mathcal{M}(F) \times_{\mathrm{Spec}\,\mathbb{F}_p} \mathrm{Spec}\,\overline{\mathbb{F}_p})^{/x} = \prod_{\wp|p} \underline{\mathrm{Hom}}_{\mathcal{O}_\wp}(\mathrm{T}_p(A_x[\wp^\infty]^{\text{ét}}), A_x[\wp^\infty]^{\mathrm{mult}})\,.$$

Notice that each factor $\underline{\mathrm{Hom}}_{\mathcal{O}_\wp}(\mathrm{T}_p(A_x[\wp^\infty]^{\text{ét}}), A_x[\wp^\infty]^{\mathrm{mult}})$ is an $[\mathcal{O}_{D,\wp} : \mathbb{Z}_p]$-dimensional formal torus, with a natural action by $\mathcal{O}_\wp$.

(8.2) Given a totally indefinite quaternion division algebra D over a totally real number field E, we choose and fix a maximal order $\mathcal{O}_D$ of D. Denote by $\mathcal{M}(D)_{/\mathbb{Z}}$ the moduli scheme such that for every scheme T, each element of $\mathcal{M}(D)_{/\mathbb{Z}}(T)$ corresponds to the isomorphism class of a triple $(A \to T, \iota, \eta)$, where

- $A \to T$ is an abelian scheme of relative dimension $2\,[E:\mathbb{Q}]$,
- $\iota : \mathcal{O}_D \to \mathrm{End}_T(A)$ is a ring homomorphism such that

$$\mathrm{Tr}_{\mathcal{O}_T}(\iota(d)|\,\mathrm{Lie}(A/T)) = \text{the image of } \mathrm{Tr}_{E/\mathbb{Q}}(\mathrm{Tr}^0_{D/E}(d)) \text{ in } \mathcal{O}_T \qquad \forall\, d \in \mathcal{O}_D,$$

  where $\mathrm{Tr}^0_{D/E} : D \to E$ is the reduced trace, and
- η is a principal level-N structure for some auxiliary integer $N \geq 3$ with $(N,p)=1$, not specified in the notation $\mathcal{M}(D)_{/\mathbb{Z}}$.

(8.2.1) Let $\mathcal{M}(D) = \mathcal{M}(D)_{/\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{F}_p$ be the reduction of $\mathcal{M}(D)_{/\mathbb{Z}}$ modulo p. Denote by $\mathcal{M}(D)^{\mathrm{ord}}$ the ordinary locus in $\mathcal{M}(D)$, that is the largest open subscheme of $\mathcal{M}(D)$ over which the universal abelian scheme is ordinary. It is known that the ordinary locus $\mathcal{M}(D)^{\mathrm{ord}}$ is non-empty if and only if the quaternion algebra D is unramified at every prime $\wp$ of $\mathcal{O}_E$ above p; see the proof of 8.3.1 for the "only if" part.

(8.2.2) Assume that the quaternion algebra D is unramified at every prime $\wp$ of $\mathcal{O}_E$ above p. For any point $x \in \mathcal{M}(D)^{\mathrm{ord}}(\overline{\mathbb{F}_p})$, let A_x be the corresponding abelian variety with multiplication by $\mathcal{O}_D$. The Barsotti-Tate group $A_x[p^\infty]$ splits canonically into a direct sum

$$A_x[p^\infty] = \oplus_{\wp|p}\, A_x[\wp^\infty]$$

where $\wp$ runs through primes $\wp$ of $\mathcal{O}_E$ above p. Each $A_x[\wp^\infty]$ is a Barsotti-Tate group of height $4\,[\mathcal{O}_\wp : \mathbb{Z}_p]$, and sits in the middle of a short exact sequence

$$0 \to A_x[\wp^\infty]^{\mathrm{mult}} \to A_x[\wp^\infty] \to A_x[\wp^\infty]^{\text{ét}} \to 0\,,$$

where $A_x[\wp^\infty]^{\mathrm{mult}}$ and $A_x[\wp^\infty]^{\text{ét}}$ denotes the multiplicative part and the maximal étale quotient of $A_x[\wp^\infty]$ respectively. The p-adic Tate module $\mathrm{T}_p(A_x[\wp^\infty]^{\text{ét}})$ of $A_x[\wp^\infty]^{\text{ét}}$ is a module over $\mathcal{O}_{D,\wp} := \mathcal{O}_D \otimes_{\mathcal{O}_E} \mathcal{O}_\wp$, isomorphic to the standard representation of $\mathcal{O}_{D,\wp} \cong M_2(\mathcal{O}_\wp)$. The same statement holds for the character group of $A_x[\wp^\infty]^{\mathrm{mult}}$. The formal completion at x of the moduli scheme $\mathcal{M}(F) \times_{\mathrm{Spec}\,\mathbb{F}_p} \mathrm{Spec}\,\overline{\mathbb{F}_p}$ has a canonical product structure

$$(\mathcal{M}(F) \times_{\mathrm{Spec}\,\mathbb{F}_p} \mathrm{Spec}\,\overline{\mathbb{F}_p})^{/x} = \prod_{\wp|p} \underline{\mathrm{Hom}}_{\mathcal{O}_{D,\wp}}(\mathrm{T}_p(A_x[\wp^\infty]^{\text{ét}}), A_x[\wp^\infty]^{\mathrm{mult}})\,.$$

