Magnetic Resonance Imaging

I, Basic Concepts

Charles L. Epstein

Departments of Mathematics
and
Applied Math and Computational Science
University of Pennsylvania

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In these lectures we present the basic concepts of magnetic resonance imaging and give some idea of the scope of this remarkable technology. We will cover the following topics:

1. Basic concepts of NMR and its application to imaging.
2. Selective excitation.
3. Contrast Mechanisms.
4. Parallel Imaging (If time permits).

Imaging Modalities

There are several successful medical imaging modalities, each is based on a different physical principle and thereby the contrasts in the images reflect different physical or chemical properties of the object under consideration.
In all modalities there are four intertwined considerations:

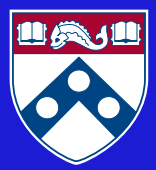
1. Contrast (numerical precision)
2. Spatial Resolution and Field of View (sampling)
3. Noise (measurement)
4. Safety! (this limits the number of allowable measurements)

For an ill-posed problem it is very difficult to attain high resolution and contrast simultaneously because of the need to suppress noise.

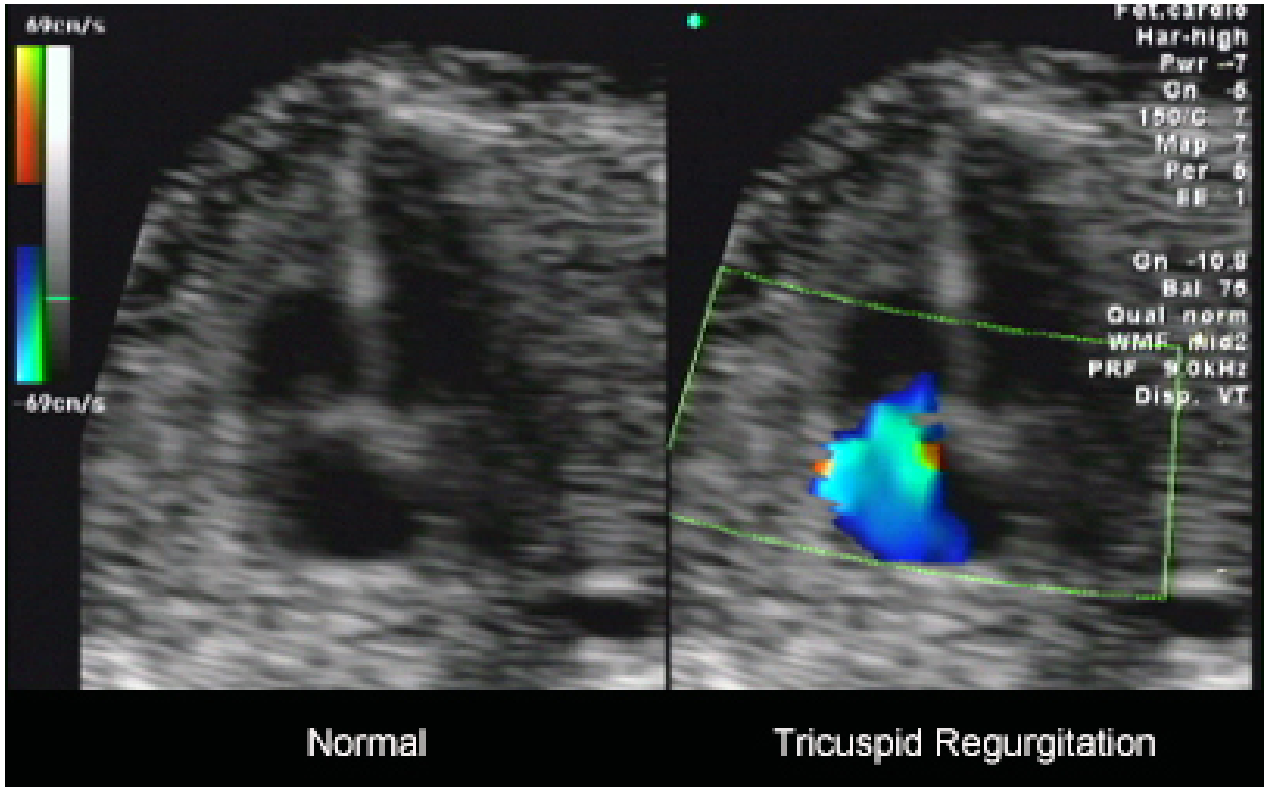
Ultrasound

Ultrasound images using high frequency sound waves as the "probe." At first glance contrast is provided by variations in sound speed and absorption, but the Doppler effect makes it possible to visualize motion (like moving blood) as well. The exact inverse problem is highly non-linear and not yet solved, even in principle. This is avoided by careful selection of data. Most of the available data is discarded!

This modality is plagued by low SNR, limited penetration depth, etc. It is inexpensive, safe, portable and adequate for many applications, so is in wide use.



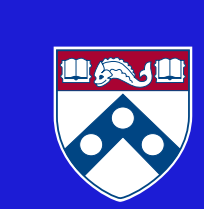
An Ultrasound Image



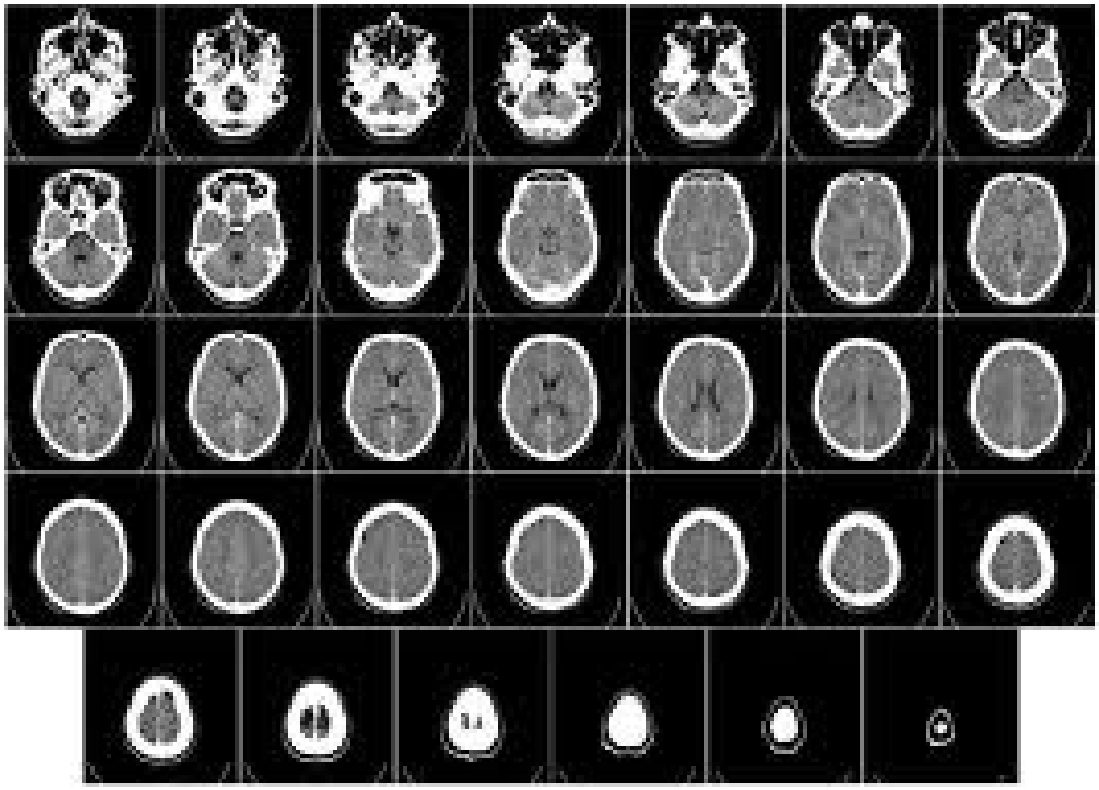
Two Ultrasound images of a fetal heart showing the importance of the contrast provided by the Doppler effect.
From:http://www.fetal.com/Genetic Sono

