Hypersymmetric abelian varieties

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§1. Notation

(1.1) Let (A, λ) be a g-dimensional polarized abelian varieties over $\overline{\mathbb{F}_p}$. Assume that A is *hypersymmetric*, i.e.

$$\mathrm{End}_{\overline{\mathbb{F}_p}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{F}_p}}(A[p^\infty])\,,$$

or equivalently

$$\dim_{\mathbb{Q}} \left(\mathrm{End}_{\overline{\mathbb{F}_p}}(A)\right) = \dim_{\mathbb{Q}_p} \left(\mathrm{End}_{\overline{\mathbb{F}_p}}(A[p^\infty])\right)\,,$$

Here $A[p^\infty]$ is the Barsotti-Tate group attached to A.

(1.1.1) Let $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_a \le 1$ be the Frobenius slopes of A, and let n_j be the multiplicity of the slope λ_j, $j = 1, \ldots, a$, in the sense that the height of the slope-λ_j-part of $A[p^\infty]$ is equal to $n_j d_j$, where d_j is the denominator of λ_j. It is well-known that $\lambda_j + \lambda_{a+1-j} = 1$ and $n_j = n_{a+1-j}$ for all $j = 1, \ldots, a$. The above duality statements implies that none of the slopes is equal to $\frac{1}{2}$ if a is even, while $\lambda_{\frac{a+1}{2}} = \frac{1}{2}$ if a is odd.

(1.1.2) By the Manin-Dieudonné classification, the Barsotti-Tate group of $A[p^\infty]$ of A decomposes up to isogeny to isoclinic pieces X_j, where X_j is a Barsotti-Tate group over $\overline{\mathbb{F}_p}$ with slope λ_j and height $n_j d_j$.

(1.1.3) It is well-known that $\mathrm{End}_{\overline{\mathbb{F}_p}}(X_j) \otimes \mathbb{Q}_p$ is isomorphic to $\mathrm{M}_{n_j}(D_j)$, where D_j is the central division algebra over $\mathbb{Q}_p$ with Brauer invariant $-\lambda_j$, $j = 1, \ldots, a$.

(1.2) Let $E = \mathrm{End}_{\overline{\mathbb{F}_p}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $\mathrm{Z}(E)$ be the center of E. According to the p-adic Tate conjecture, proved for $\overline{\mathbb{F}_p}$ by Tate, we have a canonical isomorphism $\mathrm{End}_{\overline{\mathbb{F}_p}}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{F}_p}}(A[p^\infty])$, hence $E \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{F}_p}}(A[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \prod_{j=1}^{a} \mathrm{M}_{n_j}(D_j)$. Using the theorem of Honda-Tate, one easily verify the following statements.

(i) $[\mathrm{Z}(E) : \mathbb{Q}] = a$, and the prime p splits completely in E. In other words, $\mathrm{Z}(E) \otimes \mathbb{Q}_p$ is isomorphic to the product of a copies of $\mathbb{Q}_p$.

(ii) If a is even, then the center $\mathrm{Z}(E)$ of E is a product of CM-fields K_i, $i = 1, \ldots, b$, and $[\mathrm{Z}(E) : \mathbb{Q}] = a$. The semisimple algebra E decomposes into a product $\prod_{i=1}^{b} E_i$, where each E_i is a central simple algebra over K_i.

(iii) If a is odd, then E splits into a product of two semisimple algebras E_0 and E' over $\mathbb{Q}$, such that E_0 is isomorphic to $\mathrm{M}_m(D)$, where $m = n_{\frac{a+1}{2}}$, and D is the quaternion division algebra over $\mathbb{Q}$ ramified only at p and ∞. The center $\mathrm{Z}(E')$ of E' is a product of CM-fields K_i, $i = 1, \dots, b$, and $[\mathrm{Z}(E') : \mathbb{Q}] = a - 1$. The semisimple algebra E' is a direct product of central simple algebras E_i over K_i, $i = 1, \dots, b$

(iv) Notation as in (ii) and (iii) above. For each $i = 1, \dots, b$ and each place $\wp$ of K_i above p, there exists a unique integer $j = j(i, \wp)$, $1 \le j \le a$, $2j \ne a + 1$, such that $E_i \otimes_{K_i} K_{i,\wp} \cong \mathrm{M}_{n_j}(D_j)$ in the decomposition $E \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{j=1}^{a} M_{n_j}(D_j)$. The set of natural numbers $\{j \,|\, 1 \le j \le a,\ 2j \ne a+1\}$ is the disjoint union of singletons $\{j(i, \wp)\}$, where $i = 1, \dots, b$ and $\wp$ runs through all places of K_i above p.

