Math 609
Problem set 4 due February 14, 2008
Dr. Epstein

Reading: Chapter 3 of Stein-Shakarchi.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Suppose that f is a holomorphic function in $D_r^+(0)$, continuous up to $\overline{D_r^+(0)} \cap \mathbb{R}$, such that $\lim_{y\to 0^+} |f(x+iy)| = 1$. Show that

$$F(z) = \begin{cases} f(z) & \text{if } z \in D_r^+(0) \\ \frac{1}{\overline{f(\bar{z})}} & \text{if } z \in D_r^-(0), \end{cases} \tag{1}$$

defines a holomorphic extension of f to the lower half of $D_r(0)$. Hint: Consider the behavior of the holomorphic function $h(z) = i\frac{1+z}{1-z}$, along $|z| = 1$.

2. Stein-Shakarchi page 69, problem 4.

3. Prove that a polynomial in (x, y) is harmonic if and only if each homogeneous part is harmonic. For each $n \in \mathbb{N}$ find a basis for the two dimensional, real vector space, $\mathcal{H}_n$, of homogeneous, harmonic polynomials of degree n. You must prove that $\dim \mathcal{H}_n = 2$.

4. Prove a version of the Runge theorem for harmonic functions: Suppose that K is a compact simply connected subset of $\mathbb{C}$ and u is harmonic in a neighborhood of K. There exists a sequence of harmonic polynomials $< p_n(x, y) >$ such that

$$\|u - p_n\|_{L^\infty(K)} \leq \frac{1}{n}. \tag{2}$$

5. If $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, then

$$u(x, y) = \frac{1}{4\pi} \int\limits_{\mathbb{R}^2} \varphi(a, b) \log[(x-a)^2 + (y-b)^2] da db, \tag{3}$$

is a $\mathcal{C}^\infty$-function, which solves $\Delta u = \varphi$. Show that if $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2)$, then there also exists a solution to $\Delta u = \varphi$.

6. Suppose that $F(z, w)$ is a power series in z, w,

$$F(z, w) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} z^j w^k, \tag{4}$$

that converges absolutely for all $(z, w) \in \mathbb{C}^2$. If $f(z)$ is holomorphic in $\Omega \subset \mathbb{C}$, then show that $F(z, f(z))$ is also holomorphic in Ω. Suppose that g is an analytic continuation of f to $\Omega' \supset \Omega$. If $F(z, f(z)) = 0$ in Ω, then show that $F(z, g(z)) = 0$ in Ω'.

7. Let $U \subset \mathbb{C} \setminus \{0\}$, be a simply connected open set. Show that, for each $n \geq 2$, the function $f_n(z) = z^n$, has an analytic inverse $g_n(z)$ defined in U. If $V \subset \mathbb{C} \setminus \{0\}$ is also simply connected, and $U \cap V$ is non-empty and connected, then g_n has an analytic continuation to V, which is also an inverse to f_n. Give an example to show that if $U \cap V$ is not connected, then such an analytic continuation need not exist. Prove that f_n does not have an analytic inverse defined in $\mathbb{C} \setminus \{0\}$.