

Envelopes of Horospheres and Weingarten Surfaces  
in Hyperbolic 3-Space

by

Charles L. Epstein\*  
Department of Mathematics  
Princeton University  
Princeton, N.J.

Abstract

We derive basic differential geometric formulae for surfaces in hyperbolic space represented as envelopes of horospheres. The dual notion of parallel hypersurfaces is also studied. The representation is applied to prove existence and regularity theorems for Weingarten surfaces in  $\mathbb{H}^3$  which satisfy

$$(1-\alpha)K = \alpha(2-H) \quad \alpha < 0$$

and have a specified boundary curve at infinity. These surfaces are shown to be closely connected to conformal mappings of domains in  $S^2$  into the unit disk and provide Riemannian interpretations for some conformal invariants associated to such mappings.

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The theory of immersed surfaces in hyperbolic space is a rich and largely unexplored subject. Recently several authors have studied surfaces of constant mean curvature, see: [Br], [Do,La], [Mi], [Uhl]. In this paper we present a representation for hypersurfaces in  $\mathbb{H}^n$  as graphs over the ideal boundary of  $\mathbb{H}^n$ .

In three dimensions three classes of surfaces are distinguished in this representation by the simplicity of their defining equations. If  $k_1, k_2$  denote the principal curvatures of the immersed surface  $\Sigma$  then the Gauss and Mean curvatures are:

$$K = k_1 k_2 - 1$$

$$H = k_1 + k_2$$

The three distinguished classes are given by the curvature conditions:

- A.  $K = 0$
- B.  $H = 2$
- C.  $(1-\alpha)K = \alpha(2-H)$  ;  $\alpha \in \mathbb{R} \setminus [0,1]$

Surfaces of type A. have not been considered in print. They are, in Thurston's language, flat orbifolds. Surfaces of type B. were considered by Bryant in [Br] from a point of view which is related to ours though arrived at independently. Other results on surfaces of this type can be found in [DoC,G,T], [Do,LA]. We will consider the third type of surface.

This paper is divided into three parts. In the first part, §§2-3, we present the representation theory for hypersurfaces in  $\mathbb{H}^n$  and derive the basic differential geometric formulae in this representation. In the second part, §§4-6, refinements and extensions of the theory in §§2-3 available for surfaces in  $\mathbb{H}^3$  are explored. Finally, in §§7-8 the theory developed in the first parts is applied to study a Dirichlet problem for

surfaces which satisfy:

$$(1-\alpha)K = \alpha(2-H) \quad \alpha < 0 \quad . \quad (1.1)$$

We will call such a surface an  $\alpha$ -Weingarten surface.

The results in the latter sections are obtained through a connection between  $\alpha$ -Weingarten surfaces and conformal maps of  $S^2$  into  $\mathbb{E}$ . This is somewhat in the same spirit as the work in [Br]. We will obtain Riemannian interpretations for various conformal properties of such maps.

#### Definitions

Hyperbolic  $(n+1)$ -space,  $\mathbb{H}^{n+1}$  will be represented as the interior of the unit ball,  $\mathbb{E}^{n+1}$  in  $\mathbb{R}^{n+1}$  with the metric:

$$ds^2 = \frac{4(dx_1^2 + \dots + dx_{n+1}^2)}{(1-r^2)^2} .$$

The ideal boundary of hyperbolic space,  $\partial\mathbb{H}^{n+1}$  is naturally identified with the unit sphere,  $S^n$ , in  $\mathbb{R}^{n+1}$ . The basic facts of hyperbolic geometry will be taken for granted; as a reference one can consult [Th], [Be], [Sp].

In what follows the horospheres play a central role. They are the simply connected, complete, flat hypersurfaces in  $\mathbb{H}^{n+1}$ . In the ball model they are represented by Euclidean spheres internally tangent to the unit sphere

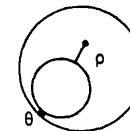


Fig. 1 The horosphere  $H(\theta, \rho)$  in  $\mathbb{H}^2$ .

They are parameterized by  $\theta$ , the point of tangency with  $S^n$  and  $\rho$ , the smallest hyperbolic distance between the horosphere and the point  $(0, \dots, 0)$  in  $\mathbb{H}^{n+1}$ . Here  $\rho$  is positive if  $(0, \dots, 0)$  is in the exterior of the horosphere and negative otherwise. We will denote this horosphere by  $H(\theta, \rho)$ .

Recall also that the geodesics in the ball model are the circular arcs in  $\mathbb{H}^{n+1}$  that meet  $S^n$  normally. Every pair of points on  $S^n$  uniquely determines a geodesic and vice versa. We will use the notation  $\psi^t(p, X)$  to denote the geodesic with initial point  $p$  and velocity  $X$ ; this is usually denoted  $\exp_p(tX)$ .

If  $\Sigma$  is an oriented hypersurface, immersed in  $\mathbb{H}^{n+1}$ , then there is a globally defined unit normal field,  $N$ . Using the ball model, we can define a Gauss map for  $\Sigma$  by:

$$\mathcal{L}_\Sigma(p) = \lim_{t \rightarrow \infty} \psi^t(p, N)$$

$\mathcal{L}_\Sigma$  has certain properties which are analogous to the Euclidean Gauss map. These were studied independently in [Br]. Note that one could also use  $-N$  as the unit field on  $\Sigma$ . The mapping one obtains can be quite different. If  $A$  is an isometry of hyperbolic space, then  $A \cdot \Sigma$  is also an oriented immersed surface and

$$\mathcal{L}_{A \cdot \Sigma}(Ap) = A \cdot \mathcal{L}_\Sigma(p).$$

Notation

Some of the notation will not be used until the end of the paper; we include it for the convenience of the reader.

$|\cdot|$  - Length in the Euclidean metric.

$\langle, \rangle$  - The inner product on  $T\mathbb{H}^{n+1}$ .

$D$  - The gradient w.r.t. the round metric on  $S^n$ .

$\nabla_X Y$  - Covariant differentiation in  $\mathbb{H}^{n+1}$  w.r.t.  $\langle, \rangle$ .

$$\begin{aligned} R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_X \nabla_X Z - \nabla_{[X, Y]} Z \\ &= -[\langle X, Z \rangle Y - \langle Y, Z \rangle X], \text{ the Riemann tensor of } \nabla_X Y. \end{aligned}$$

$\Delta_S$  - The Laplace-Beltrami operator on  $S$ .

$\Sigma$  - an immersed hypersurface.

$X_1, \dots, X_n$  - The vector fields spanning  $T\Sigma$  defined by an immersion.

$N$  - The unit normal field of  $\Sigma$ .

$g_{ij} = \langle X_i, X_j \rangle$  - The induced metric on  $T\Sigma$ .

$g^{ij}$  - The dual metric:  $g^{ik} g_{kj} = \delta_j^i$ .

$$\pi_{ij} = \langle \nabla_{X_j} X_i, N \rangle$$

$\pi_j^i = g^{ik} \pi_{kj}$  - The second fundamental form of  $\Sigma$ .

$K(g)$  - The Gauss curvature of the metric  $g$ .

$D_r$  - The disk of radius  $r$  in  $E$ .

$\Omega$  - A domain in  $S^n$ .

$d\sigma^2$  - The curvature +1 metric on  $S^n$ .

$z$  - A local conformal parameter on  $S^n$ .

' - Differentiation w.r.t.  $z$ .

$$\gamma - d\sigma^2 = \gamma^2 |dz|^2.$$

$\rho$  - A function defined on a domain in  $S^n$ .

$\rho_\Omega$  - The hyperbolic metric on  $\Omega \subset S^2$  is  $e^{2\rho_\Omega} d\sigma^2$ .

$\Sigma(\rho)$  - The surface generated by  $\rho$ .

$\Sigma_t$  - The parallel surface at distance  $t$  from  $\Sigma$ .

$f_\Omega$  - The conformal map from  $\Omega$  to  $D_1$ .

$$\left. \begin{aligned} \partial &= 1/2(\partial_x - i\partial_y) \\ \bar{\partial} &= 1/2(\partial_x + i\partial_y) \end{aligned} \right\} \begin{array}{l} \text{The complex derivatives w.r.t. a conformal} \\ \text{parameter } z = x + iy. \end{array}$$

$$\mathcal{J}_f(z) = \left[ \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 \right] (z)$$

- The Schwarzian derivative of  $f$ .

$\mu_\Omega - \mu_\Omega |dz|^2$  is the hyperbolic metric on  $\Omega$  w.r.t. the conformal parameter  $z$ .

$$\mu_\Omega = 4 |f'_\Omega|^2 (1 - |f_\Omega|^2)^{-2}$$

Note:  $\mu_\Omega |dz|^2 = e^{2\rho_\Omega} d\sigma^2$  if  $d\sigma^2$  is represented w.r.t.  $z$ .

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2. Hyperbolic Graphs and Parallel Surfaces:

Let  $\Omega$  be a domain on  $S^n$  and  $\rho(\theta)$  a differentiable function defined in  $\Omega$ . Recall that if  $\mathcal{F}$  is a family of hypersurfaces in  $\mathbb{R}^n$  then the envelope of  $\mathcal{F}$  is a hypersurface  $\Sigma$  which is everywhere tangent to hypersurfaces in  $\mathcal{F}$ . This notion is independent of the metric on  $\mathbb{R}^n$  as it is simply a statement about identity of tangent spaces.