X-Ray CT

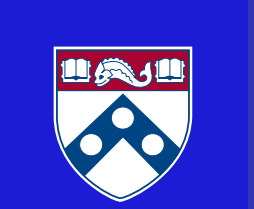
X-ray CT uses X-rays to probe the internal anatomy. Because of the very high energies involved (50-200KeV), there are very limited opportunites for contrast. The image is formed by using an approximation of the inverse Radon transform to reconstruct an approximation of the "X-ray attenuation coefficient."
There is very little contrast between different soft tissues, amounting to about 1-2% of the attainable dynamic range. There are significant non-linearities, which violate the measurement model, when bone is present. The resolution is limited due to safety constraints. There are good injectible contrast agents, which can give some information about physiology, but the images are fundamentally anatomical.



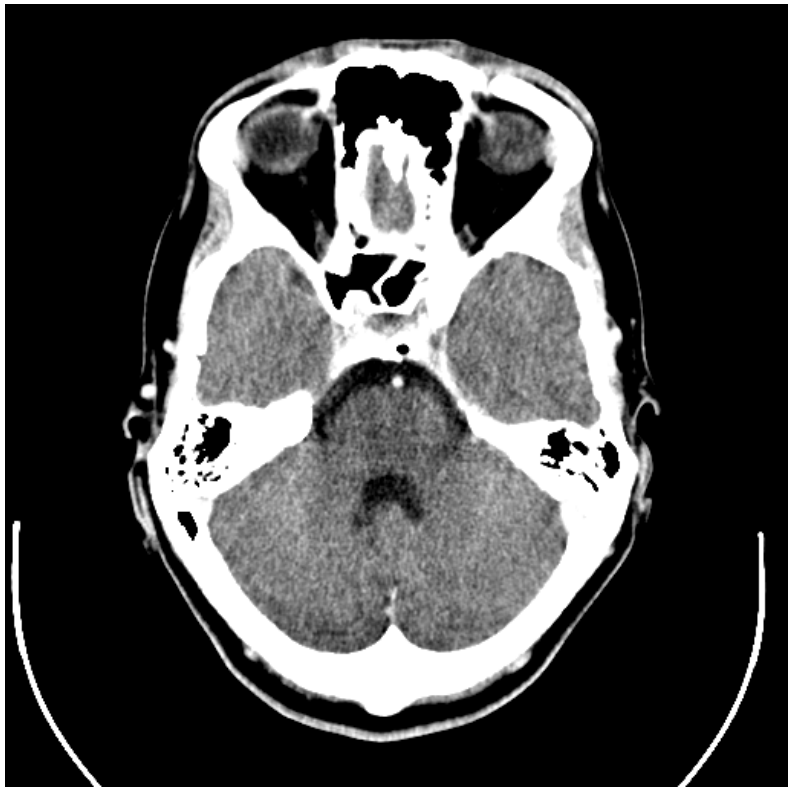
An X-Ray CT Image, I

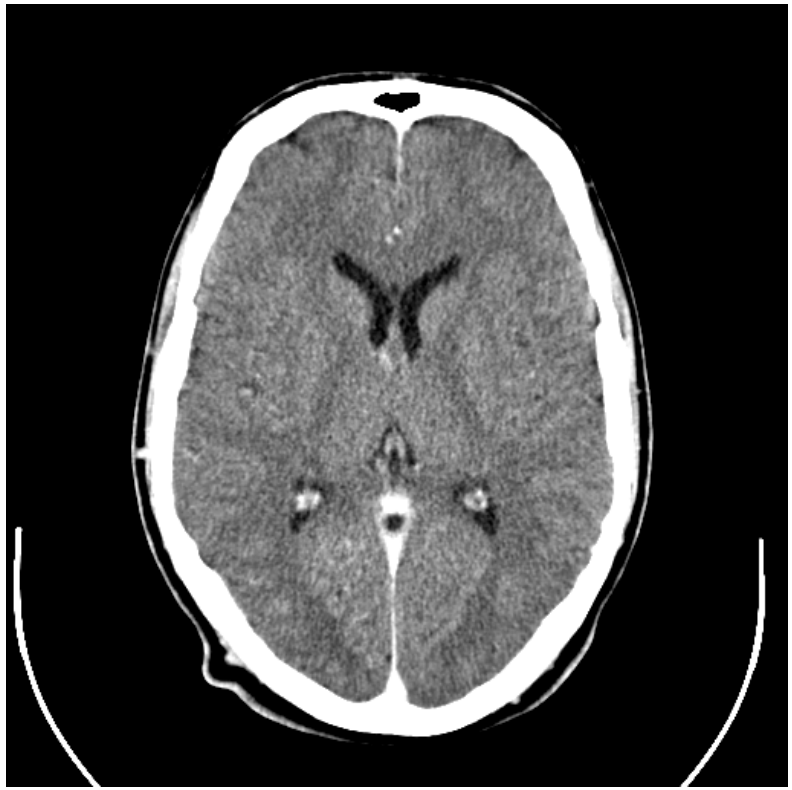


X-ray Computer tomography of human brain, from base of the skull to top. Taken with intravenous contrast medium.
From Wikipedia Commons: Radiology, Uppsala University Hospital. Brain supplied by Mikael Häggström.