(1.3) Let $*$ be the Rosati involution on E induced by the polarization λ of A. It is well-known that the Rosati involution is positive definite. Let $G = \mathrm{GU}(E, *)$ be the group of unitary similitudes attached to $(E, *)$, defined as follows. For every commutative $\mathbb{Q}$-algebra R,

$$G(R) = \left\{(x, a) \in (E \otimes_{\mathbb{Q}} R)^\times \times R^\times \,|\, x \cdot {}^*x \;=\; {}^*x \cdot x = a\right\} .$$

The linear algebraic group G over $\mathbb{Q}$ is reductive. Denote by G^{der} the derived group of G. In the rest of this subsection, we provide some additional information about G. The proofs are left as exercises. Alternatively, §7 of [1] provides more than sufficient resources for interested readers to fill in the details.

(1.3.1) The involution $*$ induces a positive definite involution $*_i$ on E_i, $i = 1, \dots, b$, and also an involution $*_0$ on E_0 if a is odd. Let H_i be the unitary group attached to $(E_i, *_i)$, where $1 \le i \le b$ when a is even, and $0 \le i \le b$ when a is odd. Let H_i^{der} be the derived group of H_i. The group G^{der} is simply connected, and is canonically isomorphic to the direct product $\prod_i H_i^{\mathrm{der}}$, where i runs from 1 to b if a is even, from 0 to b if a is odd.

(1.3.2) Suppose a is odd, then H_0^{der} is isomorphic to $\mathrm{SU}(E_0, *_0)$. The group H_0^{der} an inner twist of a symplectic group of type C_m, $m = n_{\frac{a+1}{2}}$, compact at ∞, and is split at every finite place $\ell \ne p$.

(1.3.3) For $1 \le i \le b$, the group H_i^{der} is the restriction of scalars to from F_i to $\mathbb{Q}$ of an absolutely simple group S_i over F_i, where F_i maximal totally real subfield F_i of K_i. The group S_i is an inner twist of a quasi-split special unitary group with respect to the quadratic extension K_i/F_i. Each place v of F_i above p is split over $\mathbb{Q}_p$, and $S_i \times_{\mathrm{Spec}(F_i)} \mathrm{Spec}(F_{i,v})$ is isomorphic to SL_n, $n = n_{j(i,\wp)} \cdot d_{j(i,\wp)}$, where $\wp$ is either one of the two places of K_i above v.

(1.3.4) Let $D = G/G^{\mathrm{der}}$, an algebraic torus over $\mathbb{Q}$. We have a short exact sequence of tori

$$1 \to \prod_{i=1}^{b} D_i \to D \to \mathbb{G}_{\mathrm{m}} \to 1\,,$$

over $\mathbb{Q}$, where $D_i = \mathrm{Ker}\left(\mathrm{Nm}_{K_i/F_i} : \mathrm{Res}_{K_i/\mathbb{Q}}\mathbb{G}_{\mathrm{m}} \to \mathrm{Res}_{F_i/\mathbb{Q}}\mathbb{G}_{\mathrm{m}}\right)$.

(1.3.5) Corollary *The reductive groups G^{der} and G over $\mathbb{Q}$ satisfy the Hasse principle for H^1.*

PROOF. The group G^{der} satisfies the Hasse principle since it is simply connected. It is well known that each torus D_i satisfies the Hasse principle: any element of $F_i^\times$ which is a local K_i/F_i norm everywhere is a global norm. Therefore D and G also satisfy the Hasse principle. ∎

§2. The bitorsor of symplectic isomorphisms

Let (A_1, λ_1) and (A_2, λ_2) be polarized abelian varieties over $\overline{\mathbb{F}_p}$ such that A_1 and A_2 are both hypersymmetric. Denote by $*_i$ the Rosati involution on $\mathrm{End}(A_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by λ_i, $i = 1, 2$. Let $G_i = \mathrm{GU}(E_i, *_i)$, $i = 1, 2$. Each G_i is a connected reductive linear algebraic group over $\mathbb{Q}$. Assume moreover that A_1 and A_2 are isogenous.

