Math 644, Problem set 4 due November 6, 2007
Dr. Epstein

Reading: Taylor sections 3.1–3.5

1. Prove that convolution

$$(f,g) \mapsto f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy \tag{1}$$

is a continuous mapping from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Show that if $1 \leq p < \infty$, then convolution extends to define a continuous map $L^1(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, hint: For $f, g \in \mathcal{S}(\mathbb{R}^n)$, prove the estimate

$$\|f * g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}. \tag{2}$$

What happens in the $p = \infty$ case?

2. Show that if $\{p_j : j = 1, 2, \ldots\}$ are semi-norms, then

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(f-g)}{1 + p_j(f-g)} \tag{3}$$

defines a metric.

3. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, and $\psi \in \mathcal{S}(\mathbb{R}^n)$, with $\varphi(0) = 1$. Show that

$$\psi_j(x) = \varphi\left(\frac{x}{j}\right)\psi(x), \quad j = 1, 2, \ldots \tag{4}$$

converges to ψ in $\mathcal{S}(\mathbb{R}^n)$. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfy

$$\int_{\mathbb{R}^n} \varphi(x)dx = 1. \tag{5}$$

Let $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(\epsilon^{-1}x)$. If $\psi \in \mathcal{S}(\mathbb{R}^n)$, then show that $\varphi_\epsilon * \psi$ converges to ψ in $\mathcal{S}(\mathbb{R}^n)$.

4. Let $\mathscr{C}_b^\infty(\mathbb{R}^n)$ denote the set of smooth functions on $\mathbb{R}^n$ with all derivatives bounded. That is, $\varphi \in \mathscr{C}_b^\infty(\mathbb{R}^n)$ if for each multi-index α there is a C_α so that $|\partial_x^\alpha \varphi(x)| \leq C_\alpha$.

   (a) Show that if $\varphi \in \mathscr{C}_b^\infty(\mathbb{R}^n)$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, then $\varphi\psi \in \mathscr{S}(\mathbb{R}^n)$.

   (b) If $u \in \mathscr{S}'(\mathbb{R}^n)$, and $\varphi \in \mathscr{C}_b^\infty(\mathbb{R}^n)$, then show that $\varphi u \in \mathscr{S}'(\mathbb{R}^n)$.

5. Taylor page 202, problems 3 and 4.

6. Taylor page 203, problems 14 and 15.

7. Taylor page 214, problems 1 and 2.

8. Taylor page 215, problem 10.

9. Taylor page 215, problem 11

10. Taylor page 216, problem 15.