An X-Ray CT Image, II





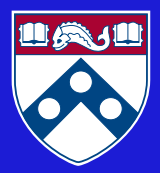
Two slices of Mikael Häggström's brain. Taken with intravenous contrast medium.
From Wikipedia Commons: Radiology, Uppsala University Hospital. Brain supplied by Mikael Häggström.

Positron Emission Tomography

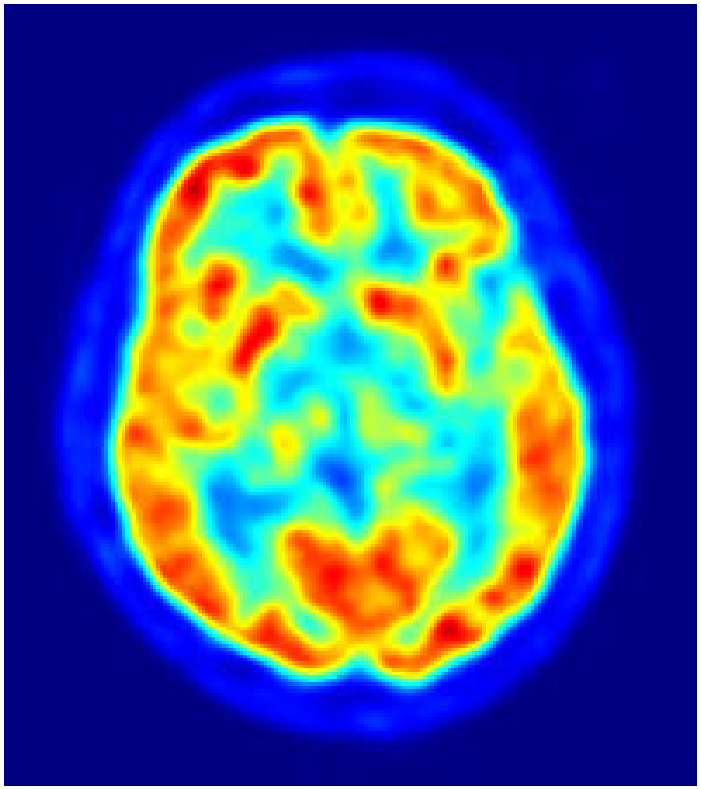
In this modality a radioactive tracer is bound to a metabolite. It is differentially taken up by pathological tissues. It then decays, producing .5MeV gamma-rays. Using a collimated gamma-ray detector to localize the decay events, a probabilitic algorithm allows for an estimation of the distribution of the marked metabolite. The importance of this modality is that it allows for a direct, spatially resolved *observation of metabolism*.

The energy of the gamma-rays is very high and this severely limits the amount of tracer that can be used, which leads to a very noisy measurement, and hence a fairly low resolution image. By physically coupling the PET detector with an X-ray CT machine, the utility of this modality has been greatly enhanced.

SPECT is a related, though somewhat less common modality. In principle these modalities could use a Radon-like inversion formula, but the data is too noisy.

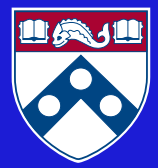


Positron Emission Tomography Image

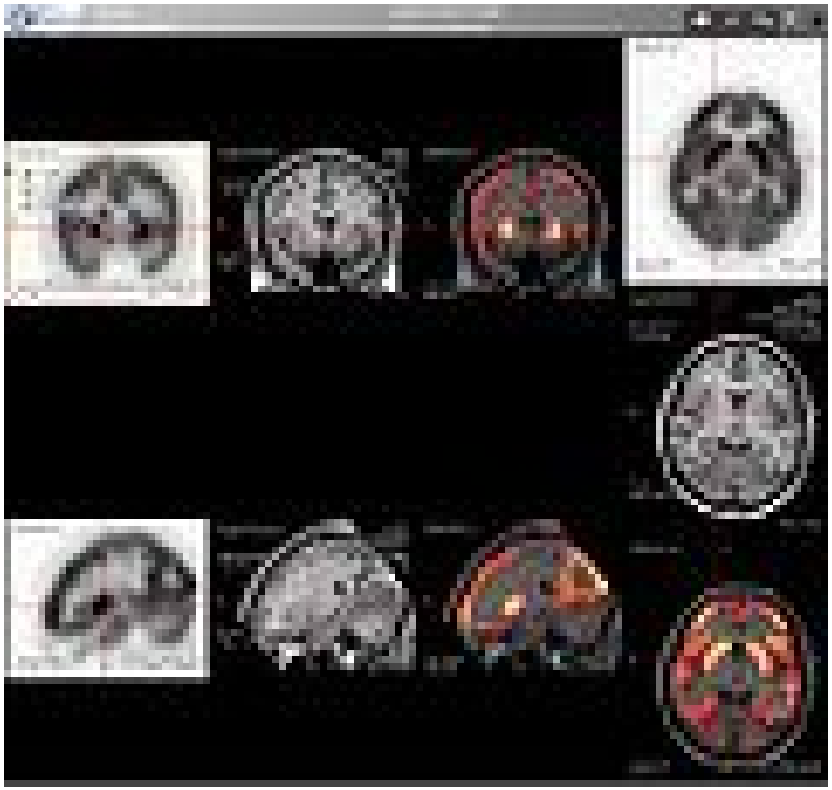


This is a transaxial slice of the brain of a 56 year old patient (male) taken with positron emission tomography (PET). The image was generated from a 20-minute measurement with an ECAT Exact HR+ PET Scanner. Red areas show more accumulated tracer substance (18F-FDG) and blue areas are regions where low to no tracer have been accumulated.

From Wikipedia Commons: Jens Langner (http://www.jens-langner.de/)



Positron Emmision/MRI Fusion Image



From Wikipedia Commons.

This is an image taken with a machine that simultaneously measured the PET data and the MRI data. This way there is no problem registering the two images. This compensates in part for the low resolution attainable with PET alone.

Magnetic Resonance Imaging, I

The remainder of these lectures will be concerned with Nuclear Magentic Resonance Imaging. This techniques takes advantage of the fact that spin-1/2 particles, like the protons in hydrogen nuclei, can be polarized in a magnetic field. By "selectively exciting" them, these polarized spins can be made to precess, and this produces a measurable signal at a very specific (and convenient) frequency, i.e. a resonance.

In its simplest form MR attempts to reconstruct $\rho(\boldsymbol{x})$, the density of water molecules in an object, from *direct* measurements of its Fourier transform $\widehat{\rho}(\xi)$.

Magnetic Resonance Imaging, II

The reconstruction algorithm is therefore essentially the Fourier transform:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-i\boldsymbol{x}\cdot\xi} d\boldsymbol{x} \quad f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{i\boldsymbol{x}\cdot\xi} d\xi, \quad (1)$$

which is not only well-conditioned, but actually unitary. There are extremely fast algorithms to approximately compute this transform, with *spectral accuracy*.