(2.1) Definition Let (A_1, λ_1) and (A_2, λ_2) be as above.

(i) Define $\mathbb{Q}$-linear maps

$$\star_{2,1} : \mathrm{Hom}_{\overline{\mathbb{F}_p}}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Hom}_{\overline{\mathbb{F}_p}}(A_2, A_1) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$\star_{1,2} : \mathrm{Hom}_{\overline{\mathbb{F}_p}}(A_2, A_1) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Hom}_{\overline{\mathbb{F}_p}}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Q}$$

by

$$^{\star_{2,1}}\gamma_1 = \lambda_1^{-1} \circ \gamma_1^t \circ \lambda_2 \qquad \forall\, \gamma_1 \in \mathrm{Hom}_{\overline{\mathbb{F}_p}}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$^{\star_{1,2}}\gamma_2 = \lambda_2^{-1} \circ \gamma_2^t \circ \lambda_1 \qquad \forall\, \gamma_2 \in \mathrm{Hom}_{\overline{\mathbb{F}_p}}(A_2, A_1) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where $\gamma_1^t : A_2^t \to A_1^t$ and $\gamma_2^t : A_1^t \to A_2^t$ are the homomorphisms between the dual abelian varieties induced by γ_1 and γ_2.

(ii) Similarly one defines $\mathbb{Q}_p$-linear maps

$$\star_{2,1} : \mathrm{Hom}(A_1[p^\infty], A_2[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Hom}(A_2[p^\infty], A_1[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$
$$\star_{1,2} : \mathrm{Hom}(A_2[p^\infty], A_1[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Hom}(A_1[p^\infty], A_2[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

by the same formulae as in (i), suitably interpreted in their respective context. This definition coincides with the $\mathbb{Q}_p$-linear extension of the corresponding maps in (i) above, if we identify $\mathrm{Hom}(A_1[p^\infty], A_2[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with $\mathrm{Hom}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ and also identify $\mathrm{Hom}(A_2[p^\infty], A_1[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with $\mathrm{Hom}(A_2, A_1) \otimes_{\mathbb{Z}} \mathbb{Q}_p$.

(2.1.1) Remark One can still define the maps $\star_{2,1}, \star_{1,2}$ without assuming that A_1, A_2 are hypersymmetric, but one has to make the following modifications in 2.1 (ii):

- Replace $\mathrm{Hom}(A_1[p^\infty], A_2[p^\infty])$ by $\lim_{n\to\infty} \mathrm{Hom}_{\mathbb{F}_{p^n}}(A_{1/\mathbb{F}_{p^n}}[p^\infty], A_{2/\mathbb{F}_{p^n}}[p^\infty])$,

- replace $\mathrm{Hom}(A_2[p^\infty], A_1[p^\infty])$ by $\lim_{n\to\infty} \mathrm{Hom}_{\mathbb{F}_{p^n}}(A_{2/\mathbb{F}_{p^n}}[p^\infty], A_{1/\mathbb{F}_{p^n}}[p^\infty])$.

The inductive limits above are understood as follows. Both (A_1,λ_1) and (A_2,λ_2) are defined over $\overline{\mathbb{F}_p}$, hence each is defined over $\mathbb{F}_{p^n}$ for some $n\in\mathbb{N}$ and gives rise to a well-defined inductive object indexed by a cofinal subset of finite extensions of $\mathbb{F}_p$.

(2) The $\mathbb{Q}_\ell$-linear extension of the maps defined in 2.1 (i) can be identified with

$$\star_{2,1} : \lim_{n\to\infty} \mathrm{Hom}_{\mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_{p^n})]}(\mathrm{V}_\ell(A_{1/\mathbb{F}_{p^n}}), \mathrm{V}_\ell(A_{2/\mathbb{F}_{p^n}})) \to \lim_{n\to\infty} \mathrm{Hom}_{\mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_{p^n})]}(\mathrm{V}_\ell(A_{2/\mathbb{F}_{p^n}}), \mathrm{V}_\ell(A_{1/\mathbb{F}_{p^n}}))\,,$$

$$\star_{1,2} : \lim_{n\to\infty} \mathrm{Hom}_{\mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_{p^n})]}(\mathrm{V}_\ell(A_{2/\mathbb{F}_{p^n}}), \mathrm{V}_\ell(A_{1/\mathbb{F}_{p^n}})) \to \lim_{n\to\infty} \mathrm{Hom}_{\mathbb{Z}_\ell[\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_{p^n})]}(\mathrm{V}_\ell(A_{1/\mathbb{F}_{p^n}}), \mathrm{V}_\ell(A_{2/\mathbb{F}_{p^n}}))$$