Notice that each factor $\underline{\mathrm{Hom}}_{\mathcal{O}_{D,\wp}}(\mathrm{T}_p(A_x[\wp^\infty]^{\text{ét}}), A_x[\wp^\infty]^{\mathrm{mult}})$ is a $[\mathcal{O}_\wp : \mathbb{Z}_p]$-dimensional formal torus, with a natural action by $\mathcal{O}_\wp$.

(8.3) Let $Z \subseteq \mathcal{M}(F)^{\mathrm{ord}} \times_{\mathrm{Spec}\,\mathbb{F}_p} \mathrm{Spec}\,\overline{\mathbb{F}_p}$ be a smooth irreducible subscheme of the ordinary locus of $\mathcal{M}(F)_{\overline{\mathbb{F}_p}} = \mathcal{M}(F) \times_{\mathrm{Spec}\,\mathbb{F}_p} \mathrm{Spec}\,\overline{\mathbb{F}_p}$. We assume that $\dim(Z) \geq 1$, and that for each closed point $x \in Z$, the formal completion $Z^{/x}$ is a formal subtorus of the Serre-Tate formal torus $\mathcal{M}(F)^{/x}_{\overline{\mathbb{F}_p}}$. The goal of this section is to prove that Z is a Shimura subvariety of $\mathcal{M}(F)^{\mathrm{ord}}_{\overline{\mathbb{F}_p}}$.

The last statement means that there exists a Shimura subvariety W of $\mathcal{M}(F) \times_{\mathrm{Spec}\, Z} \mathrm{Spec}\, \overline{\mathbb{Q}}$ such that there is one irreducible component of the ordinary locus of the reduction modulo p of W which coincides with Z.

(8.3.1) Lemma *Let K be a field of characteristic p. Let A be an ordinary abelian variety over a field K, with $\dim(A) = [F : \mathbb{Q}]$. Suppose that there exists an embedding $F \hookrightarrow \mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Assume moreover that $\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ does not contain any commutative semisimple subalgebra of dimension $2\dim(A)$ over $\mathbb{Q}$. Then A is K-isogenous to B^n, where n is a positive integer and B is a K-simple abelian variety over K. Let $D = \mathrm{End}_K(B) \otimes_{\mathbb{Z}} \mathbb{Q}$, so that $\mathrm{End}_K(A) \otimes_{\mathbb{Q}} \cong \mathrm{M}_n(D)$. There are two possibilities.*

- (Type I) *The algebra D is a totally real number field E, $[E : \mathbb{Q}] = \dim(B)$, F contains E, and $[F : E] = n$.*
- (Type II) *The algebra D is a totally indefinite quaternion division algebra over a totally real number field E, $\dim(B) = 2\,[E : \mathbb{Q}]$, F contains E, and $[F : E] = 2n$. Moreover the quaternion algebra D is unramified at every prime $\wp$ of $\mathcal{O}_E$ above p.*

Proof. Everything except the last sentence in the (Type II) case follows from [5] p. 464, Lemma 6. The possibilities of (Type III) and (Type IV) do not occur because A is ordinary and does not have "sufficiently many complex multiplications".

Suppose that D is of type III. We may and do assume that $\mathcal{O}_E$ operates on B. Then for every prime $\wp$ of E above p, the ordinary Barsotti-Tate group $B[\wp^\infty]$ has height $4\,[E_\wp : \mathbb{Q}_p]$, and its multiplicative part and eétale quotient are Barsotti-Tate groups of height $2\,[E_\wp : \mathbb{Q}_p]$. So the injection $D \otimes_E E_\wp \hookrightarrow \mathrm{End}_K(B[\wp^\infty]) \otimes_{\mathbb{Z}} \mathbb{Q}$ forces D to be unramified at $\wp$. ∎

(8.3.2) Remark In Lemma 6 on p. 464 of [5], the case of Type III (b) does not occur under the assumptions there, because the totally real field F cannot be embedded into the symmetric part of $M_n(D)$.