Magnetic Resonance Imaging, III

Unlike other modalities, the reconstruction problem in MR is trivial and well-conditioned, and the acquisition of data is fundamentally safe. Thus the emphasis in this field is on the search for useful contrasts,

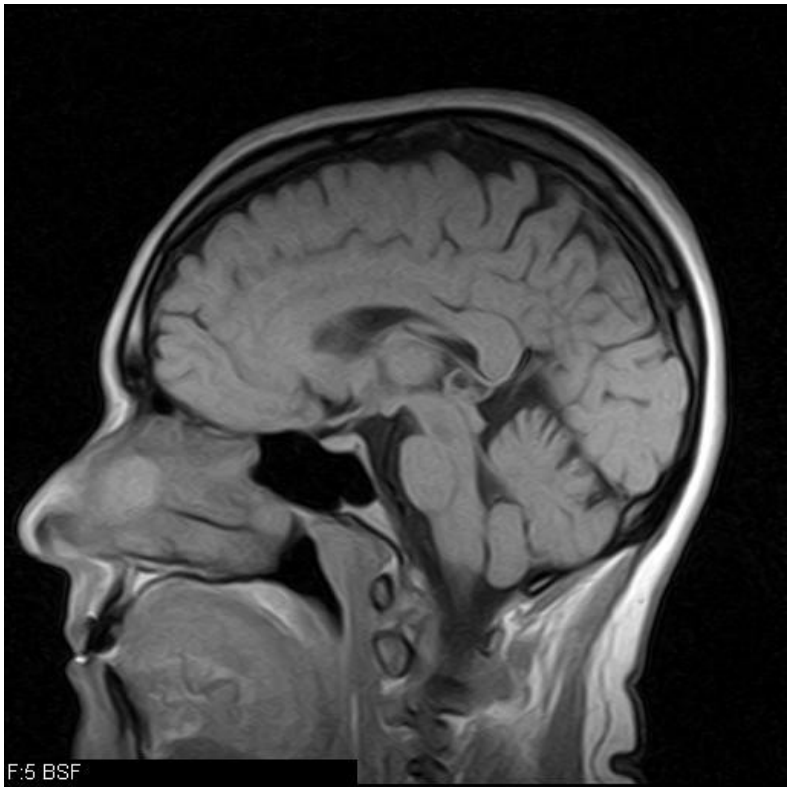
This spin-resonance phenomenon is sensitive to an enormous range of physical and chemical effects, and so can be used to obtain a tremendous variety of contrast mechanisms: from proton-density, to local magnetic susceptibility, to NMR spectrum, to diffusivity.

The enemy in MR is time: for deep physical reasons, there are limits to how fast the data can be acquired.

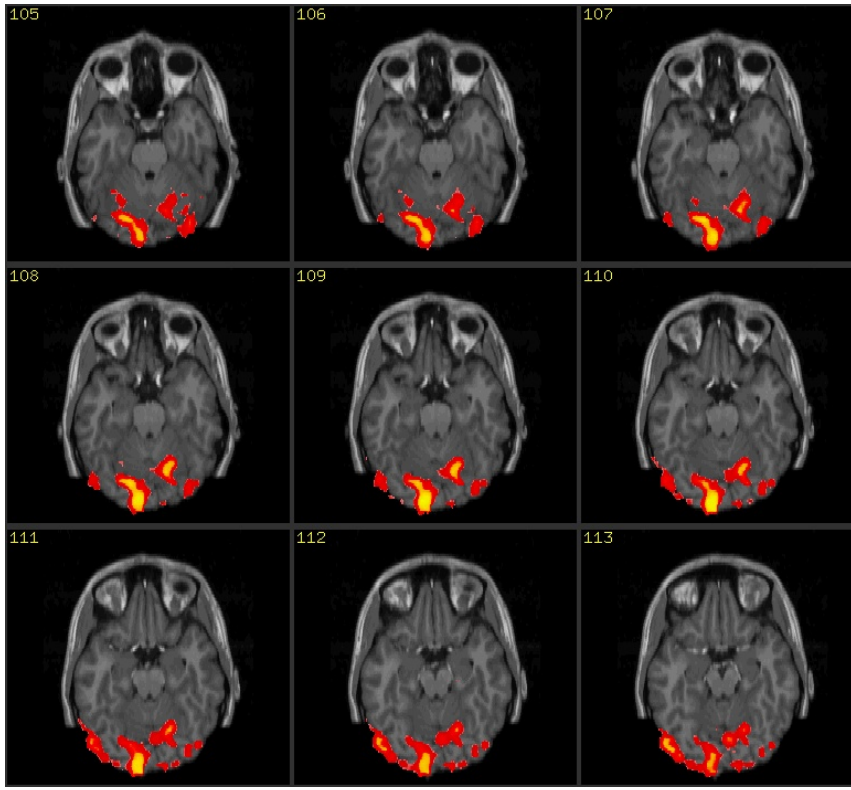
Why MRI is Safe (almost)

The frequencies encountered in MRI are in the 50-300 MhZ range. Using Planck's relation $E = h\nu$, we see that these energies are about 10^{-7}eV. This should be compared to X-rays which are typically in the 100KeV range, or PET in the .5MeV range. It explains why MRI is an extremely safe imaging modality: the energies involved are much too small to break chemical bonds. It's most significant limitation is the long time needed to acquire the data, which is also a consequence of spin physics.
These days the magnets in clinical use have field strengths of about 7 Tesla. The resonance frequency is around 300 MHz. At these frequencies the scanner can easily become a microwave oven, so you need to be careful not to cook the subject.... In MR this is called *SAR*, or specific absorption rate. It may also be a problem with cell-phones.

MR Images



From: http://www.melissa-memorial.org/CMS/



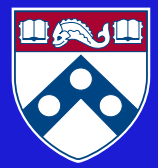
From: http://www.csulb.edu/ cwal-lis/482/fmri/fmri.html

Nuclear Magnetic Resonance, I

- A proton is a particle of spin $\frac{1}{2}$. This means that it has an intrinsic spin $\boldsymbol{\mu}_p$ and also an intrinsic angular momentum $\boldsymbol{J}_p$. As a consequence of Wigner-Eckert theorem these two quantities must be proportional: $\boldsymbol{\mu}_p = \gamma_p \boldsymbol{J}_p$.
- Empirically, this means that if a single proton could somehow be isolated and placed in a magnetic field $\boldsymbol{B}_0$, then it would act like a little bar-magnet: the expected value of its magnetic moment would satisfy