$\rho(\theta)$  defines a surface in  $\mathbb{H}^{n+1}$  by the prescription:

$$\Sigma(\rho) = \text{Outer envelope of the family of horospheres: } \{H(\theta, \rho(\theta)) : \theta \in \Omega\} .$$

(Outer envelope refers to the envelope which is not the unit sphere, which is obviously an envelope for any smooth family of horospheres).

The envelope of a smooth family of horospheres may fail to exist in that  $\Sigma(\rho)$  may fail to be smooth. Notwithstanding, we will derive a formula for  $\Sigma(\rho)$  which gives the usual envelope wherever it exists. At other points it defines a continuous mapping from  $\Omega$  into  $\mathbb{H}^{n+1}$ .

We need an explicit formula for the horosphere  $H(\theta, \rho)$ . Let

$$r = \frac{e^\rho - 1}{e^\rho + 1} \tag{2.1}$$

$$H(\theta, \rho) = \left\{ \frac{1+r}{2} X(\theta) + \frac{1-r}{2} Y : Y \in S^n \right\} \hookrightarrow \mathbb{R}^{n+1}$$

$X(\theta)$  is the point  $\theta$  on the unit sphere in  $\mathbb{R}^{n+1}$ . We will also use  $\theta$  as notation for a coordinate.

To derive the formula for the envelope we let  $Y \in S^n$  be represented parametrically as  $Y(\alpha)$  where  $\alpha = (\alpha^1, \dots, \alpha^n)$  ranges over an open set in  $\mathbb{R}^n$ .  $\theta = (\theta_1, \dots, \theta_n)$  also ranges over an open set in  $\mathbb{R}^n$ . Let

$$R(\theta, \alpha) = \frac{1+r(\theta)}{2} X(\theta) + \frac{1-r(\theta)}{2} Y(\alpha)$$

Formally to solve for the envelope we should solve for  $\alpha(\theta)$ . In fact this will be unnecessary as we can solve for  $Y(\alpha(\theta))$  directly. For the remainder of the derivation we will assume that the envelope exists, i.e., that one could solve for  $\alpha(\theta)$  and it would be differentiable.

The tangent space to  $\Sigma(\rho)$  at  $R(\theta, \alpha(\theta))$  is spanned by:

$$2R_{\theta_i} = r_{\theta_i} X + (1+r)X_{\theta_i} - r_{\theta_i} Y + (1-r) \sum_{j=1}^n Y_j \alpha_{\theta_i}^j \quad i=1, \dots, n .$$

The tangent space to  $H(\theta, \rho(\theta))$  is spanned by:

$$2R_{\alpha^j} = (1-r)Y_{\alpha^j} \quad j=1, \dots, n .$$

The conditions defining the envelope are

$$\text{Span}\{R_{\theta_i}(\theta, \alpha(\theta))\} = \text{Span}\{R_{\alpha^j}(\theta, \alpha(\theta))\} . \tag{2.2}$$

This is true at a regular point if and only if:

$$R_{\theta_i}(\theta, \alpha(\theta)) \perp R_{\alpha^1} X \dots X R_{\alpha^n}$$

where  $\perp$  and  $\times$  are with respect to the Euclidean inner product which we denote by  $X \cdot Y$ .

An elementary calculation shows that

$$R_{\alpha^1} X \dots X R_{\alpha^n} = \lambda Y$$

for some  $\lambda \neq 0$ . We rewrite the conditions, (2.2) as:

$$r_{\theta_i} X \cdot Y + (1+r)X_{\theta_i} \cdot Y = r_{\theta_i} \quad i=1, \dots, n \quad (2.2')$$

These follow as  $Y \cdot Y = 1$  and  $Y \cdot Y_{\alpha^j} = 0$ . This is an inhomogeneous system of linear equations for  $Y(\theta)$ . If we compute the Gramian matrix associated to the system we obtain:

$$G = (1+r)^2 I + r_{\theta} \otimes r_{\theta}$$

Since  $G$  is always of rank  $n$ , the kernel of (2.2') is one dimensional.

It is generated by the vector:

$$Z = X - (r+1)^{-1} \sum_{i=1}^n r_{\theta_i} X_{\theta_i} (X_{\theta_i} \cdot X_{\theta_i})^{-1}$$

A particular solution to (2.2') is  $Y_p = X$ , so the general solution is

$$X + \mu Z \quad \mu \in \mathbb{R}.$$

The condition which we use to determine  $\mu$  is  $Y \cdot Y = 1$ . The possible solutions are:

$$Y = X \quad \text{and}$$

$$Y = \frac{|Dr|^2 - (1+r)^2}{|Dr|^2 + (1+r)^2} X + \frac{2(1+r)Dr}{|Dr|^2 + (1+r)^2} \quad (2.3)$$

where:

$$Dr = \sum r_{\theta_i} X_{\theta_i} (X_{\theta_i} \cdot X_{\theta_i})^{-1} \quad \text{and}$$

$$|Dr|^2 = Dr \cdot Dr.$$

"D" is the gradient on  $S^n$  with respect to the round metric.  $Y = X$  is clearly the inner envelope and thus using (2.1) we obtain a formula for  $R_{\rho}$ :

$$R_{\rho}(\theta) = \frac{|D\rho|^2 + (e^{2\rho} - 1)}{|D\rho|^2 + (e^{\rho} + 1)^2} X(\theta) + \frac{2D\rho}{|D\rho|^2 + (e^{\rho} + 1)^2}. \quad (2.4)$$

We will use  $R_{\rho}(\theta)$  to denote the parametric representation of the mapping from  $\Omega$  into  $\mathbb{H}^{n+1}$  defined by  $\rho$ . The parametrization is determined by the parametrization  $X(\theta)$  of the unit sphere.

From the derivation it is clear that  $R_{\rho}(\theta)$  coincides with the envelope of  $H(\theta, \rho(\theta))$  whenever it is possible to solve for  $\alpha(\theta)$ . For under this assumption there is a unique point on each horosphere which lies in the envelope. We will use  $\Sigma(\rho)$  to denote the hypersurface generated by  $\rho$ . As we will see in §5, if  $\rho$  is twice differentiable then  $\alpha(\theta)$  exists wherever  $\Sigma(\rho)$  is smooth. Clearly  $R_{\rho}(\theta)$  is continuous if  $\rho(\theta)$  is continuously differentiable and thus the envelope is connected if  $\Omega$  is connected.

If  $\rho \in C^k(\Omega)$ , then  $\Sigma(\rho)$  will be a  $C^{k-1}$ -surface wherever  $R_{\rho}$  is an immersion. An interesting feature of this representation is that the principal curvatures of  $\Sigma(\rho)$  are expressions involving  $\rho$  and its first and second derivatives. As the formula for  $R_{\rho}$  involves  $\rho$  and its first derivative, we would expect the principal curvatures to depend on the third derivatives of  $\rho$  as well.

Closely connected to the construction of hypersurfaces as envelopes of horospheres is the family of hypersurfaces parallel to a given hypersurface. Let  $N$  denote the outward unit normal field on  $\Sigma$ . Then the parallel surface at distance  $t$  is defined to be:

$$\Sigma_t = \{ \psi^t(p, N) : p \in \Sigma \}.$$

The dynamical system  $\Sigma \longrightarrow \Sigma_t$  will be referred to as the parallel flow.

The connection between parallel surfaces and envelopes of horospheres is contained in the next theorem. This fact was observed by Thurston:

**Theorem 2.1:** The parallel surface at distance  $t$  from  $\Sigma(\rho)$  is the surface generated by the function  $\rho+t$ ; more succinctly:

$$\Sigma_t(\rho) = \Sigma(\rho+t).$$

**Remark:**  $\Sigma(\rho)$  need not be smooth; at a non-smooth point there is no normal vector. However, the representation  $R_\rho$  allows us to define a subspace of  $\mathbb{R}^{n+1}$  at non-smooth points composed of "normal" vectors to  $\Sigma(\rho)$ . If  $\mathcal{M}_\rho$  is this family of vectors then

$$S_t = \{\psi^t(p, N) : N \in \mathcal{M}_\rho\}$$

will be a smooth hypersurface for small non-zero  $t$ . Fix such a  $t = t_0$ . Then  $p \in \Sigma(\rho)$  will be a focal point of the surface  $S_{t_0}$  (see §3). At smooth points on  $\Sigma(\rho)$  the statement of the theorem is rigorously correct. We will make further remarks after the proof.