(8.3.3) Proposition *Let Z be an irreducible smooth Tate-linear subscheme of the ordinary locus $\mathcal{M}(F)^{\mathrm{ord}} \times_{\mathrm{Spec}\, \mathbb{F}_p} \mathrm{Spec}\, \overline{\mathbb{F}_p}$ of the Hilbert modular scheme as in the beginning of 8.3.*

(I) *Suppose that $\mathrm{End}_Z(A_{/Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathrm{M}_n(E)$ for a subfield E of F as in the Type I case of Lemma 8.3.1. Then there exists a finite flat correspondence*

$$\mathcal{M}(E)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}} \overset{\pi}{\leftarrow} \widetilde{\mathcal{M}(E)}^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}} \overset{\beta}{\rightarrow} \mathcal{M}(F)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}}$$

from $\mathcal{M}(E)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}} \overset{\pi}{\leftarrow}$ to $\mathcal{M}(F)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}}$ and an irreducible smooth subvariety W of $\mathcal{M}(E)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}}$ satisfying the following properties.

(i) *The morphism π is the projection map of a Igusa-type level-structure. In particular it is a finite flat morphism.*

(ii) *The morphism β is a finite.*

(iii) *The subvariety W is Tate-linear. That is for each closed point $x \in W(\overline{\mathbb{F}_p})$, the formal completion is $W^{/x}$ is a formal subtorus of the Serre-Tate torus reviewed at the end of 8.1.3.*

(iv) *There exists an irreducible component $\widetilde{W}$ of $\pi^{-1}(W)$ such that $\beta(W) = Z$.*

(v) *Let $B \to W$ be the restriction to W of the universal abelian scheme over $\mathcal{M}(E)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}}$. Then $\mathrm{End}_W(B) \otimes_{\mathbb{Z}} \mathbb{Q} = E$.*

(II) *Suppose that $\mathrm{End}_Z(A_{/Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathrm{M}_n(D)$ for a totally indefinite quaternion division algebra over a subfield E of F as in the Type II case of Lemma 8.3.1. Then there exists a finite correspondence*

$$\mathcal{M}(D)^{or}_{/\overline{\mathbb{F}_p}} \xleftarrow{\pi} \widetilde{\mathcal{M}(D)}^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}} \xrightarrow{\beta} \mathcal{M}(F)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}}$$

from $\mathcal{M}(D)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}} \xleftarrow{\pi}$ to $\mathcal{M}(F)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}}$ and an irreducible smooth Tate-linear subvariety W of $\mathcal{M}(D)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}}$ satisfying the following properties.

(i) *The morphism π is the projection map of a Igusa-type level-structure. In particular it is a finite flat morphism.*

(ii) *The morphism β is a finite.*

(iii) *The subvariety W is Tate-linear. That is for each closed point $x \in W(\overline{\mathbb{F}_p})$, the formal completion is $W^{/x}$ is a formal subtorus of the Serre-Tate torus reviewed at the end of 8.2.2.*

(iv) *There exists an irreducible component $\widetilde{W}$ of $\pi^{-1}(W)$ such that $\beta(W) = Z$.*

(v) *Let $B \to W$ be the restriction to W of the universal abelian scheme over $\mathcal{M}(E)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}}$. Then $\mathrm{End}_W(B) \otimes_{\mathbb{Z}} \mathbb{Q} = D$.*

PROOF. This Proposition is essentially a corollary of Lemma 8.3.1. ∎

(8.3.4) Remark Prop. 8.3.3 is a reduction step for showing that the Tate-linear subvariety Z of $\mathcal{M}(F)^{\mathrm{ord}}$ is the reduction of a Shimura subvariety. Roughly, 8.3.3 produces the smallest Shimura subvariety of $\mathcal{M}(F)^{\mathrm{ord}}$ containing Z, which is defined by endomorphisms of abelian varieties. The two cases of Prop. 8.3.3 will be referred as Case I and Case II respectively.

We record a technical result 8.4 on Barsotti-Tate groups before giving further properties of the Tate-linear subvariety W in Prop. 8.3.3.

(8.4) Proposition *Let R be a reduced excellent commutative ring such that $p = 0$ in R. Let $S = \mathrm{Spec}\, R$.*

(i) *Let $X \to S$ be a truncated Barsotti-Tate group of level n over S, which is an extension of an étale BT_n group $X^{\text{ét}} \to S$ by a toric BT_n group $X^{\text{mult}} \to S$. Suppose that s is a point of S such that the extension*

$$0 \to X^{\text{mult}} \times_S S^{/s} \to X \times_S S^{/s} \to X^{\text{ét}} \times_S S^{/s} \to 0$$

splits over the formal completion of S at s. Then there exists an open neighborhood U of s in S such that the extension

$$0 \to X^{\text{mult}} \times_S U \to X \times_S U \to X^{\text{ét}} \times_S U \to 0$$

splits over U. Notice that the above splitting is unique because $\mathcal{O}_S$ is reduced.