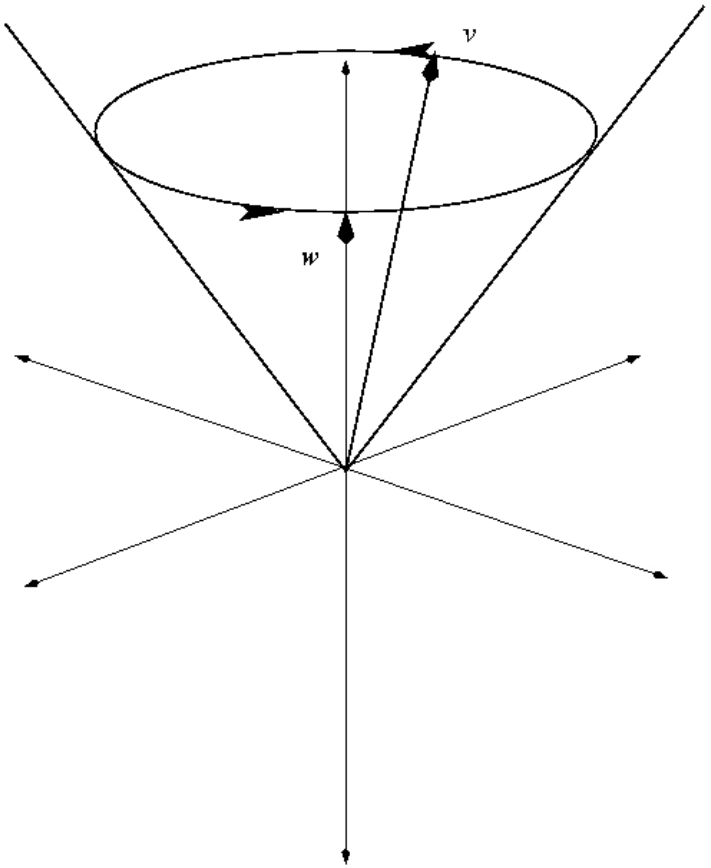
$$\frac{d\langle \boldsymbol{\mu}_p \rangle}{dt} = \gamma_p \langle \boldsymbol{\mu}_p \rangle \times \boldsymbol{B}_0. \tag{2}$$

Here $\gamma_p \simeq 2.675 \times 10^8 \text{rad/sT}$ (42.5 MhZ) is the "gyromagnetic ratio."



Nuclear Magnetic Resonance, II

This equation predicts that the little bar-magnet will precess about the field $\boldsymbol{B}_0$ with angular frequency $\omega_0 = \gamma_p \|\boldsymbol{B}_0\|$. This is the "resonance" in nuclear magnetic resonance. This frequency is called the Larmor frequency.



Don’t ask your guru

- I do not recommend asking your guru to explain spin.
- It is no more or less understandable than charge or mass.
- It is simply a fact of life that certain elementary particles possess spin, and you should understand it in terms of how it manifests itself physically.
- That said: spin is really weird!

Ensembles of MANY Spins

- The are a lot of water molecules H_2O in a living thing. Each of the hydrogens is a spin-$\frac{1}{2}$ particle. If you place this huge ensemble of spin-$\frac{1}{2}$ particles in a static magnetic field, $\boldsymbol{B}_0$, then thermodynamical considerations show that these nuclear spins will become polarized, leading to a bulk magnetization $\boldsymbol{M}_0(x) = \epsilon\rho(x)\boldsymbol{B}_0(x)$. Here $\rho(x)$ is the density of water molecules at position x.
- The proportionality constant, ϵ, is very small: at room temperature about 1 in a million spins is aligned with the field. This is why we use a very large field (1.5-7 Tesla) to do imaging experiments. Even so, the *static* field perturbation is essentially undetectable.
- The Earth's field is $.5 \times 10^{-4}$ Tesla.

The Bloch Equation, I

- To a first approximation, so long as these water molecules are in the liquid state, this ensemble responds to a magnetic field, $\boldsymbol{B}(\boldsymbol{x};t)$, according to the same equation as a single spin:

$$\frac{d\boldsymbol{M}}{dt}(x,t) = \gamma\, \boldsymbol{M}(x;t) \times \boldsymbol{B}(x;t), \tag{3}$$

  for a constant $\gamma \approx \gamma_p$, that depends on a variety of physical things. $\boldsymbol{M}$ is called the magnetization.
- Notice that spatial position is a parameter in this equation. The spin degrees of freedom are *uncoupled* from the spatial degrees of freedom.
- This is essentially the same as the equation describing the motion of a gyroscope in a gravitational field.
- The magnetic field is allowed to depend on both space and time. Felix Bloch introduced a more realistic model that forms the basis of most analysis in imaging applications

The Bloch Equation, II

- In the model above, there is no way for spins to become polarized.
- To get more accurate predictions we model the coupling of the spins with the "spin environment" by adding relaxation terms:

$$\frac{d\boldsymbol{M}}{dt}(x;t) = (1-\sigma(x))\gamma_p \boldsymbol{M}(x;t)\times \boldsymbol{B}(x;t) - \frac{1}{T_2}\boldsymbol{M}^{\perp} - \frac{1}{T_1}(\boldsymbol{M}^{\parallel} - \boldsymbol{M}_0). \quad (4)$$

Here $\sigma(x)$ is the *chemical shift*; it is needed because the local electronic environment in a molecule changes the magnetic field in a neighborhood of the nuclei. It usually lies in the range $0 - 100$ p.p.m.

The Bloch Equation, III

- Here $\boldsymbol{B}_0$ is a large static field, usually spatially homogeneous, e.g. $\boldsymbol{B}_0 = (0, 0, b_0)$, and $\boldsymbol{M}_0 \propto \boldsymbol{B}_0$, is the equilibrium field; $\boldsymbol{M}^{\perp}$ is the part of $\boldsymbol{M}$ orthogonal to $\boldsymbol{B}_0$ and $\boldsymbol{M}^{\parallel}$ the projection along $\boldsymbol{B}_0$.
- Here $\boldsymbol{B}(x,t) = \boldsymbol{B}_0(x) + \boldsymbol{G}(x,t) + \boldsymbol{B}_1(x,t)$; the $\boldsymbol{G}$ are quasi-static *gradient* fields, typically much smaller than $\boldsymbol{B}_0$. These fields are needed to get spatially resolved images.
- The terms $\frac{1}{T_2}\boldsymbol{M}^{\perp} - \frac{1}{T_1}(\boldsymbol{M}^{\parallel} - \boldsymbol{M}_0)$ are relaxation terms that capture the interaction of the spins with their environment.

The Bloch Equation, IV

The $\boldsymbol{B}_1$-field is an RF-field, with $\|\boldsymbol{B}_1\| << \|\boldsymbol{B}_0\|$. If $\boldsymbol{B}_0$ points along the z-axis, then usually

$$\boldsymbol{B}_1(t) = (a(t)e^{i\omega t}, 0) \text{ where } \omega = \gamma \|\boldsymbol{B}_0\|. \tag{5}$$

We often represent vectors in $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$ as a pair $(x + iy, z)$. The function $a(t)$ is a complex valued envelope.

