Math 644, Problem set 6 due December 6, 2007
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Reading: Taylor sections 4.1, 4.4, 4.5, 5.1

1. Taylor page 274, problem 1.
2. Taylor page 275, problem 9.
3. Suppose that $f \in H^{k+1}(\mathbb{R}^n)$ and $g \in H^k(\mathbb{R}^n)$. Show that the unique solution to the wave equation
$$u_{tt} = \Delta u \text{ on } \mathbb{R} \times \mathbb{R}^n \text{ with } u(0,x) = f(x) \text{ and } u_t(u,x) = g(x), \tag{1}$$
satisfies $u(t,\cdot) \in H^{k+1}(\mathbb{R}^n)$ and $u_t(t,\cdot) \in H^k(\mathbb{R}^n)$ for all $t \in \mathbb{R}$.
4. If $f \in H^k(\mathbb{R}^n)$, $k \geq 2$, then the distributional solution to
$$u_t = \Delta u \text{ with } u(0,x) = f(x), \tag{2}$$
satisfies $u(t,\cdot) \in C^0([0,\infty); H^k(\mathbb{R}^n)) \cap C^1([0,\infty); H^{k-2}(\mathbb{R}^n))$.
5. Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$ and let $u = \delta(x)f(y)$ define an element of $\mathcal{S}'(\mathbb{R}^2)$. Show that the equation
$$\partial_{\bar{z}} v = u, \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y), \tag{3}$$
has a solution $v \in \mathcal{S}'(\mathbb{R}^2)$ given by
$$v = E * u, \text{ where } E(x,y) = \frac{1}{\pi(x+iy)}. \tag{4}$$
Prove that v is a holomorphic function in $\mathbb{C} \setminus \{x = 0\}$, and give a simple explicit formula for v in the components of $\mathbb{C} \setminus \{x = 0\}$. Show that, at least in the sense of 1-dimensional distributions, we have
$$\lim_{\epsilon \to 0^+} [v(\epsilon, y) - v(-\epsilon, y)] = f(y). \tag{5}$$
What is $\operatorname{singsupp} v$? Can you show that v has a continuous extension to each of the half planes $\{x \leq 0\}$ and $\{x \geq 0\}$?

6. Let $u(t,x,y)$ solve the 2-dimensional wave equation

$$u_{tt} = u_{xx} + u_{yy} \text{ with } u(0,x,y) = 0, \quad u_t(0,x,y) = g(x,y) \in \mathcal{S}'(\mathbb{R}^2). \tag{6}$$

For each t show that

$$\begin{aligned} &\operatorname{singsupp} u(t,\cdot) \\ &\subset \{(x,y): \text{ there is a point } (a,b) \in \operatorname{singsupp} g, \text{ with } \|(x,y)-(a,b)\| = |t|\}. \end{aligned} \tag{7}$$

7. The function

$$g(x,t) = \begin{cases} \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{8}$$

defines an element of $\mathcal{S}'(\mathbb{R}^2)$. What is $\operatorname{singsupp} g$? If $f \in C^0([0,\infty);\mathcal{S}(\mathbb{R}))$, then show that

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} g(t-s, x-y) f(s,y) dy ds, \tag{9}$$

solves $u_t - u_{xx} = f$ in $(0,\infty)\times\mathbb{R}$ and $u(0,x) = 0$. Prove that $u \in C^1([0,\infty);\mathcal{S}(\mathbb{R}))$. Hint: Use the Parseval relation to re-express u.