(ii) *Let $Y \to S$ be a Barsotti-Tate group over S which is an extension of an étale BT group $Y^{\text{ét}} \to S$ by a toric BT group $Y^{\text{mult}} \to S$. Suppose that s is a point of S such that the extension*

$$0 \to Y^{\text{mult}} \times_S S^{/s} \to Y \times_S S^{/s} \to Y^{\text{ét}} \times_S S^{/s} \to 0$$

splits over the formal completion of S at s. Then there exists an open neighborhood U of s in S such that the extension

$$0 \to Y^{\text{mult}} \times_S U \to Y \times_S U \to Y^{\text{ét}} \times_S U \to 0$$

splits over U. The above splitting is unique because $\mathcal{O}_S$ is reduced.

PROOF. Clearly the statement (ii) follows from the statement (i). So it suffice to prove the statement (i). According to Artin's approximation theorem in [1], the assumption of (i) tells us that there exists an étale neighborhood $U_1 \to S$ of s such that

$$0 \to X^{\text{mult}} \times_S U_1 \to X \times_S U_1 \to X^{\text{ét}} \times_S U_1 \to 0$$

splits over U_1. The two pull-backs of the splitting over U_1 to $U_1 \times_S U_1$ by the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : U_1 \times_S U_1 \to U_1$ coincide since the $U_1 \times_S U_1$ is reduced. So the unique splitting over U_1 descends to a splitting over a Zariski neighborhood U of s. Another way to prove (i) is to apply the fpqc decent to $S^{/s} \to \operatorname{Spec} \mathcal{O}_{S,s}$, using the fact that $\mathcal{O}^{\wedge}_{S,s} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^{\wedge}_{S,s}$ is reduced, instead of the approximation theorem. Here $\mathcal{O}^{\wedge}_{S,s}$ denotes the completion of the local ring $\mathcal{O}_{S,s}$. ∎

(8.4.1) Remark In the case when R is a reduced excellent local ring containing $\mathbb{F}_p$, $X^{\text{ét}}$ is the constant group $\mathbb{Z}/p^n\mathbb{Z}$ over $\operatorname{Spec} R$ and $X^{\text{mult}} = \mu_{p^n}$ over $\operatorname{Spec} R$, Prop. 8.4 (i) becomes the following consequence of Lemma 2.1.6. If a unit $u \in R$ is equal to the p^n-th power of an element in the formal completion $R^{\wedge}$ of R, then there exists an element $v \in R$ such that $v^{p^n} = u$.

(8.5) Proposition *Notation as in Prop. 8.3.3.*

(i) *In Case I, there exists a dense open subscheme U of W and a non-empty subset J of the set of all primes of $\mathcal{O}_E$ above p, such that*

$$W^{/x} = \prod_{\wp\in J} \underline{\mathrm{Hom}}_{\mathcal{O}_\wp}(\mathrm{T}_p(B_x[\wp^\infty]^{\text{ét}}), B_x[\wp^\infty]^{\text{mult}})$$

for each closed point $x \in U(\overline{\mathbb{F}_p})$. Here B_x denotes the fiber at x of the universal abelian scheme $B \to \mathcal{M}(E)$.

(ii) *In Case II, there exists a dense open subscheme U of W and a non-empty subset J of the set of all primes of $\mathcal{O}_E$ above p, such that*

$$W^{/x} = \prod_{\wp\in J} \underline{\mathrm{Hom}}_{\mathcal{O}_{D,\wp}}(\mathrm{T}_p(B_x[\wp^\infty]^{\text{ét}}), B_x[\wp^\infty]^{\text{mult}})$$

for each closed point $x \in U(\overline{\mathbb{F}_p})$. Here B_x denotes the fiber at x of the universal abelian scheme $B \to \mathcal{M}(D)$.

PROOF. We provide a proof for Case I; the proof for Case II is similar and is left to the reader. The argument consists of an application of Zarhin's theorem on Tate's conjecture, combined with Chabotarev's density theorem. See [27], [28] for Zarhin's theorem, and the remark in 8.5.1 (i).

In Case I we are given an irreducible Tate-linear subvariety W in $\mathcal{M}(E)^{\text{ord}} \times_{\operatorname{Spec}\mathbb{F}_p} \operatorname{Spec}\overline{\mathbb{F}_p}$ such that endomorphism ring of the restriction $B \to W$ to W of the universal abelian scheme over $\mathcal{M}(E)$ is equal to $\mathcal{O}_E$. Let $\ell \neq p$ be a prime number. Let q be a power of p such that the subvariety W of $\mathcal{M}(E)^{\text{ord}}$ is the base change from $\mathbb{F}_q$ to $\overline{\mathbb{F}_p}$ of a subscheme W_1 of $\mathcal{M}(E)^{\text{ord}} \times_{\operatorname{Spec}\mathbb{F}_p} \operatorname{Spec}\mathbb{F}_q$. Let η_1 be the generic point of W_1, and let $\overline{\eta}$ be a geometric generic point of W. Let $\rho_\ell : \operatorname{Gal}(\overline{\eta}/\eta_1) \to \mathrm{GL}_{\mathcal{O}_E\otimes_\mathbb{Z}\mathbb{Z}_\ell}(\mathrm{T}_\ell(B_{\overline{\eta}}))$ be the Galois representation attached to the ℓ-adic Tate module of $B \to W_1$.

