Math 644, Problem set 3 due October 23, 2007
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Reading: Taylor sections 2.5, 2.6, 2.8

1. Let a, b, c be constants and suppose that $ac - b^2 > 0$, explain how the solutions of the equation $au_{xx} + 2bu_{xy} + cu_{yy} = 0$ are related to harmonic functions. If $ac - b^2 < 0$, then explain how solutions of the equation $au_{xx} + 2bu_{xy} + cu_{yy} = 0$ are related to solutions of the 1d wave equation.

2. Let $a(q), b(q), c(q), d(q), e(q)$ be continuous functions in a bounded set $q \in \overline{\Omega} \subset \mathbb{R}^2$. Suppose that there is a constant $M > 0$ so that, for all (ξ, η), we have

$$a(q)\xi^2 + 2b(q)\xi\eta + c(q)\eta^2 \geq M(\xi^2 + \eta^2) \text{ for all } q \in \Omega. \tag{1}$$

   (a) Suppose that u is continuous in $\overline{\Omega}$ and

$$Lu = a(q)u_{xx} + 2b(q)u_{xy} + c(q)u_{yy} + d(q)u_x + e(q)u_y > 0. \tag{2}$$

   Show that $\sup_{q\in\Omega} u(q) = \sup_{q\in b\Omega} u(q)$.

   (b) Now assume that u is continuous in $\overline{\Omega}$ and

$$Lu = a(q)u_{xx} + 2b(q)u_{xy} + c(q)u_{yy} + d(q)u_x + e(q)u_y = 0. \tag{3}$$

   Show that $\sup_{q\in\Omega} u(q) = \sup_{q\in b\Omega} u(q)$. Hint: consider $u + \epsilon v$, for an apprpriate choice of v.

   (c) Show that, if it exists, the solution to the Dirichlet problem $Lu = f$, in Ω, $u \restriction_{b\Omega} = g$, is unique.

3. Let $\Omega \subset \mathbb{R}^n$, $x_0 \in \Omega$ and suppose that $\Delta u = 0$ in $\Omega \setminus \{x_0\}$. Suppose that $u(x) = o(\|x - x_0\|^{2-n})$. Prove that u has a smooth extension to Ω, satisfying $\Delta u = 0$.

4. For each real number s define the hypersurface in $\mathbb{R}^4$

$$H_s = \{(x, y, z, t) : x^2 + y^2 + z^2 - t^2 = s\}.$$

   Show that H_s is spacelike if and only if $s < 0$.

5. A solution to the wave equation is called a "plane wave" if $u(x,t) = f(x \cdot \xi - t)$, for a constant vector ξ. Describe all plane wave solutions. Why are they called "plane waves?"

6. Prove that there is no classical solution to $u_{xx} - u_{tt} = 0$ supported in the exterior of the region $\{(x,t) : |x| \leq |t|\}$.

7. Suppose that u solves $u_{tt} = \Delta u$ in $\mathbb{R}^3 \times \mathbb{R}$. *Use energy estimates* to show that $u(0,0,0,T)$ is determined by the initial data $(u(x,0), u_t(x,0))$ in the ball $\|x\| \leq T$.

8. If the initial data for the wave equation in $\mathbb{R}^n$ is radial, i.e. $u(x,0) = f(\|x\|)$, $u_t(x,0) = g(\|x\|)$, then show that the solution, with this initial data, must also be radial in x for all latter times.

   (a) For $n = 3$ show that the most general radial solution to the wave equation is of the form:
   $$u(x,t) = \frac{F(\|x\| + t) + G(\|x\| - t)}{\|x\|}. \tag{4}$$

   (b) If the initial data is $u(x,0) = 0$, $u_t(x,0) = g(\|x\|)$, then
   $$u = \frac{1}{2r} \int_{r-t}^{r+t} s g(s) ds, \text{ where } r = \|x\|. \tag{5}$$

9. Solve the initial value problem for the PDE $w_{tt} = w_{xx} - \lambda^2 w$ with $w(x,0) = 0$, $w_t(x,0) = g(x)$ by descent from the two dimensional wave equation. That is consider $u(x,y,t) = \cos(\lambda y) w(x,t)$. Find an explicit formula for the solution in terms of the function:
   $$J_0(z) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(z \sin(\theta)) d\theta. \tag{6}$$

   Does finite propagation speed hold for this equation?

10. A matrix A defines a Lorentz transformation if, for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$,
    $$\langle A(x,t), A(x,t) \rangle = \langle (x,t), (x,t) \rangle, \tag{7}$$
    where $\langle (x,t), (x,t) \rangle = \|x\|^2 - t^2$. If A is a Lorentz transformation and $u(x,t)$ solves the wave equation in $\mathbb{R}^3 \times \mathbb{R}$, then show that $v(x,t) = u(A(x,t))$ does as well.