where $\mathrm{V}_\ell(A_i)$ denotes the $\mathbb{Q}_\ell$-Tate module for A_i, endowed with the action of $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_{p^n})$ if A_i is defined over $\mathbb{F}_{p^n}$, and the inductive object $\lim_{n\to\infty} A_{i/\mathbb{F}_{p^n}}$ attached to the abelian variety A_i over $\overline{\mathbb{F}_p}$ is explained in (1) above.

(2.1.2) Lemma (i) $\star_{1,2}\circ\star_{2,1} = \mathrm{Id}_{\mathrm{End}(A_1)\otimes_{\mathbb{Z}}\mathbb{Q}}$, $\star_{2,1}\circ\star_{1,2} = \mathrm{Id}_{\mathrm{End}(A_2)\otimes_{\mathbb{Z}}\mathbb{Q}}$.

(ii) *We have* ${}^{\star_{2,1}}(\gamma_1\circ\alpha) = {}^{*_1}\alpha\circ{}^{\star_{2,1}}\gamma_1$ *and* ${}^{\star_{2,1}}(\beta\circ\gamma_1) = {}^{\star_{2,1}}\gamma_1\circ{}^{*_2}\beta$, *for all* $\gamma_1\in \mathrm{Hom}(A_1,A_2)\otimes_{\mathbb{Z}}\mathbb{Q}$, *all* $\alpha\in\mathrm{End}(A_1)\otimes_{\mathbb{Z}}\mathbb{Q}$, *and all* $\beta\in\mathrm{End}(A_2)\otimes_{\mathbb{Z}}\mathbb{Q}$. *A similar statement holds for* $\star_{1,2}$.

PROOF. The proofs are straight-forward, hence omitted. ∎

(2.2) Definition Define $\mathcal{T}=\mathcal{T}(A_1,A_2)$ to be the affine scheme over $\mathbb{Q}$ such that for every commutative $\mathbb{Q}$-algebra R, $\mathcal{T}(R)$ is the set of all elements $(x,a)\in(\mathrm{Hom}(A_1,A_2)\otimes_{\mathbb{Z}}R)\times R^\times$ such that

$${}^{\star_{2,1}}x\circ x = a\cdot\mathrm{Id}_{\mathrm{End}(A_1)\otimes_{\mathbb{Z}}R} \quad\text{and}\quad x\circ{}^{\star_{2,1}}x = a\cdot\mathrm{Id}_{\mathrm{End}(A_2)\otimes_{\mathbb{Z}}R}\,.$$

The group G_1 operates on the right of $\mathcal{T}$ by pre-composition, and the group G_2 operates on the left of $\mathcal{T}$ by post-composition. Clearly the left G_1-action and right G_2 action on $\mathcal{T}$ above are compatible.

(2.2.1) Lemma (i) *The* $\mathbb{Q}$*-scheme* $\mathcal{T}$ *is a right* G_1*-torsor and is also a left* G_2*-torsor. Moreover these two torsor structures are compatible.*

(ii) *The set of* $\mathbb{R}$*-valued points* $\mathcal{T}(\mathbb{R})$ *of* $\mathcal{T}$ *is not empty.*

PROOF. The statement (ii) follows from the positive definiteness of the involutions $*_1$ and $*_2$. The statement (i) is immediate from the definition if one replaces "torsor" by "quasi-torsor" in it; i.e. $\mathcal{T}$ is a torsor if it is not empty. Clearly (ii) implies that $\mathcal{T}$ is non-empty, hence $\mathcal{T}$ is a torsor for both G_1 and G_2. ∎

(2.2.2) Lemma (i) *The (possibly empty) set $\mathcal{T}(A_1, A_2)(\mathbb{Q}_p)$ of $\mathbb{Q}_p$-points of the torsor $\mathcal{T}$ consists of all homomorphisms $h \in \mathrm{Hom}(A_1[p^\infty], A_2[p^\infty])$ such that*

$$^{\star_{2,1}}h \circ h = a \cdot \mathrm{Id}_{\mathrm{End}(A_1[p^\infty])} \quad \text{and} \quad h \circ {}^{\star_{2,1}}h = a \cdot \mathrm{Id}_{\mathrm{End}(A_2[p^\infty])}$$

for some element $a \in \mathbb{Q}_p^\times$.