By Zarhin's theorem, the Zariski closure of the image of ρ_ℓ in $\mathrm{GL}_{E\otimes\mathbb{Q}_\ell}(\mathrm{V}_\ell(B_{\overline{\eta}})) \cong \mathrm{GL}_2(E \otimes_\mathbb{Q} \mathbb{Q}_\ell)$ is a reductive subgroup G_ℓ of $\mathrm{GL}_{\mathcal{O}_E\otimes_\mathbb{Z}\mathbb{Z}_\ell}(\mathrm{T}_\ell(B_{\overline{\eta}}))$, and $\mathrm{End}_{G_\ell}(\mathrm{V}_\ell(B_{\overline{\eta}})) = E \otimes_\mathbb{Q} \mathbb{Q}_\ell$. Hence the derived group of G_ℓ is equal to $\mathrm{SL}_{\mathcal{O}_E\otimes_\mathbb{Z}\mathbb{Z}_\ell}(\mathrm{T}_\ell(B_{\overline{\eta}}))$

A standard argument, using [4, §7, Cor. 7.9], shows that the image of ρ_ℓ is an open subgroup of $G_\ell(\mathbb{Q}_\ell)$. By Chabotarev's density theorem, the subset $\Sigma \subset |W_1|$ consisting of all closed points w of W_1 such that the Zariski closure of the subgroup generated by Fr_w is a maximal torus of G_ℓ^0 has positive density.

Let y be a closed point in the subset Σ above. The formal completion $W_1^{/y}$ of W_1 at y is a formal subtorus of the Serre-Tate formal torus

$$\mathcal{M}(E)^{/y} = \prod_{\wp|p} \underline{\mathrm{Hom}}_{\mathcal{O}_\wp}(\mathrm{T}_p(B_y[\wp^\infty]^{\text{ét}}), B_y[\wp^\infty]^{\text{mult}})\,.$$

Both formal tori above are defined over the finite field $\kappa(y)$. So the cocharacter group $X_*(W_1^{/y})$ of $W_1^{/y}$, a subgroup of the cocharacter group $X_*(\mathcal{M}(E)^{/y})$ of $\mathcal{M}(E)^{/y}$, is stable under the action of the Frobenius element Fr_y.

Let $q_y = \mathrm{Card}(\kappa(y))$. The Frobenius element Fr_y generates a commutative semisimple subalgebra $\mathbb{Q}(\mathrm{Fr}_y)$ of $\mathrm{End}^0_{\kappa(y)}(B_y) := \mathrm{End}_{\kappa(y)}(B_y) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $L := E(\mathrm{Fr}_y)$ is a totally imaginary quadratic extension of the totally real field E. Let H be the linear algebraic group over $\mathbb{Q}$ such that $H(\mathbb{Q}) = (\mathrm{End}^0_{\kappa(y)}(B_y))^\times$, and let T be the $\mathbb{Q}$-Zariski closure of the cyclic subgroup $\mathrm{Fr}_y^{\mathbb{Z}}$ of $H(\mathbb{Q})$ generated by Fr_y. Since $\mathrm{Fr}_y \in H(\mathbb{Q})$, $T \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{Q}_{\ell'}$ is equal to the $\mathbb{Q}_{\ell'}$-Zariski closure of $\mathrm{Fr}_y^{\mathbb{Z}}$ in $H \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{Q}_{\ell'}$, for all prime numbers ℓ', including p. We know that the $\mathbb{Q}_\ell$-Zariski closure of the cyclic subgroup $\mathrm{Fr}_y^{\mathbb{Z}}$ of $\mathrm{GL}_{E\otimes_{\mathbb{Q}}\mathbb{Q}_\ell}(\mathrm{V}_\ell(B_y))(\mathbb{Q}_\ell)$ is a maximal torus of a reductive subgroup G_ℓ of $\mathrm{GL}_{E\otimes_{\mathbb{Q}}\mathbb{Q}_\ell}(\mathrm{V}_\ell(B_y))$, and the derived group of G_ℓ is equal to $\mathrm{SL}_{E\otimes_{\mathbb{Q}}\mathbb{Q}_\ell}(\mathrm{V}_\ell(B_y))$. This implies that the $\mathbb{Q}$-torus T contains the norm-one torus T_1 of the induced torus $\mathrm{Res}_{L/\mathbb{Q}}(\mathbb{G}_{\mathrm{m}})$, characterized by the property that $T_1(\mathbb{Q})$ consists of all elements $t \in L^\times$ such that $\mathrm{Nm}_{L/E}(t) = 1$.