**Proof of Theorem 2.1:** Consider the formula for  $R_{\rho+t}(\theta)$ :

$$R_{\rho+t}(\theta) = \frac{|D\rho|^2 + (e^{2(\rho+t)} - 1)}{|D\rho|^2 + (e^{\rho+t} + 1)^2} X(\theta) + \frac{2 D\rho}{|D\rho|^2 + (e^{\rho+t} + 1)^2}$$

Elementary but tedious calculations show that for fixed  $\theta$ :

$$1) \left| R_{\rho+t}(\theta) - \left( X(\theta) + \frac{D\rho(\theta)}{|D\rho(\theta)|^2} \right) \right| \text{ is constant}$$

$|\cdot|$  is Euclidean distance.

$$2) \lim_{t \rightarrow \infty} e^t \frac{dR_{\rho+t}(\theta)}{dt} = \lim_{t \rightarrow \infty} 2R_{\rho+t}(\theta) = 2X(\theta)$$

and

$$\lim_{t \rightarrow -\infty} e^{-t} \frac{dR_{\rho+t}(\theta)}{dt} = \lim_{t \rightarrow -\infty} -2R_{\rho+t}(\theta)$$

$$= -2 \left[ \frac{|D\rho|^2 - 1}{|D\rho|^2 + 1} X(\theta) + \frac{2 D\rho}{|D\rho|^2 + 1} \right]$$

The limits are taken in the Euclidean topology of  $\mathbb{R}^{n+1}$ .

$$3) \left\langle \frac{dR_{\rho+t}}{dt}(\theta), \frac{dR_{\rho+t}}{dt}(\theta) \right\rangle = 1 \text{ for all } t.$$

From 1), 2) and 3) it follows that  $R_{\rho+t}(\theta)$  is a unit speed parametrization of a hyperbolic geodesic normal to the family of horospheres  $\{H(\theta, \rho(\theta) + t) : t \in \mathbb{R}\}$ . If  $\Sigma(\rho)$  is smooth at  $p$  then  $\Sigma(\rho)$  is tangent to  $H(\theta, \rho(\theta))$  at  $p$  and thus the unit normal vector to  $\Sigma(\rho)$  at  $p$  is:

$$N = \left. \frac{dR_{\rho+t}}{dt}(\theta) \right|_{t=0}$$

From the definition of a parallel surface, it is clear that if  $p = R_\rho(\theta)$  is a smooth point on  $\Sigma(\rho)$  then:

$$R_{\rho+t}(\theta) = \psi^t(p, N)$$

Thus the theorem follows wherever  $\Sigma(\rho)$  is smooth.

If  $\Sigma(\rho)$  is not smooth at  $R_\rho(\theta)$  then  $R_{\rho+t}(\theta)$  is a point a distance  $t$  from  $\Sigma(\rho)$ . If  $\Sigma(\rho+t)$  is smooth at this point then the normal vector to  $\Sigma(\rho+t)$  at  $R_{\rho+t}(\theta)$  is

$$\tilde{N} = \left. \frac{dR_{\rho+t+s}}{ds}(\theta) \right|_{s=0}$$

$R_\rho(\theta)$  is the point on  $\Sigma_{-t}(\rho+t)$  given by:

$$\psi^{-t}(p, \tilde{N})$$

$R_\rho(\theta)$  is in the focal set of  $\Sigma(\rho+t)$ . We will discuss this further in §3.

Remark: From the argument in the proof it is clear that the family of hypersurfaces:  $\{\Sigma_t(\rho) : t \in \mathbb{R}\}$  agrees insofar as it is defined with the family of hypersurfaces:

$$\{\Sigma(\rho+t) : t \in \mathbb{R}\} .$$

Moreover, the second family provides a consistent completion for the first family.

One could eliminate this discussion altogether by letting  $\rho$  define a mapping into the unit tangent bundle,  $T_1 \mathbb{H}^{n+1}$  by

$$\theta \longmapsto \left( R_\rho(\theta), \left. \frac{dR_{\rho+t}}{dt}(\theta) \right|_{t=0} \right) = T\Sigma(\rho) .$$

The parallel flow is defined in an obvious way on  $T_1 \mathbb{H}^{n+1}$  and is everywhere smooth; it is the projection into  $\mathbb{H}^{n+1}$  that produces the singularities. In this context the statement:

$$T\Sigma_t(\rho) = T\Sigma(\rho+t)$$

is rigorously correct at every point. As this is not important for what follows, we will not pursue this. For more details see [ ].



### 3. Equations of Motion for Parallel Surfaces

In this section we will derive equations that describe the evolution of the fundamental geometric quantities of an immersed hypersurface under the parallel flow. In particular, we will derive equations of motion for the principal curvatures. The rather startling fact is that these equations are decoupled; consequently, the principal curvatures evolve independently of one another. This allows a good qualitative understanding of the formation of singularities under the parallel flow.

We will not employ the representation discussed in §2, rather we will consider hypersurfaces as given locally by embeddings of open sets  $U \subset \mathbb{R}^n$  :

$$i : U \longrightarrow \mathbb{H}^{n+1}$$

If  $(x_1, \dots, x_n)$  are coordinates on  $U$  then the vector fields:

$$X_j = i_* \frac{\partial}{\partial x_j} \quad j = 1, \dots, n$$

span the tangent space of  $i(U)$ .  $N$  will denote the outward unit normal vector field. The first fundamental form is:

$$g_{ij} = \langle X_i, X_j \rangle$$

the second fundamental form is:

$$\pi_j^i = g^{ik} \pi_{kj}$$

where

$$\pi_{kj} = \langle \nabla_{X_j} X_i, N \rangle$$

$X_i$   $i = 1, \dots, n$  and  $N$  will also be used to denote respectively the coordinate and normal vector fields of the parallel hypersurfaces. The  $X_i$  are defined by the immersion:

$$\Psi^t(i(x), N_i(x)) \quad x \in U$$

As a coordinate vector field

$$N = \frac{d\Psi^t}{dt}(i(x), N_i(x))$$

and it satisfies the geodesic equation

$$\nabla_N N = 0$$

All the commutators,  $[X_i, X_j]$   $i, j = 1, \dots, n$  and  $[X_i, N]$   $i = 1, \dots, n$  vanish. From this we easily conclude that

$$\langle X_i, N \rangle = 0$$

for all  $t$ .

The tangential vector fields,  $\{X_1, \dots, X_n\}$  satisfy the Jacobi equation:

$$\nabla_N^2 X_i + R(N, X_i)N = 0 \quad (3.1)$$

The quantities on the parallel hypersurfaces corresponding to those introduced above will also be denoted by  $g_{ij}$ ,  $\pi_{ij}$  and  $\pi_j^i$  respectively. Whenever we want to emphasize the dependence on a particular variable, we will write a quantity as explicitly depending on that variable. For instance  $g_{ij}(t)$ .  $\frac{d}{dt}$  denotes the action of the vector field  $N$ .

**Theorem 3.1:** The first and second fundamental forms satisfy the following equations under the parallel flow:

$$\begin{aligned}
 \text{a)} \quad \frac{dg_{ij}}{dt} &= -2 \pi_{ij} \\
 \text{b)} \quad \frac{d\pi_{ij}^i}{dt} &= \pi_{ik}^i \pi_{kj}^k - \delta_j^i \\
 \text{c)} \quad \frac{d^2 \pi_{ij}}{dt^2} &= 4 \pi_{ij}
 \end{aligned} \tag{3.2}$$

Remark: The second fundamental form is  $\pi_j^i$  and it evolves independently of  $g_{ij}$ . Note however, that a and c are linear while b is non-linear. For this reason it is sometimes advantageous to use a and c instead of b.

Proof of Theorem 3.1: The proofs are elementary calculations using the properties of the Levi-Civita connection:

$$\begin{aligned}
 \text{a)} \quad \frac{dg_{ij}}{dt} &= N \langle X_i, X_j \rangle \\
 &= \langle \nabla_N X_i, X_j \rangle + \langle X_i, \nabla_N X_j \rangle \\
 &= \langle \nabla_{X_i} N, X_j \rangle + \langle X_i, \nabla_{X_j} N \rangle
 \end{aligned}$$

as  $\langle X_i, N \rangle = 0 \quad i=1, \dots, N$ . This equals

$$\begin{aligned}
 &= -[\langle N, \nabla_{X_i} X_j \rangle + \langle N, \nabla_{X_j} X_i \rangle] \\
 &= -2 \pi_{ij}
 \end{aligned}$$

b) To prove part b it is more convenient to begin with an orthonormal frame adapted to the hypersurfaces. Let  $\Sigma$  denote the initial surface and  $\{Y_1, \dots, Y_n\}$  denote an orthonormal frame field tangent to  $\Sigma$ . Extend this frame to  $\Sigma_t$  by parallel translation along the normal geodesics to  $\Sigma$ .