Understanding the Bloch Equation, 0

The vast majority of the theoretical analysis done in MR-imaging involves a detailed understanding of the solutions of this very simple system of ODEs. MR-scientists have devised many very clever methods for solving both the forward and inverse problems, albeit approximately, for this system of equations.
See, for example:
Echoes -How to Generate, Recognize, Use or Avoid Them in MR-Imaging Sequences Part I: Fundamental and Not So Fundamental Properties of Spin Echoes, by Jürgen Hennig in Concepts in Magnetic Resonance, 3(1991), 125-143.
and *Inversion of the Bloch Equation*, by M. Shinnar and J.S. Leigh in J. Chem. Phys.,98(1993) pp. 6121-6128.

Understanding the Bloch Equation, I

- If $\boldsymbol{B}(x) = \boldsymbol{B}_0$, a time independent uniform field, then the Bloch equation predicts that the *transverse* components $\boldsymbol{M}^{\perp}$ will decay exponentially, like $e^{-\frac{t}{T_2}}$, while the *longitudinal* components, $\boldsymbol{M}^{\parallel}$, will approach $\boldsymbol{M}_0$ like $e^{-\frac{t}{T_1}}$.
- The physical processes that produce these relaxation effects are different. The process that returns the longitudinal component to its equilibrium state is called *spin-lattice* relaxation, whereas the process that destroys the phase coherence leading to a non-zero transversal component is called *spin-spin* relaxation. typically $T_2 < T_1$.
- The time constants governing these processes play a big role in determining how long it takes to collect the data, and the quality and contrast of an MR-image. In most soft tissue they lie in the 10ms-1s range.

Free Precession

- Suppose that $\boldsymbol{B}(x)$ is a static field.
- If we ignore relaxation, and imagine that we start with some initial $\boldsymbol{M}(x;0)$, then Bloch's equation predicts that the magnetization will precess around $\boldsymbol{B}$ with an angular frequency $\omega(x) = \gamma \, \|\boldsymbol{B}(x)\|$.
- If $\boldsymbol{B} = (0,0,b_0)$, then $\omega_0 = \gamma \, b_0$, and

$$\boldsymbol{M}(x;t) = \begin{pmatrix} \cos\omega_0 t & -\sin\omega_0 t & 0 \\ \sin\omega_0 t & \cos\omega_o t & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{M}(x;0). \qquad (6)$$

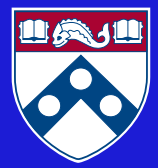
- Notice that the chemical shift replaces γ with $(1-\sigma)\gamma$, which changes the local resonance frequency. This is the basis of NMR spectroscopy.

The FID and Faraday's Law

- The solution on the previous slide is called "free precession." If a coil is placed near the object, then this changing field will produce an EMF, which is the measured signal in MRI.
- The fact that the precession frequency $\omega_0 = \gamma \|\boldsymbol{B}_0\|$, is the resonance phenomenon in nuclear magnetic *resonance*. Faraday's law states that the measured signal in a loop, ℓ of wire is

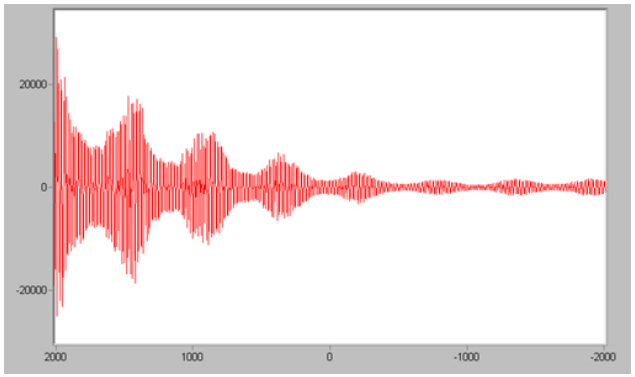
$$\text{E.M.F.} \propto \frac{d}{dt} \int_{\Sigma} \boldsymbol{M}(x;t) \cdot \boldsymbol{n} dS \tag{7}$$

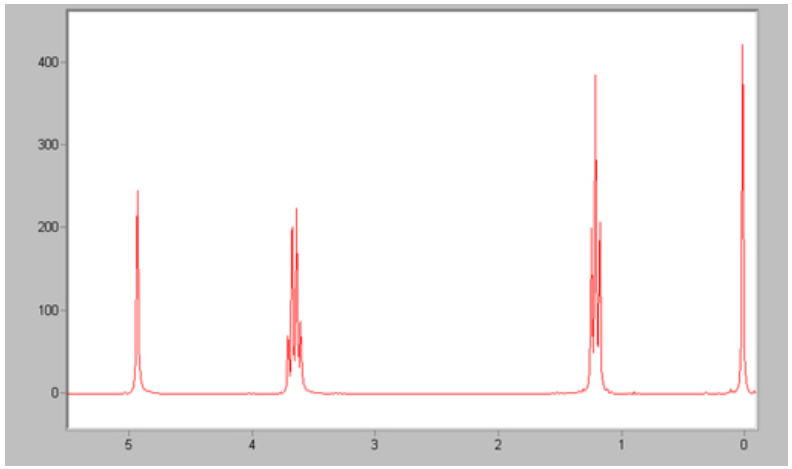
  where $b\Sigma = \ell$. It is proportional to ω_0^2.
- If there are different chemical shifts, then different parts of the sample will resonate at different frequencies.



NMR Spectroscopy

If our sample (ethyl alcohol $H_3C\text{-}CH_2\text{-}OH$) is a compound with several chemical shifts then the measured signal, and its Fourier transform resembles:





The widths of the peaks on the right are a measure of the T_2-relaxation rates, shorter T_2s produce broader peaks. There is a lot of fine structure, which gives a tool for determining the detailed structure of the molecule.

From: http://pslc.ws/macrog/nmrsft.htm

Spin Dynamics

To design MR-imaging experiments we need to somehow select time dependent fields $\boldsymbol{B}_1(x;t)$ to manipulate the arrangement of the spins. This turns out to be a problem in classical *inverse scattering* theory, which was solved, in principle, in the early 1970s by Zhaharov and Sabat.

Much earlier than that, NMR spectroscopists solved many special cases of these problems using physical intuition and ad hoc methods. This is a case where the ill-conditioning of the inverse scattering problem works to the advantage of the experimental scientists: within experimental error many different potentials produce the same result!