Since the abelian variety B_y is ordinary, the quadratic extension L of E splits over every prime ideal $\wp$ of $\mathcal{O}_E$ containing p Write $L_\wp = L_{\tilde\wp_1} \times L_{\tilde\wp_2}$, where $\tilde\wp_1, \tilde\wp_2$ are the two prime ideals of $\mathcal{O}_L$ above $\wp$, such that the image of Fr_y in $L_\wp$ has the form $(u_\wp, u'_\wp)$, with $u_\wp \in \mathcal{O}^\times_{\tilde\wp_1}$ and $u'_\wp \in q_y \cdot \mathcal{O}^\times_{\tilde\wp_2}$. We have canonical isomorphisms $E_\wp \xrightarrow{\sim} L_{\tilde\wp_i}$, $i = 1, 2$. Under these canonical isomorphisms, we can regard $u_\wp$ as a unit of $\mathcal{O}_\wp$, and identify $u'_\wp$ with the element $q_y \cdot u_\wp^{-1}$ in $\mathcal{O}_\wp$. Recall that the p-adic Tate module of $B_x[\wp^\infty]^{\text{ét}}$ and the cocharacter group of $B_x[\wp^\infty]^{\text{mult}}$ are both free $\mathcal{O}_\wp$-modules of rank one. The Frobenius element Fr_y operates on $\mathrm{T}_p(B_x[\wp^\infty]^{\text{ét}})$ via the unit $u_\wp$ of $\mathcal{O}_\wp$, and it operates on the cocharacter group of $B_x[\wp^\infty]^{\text{mult}}$ via the element $u'_\wp \in \mathcal{O}^\times_\wp$. We refer to [11] for the facts used here.

For each prime ideal $\wp$ of $\mathcal{O}_E$, let $\mathrm{R}_\wp$ be the induced torus $\mathrm{Res}_{E_\wp/\mathbb{Q}_p}(\mathbb{G}_{\mathrm{m}})$ over $\mathbb{Q}_p$. Let $\mathrm{R}_p = \prod_{\wp|p} \mathrm{R}_\wp$. The fact that the $\mathbb{Q}$-torus T contains the norm-one torus T_1 implies that the cyclic subgroup $u_p := (u_\wp) \in (\prod_{\wp|p} \mathrm{R}_\wp)(\mathbb{Q}_p) = \mathrm{R}_p(\mathbb{Q}_p)$ is Zariski dense in R_p. Hence the cyclic subgroup generated by u_p^2 is also Zariski dense in T_p.

Recall that $W_1^{/y}$ is a formal subtorus of the Serre-Tate formal torus

$$\prod_{\wp|p} \underline{\mathrm{Hom}}_{\mathcal{O}_\wp}(\mathrm{T}_p(B_y[\wp^\infty]^{\text{ét}}), B_y[\wp^\infty]^{\text{mult}})\,.$$

over $\kappa(y)$, hence its cocharacter group is a $\mathbb{Z}_p$-direct summand of the cocharacter group of the Serre-Tate formal torus stable under the action of the Frobenius element Fr_y. Moreover the cocharacter group $\mathrm{X}_*(\mathcal{M}(E)^{/y})$ of the Serre-Tate formal torus has a natural structure as a free $\prod_{\wp|p} \mathcal{O}_\wp$-module of rank one. We have seen that Fr_y operates on the Serre-Tate formal torus via the element $u_p^2 \in \prod_{\wp|p} \mathcal{O}_\wp^\times \subset T_p(\mathbb{Q}_p)$. So the Zariski density of u_p^2 in R_p implies that the cocharacter group $\mathrm{X}_*(W_1^{/y})$ of the formal torus $W_1^{/y}$, as a $\mathbb{Z}_p$-direct summand of the cocharacter group of $\mathrm{X}_*(\mathcal{M}(E)^{/y})$ of the Serre-Tate formal torus, and is stable under the action of $\prod_{\wp|p} \mathcal{O}_\wp$. Therefore there exists a non-empty subset J of primes of $\mathcal{O}_E$ above p such that $\mathrm{X}_*(W_1^{/y}) = \left(\prod_{\wp\in J} \mathcal{O}_\wp\right) \cdot \mathrm{X}_*(\mathcal{M}(E)^{/y})$. Equivalently, $W_1^{/y} = \prod_{\wp\in J} \underline{\mathrm{Hom}}_{\mathcal{O}_\wp}(\mathrm{T}_p(B_y[\wp^\infty]^{\text{ét}}), B_y[\wp^\infty]^{\text{mult}})$. An application of Prop. 8.4 finishes the proof of 8.5. ∎

(8.5.1) Remark (i) The main results in [27], [28] are stated for function fields of characteristic $p \neq 2$. The restriction $p \neq 2$ can be removed; see Thm. 4.7, chap. V of [14].