(ii) *For any prime number $\ell \neq p$, the (possibly empty) set $\mathcal{T}(A_1, A_2)(\mathbb{Q}_\ell)$ consists of all $\mathbb{Q}_\ell$-linear homomorphisms $h \in \lim_{n\to\infty} \mathrm{Hom}_{\mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_{p^n})]}(\mathrm{V}_\ell(A_{1/\mathbb{F}_{p^n}}), \mathrm{V}_\ell(A_{2/\mathbb{F}_{p^n}}))$ such that*

$$^{\star_{2,1}}h \circ h = a \cdot \mathrm{Id}_{\mathrm{End}(\mathrm{V}_\ell(A_1))} \quad \text{and} \quad h \circ {}^{\star_{2,1}}h = a \cdot \mathrm{Id}_{\mathrm{End}(\mathrm{V}_\ell(A_2))}$$

for some element $a \in \mathbb{Q}_\ell^\times$.

PROOF. Immediate from the p-adic and ℓ-adic version of the Tate conjecture for abelian varieties over finite fields and the assumption that A_1 and A_2 are hypersymmetric. ∎

(2.2.3) Definition (i) Denote by $U_p(A_1, A_2)$ the open (and possibly empty) subset of $\mathcal{T}(A_1, A_2)(\mathbb{Q}_p)$, consisting of all isomorphisms $h \in \mathrm{Isom}(A_1[p^\infty], A_2[p^\infty])$ such that

$$^{\star_{2,1}}h \circ h = a \cdot \mathrm{Id}_{\mathrm{End}(A_1[p^\infty])} \quad \text{and} \quad h \circ {}^{\star_{2,1}}h = a \cdot \mathrm{Id}_{\mathrm{End}(A_2[p^\infty])}$$

for some element $a \in \mathbb{Z}_p^\times$.

(ii) For any prime number $\ell \neq p$, denote by $U_\ell(A_1, A_2)$ the (possibly empty) open subset of $\mathcal{T}(A_1, A_2)(\mathbb{Q}_\ell)$, consisting of all $\mathbb{Z}_\ell$-linear isomorphisms

$$h \in \mathrm{Isom}_{\mathbb{Z}_\ell[\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_{p^n})]}(\mathrm{T}_\ell(A_{1/\mathbb{F}_{p^n}}), \mathrm{T}_\ell(A_{2/\mathbb{F}_{p^n}}))$$

such that

$$^{\star_{2,1}}h \circ h = a \cdot \mathrm{Id}_{\mathrm{End}(\mathrm{T}_\ell(A_1))} \quad \text{and} \quad h \circ {}^{\star_{2,1}}h = a \cdot \mathrm{Id}_{\mathrm{End}(\mathrm{T}_\ell(A_2))}$$

for some element $a \in \mathbb{Z}_\ell^\times$. Here $\mathrm{T}_\ell(A_i)$ denotes the $\mathbb{Z}_\ell$-Tate module of A_i, endowed with an action of $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_{p^n})$ if A_i is defined over $\mathbb{F}_{p^n}$.

§3. Hecke correspondences

(3.1) Lemma *Let (A_1, λ_1), (A_2, λ_2) be hypersymmetric g-dimensional principally polarized abelian varieties over $\overline{\mathbb{F}_p}$ such that A_1 and A_2 are isogenous.*

(i) *For every prime number $\ell \neq p$, there exists a principally polarized abelian variety (A_3, λ_3) over $\overline{\mathbb{F}_p}$, an ℓ-power isogeny $\beta_\ell : A_2 \to A_3$, and a non-negative integer r such that $\beta_\ell^t \circ \lambda_3 \circ \beta_\ell = \ell^r \lambda_2$ and $\mathcal{T}(A_1, A_3)(\mathbb{Q}_\ell) \neq \emptyset$.*

(ii) *There exists a principally polarized abelian variety* (A_3, λ_3) *over* $\overline{\mathbb{F}_p}$*, a prime-to-p isogeny* $\beta : A_2 \to A_3$*, and a positive integer n prime to p such that* $\beta^t \circ \lambda_3 \circ \beta = n\lambda_2$ *and* $\mathcal{T}(A_1, A_3)(\mathbb{Q}_\ell) \neq \emptyset$ *for every prime number* $\ell \neq p$.

