# Math 584, Problem set 5 due April 11, 2017
# Dr. Epstein

**Reading:** Read 7.1–7.3.3, 7.4–7.5.4, 7.6. Go through the MATLAB worksheets on the Hilbert transform and the Gibbs Phenomenon. As always, you will get more out of the worksheets if you try varying their parameters. This is a long problem set, try to do as much as you can.

1. Do problem 6.2.3 from the text.
2. Do exercises 6.3.1, 6.3.5, 6.3.6
3. Compute the Fourier coefficients of the following functions
   (a) The 1-periodic function defined by
   $$f(x) = \begin{cases} 1 & \text{for } x \in [0, \frac{1}{2}], \\ -1 & \text{for } x \in (\frac{1}{2}, 1]. \end{cases}$$
   (b) The 1-periodic function
   $$f(x) = x(1-x) \text{ for } x \in [0,1].$$
   (c) The $2\pi$-periodic function given by
   $$f(x) = \begin{cases} \cos(x) & \text{for } x \in [0, \pi), \\ \cos(x-\pi) & \text{for } x \in [\pi, 2\pi). \end{cases}$$
4. Define a $2\pi$-periodic function by setting
   $$f(x) = x \text{ for } x \in [-\pi, \pi).$$
   The Fourier coefficients of $f$ are given by
   $$\hat{f}(n) = \begin{cases} 0 & \text{if } n = 0, \\ \frac{(-1)^n 2\pi i}{n}. \end{cases}$$
   Use the Parseval formula to determine the value of the infinite sum
   $$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$
5. We can define a discrete analogue of the Hilbert transform by setting
   $$\widehat{\mathcal{H}f}(n) = \operatorname{sgn}(n)\hat{f}(n),$$
   so that
   $$\mathcal{H}f(x) = \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)\hat{f}(n)e^{inx},$$

whenever the sum makes sense. We let $\operatorname{sgn}(n) = 1$ if $n > 0$, $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(n) = -1$ if $n < 0$. As before we can express this as a principal value integral.

$$\mathcal{H}f(x) = \text{P. V.} \int_0^{2\pi} h(x-y) f(y) dy.$$

(a) By considering the approximate Hilbert transforms defined by

$$\widehat{\mathcal{H}_\epsilon f}(n) = e^{-\epsilon|n|} \operatorname{sgn}(n) \hat{f}(n)$$

find the function $h(x)$ appearing in the formula above.

(b) Show that if

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

then

$$\mathcal{H}f(x) = -i \sum_{n=1}^{\infty} a_n \cos(nx).$$

6. Do exercises 7.3.3, 7.3.6, 7.3.8.
7. Do exercise 7.3.14.
8. Do exercise 7.4.11.