The Rotating Frame, I

If $\boldsymbol{B}_0 = (0, 0, b_0)$ and $B_1(t) = (\alpha(t)e^{i\omega_0 t}, 0)$, then the solution operator for Bloch's equation, without relaxation is $U(t) = U_p(t)U_n(t)$ with

$$U(t) = \begin{pmatrix} \cos\omega_0 t & -\sin\omega_0 t & 0 \\ \sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & -\sin\theta(t) \\ 0 & \sin\theta(t) & \cos\theta(t) \end{pmatrix}, \tag{8}$$

where $\theta(t) = \int_0^t \alpha(s)ds$. To simplify the discussion we remove the "laboratory precession" part, and let:

$$\boldsymbol{M}(t) = U_p(t)\boldsymbol{m}(t) \tag{9}$$

This defines the *rotating reference frame*. In this frame the initial vector $(0, 1, 0)$ is rotated about the x-axis through $\theta(t)$ radians.

The Rotating Frame, II

In the rotating reference frame the Bloch equations take the same form, but $\boldsymbol{B}$ is replaced by

$$\boldsymbol{B}_{\text{eff}} = U_p^{-1}(t)\boldsymbol{B}(x;t) - (0,0,b_0). \tag{10}$$

In the case we were considering this reduces to $\boldsymbol{B}_{\text{eff}} = (\alpha(t),0,0)$. In essentially all cases it is easier to predict what will happen in the rotating frame.
Practically speaking the experiment described above is accomplished by placing the object in the $\boldsymbol{B}_0$-field until the spins are polarized, and then "turning on" the RF-field for a certain amount of time.

The signal equation, I

- If $\boldsymbol{M}(x;0)$ is just $\boldsymbol{M}_0(x) = \epsilon\rho(x)\boldsymbol{B}_0$ rotated (coherently) through an angle Θ, then, ignoring relaxation, the measured signal is

$$s(t) \propto \sin(\Theta)\omega_0^2 e^{-i\omega_0 t}\int \rho(x)dx. \tag{11}$$

- After demodulating, (i.e. multiplication by $e^{i\omega_0 t}$) this gives us a measure of the total spin density.

The signal equation, II

- The rate precession at x depends only on the strength of the static field at x. If we turn on a *gradient field*, $\boldsymbol{G} = (\star, \star, \langle \boldsymbol{g}, (x, y, z)\rangle)$, with $\boldsymbol{g} = (g_1, g_2, g_3)$, then the measured signal becomes:

$$s(t) \propto \sin(\alpha)\omega_0^2 e^{-i\omega_0 t} \int e^{-it\gamma \langle \boldsymbol{g},(x,y,z)\rangle} \rho(x) dx. \tag{12}$$

- In other words: we can essentially measure the Fourier transform of ρ at the frequencies along a ray $\boldsymbol{k} = t\gamma\, \boldsymbol{g}$!
- An image with just $\rho(x)$ used to define the contrast is called a *proton density image*.
- For a variety of practical reasons this is not, in fact, how imaging is usually done. This description of the measurement also leaves out the relaxation terms.

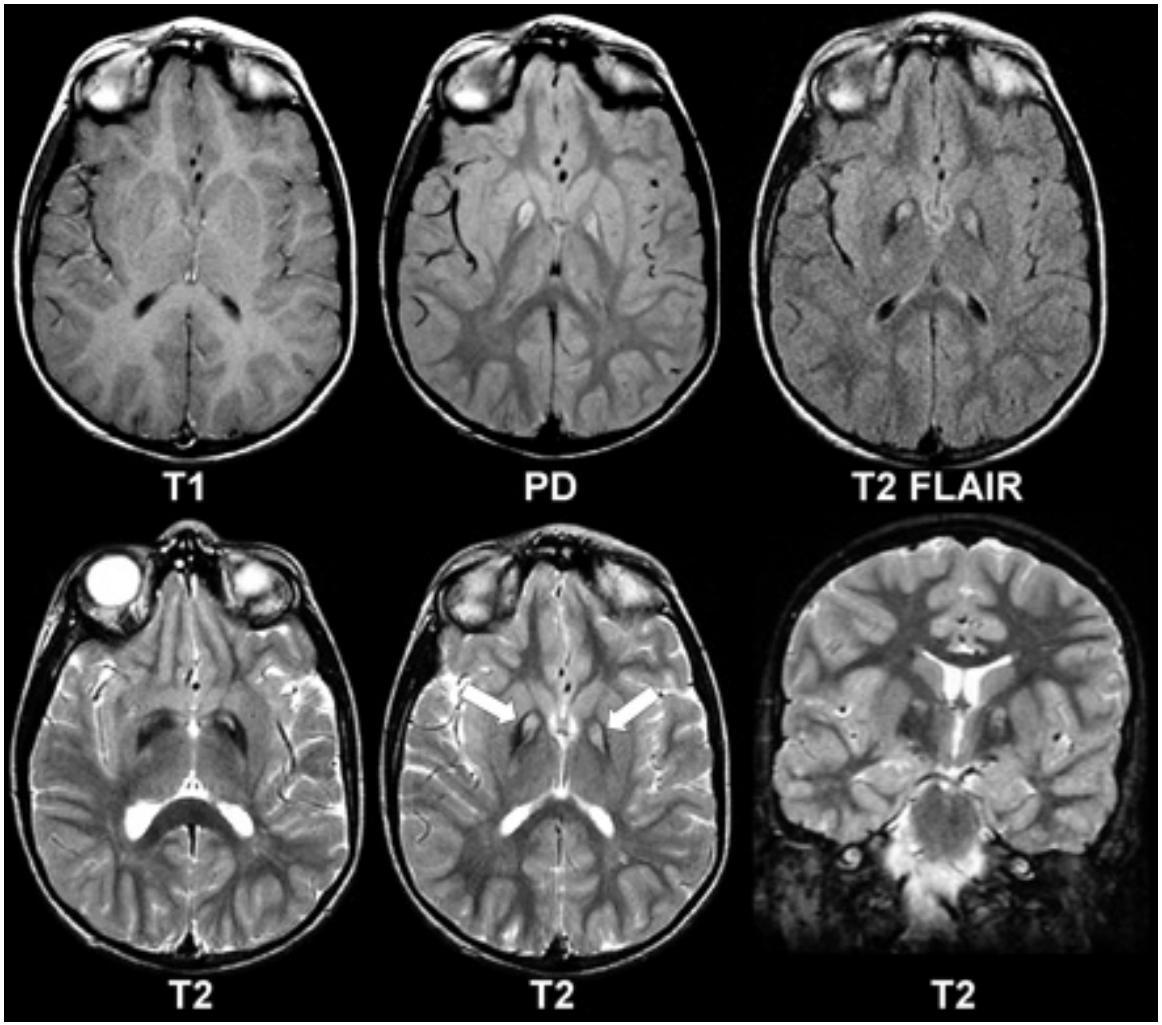
Different MR-contrasts, I

- A more accurate description of the signal would be:

$$s(t) \propto \sin(\alpha)\omega_0^2 e^{-i\omega_0 t} \int e^{-it\gamma\langle \boldsymbol{g},(x,y,z)\rangle} \rho(x) e^{-\frac{t}{T_2(x)}} dx. \quad (13)$$

- Notice that the image is being "modulated" by the spatially dependent, decaying exponential $e^{-t/T_2(x)}$. This provides a simple, but very useful contrast. White and grey matter in the brain exhibit different T_2 decay rates.
- This is just one of many possible ways to obtain contrast in an MR image; we will discuss this in the next lecture.
- See http://spinwarp.ucsd.edu/NeuroWeb/Text/br-100.htm

Different MR-contrasts, II



From: http://www.ajnr.org/cgi/content/full/27/6/1230/F1.