(ii) Actually the statements (i), (ii) hold for all point $x \in W(\overline{\mathbb{F}_p})$, but we do not need this fact for Thm. 8.6.

(iii) After Prop. 8.5, our goal is to show that the subset J is equal to the set of all primes of $\mathcal{O}_E$ above p. Clearly this implies that the Tate-linear subvariety $Z \subseteq \mathcal{M}(F)^{\mathrm{ord}}_{/\overline{\mathbb{F}_p}}$ comes from the Shimura subvariety $\mathcal{M}(E)^{\mathrm{ord}}$ in Case I, and from the Shimura subvariety $\mathcal{M}(D)^{\mathrm{ord}}$ in Case II.

(8.6) Theorem *Notation as in 8.3.3 and 8.5. Then the subset* $J \subseteq \{\wp : \wp \text{ lies above } p\}$ *is equal to the set of all prime ideals of* $\mathcal{O}_E$ *containing* p. *In other words, the linear subvarieties* U *and* W *are both open subscheme of* $\mathcal{M}(E)^{\mathrm{ord}}$ *in Case I, and are open subschemes of* $\mathcal{M}(D)^{\mathrm{ord}}$ *in Case II.*

PROOF. We give a proof for Case I here; the same argument applies to Case II after obvious modifications. After shrinking Z and W in 8.3.3 and 8.5, we may and do assume that $U = W$.

Let J' be the set of all prime ideals $\wp$ of $\mathcal{O}_E$ above p such that $\wp \notin J$, and we assume that $J' \neq \emptyset$. Let $B[p^\infty] \to W$ be the Barsotti-Tate group attached to the restriction to W of the universal abelian variety $B \to \mathcal{M}(E)$. For each prime $\wp$ of $\mathcal{O}_E$, denote by $Y_\wp \to W$ the factor $(B \to W)[\wp^\infty]$ of $B[p^\infty] \to W$. Consider the decomposition

$$B[p^\infty]_{/W} = \left(\prod_{\wp \in J} Y_\wp\right) \oplus \left(\prod_{\wp \in J'} Y_\wp\right)$$

of the Barsotti-Tate group $B[p^\infty] \to W$ over W. For each closed point x of W and each $\wp \in J'$, the Barsotti-Tate group $Y_\wp \times_W W^{/x}$ over the formal completion of $W^{/x}$ splits into the direct sum of its toric part and its maximal étale quotient. By Prop. 8.4, we deduce that $Y_\wp$ splits uniquely as the sum of its multiplicative part and its maximal étale quotient. This splitting of $Y_\wp$ gives two orthogonal idempotents $e_{\wp,\mathrm{mult}}$ and $e_{\wp,\text{ét}}$ in $\mathrm{End}_W(Y_\wp)$, with the following properties.

- The idempotents $e_{\wp,\mathrm{mult}}$ and $e_{\wp,\text{ét}}$ commute with the action of $\mathcal{O}_\wp$ on $Y_\wp$,
- $e_{\wp,\mathrm{mult}} + e_{\wp,\text{ét}} = \mathrm{Id}_{Y_\wp}$,
- The image of $e_{\wp,\mathrm{mult}}$ is the multiplicative part of $Y_\wp$, and the image of $e_{\wp,\text{ét}}$ is naturally isomorphic to the maximal étale quotient of $Y_\wp$.

In particular, we see that

$$\mathrm{End}_W(B[p^\infty]) \supseteq (\oplus_{\wp \in J} \mathcal{O}_\wp) \oplus (\oplus_{\wp \in J'} (\mathcal{O}_\wp \cdot e_{\wp,\mathrm{mult}} \oplus \mathcal{O}_\wp \cdot e_{\wp,\text{ét}})) \supsetneq \oplus_{\wp | p}\, \mathcal{O}_\wp = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p\,.$$

On the other hand, Theorem 2.6 of [10] tells us that

$$\mathrm{End}_W(B[p^\infty]) = \mathrm{End}_W(B) \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p \,.$$

This is a contradiction, hence $J' = \emptyset$. ∎

(8.6.1) Remark It is known that $\mathcal{M}(E)^{\mathrm{ord}}$ is dense in $\mathcal{M}(E)$ and is geometrically irreducible; see [13]. Hence U and W are both dense open subschemes of $\mathcal{M}(E)^{\mathrm{ord}} \times_{\mathrm{Spec}\,\mathbb{F}_p} \mathrm{Spec}\,\overline{\mathbb{F}_p}$. It is a folklore that the method of [13] can be used to show, as a "routine exercise", that $\mathcal{M}(D)^{\mathrm{ord}}$ is geometrically irreducible if D is unramified at every prime $\wp$ of $\mathcal{O}_E$ above p. However this statement does not seem to have been documented in the literature.