Denote the extended vector fields by  $\{Y_1, \dots, Y_n\}$  as well. They are orthonormal and satisfy:

$$\begin{aligned}
 \nabla_N Y_i &= 0 \\
 \langle N, Y_i \rangle &= 0
 \end{aligned} \quad i=1, \dots, n \tag{3.3}$$

In terms of such a frame:

$$\pi_j^i = \langle \nabla_{Y_i} Y_j, N \rangle \tag{3.4}$$

We differentiate (3.4) to obtain:

$$\frac{d\pi_j^i}{dt} = \langle \nabla_N \nabla_{Y_i} Y_j, N \rangle$$

Using the definition of the Riemann tensor, this can be rewritten as

$$\begin{aligned}
 \frac{d\pi_j^i}{dt} &= \langle \nabla_{Y_i} \nabla_N Y_j, N \rangle + \langle \nabla_{[N, Y_i]} Y_j, N \rangle \\
 &\quad + \langle R(N, Y_i) Y_j, N \rangle \\
 &= \langle \nabla_{[N, Y_i]} Y_j, N \rangle - \delta_j^i
 \end{aligned} \tag{3.5}$$

We have used (3.3) and the formula for the Riemann tensor of  $\mathbb{H}^{n+1}$  given in the introduction. The connection is symmetric thus:

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

Hence it follows from (3.3) that

$$[N, Y_i] = - \nabla_{Y_i} N$$

$V_N$  is tangent to  $\Sigma_t$  and so can be expressed in terms of the frame  $\{Y_1, \dots, Y_n\}$ , by

$$[N, Y_i] = \bar{\pi}_k^i Y_k \quad (3.6)$$

Using (3.6) in (3.5) we obtain:

$$\frac{d\bar{\pi}_i^i}{dt} = \bar{\pi}_k^i \bar{\pi}_j^k - \delta_j^i$$

as asserted. To complete the derivation of b we introduce matrix notation:

$$\bar{P} = \bar{\pi}_j^i$$

$$P = \pi_j^i$$

$$Q = \pi_{ij}$$

$$G = g_{ij}$$

In this notation

$$P = G^{-1}Q \quad (3.7)$$

As  $\{Y_1, \dots, Y_n\}$  is a basis for each tangent space  $T_p \Sigma_t$ , there is a matrix  $A_{ij}$  such that

$$X_i = \sum_j A_{ij} Y_j \quad (*)$$

We can express  $G$ ,  $Q$  and  $P$  in terms of  $A$  and  $\bar{P}$ :

$$G = AA^t$$

$$Q = A\bar{P}A^t$$

$$P = A^{-1}\bar{P}A^t$$

We differentiate (\*) with respect to time to obtain:

$$V_N X_i = \sum_j \frac{dA_{ij}}{dt} Y_j$$

By taking the inner product with  $Y_k$  and applying (\*) we obtain:

$$\frac{dA}{dt} = -A\bar{P}$$

Using these formulae and the equation satisfied by  $\bar{P}$  one easily derives

$$\frac{dP}{dt} = P^2 - I$$

which completes the proof of b.

To derive c we use the matrix notation introduced above:

$$\begin{aligned} \frac{dP}{dt} &= -G^{-1} \frac{dG}{dt} G^{-1}Q + G^{-1} \frac{dQ}{dt} \\ &= 2P^2 + G^{-1} \frac{dQ}{dt} \end{aligned}$$

Recall that

$$\frac{dP}{dt} = P^2 - I$$

thus

$$\frac{dQ}{dt} = -(PQ + G) \quad (3.8)$$

We differentiate (3.8) to obtain:

$$\frac{d^2Q}{dt^2} = -\frac{dQ}{dt}P + Q\frac{dP}{dt} + \frac{dG}{dt}$$

Using (3.8) and (3.2) a and b we obtain:

$$\frac{d^2Q}{dt^2} = 4Q \quad \blacksquare$$

Using (3.2) a and c it is possible to write an explicit formula for  $g_{ij}$  and  $\pi_{ij}$ :

$$\pi_{ij}^{\pm} = e^{2t} \pi_{ij}^{+} + e^{-2t} \pi_{ij}^{-}$$

$$g_{ij} = \Gamma_{ij} - e^{2t} \pi_{ij}^{+} + e^{-2t} \pi_{ij}^{-}$$

$\pi_{ij}^{\pm}$  and  $\Gamma_{ij}$  are functions of position on the initial hypersurface.

The principal curvatures  $\{k_1, \dots, k_n\}$  are the eigenvalues of  $\pi_j^i$ , they are real as  $\pi_j^i$  is symmetric in an orthonormal frame.

The eigenvectors of  $\pi_j^i$  are the principal directions. Corollary 3.2 follows easily from equation (3.2), b:

Corollary 3.2: a) The principal curvatures satisfy the equations:

$$\frac{dk_i}{dt} = k_i^2 - 1 \quad i=1, \dots, n \quad (3.9)$$

b) The principal directions on  $\Sigma_t$  are the images under the parallel flow of the principal directions on  $\Sigma$ .

Proof of Corollary 3.2: As  $\pi_j^i$  is a  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ -tensor, it is diagonal if and only if it is represented by a frame composed of eigenvectors, that is principal directions. In this case:

$$\pi_j^i = k_i \delta_j^i .$$

The inhomogeneity in (3.2),b is a diagonal matrix. Thus a solution with diagonal initial data is diagonal for all time. The diagonal entries satisfy the equation

$$\frac{dk_i}{dt} = k_i^2 - 1 \quad i = 1, \dots, n$$

proving the assertion in part a

To prove part b we observe that the coordinates at a point  $p$  on  $\Sigma$  can be chosen so that the initial coordinate vector fields  $\{X_1, \dots, X_n\}$  are the principal directions at  $p$ . Thus  $\pi_j^i(0, p)$  will be diagonal and so it will be diagonal for all time.  $\pi_j^i(t, p)$  is represented with respect to the extended coordinate vector fields  $\{X_1(t, p), \dots, X_n(t, p)\}$ . From the remarks in the proof of part a it follows that  $\{X_1(t, p), \dots, X_n(t, p)\}$  must be a frame of principal directions for  $\Sigma_t$  at  $\psi^t(p, N)$ . ■

If the principal curvatures are distinct at a point  $p \in \Sigma$  then we can construct smooth vector fields  $\{P_1, \dots, P_n\}$  composed of principal directions in a neighborhood of  $p$ . The integral curves to these vector fields are called the lines of curvature of  $\Sigma$ . Part b of the Corollary could be rephrased: The lines of curvature on  $\Sigma_t$  are the images under the parallel flow of the lines of curvature on  $\Sigma$ .

Using equation (3.9) we can derive a formula for  $k_i(t)$  :

$$k_i(t) = \frac{k_i(0)\csc t - \sec t}{-k_i(0)\sec t + \csc t} \quad (3.10)$$

From this it is evident that  $k_i(t)$  will be infinite at some time if and only if:

$$|k_i(0)| > 1$$

This is a reflection of the non-linear evolution equation governing the second fundamental form. On the other hand (3.2)a and c are linear equations and therefore  $g_{ij}$  and  $\pi_{ij}$  are smooth and bounded for all finite times.  $g_{ij}$  may fail to be invertible. At such points  $\pi_j^i$  will be unbounded. To discuss this further we require a definition:

Definition: Let  $\Sigma$  be a smooth immersed surface with unit normal field  $N$ . The set of points:

$$\mathcal{F} = \{\psi^t(p, N) : p \in \Sigma, t \in \mathbb{R} \text{ and } \det g_{ij}(t, p) = 0\}$$

in  $\mathbb{H}^{n+1}$  is called the focal manifold of  $\Sigma$ .

The forward focal manifold,  $\mathcal{F}_+$  is the subset of  $\mathcal{F}$  with  $t \geq 0$ .

The backward focal manifold,  $\mathcal{F}_-$  is the subset of  $\mathcal{F}$  with  $t \leq 0$ .

Let  $i : U \rightarrow \mathbb{H}^{n+1}$  be an immersion defining  $\Sigma$  locally. As is evident from the definition, the composed mapping

$$i_t(x) = \psi^t(i(x), N_{i(x)})$$

fails to be an immersion if  $\psi^t(i(x), N_{i(x)}) \in \mathcal{F}$ . We will show that the surface  $\Sigma_t$  actually is singular on the focal set:

Proposition 3.3: If  $\mathcal{F} \cap \Sigma_t$  is not empty then  $\Sigma_t$  is not a smooth hypersurface. The singular locus is the image under the parallel flow of the lines of curvature on  $\Sigma$  where

$$k_i = \frac{cht}{\text{sh}t} \text{ for some } i .$$

Proof of Proposition 3.3: We need to show that whenever  $\det g_{ij}(t)$  is zero some  $k_i(t)$  is infinite.

Fix a point  $p$  on  $\Sigma$  and introduce coordinates on  $\Sigma$  so that the coordinate vector fields  $\{X_1, \dots, X_n\}$  are the principal directions at  $p$ . As usual  $X_i$  will also denote the extended coordinate vector fields. An easy calculation shows that at  $p$ :

$$\begin{aligned} \nabla_N X_i \Big|_{t=0} &= -k_i X_i \\ \langle X_i, N \rangle &= 0 \end{aligned} \tag{3.11}$$

Recall that the vector fields  $\{X_1, \dots, X_n\}$  are solutions of the Jacobi equation, (3.1). As  $X_i(0, p)$  satisfies the initial conditions in (3.11), it can be expressed as:

$$X_i(t, p) = 1/2[(1-k_i(0))e^{ct} + (k_i(0)+1)e^{-ct}] \tilde{X}_i(t) \tag{3.12}$$

where  $\tilde{X}_i(0) = X_i(0)$  and

$$\nabla_N \tilde{X}_i = 0 .$$

Thus we see that  $\langle X_i, X_i \rangle = 0$  if and only if:

$$e^{2t} = \frac{k_i(0) + 1}{k_i(0) - 1} .$$

Solving for  $k_i(0)$  we obtain:

$$k_i(0) = \frac{cht}{\text{sh}t}$$

as asserted. From formula (3.10) it is evident that  $\langle X_i(t, p), X_i(t, p) \rangle = 0$  if and only if  $k_i(t) = \infty$ . As  $\{X_1(t, p), \dots, X_n(t, p)\}$  are a frame of principal directions, they are orthogonal; hence,

$$\det g_{ij}(t, p) = \prod_{i=1}^n \langle X_i(t, p), X_i(t, p) \rangle .$$

The proposition follows easily from this and Corollary 3.2. ■

If  $\mathcal{F}_+$  is empty we say that  $\Sigma$  is forward convex and backward convex if  $\mathcal{F}_-$  is empty. Evidently  $\Sigma$  is forward convex if and only if:

$$k_i \leq 1 \quad i = 1, \dots, n \tag{3.13), a}$$

and backward convex if and only if

$$k_i \geq -1 \quad i = 1, \dots, n . \tag{3.13), b}$$

A remarkable feature of hyperbolic space is the existence of surfaces which are both forward and backward convex. This occurs if and only if

$$|k_i| \leq 1 \quad i = 1, \dots, n . \tag{3.14}$$

Note: These notions of convexity are different from geodesic convexity.

