Math 644, Problem set 2 due October 9, 2007
Dr. Epstein

Reading: Taylor sections 2.1, 2.2, 2.4

1. The function $G_2(x, y) = [2\pi]^{-1} \log \|x - y\|$ is defined for $x \neq y$ in $\mathbb{R}^2$. Show that if $\varphi \in C^2(\mathbb{R}^2)$ has compact support, then

$$u(x) = \int\limits_{\mathbb{R}^2} G_2(x, y)\varphi(y)dy \tag{1}$$

is a C^2-function satisfying $\Delta u = \varphi$. Show that limit

$$\gamma = \lim_{x\to\infty} \frac{u(x)}{\log r}$$

exists and give a formula for its value. Under what conditions does $\gamma = 0$?

2. Let r_θ be the rotation of $\mathbb{R}^2$ defined by the matrix

$$r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We let R_θ denote the operator defined by $(R_\theta u)(x) = u(r_\theta x)$.

   (a) Show that $\Delta \circ R_\theta = R_\theta \circ \Delta$.

   (b) Show that if u is a harmonic function defined in $B_R(0)$ then

$$v(x) = \frac{1}{2\pi} \int\limits_0^{2\pi} u(r_\theta x)d\theta$$

   is harmonic in $B_R(0)$ and rotationally invariant, that is $R_\theta v = v$ for any θ.

   (c) Give a formula for v (which you must prove) and use it to give a new proof of the mean value property for harmonic functions.

3. Show that if f is a holomorphic function in $\Omega \subset \mathbb{R}^2$, then $u = \log|f|$ satisfies the mean value inequality

$$u(x) \leq \frac{1}{\pi r^2} \iint\limits_{B_r(x)} u\,dA,$$

provided $\overline{B_r(x)} \subset \Omega$. Show that if $f^{-1}(\{0\}) \neq \emptyset$, then the inequality is sometimes strict.

4. Show that if f is a C^2-function defined in an open subset of $\mathbb{R}^n$, which satisfies the mean value property, then $\Delta f = 0$.

5. Exercise 3 (a)–(c) on page 139 of Taylor.

6. Show that, for $n \geq 3$, there is a constant c_n, so that

$$g_n(x) = \frac{c_n}{\|x\|^{n-2}}$$

is a fundamental solution for the Laplace operator on $\mathbb{R}^n$. That is, we define the operator

$$G_n f(x) = \int\limits_{\mathbb{R}^n} g_n(x-y) f(y) dy;$$

if $f \in C_c^2(\mathbb{R}^n)$, then $\Delta \circ G_n f = G_n \circ \Delta f = f$. Describe c_n geometrically. Prove that for every $f \in C_c^2(\mathbb{R}^n)$ the equation $\Delta u = f$ has a unique, bounded solution.

7. Consider the operator $L = \Delta + c^2$ acting on functions defined in $\mathbb{R}^3$. Here $c \in \mathbb{R}$.

   (a) Find all solutions to the equation

   $$Lu = 0$$

   with spherical symmetry, that is depending only on $\|x\|$.

   (b) Prove that

   $$k(x, y) = -\frac{\cos(c\|x-y\|)}{4\pi \|x-y\|}$$

   is a fundamental solution for L; that is if we define

   $$Kf(x) = \int\limits_{\mathbb{R}^3} k(x, y) f(y) dy,$$

   then for $f \in C_c^2(\mathbb{R}^3)$ we have

   $$K \circ Lf = L \circ Kf = f.$$

8. Assume that in a neighborhood of $(x_0, y_0) \in \mathbb{R}^2$, $u(x, y)$ is a smooth solution to the quasi-linear PDE:

$$a(u_x, u_y)u_{xx} + 2b(u_x, u_y)u_{xy} + c(u_x, u_y)u_{yy} = 0.$$

Suppose that in some neighborhood of (x_0, y_0) the variables

$$\xi = u_x(x, y) \quad \eta = u_y(x, y)$$

define local coordinates; set $\phi = xu_x + yu_y - u$. Prove that, as a function of (ξ, η), ϕ satisfies $x = \phi_\xi$, $y = \phi_\eta$ and the *linear* PDE:

$$a(\xi, \eta)\phi_{\eta\eta} - 2b(\xi, \eta)\phi_{\xi\eta} + c(\xi, \eta)\phi_{\xi\xi} = 0.$$