(8.6.2) Remark The method used in the proof of Thm. 8.6 can be applied to prove the other conjectures in §7 in the case of Hilbert modular varieties; that is, when the base scheme S is contained in the ordinary locus $\mathcal{M}(F)^{\mathrm{ord}}$ of a Hilbert modular variety, and the family of ordinary abelian varieties is the restriction to S of the universal abelian scheme over $\mathcal{M}(F)^{\mathrm{ord}}$. We leave the proof to the interested readers.

References

[1] M. Artin. Algebraic approximation of structures over complete local rings. *Publ. Math. I.H.E.S.*, 36(23–58), 1969.

[2] P. Berthelot. Cohomologie rigide et cohomologie à support propres. première partie (version provisoire 1991). Jan. 1996.

[3] P. Berthelot, L. Breen, and W. Messing. *Théorie de Dieudonné Cristalline II*, volume 930. Springer-Verlag, 1982.

[4] A. Borel. *Linear Algebraic Groups.* Benjamin, 1969. Notes taken by Hyman Bass.

[5] C.-L. Chai. Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli. *Invent. Math.*, 121:439–479, 1995.

[6] C.-L. Chai. Local monodromy for deformations of one-dimensional formal groups. *J. reine angew. Math.*, 524:227–238, 2000.

[7] R. Crew. F-isocrystals and their monodromy groups. *Ann. Sci. Éc. Norm. Sup.*, 25:429–464, 1992.

[8] A. J. de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Publ. Math. IHES*, 82:5–96, 1995.

[9] A. J. de Jong. Smoothness, semi-stability and alterations. *Publ. Math. IHES*, 83:51–93, 1996.

[10] A. J. de Jong. Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic. *Invent. Math.*, 134:301–333, 1998.

[11] P. Deligne. Variétés abéliennes ordinaires sur un corps fini. *Invent. Math.*, 8:238–243, 1969.

[12] P. Deligne and L. Illusie. Cristaux ordinaire et coordonées canoniques, with an appendix by N. Katz. In J. Giraud, L. Illusie, and M. Raynaud, editors, *Surfacecs Algébriques*, volume 868 of *Lecture Notes in Mathematics*, pages 80–127. Springer-Verlag, 1981. Séminaire de Géométrie Algébrique d'Orsay 1976–78, Exposé V.

[13] P. Deligne and G. Pappas. Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant. *Compos. Math.*, 90:59–79, 1994.

[14] G. Faltings and C.-L. Chai. *Degeneration of Abelian Varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiet, 3 Folge*. Springer-Verlag, 1990.

[15] A. Grothendieck. *Groupes de Barsotti-Tate et Cristaux de Dieudonné*, volume 45 of *Sém. Math. Sup.* Presses de l'Université de Montreal, 1974.

[16] A. Grothendieck and J. Dieudonné. *Elément de Géométrie Algébrique (EGA)*, volume 4, 8, 11, 17, 20, 24, 28, 32 of *Publ. Math. IHES*. IHES, 1961–1967.

[17] L. Illusie. Déformation de groupes de Barsotti-Tate. In L. Szpiro, editor, *Séminaire Sur Les Pinceaux Arithm'etiques: La Conjecture De Mordell*, volume 127 of *Astérisque*, pages 151–198, 1985.

[18] N. M. Katz. Serre-Tate local moduli. In *Surface Algébriques*, volume 868 of *Lecture Notes in Math.*, pages 138–202. Springer-Verlag, 1981. Séminaire de Géométrie Algébrique d'Orsay 1976-78, Exppposé Vbis.

[19] R. E. Kottwitz. Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.*, 2:373–444, 1992.

[20] H. Matsumura. *Commutative Algebra*. Benjamin/Cummings, 1980.

[21] W. Messing. *The Crystals Associated to Barsotti-Tate Groups: with Applications to Abelian Schemes*, volume 264 of *Lecture Notes in Math.* Springer-Verlag, 1972.

[22] B. Moonen. *Special Points and Linearity Properties of Shimura Varieties.* PhD thesis, University Utrecht, 1995.

[23] B. Moonen. Linearity properties of shimura varieties. I. *J. Alg. Geom.*, 7:539–567, 1998.

[24] B. Moonen. Linearity properties of shimura varieties. II. *Compositio Math.*, 114:3–35, 1998.

[25] A. Ogus. F-isocrystals and de Rham cohomology II–convergent isocrystals. *Duke Math. J.*, 51(4):765–850, 1984.

[26] M. Rapoport. Compactifications de l'espace de modules de Hilbert-Blumenthal. *Compos. Math.*, 36:255–335, 1978.

[27] J. G. Zarhin. Isogenies of abelian varieties over fields of finite characteristics. *Math. USSR Sbornik*, 24:451–461, 1974.

[28] J. G. Zarhin. Endomorphisms of abelian varieties over fields of finite characteristics. *Math. USSR Izvestija*, 9(2):255–260, 1975.