We close this section with a theorem on hypersurfaces which satisfy (3.14). The proof was suggested by Thurston.

Theorem 3.4: If  $\Sigma$  is a complete hypersurface in  $\mathbb{H}^{n+1}$  whose principal curvatures satisfy (3.14), then  $\Sigma$  has no self intersections.

Proof of Theorem 3.4: If this were not the case the completeness of  $\Sigma$  would imply the existence of a non-trivial geodesic loop. That is an arclength parametrized curve  $c(t)$  on  $\Sigma$  such that

$$\nabla_T T = KN \quad (3.15)$$

where  $T = \dot{c}(t)$  and  $N$  is the normal vector to  $\Sigma$  at  $c(t)$ ;  $K$  is the geodesic curvature of  $c(t)$  as a curve in  $\mathbb{H}^{n+1}$ . Moreover

$$c(0) = c(t_0)$$

for some  $t_0 > 0$ . Taking the inner product of (3.15) with  $N$  we obtain:

$$\langle \nabla_T T, N \rangle = K$$

or

$$\pi(T, T) = K$$

$\pi(\cdot, \cdot)$  is the second fundamental form of  $\Sigma$ . The principal curvatures are the eigenvalues of  $\pi$ ; the Courant min-max principle implies that:

$$\begin{aligned} |\pi(T, T)| &\leq \max |k_i| \\ &\leq 1 \end{aligned}$$

But this is impossible as we will show in Lemma 3.5. The proof of the theorem will be complete once we've proved:

Lemma 3.5: Let  $c(t)$  be an arclength parametrized curve in  $\mathbb{H}^{n+1}$  with geodesic curvature  $K$ . Let  $p$  denote a fixed point in  $\mathbb{H}^{n+1}$ . If

$|K| \leq 1$  then

$$f(t) = \cosh(d(p, c(t)))$$

is a convex function. ( $d(\cdot, \cdot)$  is hyperbolic distance).

An immediate corollary is:

Corollary 3.6: If  $c(t)$  is as described in Lemma 3.5, then  $c(t)$  has no self intersections.

Proof of Lemma 3.5: First we note that in the upper half space model  $d(\cdot, \cdot)$  is the inverse hyperbolic cosine of a smooth function. Thus  $f(t)$  is as smooth as  $c(t)$ . Let  $T = \dot{c}(t)$  be the tangent vector. Let  $p$  be the center of a geodesic normal coordinate system,  $(r, \omega)$  for  $\mathbb{H}^{n+1}$ . In such a coordinate system:

$$ds^2 = dr^2 + sh^2 r d\sigma^2$$

$d\sigma$  is the line element on the  $n$ -sphere of curvature  $+1$ . Let

$$c(t) = (r(t), \omega(t))$$

$$T = \dot{r} \partial_r + \dot{\omega}_i X_i$$

where  $\{X_1, \dots, X_n\}$  is an orthogonal frame of coordinate vector fields on  $S^n$ .

$$\langle X_i, X_j \rangle = g_{ij} \delta_j^i$$

$$\langle X_i, \partial_r \rangle = 0 \quad i = 1, \dots, n$$

As  $c(t)$  is parametrized by arclength,

$$\dot{r}^2 + sh^2 r |\dot{\omega}|^2 = 1 \quad (3.16)$$

The first Frenet-Serret equation reads:

$$\nabla_T T = KN \quad (3.17)$$

$N$  is a unit vector field along  $c(t)$ . We will need only the  $\partial_r$  component of this equation. As  $S^n$  is totally umbilic, it is easy to see that

$$\langle \nabla_{X_i} X_j, \partial_r \rangle = 0 \text{ if } i \neq j \tag{3.18}$$

From the form of the metric it follows that:

$$\begin{aligned} \langle \nabla_{X_i} X_i, \partial_r \rangle &= -g_{ii} \text{shr chr} \\ \langle \nabla_{\partial_r} X_i, \partial_r \rangle &= 0 \\ \nabla_{\partial_r} \partial_r &= 0. \end{aligned} \tag{3.19}$$

Taking the inner product of (3.17) with  $\partial_r$  and using (3.18) and (3.19), we obtain:

$$\ddot{r} = |\dot{w}|^2 \text{shr chr} + K \langle \partial_r, N \rangle.$$

Thus

$$\begin{aligned} \frac{d^2 r}{dt^2} &= \ddot{r} \text{shr} + \dot{r}^2 \text{chr} \\ &= \text{chr} (\dot{r}^2 + |\dot{w}|^2 \text{sh}^2 r) + K \langle \partial_r, N \rangle \text{shr} \\ &\geq \text{chr} - |K| \text{shr} \\ &> 0 \end{aligned}$$

whenever  $|K| \leq 1$ . The inequality above follows from (3.16) and the fact that  $\partial_r$  and  $N$  are unit vectors. ■

Remark: As a final remark we note that the conditions:

$$|k_i| \leq 1 \quad i = 1, \dots, n$$

are preserved under the flow defined on  $(k_1, \dots, k_n)$  by the system of equations (3.9). This is because the locus of points where

$$|k_i| = 1 \quad i = 1, \dots, n$$

consists of critical points. Thus, no trajectory is transverse to any of the hyperplanes defined by

$$k_i = \pm 1 \quad i = 1, \dots, n.$$

#### 4. Surfaces in $\mathbb{H}^3$

In this section and for the remainder of the paper we will consider the classical case of two dimensional surfaces in  $\mathbb{H}^3$ . Recall that the Gauss and mean curvatures are expressed in terms of the principal curvatures by:

$$K = k_1 k_2 - 1 \tag{4.1}$$

$$H = k_1 + k_2 .$$

Let  $g$  denote  $\det g_{ij}$ . The area form with respect to the parameters  $(x, y)$  is

$$dA = \sqrt{g} \, dx \wedge dy .$$

In this section we derive equations of motion for  $K, H$  and  $g$ .

Proposition 4.1: Under the parallel flow the functions  $K, H$  and  $g$  satisfy the equations:

- a)  $\frac{dK}{dt} = KH$
- b)  $\frac{dH}{dt} = H^2 - 2K - 4$  (4.2)
- c)  $\frac{dg}{dt} = -2gH$

Proof of Proposition 4.1: a and b follow immediately from (4.1) and (3.9). To prove c we use the standard result on ordinary differential equations:

Lemma: If  $A$  is an  $n \times n$  matrix function of  $t$  and  $a = \det A$  then

$$\frac{da}{dt} = a \operatorname{tr} \left[ A^{-1} \frac{dA}{dt} \right] ,$$

see [C-L].

Therefore:

$$\begin{aligned} \frac{dg}{dt} &= g \operatorname{tr} \left[ G^{-1} \frac{dG}{dt} \right] \\ &= -2 \operatorname{tr} G^{-1} Q \\ &= -2gH . \end{aligned}$$

The last equality follows as  $G^{-1}Q$  is the second fundamental form and thus:

$$\operatorname{tr} G^{-1} Q = (k_1 + k_2) .$$

Using Cramer's rule to express  $G^{-1}$  it is evident that  $gG^{-1}$  is well defined and differentiable even if  $G$  is not invertible. ■

An important corollary of Proposition 4.1 is:

Corollary 4.2: As long as  $K(t)$  is finite

$$\frac{dK^2}{dt} = 0 \tag{4.3}$$

Proof of Corollary 4.2: We differentiate and substitute from (4.2)

to obtain:

$$\begin{aligned} \frac{dK^2}{dt} &= 2K \frac{dK}{dt} + K^2 \frac{dg}{dt} \\ &= 2K^2 Hg - 2K^2 gH \\ &= 0 \end{aligned}$$



In fact  $K^2(t)g(t)$  is constant whether or not  $K$  becomes infinite. As we saw in §3,  $K(t)$  becomes infinite if and only if  $g(t)$  goes to zero. An easy asymptotic analysis using formulae for  $g_{ij}(t)$  and  $K(t)$  following from (3.12) and (3.10) respectively shows that the constant value of  $K^2g$  does not change across a singularity of  $K$ .

Using formula (3.12), we get an even clearer picture of the behavior of the surface at a focal point. If a single principal curvature becomes infinite, then the corresponding principal vector  $X_i(t)$  goes to zero. The orientation of the surface is therefore reversed as it passes through the focal point. If both curvatures become infinite, then the orientation is preserved. As these sign changes are mirrored by sign changes in  $K$  Corollary 4.2 can be restated:

Corollary 4.2':  $KdA$  is invariant under the parallel flow.

This is important for it allows one to compute the Gauss curvature of  $\Sigma(\rho)$  in terms of the first derivatives of  $R_\rho$ .