PROOF. Clearly (i) implies (ii). The statement (i) follows from the following well-known fact in linear algebra. Let V_1, V_2 be two finite dimensional vector spaces over $\mathbb{Q}_\ell$. Let $\langle , \rangle_i : V_i \times V_i \to \mathbb{Q}_\ell$ be a symplectic pairing on V_i, and let Λ_i be a self-dual $\mathbb{Z}_\ell$-lattice in V_i with respect to $\langle , \rangle_i$, $i = 1, 2$. Then there exists a $\mathbb{Q}_\ell$-linear isomorphism $h : V_1 \to V_2$ such that $\beta(\Lambda_1) \subseteq \Lambda_2$ and $h^*(\langle , \rangle_2) = \ell^r \langle , \rangle_1$ for some non-negative integer r. (In fact one can impose the addition condition that $r = 0$ in the above.) ∎

(3.2) Proposition *Let* (A_1, λ_1)*,* $A_2, \lambda_2)$ *be as in 3.1. Assume moreover that the principally quasi-polarized Barsotti-Tate groups* $(A_1[p^\infty], \lambda_1[p^\infty])$*,* $(A_2[p^\infty], \lambda_2[p^\infty])$ *are isomorphic. Then there exists a* (A_3, λ_3) *be a principally polarized abelian variety over* $\overline{\mathbb{F}_p}$ *such that*

(i) *There exists a prime-to-p isogeny* $\beta : A_2 \to A_3$*, and a positive integer n prime to p such that* $\beta^t \circ \lambda_3 \circ \beta = n\lambda_2$.

(ii) *There exists an isogeny* $\gamma : A_1 \to A_3$ *over* $\overline{\mathbb{F}_p}$ *such that* $\gamma \in \mathcal{T}(A_1, A_3)(\mathbb{Q})$ *and* γ *induces an isomorphism from* $(A_1[p^\infty], \lambda_1[p^\infty])$ *to* $(A_3[p^\infty], \lambda_3[p^\infty])$.

PROOF. Choose a principally polarized abelian variety (A_3, λ_3) which satisfies 3.1 (ii). The polarized Barsotti Tate group $(A_3[p^\infty], \lambda_3[p^\infty])$ is isomorphic to $(A_2[p^\infty], \lambda_2[p^\infty])$ construction, which is isomorphic to $(A_1[p^\infty], \lambda_1[p^\infty])$ by hypothesis. Therefore $\mathcal{T}(A_1, A_3)(\mathbb{Q}_p) \neq \emptyset$. On the other hand, $\mathcal{T}(A_1, A_3)(\mathbb{Q}_\ell) \neq \emptyset$ for every prime number $\ell \neq p$ by the construction of (A_3, λ_3). And we have seen that $(\mathcal{T}(A_1, A_3)(\mathbb{C}) \neq \emptyset$, a general property of torsor of the form $\mathcal{T}(A_1, A_3)$; see 2.2.1. By Cor. 1.3.5 to $\mathcal{T}(A_1, A_3)$, we conclude that $\mathcal{T}(A_1, A_3)(\mathbb{Q}) \neq \emptyset$. Choose a point $\gamma_1 \in \mathcal{T}(A_1, A_3)(\mathbb{Q})$.

Our hypotheses implies the open subset $U_p(A_1, A_3)$ of $\mathcal{T}(A_1, A_3)(\mathbb{Q}_p)$ is not empty. By the weak approximation theorem, we can adjust the rational point γ_1 of $\mathcal{T}(A_1, A_3)$ by a element of $G_1(\mathbb{Q})$ to obtain a point $\gamma \in \mathcal{T}(A_1, A_3)(\mathbb{Q}) \cap U_p(A_1, A_3)$. Multiplying γ by a suitable positive integer close to 1 in $\mathbb{Z}_p$, we can assume that γ is an isogeny from A_1 to A_3. ∎

(3.2.1) Corollary *Notation and assumption as in 3.2. Then* (A_2, λ_2) *lies in the prime-to-p Hecke orbit of* (A_1, λ_1).

PROOF. The composition $n\beta^{-1}\alpha : A_1 \to A_2$ is a prime-to-p symplectic isogeny up to $\mathbb{Z} \cap \mathbb{Z}_p^\times$. ∎

References

[1] R. Kottwitz. Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.*, 5:373–444, 1992.