Print ISSN: 0195-6108 Online ISSN: 1936-959X.

The Problem of Selective Excitation, I

- We saw above how to rotate the magnetization from equilibrium, aligned with $\boldsymbol{B}_0$, so that it has a non-trivial transverse component. This is needed to get a measurable signal. By turning on an "RF" magnetic field of the form $B_1(t) = (qe^{i\omega_0 t}, 0)$, for a certain amount of time we can uniformly rotate $\boldsymbol{M}$ through a fixed angle.
- For a variety of reasons, it is better not to do this uniformly across the whole object, but rather to rotate the magnetization in a slice, and leave it in the equilibrium state outside a slightly larger slice.

The Problem of Selective Excitation, II

For this purpose we need time dependent magnetic fields of the form

$$\boldsymbol{B}(x;t) = (0,0,b_0) + (\star,\star,\langle \boldsymbol{g},(x,y,z)\rangle) + \boldsymbol{B}_1(t),$$

where $\boldsymbol{B}_1(t) = (q(t)e^{i\omega_0 t},0)$.
By having different resonance frequencies at different points in space, we can get a different response to a given time dependent RF-field. This is called *selective excitation*.
For simplicity we usually ignore relaxation effects when discussing selective excitation, this is fine for objects with T_2 much larger than the time required for selective excitation.

Target Magnetization

- The second term is a *gradient field;* it makes the magnetic environment different at different points in space. This makes it possible for $\boldsymbol{B}_1$ to have different effects at different points. We use a parameter $f = \gamma \|\boldsymbol{G}(x)\|$, the offset frequency, to capture this effect.
- A more general problem, than flipping the spins in a slice is to find $q(t)$ so that, starting from equilibrium, the field at a later time T follows a specified pattern: $\boldsymbol{M}(f;T) = \boldsymbol{M}_{\text{target}}(f)$.
- This is a classical inverse scattering problem. In the late 1980s various people approached the problem from this angle. (A. Grünbaum, Jack Leigh, Meir Shinnar, Patrick Le Roux, Rourke and Morris). It was realized that this problem could be rephrased in terms of the classical 2 AKNS system.

A Linear Approximation

For small flip angles, (so that $m_z \approx 1$,) we can use a linear approximation

$$\frac{d(m_x(f;t)+im_y(f;t))}{dt} = if(m_x(f;t)+im_y(f;t)) - i\gamma q(t), \tag{14}$$

with solution

$$m_x(f;T)+im_y(f;T) = -i\gamma e^{ifT}\int_0^T q(s)e^{-isf}ds. \tag{15}$$

That is, the response to the pulse is essentially the Fourier transform of the pulse envelope. This works surprisingly well, even up to flip angles of about 90°.

Spin Domain Bloch Equation

For higher flip angles, or to achieve greater control, we need to use a more exact method. For this purpose it is useful to use the spin-representation that comes from the covering $SU(2) \to SO(3)$, so that, with $\Psi = (\psi_1, \psi_2)$, taking values in $\mathbb{C}^2$, we have

$$(m_x + i m_y, m_z) = (2\psi_1^* \psi_2, |\psi_1|^2 - |\psi_2|^2). \tag{16}$$

The Spin domain Bloch equation is

$$\frac{d\Psi}{dt}(\xi; t) = \begin{pmatrix} -i\xi & -i\frac{\gamma}{2} q(t) \\ -i\frac{\gamma}{2} q^*(t) & i\xi \end{pmatrix} \Psi. \tag{17}$$

Scattering and Inverse Scattering

In applications to NMR, the potential function $q(t)$ is non-zero in an interval $[t_0, t_1]$. For $t < t_0$ the spins are in their equilibrium state $\Psi(t) = (e^{-i\xi t}, 0)$, and for $t > t_1$, the solution is of the form:

$$\Psi(\xi; t) = (a(\xi)e^{-i\xi t}, b(\xi)e^{i\xi t}). \tag{18}$$

The coeffcients $(a(\xi), b(\xi))$ are the scattering coefficients. The reflection coefficient is

$$r(\xi) = \frac{b(\xi)}{a(\xi)} = e^{-2i\xi t}\frac{(m_x(2\xi; t) + im_y(2\xi; t))}{1 + m_z(2\xi; t)}. \tag{19}$$

What appears on the right is easily computed from the target magnetization profile, whereas the left had side is the continuum scattering data for the 2×2 AKNS. In this way the basic problem of selective pulse design in NMR is reduced to the inverse scattering transform for this equation. This problem was solved constructively, in principle, in the early 1970s.

History of Practical Solutions

- Jack Leigh, Meir Shinnar, and Patrick Le Roux, found an approximate solution to this problem, using the "hard pulse" approximation, and a layer stripping method. Using this method, one can get $\|\boldsymbol{m}_{\text{target}}(f)\|$, but you loose control on the phase. This is called the SLR (Shinnar-Le Roux) method.
- Rourke and Morris used various methods to try to solve the Gelfand-Levitan-Marchenko equations. Their methods were too slow, unstable, and hard to understand. It had little impact.
- In his thesis (2004), my former student, Jeremy Magland combined these two ideas, finding a highly efficient algorithm that solves a clever discrete approximation to the true inverse scattering problem. Thus far it has had little impact, but we hope that as the field evolves our method will become more important.

An important practical lesson

While my work with Magland represented a conceptual and practical advance on a fundamental problem, it was not a problem that most people in MR felt needed solving. While their existing solutions were imperfect, they were good enough. We're hoping that advances in parallel imaging will make our techniques ability to control the phase of the excitation, as well as the magnitude, more important.

Progress in MRI is measured by improved image quality, new contrast mechanisms and reduced acquisition time, nothing else really counts!

We close today's lecture with a demonstration of our pulse design tool and some examples of pulse-sequences used in MR.

Some References

General References on MRI:
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