From the formulae (4.1) and (3.10) we derive expressions for  $K(t)$  and  $H(t)$ . Let  $K_0$  and  $H_0$  denote the respective initial values.

$$K(t) = \frac{K_0}{K_0 \text{sh}^2 t - H_0 \text{sht} \text{cht} + (\text{ch}^2 t + \text{sh}^2 t)} \quad (4.4)$$

$$H(t) = \frac{H_0 (\text{sh}^2 t + \text{ch}^2 t) - 4 \text{sht} \text{cht} - K_0 \text{sht} \text{cht}}{K_0 \text{sh}^2 t - H_0 \text{sht} \text{cht} + (\text{sh}^2 t + \text{ch}^2 t)}$$

From these formulae the following analogue of a classical theorem of Bonnet is immediate:

Proposition 4.3: If  $\Sigma$  is a surface in  $H^3$  with constant mean curvature  $H$  such that

$$|H| > 2,$$

then some parallel surface  $\Sigma_c$  has constant Gauss curvature  $K > 0$  and vice versa.

Remark: The extreme values  $|H| = 2$  and  $K = 0$  correspond to surfaces with very simple defining equations in the representation described in §3. See §5.

5. Explicit Formulae for Envelopes

In this section we return to the study of surfaces represented as envelopes of horospheres. We will derive formulae for the first and second fundamental forms of  $\Sigma(\rho)$  and use these to study the behavior of  $\Sigma_t(\rho)$  as  $t \rightarrow \infty$ .

Calculations will be done at a point  $p$  on  $S^2$  about which geodesic normal coordinates have been introduced. At such a point:

- 1)  $g_{ij}^{S^2}(p) = \delta_{ij}$
- 2)  $\Gamma_{jk}^i(p) = 0$  - the Christoffel symbols of  $S_{ij}^{S^2}$ .

In fact we will use coordinates provided by stereographic projection from the antipodal point to  $p$ ,  $p^*$ . Such coordinates satisfy 1 and 2 at  $p$ . They also define local conformal parameters for  $S^2$ .  $p$  will always go to the point  $z = 0$ . The round metric on  $S^2$  is:

$$d\sigma^2 = \frac{16|dz|^2}{(4+|z|^2)^2}.$$

By a rotation in  $\mathbb{R}^3$   $p$  can be normalized to be the point  $(0,0,-1)$ . The formula for  $R_\rho(x,y) : U \rightarrow \mathbb{H}^3$ ,  $U$  an open set in  $\mathbb{E}$ , is:

$$AR_\rho(x,y) = \left[ \frac{(4+r^2)^2}{16} (\rho_x^2 + \rho_y^2) + e^{2\rho} - 1 \right] \left( \frac{4x}{4+r^2}, \frac{4y}{4+r^2}, \frac{r^2-4}{r^2+4} \right) + 2\rho_x \left( 1 + \frac{y^2-x^2}{4}, -\frac{xy}{2}, x \right) + 2\rho_y \left( -\frac{xy}{2}, 1 + \frac{x^2-y^2}{4}, y \right) \quad (5.1)$$

where

$$r^2 = |z|^2 \quad \text{and}$$

$$A = \frac{(4+r^2)^2}{16} (\rho_x^2 + \rho_y^2) + (e^\rho + 1)^2.$$

From (5.1) we can calculate the coordinate tangent vectors  $\partial_x R_\rho$  and  $\partial_y R_\rho$ . Call them  $X$  and  $Y$  respectively. A very tedious but elementary calculation shows that:

$$\begin{pmatrix} X \cdot X & X \cdot Y \\ X \cdot Y & Y \cdot Y \end{pmatrix} \Big|_{z=0} = \begin{pmatrix} 2\rho_{xx} + \rho_y^2 - \rho_x^2 - 1 + e^{2\rho} & 2(\rho_{xy} - \rho_x \rho_y) \\ 2(\rho_{xy} - \rho_x \rho_y) & 2\rho_{yy} + \rho_x^2 - \rho_y^2 - 1 + e^{2\rho} \end{pmatrix}^2 \quad (5.2)$$

where  $A = \rho_x^2 + \rho_y^2 + (e^\rho + 1)^2 \Big|_{z=0}$ .

This is the induced Euclidean metric for the immersion. The hyperbolic metric is easily obtained as

$$\langle S, T \rangle = 4S \cdot T (1 - |R_\rho|^2)^{-2}.$$

From (5.1) we compute that:

$$4(1 - |R_\rho|^2)^{-2} \Big|_{z=0} = 1/4 e^{-2\rho} A^2. \quad (5.3)$$

Putting (5.2) and (5.3) together, we obtain:

**Proposition 5.1:** At the center,  $p$  of a geodesic normal coordinate system on  $S^2$  the metric on  $\Sigma(\rho) \hookrightarrow \mathbb{H}^3$  is given by:



$$g_{ij}(p) = \begin{pmatrix} \frac{e^\rho}{2} + (\rho_{xx} + \frac{\rho_y^2 - \rho_x^2 - 1}{2})e^{-\rho} & (\rho_{xy} - \rho_x \rho_y)e^{-\rho} \\ (\rho_{xy} - \rho_x \rho_y)e^{-\rho} & \frac{e^\rho}{2} + (\rho_{yy} + \frac{\rho_x^2 - \rho_y^2 - 1}{2})e^{-\rho} \end{pmatrix}^2. \quad (5.4)$$

To calculate the corresponding quantities for  $\Sigma_t(p)$  it follows from Theorem 2.1 that one merely replaces  $\rho$  by  $\rho+t$  in (5.4). We write this as:

$$g_{ij}(t,p) = \begin{pmatrix} \frac{e^{\rho+t}}{2} + E e^{-(\rho+t)} & F e^{-(\rho+t)} \\ F e^{-(\rho+t)} & \frac{e^{\rho+t}}{2} + G e^{-(\rho+t)} \end{pmatrix}^2 \quad (5.5)$$

where E, F and G are given by

$$\begin{aligned} E &= \rho_{xx} + \frac{1}{2}(\rho_y^2 - \rho_x^2 - 1) \\ F &= \rho_{xy} - \rho_x \rho_y \\ G &= \rho_{yy} + \frac{1}{2}(\rho_x^2 - \rho_y^2 - 1) \end{aligned}$$

According to Theorem 3.1

$$\frac{dg_{ij}}{dt} = -2\pi_{ij}.$$

Differentiating (5.5) and setting  $t = 0$  we obtain:

$$\pi_{ij} = \begin{pmatrix} \frac{e^\rho}{2} + E e^{-\rho} & F e^{-\rho} \\ F e^{-\rho} & \frac{e^\rho}{2} + G e^{-\rho} \end{pmatrix} \begin{pmatrix} E e^{-\rho} - \frac{e^\rho}{2} & F e^{-\rho} \\ F e^{-\rho} & G e^{-\rho} - \frac{e^\rho}{2} \end{pmatrix} \quad (5.6)$$

The second fundamental form equals

$$\pi_j^i = g^{ik} \pi_{kj}$$

Using (5.5) to compute  $g^{ij}$  we obtain:

Proposition 5.2: At the center,  $p$  of a geodesic normal coordinate system on  $S^2$  the second fundamental form of  $\Sigma(\rho) \hookrightarrow \mathbb{H}^3$  is:

$$\pi_j^i = D^{-1} \left[ \begin{pmatrix} \frac{E-G}{2} & F \\ F & \frac{G-E}{2} \end{pmatrix} + \left[ (EG-F^2)e^{-2\rho} - \frac{e^{2\rho}}{2} \right] I \right] \quad (5.7)$$

here  $D = (EG-F^2)e^{-2\rho} + \frac{e^{2\rho}}{4} + \frac{E+G}{2}$ .

- Remarks:
1. From formula (5.7) it is apparent that the principal directions are preserved under the parallel flow as the eigenvectors of  $\pi_j^i$  are clearly independent of  $t$ .
  2. From (5.5) it is clear that for a  $C^2$ -function  $\rho$  the  $\det g_{ij}(t,p)$  is usually non-zero. In fact it follows from our analysis of focal sets in §3 that this determinant vanishes at 0, 1 or 2 points.
  3. Therefore if  $\rho \in C^2$ , then  $\Sigma(\rho)$  is either smooth at a given point  $p$ , or  $\Sigma(\rho \pm t)$ , for small  $t$ , is smooth at a point  $q$  that

projects to  $p$  under the parallel flow. In the latter case  $p$  is in the focal set of  $\Sigma(\rho-t)$ . From this we conclude that the only singularities that can arise in an envelope  $\Sigma(\rho)$  generated by a  $C^2$ -function  $\rho$ , are those which arise as focal singularities of an immersed  $C^1$ -surface.

Using (4.4) and (5.5), we can study the asymptotic behavior of  $g_{ij}(t)$  and  $K(t)$  as  $t \rightarrow \infty$ . A quick glance at (5.5) shows that  $g_{ij}(t)$  tends uniformly to infinity as  $t \rightarrow \infty$ . Thus we must rescale to obtain a finite limit; let

$$\tilde{g}_{ij}(t) = 4e^{-2t} g_{ij}(t) \quad (5.8)$$

An elementary fact about Gauss curvature is that:

$$K(cg_{ij}) = c^{-1}K(g_{ij})$$

for  $c$  a positive constant.

Hence:

$$K(\tilde{g}_{ij}) = \frac{e^{-2t}}{4} K(g_{ij}(t)) \quad (5.9)$$

We will regard  $\tilde{g}_{ij}(t)$  as a family of metrics on a domain  $\Omega \subset S^2$ . The Gauss curvatures of these metrics are calculated using (5.9). The asymptotic behavior is as follows:

Proposition 5.3: Let  $\rho$  be a  $C^4$ -function on a domain  $\Omega \subset S^2$ . Then

- a)  $\lim_{t \rightarrow \infty} R_{\rho+t} = \text{identity}$
- b)  $\lim_{t \rightarrow \infty} \tilde{g}_{ij}(t) = e^{2\rho} d\sigma^2 \quad (5.10)$
- c)  $\lim_{t \rightarrow \infty} K(\tilde{g}_{ij}(t)) = \frac{K_0}{(K_0 + 2 - H_0)}$   
 $= \frac{(k_1 k_2 - 1)}{(1-k_1)(1-k_2)}$

Proof of Proposition 5.3: The proof of part a follows from (2.4):

$$R_{\rho+t} = \frac{|D\rho|^2 + (e^{2(\rho+t)} - 1)}{|D\rho|^2 + (e^{\rho+t} + 1)^2} X + \frac{2 D\rho}{|D\rho|^2 + (e^{\rho+t} + 1)^2}$$

As  $\rho$  is differentiable

$$\lim_{t \rightarrow \infty} R_{\rho+t} = X,$$

which is the assertion of part a.

At the center of a geodesic normal coordinate system on  $S^2$ ,  $\tilde{g}_{ij}(t)$  is given by:

$$\tilde{g}_{ij}(t) = \begin{pmatrix} e^\rho + 2Ee^{-(\rho+t)} & 2Fe^{-(\rho+t)} \\ 2Fe^{-(\rho+t)} & e^\rho + 2Ge^{-(\rho+t)} \end{pmatrix}^2$$

As  $t \rightarrow \infty$  this tends to  $e^{2\rho} I$ . Since the metric is a tensor and the uniform limit of tensors is a tensor we have shown that the tensor

$$\lim_{t \rightarrow \infty} \tilde{g}_{ij}(t) dx_i \otimes dx_j,$$

at the center of a geodesic normal coordinate system is given by:

$$e^{2\rho}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2) .$$

This tensor agrees at every point with the invariantly given tensor  $e^{2\rho}d\sigma^2$ . This establishes (5.10)b. Using (5.9) and formula (4.4) we obtain:

$$K(\tilde{g}_{ij}(t)) = \frac{K_0 e^{2t}}{(K_0 + 2 - H_0) e^{2t}} + O(e^{-2t}) .$$

Letting  $t \rightarrow \infty$  we obtain (5.10)c. It is not immediate that the limiting curvature function is the curvature of the limiting metric. We have assumed that  $\rho \in C^4(\Omega)$  and thus  $\tilde{g}_{ij}(t) \in C^2(\Omega)$ . From the form of  $\tilde{g}_{ij}(t)$  it is clear that the second derivatives of the metric are equi-continuous in  $t$ . And therefore  $\tilde{g}_{ij}(t)$  and its first two derivatives converge locally uniformly to  $e^{2\rho}d\sigma^2$ . Therefore the limiting metric has a curvature which must coincide with the limiting value obtained above. An elementary calculation shows that the two formulae given for this limit agree. ■

Remarks: 1.  $\rho \in C^4$  is probably more restrictive than necessary as the formula for the curvature only involves the first two derivatives of  $\rho$ . The functions we will be dealing with are real analytic so we will not pursue the optimal smoothness hypothesis here.

2. Henceforth we will denote the limiting curvature by  $K_\infty$  and the limiting area form by  $dA_\infty$ .

3. The virtue of formula (5.10)c. is that the left hand side is a very simple expression in  $\rho$ :

$$K_\infty = (1 - \Delta_S \rho) e^{-2\rho} . \tag{5.11}$$

Whereas the right hand side is typically a fully non-linear second order expression for most representations of immersed surfaces. The other side of the coin is that the expressions for both  $K$  and  $H$  in our representation are fully non-linear second order quantities.

4. The three special classes of surfaces discussed in the introduction are given by the conditions:

- a)  $K_\infty = 0$
- b)  $K_\infty = 1$
- c)  $K_\infty = \alpha < 0$

Putting these values into (5.11), we see that they correspond to:

- a')  $K = 0$
- b')  $H = 2$
- c')  $(1-\alpha)K = \alpha(2-H)$

Somewhat more general than c' is the condition:

$$c'') (1-K_\infty)K = K_\infty(2-H)$$

for  $K_\infty$  a negative function on  $\Omega$ . This gives rise to an

asymptotic Minkowski Problem for surfaces in  $\mathbb{H}^3$ . In a second paper we will study the regularity of the corresponding surfaces. This study requires a rather detailed analysis of the solution to (5.11) near  $\partial\Omega$ .

We will study surfaces satisfying c in the last sections of this paper. Robert Bryant has written a beautiful paper on mean curvature 2 surfaces, [Br]. There he proves a result which we obtained independently:

**Proposition 5.4:** The Gauss map for a surface  $\Sigma$  is conformal if and only if  $\Sigma$  has either mean curvature 2 or is umbilic.

In our representation the proof of Proposition 5.4 is a tedious calculation in local coordinates which we omit. We refer the interested reader to [Br] where an elegant proof using moving frames is presented.

We will consider the following Dirichlet problem:

Given a collection of curves  $\Gamma \subset S^2$  which are the oriented boundary of a domain  $\Omega \subset S^2$ , find an immersed surface

$$\Sigma \hookrightarrow \mathbb{H}^3$$

such that:

a) the principal curvatures of  $\Sigma$  satisfy:

$$(k_1 k_2 - 1) = \alpha(1 - k_1)(1 - k_2)$$

(Note if  $\alpha = -1$  this equation reduces to

$$k_1^{-1} + k_2^{-1} = 2 \quad ;$$

$\Sigma$  is a surface with mean radius of curvature 2.)

b)  $\bar{\Sigma} \cap \partial\mathbb{H}^3 = \Gamma$ ;  $\bar{\Sigma} \cap \partial\mathbb{H}^3$  is called the asymptotic boundary. It is denoted by  $\partial_\infty \Sigma$ .

$\Gamma$  must be oriented as the curvature equation involves the mean curvature. We will think of  $\Gamma$  as the boundary of a domain  $\Omega$ .

In our representation this problem becomes a singular Dirichlet problem for a complete conformal metric on  $\Omega$  with curvature  $\alpha$ : Find a  $\rho \in C^2(\Omega)$  such that

$$a') \quad (1 - \Delta_{S^2} \rho) e^{-2\rho} = \alpha \tag{5.12}$$

$$b') \quad \lim_{p \rightarrow \partial\Omega} \rho = \infty$$

If  $\Sigma(\rho)$  is smooth then its curvatures necessarily satisfy a. From formula (2.4) it follows that b' implies b.

As a final result in this section we show that the invariance of the curvature form extends to  $\infty$ :

**Proposition 5.5:** The curvature form  $K dA$  equals the asymptotic curvature form  $K_\infty dA_\infty$ .

**Proof of Proposition 5.5:** The assertion follows immediately from Corollary 4.2' and the fact that

$$K(t)dA(t) = \tilde{K}(t)d\tilde{A}(t).$$

The left hand side is computed with respect to the metric  $g_{ij}(t)$  while the right hand side is computed with respect to  $\tilde{g}_{ij}(t)$ . ■

The equation

$$K dA = K_m dA_m \quad (5.13)$$

is an analogue of Gauss' formula for the curvature of a surface immersed in  $\mathbb{R}^3$ . Proposition 5.5 could be restated:

Proposition 5.5': The pullback of the two form  $K_m dA_m$  via the Gauss map of  $\Sigma$  is the curvature form on  $\Sigma$ .

Remarks: 1. The special case  $H = 2 \iff K_m = 1$  is closest to the classical theorem. This case was also treated by Bryant.

2. (5.13) provides a relatively simple way to compute the Gauss curvature of  $\Sigma$  in terms of  $\rho$ . Using the formula for  $K_m$  in terms of the principal curvatures, we see that:

$$dA = \frac{e^{2\rho} dA_m}{(1-k_1)(1-k_2)} \quad (5.14)$$

3. We close this section with the generating functions for several well known surfaces in  $\mathbb{H}^3$ :

a) A totally geodesic surface meeting  $S^2$  in an equator:

$$\rho = \log \frac{1}{\cos \theta}$$

$\theta$  is the azimuthal angle relative to the normal direction to the plane defining the equator.

b) A horosphere:

$$\begin{aligned} \rho &= \log \frac{1}{\sin \theta} + \frac{1}{2} \log \frac{1+\cos \theta}{1-\cos \theta} + t \\ &= \log \frac{1}{(1-\cos \theta)} + t \end{aligned}$$

$\theta$  is the azimuthal angle measured from the point of tangency of the horosphere.

c) A geodesic connecting two antipodal points:

$$\rho = \log \frac{1}{\sin \theta}$$

$\theta$  is the azimuthal angle measured from either endpoint of the geodesic.



6. Conformal Parameters

If we restrict ourselves to simply connected regions,  $\Omega$  then the Dirichlet problem is solved in terms of the conformal map from  $\Omega$  to the unit disk,  $D_1$  in  $\mathbb{E}$ . It is useful to express the fundamental geometric quantities for  $\Sigma(\rho)$  in terms of the complex derivatives of  $\rho$ . As before, we use conformal parameters arising from stereographic projection. When  $z$  denotes such a parameter, the complex derivatives are defined by:

$$\partial = \frac{1}{2}(\partial_x - i\partial_y) \quad \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y). \quad (6.1)$$

The round metric on the sphere is

$$d\sigma^2 = \gamma^2 |dz|^2 \quad (6.2)$$

$$\gamma = \frac{4}{4 + |z|^2}.$$

The expression for  $g_{ij}$  in terms of complex derivatives is not particularly illuminating. However, there is a Hermitian matrix with the same determinant and trace as  $g_{ij}$  which is useful:

Proposition 6.1: The matrix:

$$h_{ij} \Big|_{z=0} = e^{2\rho} \begin{bmatrix} (2\partial\bar{\partial}\rho - 1/2)e^{-2\rho} + 1/2 & 2(\partial^2\rho - (\partial\rho)^2)e^{-2\rho} \\ 2(\bar{\partial}^2\rho - (\bar{\partial}\rho)^2)e^{-2\rho} & (2\partial\bar{\partial}\rho - 1/2)e^{-2\rho} + 1/2 \end{bmatrix} \quad (6.3)$$

has the same determinant and trace as  $g_{ij}$  at  $z=0$ .

The proof is an elementary calculation which we omit.

The formula for  $\pi_j^i$ , (5.7) is considerably simpler in terms of the complex derivatives:

Proposition 6.2: The second fundamental form is:

$$\pi_j^i \Big|_{z=0} = D^{-1} \left[ \begin{pmatrix} 2\text{Re}S & -2\text{Im}S \\ -2\text{Im}S & -2\text{Re}S \end{pmatrix} + \lambda I \right] \quad (6.4)$$

$$S = \partial^2\rho - (\partial\rho)^2$$

$$D = (EG - F^2)e^{-2\rho} + \frac{e^{2\rho}}{4} + \frac{E+G}{2}$$

$$\lambda = (EG - F^2)e^{-2\rho} - \frac{e^{2\rho}}{2}$$

Again the proof is an elementary calculation which we omit.

7. Weingarten Surfaces

In this section we will solve the Dirichlet problem for  $\rho$  and discuss the regularity of the associated Weingarten surface,  $\Sigma(\rho)$ . We will obtain formulae for the principal curvatures and identify the lines of curvature with the trajectories of a holomorphic quadratic differential in  $\Omega$ . We will suppose  $\Omega$  is simply connected in this section.

The Dirichlet problem, (5.12) is an equation for the logarithm of the conformal factor for the complete hyperbolic metric on  $\Omega \subset S^2$ . Let  $f_\Omega$  be a conformal map from  $\Omega$  onto the unit disk,  $D_1$  in  $\mathbb{E}$ . We obtain  $\rho$  by pulling back the Poincaré metric on  $D_1$  via  $f_\Omega$ . Let  $z$  denote a conformal parameter on  $\Omega$  and, differentiation with respect to  $z$  then:

$$\rho = \log(2|f'_\Omega|^{-1}(1-|f_\Omega|^2)^{-1}) \quad (7.1)$$

$\rho$  solves the equation:

$$(1 - \Delta_{S^2} \rho) e^{-2\rho} = -1$$

Using Schwarz' Lemma and the Koebe 1/4-theorem, we obtain the classical estimates, [Ah&1]:

If  $\Omega$  is a simply connected domain and  $\partial\Omega$  has more than two points then

$$1/4 \delta(z, \partial\Omega)^{-1} \leq e^\rho \leq \delta(z, \partial\Omega)^{-1} \quad (7.2)$$

$\delta(\cdot, \cdot)$  is Euclidean distance measured in the conformal parameter  $z$ . From (7.2) it follows easily that there is an exhaustion of  $\Omega$  by compact subregions  $\Omega_n \subset\subset \Omega$  such that

$$\rho \geq n \quad \text{in} \quad \Omega \setminus \Omega_n \quad (7.3)$$

which implies that  $\rho$  satisfies the boundary condition:

$$\lim_{\rho \rightarrow \partial\Omega} \rho = \infty$$

This boundary condition is equivalent to

$$\overline{\Sigma(\rho)} \cap S^2 = \partial\Omega$$

Proposition 7.1: If  $\rho \in C^2(\Omega)$  tends to  $\infty$  on  $\partial\Omega$  as in (7.3), then the surface  $\Sigma(\rho)$  tends to  $S^2$  precisely along  $\partial\Omega$ .

Remark: For the proposition to be correct  $\Sigma(\rho)$  does not need to be smooth.

Proof of Proposition 7.1: Using (2.4), we calculate the Euclidean norms of  $R_\rho$  and  $R_{\rho-X}$ :

$$|R_\rho|^2 = 1 - \frac{4e^\rho}{|D\rho|^2 + (e^\rho + 1)^2}$$

$$|R_{\rho-X}|^2 = \frac{4(e^\rho - 1)^2 + 4|D\rho|^2}{(|D\rho|^2 + (e^\rho + 1)^2)^2}$$

From the first formula it is clear that  $R_\rho(z)$  lies in the interior of  $H^3$  for  $z$  in the interior of  $\Omega$ . From the second formula we easily obtain that

$$\begin{aligned} |R_{\rho-X}| &\leq 2 \min \{ |D\rho|^{-1}, |D\rho|(e^\rho + 1)^{-2} \} \\ &\leq 2(e^\rho + 1)^{-1} \end{aligned}$$

Hence (7.3) implies:

$$|R_{\rho-X}| \leq 2e^{-n} \quad \text{for} \quad n \in \Omega \setminus \Omega_n$$

This completes the proof of the proposition. ■

To study the regularity and lines of curvature of  $\Sigma(\rho)$ , we rewrite the formulae for  $\pi_j^i$  and  $h_{ij}$  in terms of  $f_\Omega$ .

**Proposition 7.2:** At the center of a stereographic coordinate system on  $S^2$  ( $z=0$ )  $h_{ij}(t)$  and  $\pi_j^i(t)$  are given by:

$$h_{ij}(t) = e^{2(\rho+t)} \begin{bmatrix} 1/2(1+e^{-2t}) & \mathcal{S}_{f_\Omega}(0)\mu_\Omega(0)^{-1}e^{-2t} \\ \overline{\mathcal{S}_{f_\Omega}(0)\mu_\Omega(0)^{-1}e^{-2t}} & 1/2(1+e^{-2t}) \end{bmatrix}^2 \quad (7.5)$$

$$\pi_j^i(t) = D^{-1} \left[ \begin{pmatrix} \operatorname{Re} \mathcal{S}_{f_\Omega}(0) & -\operatorname{Im} \mathcal{S}_{f_\Omega}(0) \\ -\operatorname{Im} \mathcal{S}_{f_\Omega}(0) & -\operatorname{Re} \mathcal{S}_{f_\Omega}(0) \end{pmatrix} + \lambda I \right]$$

where  $\mathcal{S}_f(z) = \left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2$  is the Schwarzian derivative of  $f$  with respect to the conformal parameter  $z$ ; the conformal factor for the hyperbolic metric is:

$$\mu_\Omega = 4|f'_\Omega|^2(1-|f_\Omega|^2)^{-2}$$

**Remark:** The function  $\mathcal{S}_f(z)$  is not well defined in  $\Omega$ ; however, the quadratic differential:

$$q_\Omega = \mathcal{S}_{f_\Omega}(z) dz^2$$

is well defined. For the case of Weingarten surfaces, the Gauss map is not conformal and therefore  $q_\Omega$  does not define a holomorphic differential on  $\Sigma(\rho)$ .

**Proof of Proposition 7.2:** To derive (7.5) from (6.3) and (6.4), we need only calculate:

$$(2\partial\bar{\partial}\rho - 1/2)e^{-2\rho} \Big|_{z=0}$$

and

$$(\partial^2\rho - (\partial\rho)^2)e^{-2\rho} \Big|_{z=0}$$

in terms of  $f_\Omega$ . Using the facts that

$$\gamma \Big|_{z=0} = 1$$

and

$$\partial\gamma \Big|_{z=0} = \partial^2\gamma \Big|_{z=0} = 0$$

we easily obtain:

$$(2\partial\bar{\partial}\rho - 1/2)e^{-2\rho} = 1/2$$

and

$$e^{2\rho} \Big|_{z=0} = 4|f'_\Omega|^2(1-|f_\Omega|^2)^{-2} \Big|_{z=0}$$

A straightforward calculation shows:

$$2(\partial^2\rho - (\partial\rho)^2) \Big|_{z=0} = \mathcal{S}_{f_\Omega}(0)$$

**Remark:** The last formula appears in [Br].

As  $h_{ij}(t)$  is a matrix of the form:

$$e^{2(\rho+t)} [A e^{-2t} + 1/2],$$

A bound on the eigenvalues of  $A$  uniform in  $\Omega$  would imply that

$$\det h_{ij}(t) > 0$$

for  $t$  sufficiently large. As  $h_{ij}$  and  $g_{ij}$  have the same determinant, it would then follow that  $R_{\rho+t}$  is an immersion for sufficiently large  $t$ . The estimate we need follows from a theorem of Kraus; [Kr]:

Theorem A: If  $f$  is a univalent function in the disk, then

$$|\mathcal{L}_f(z)| \leq 6(1 - |z|^2)^{-2} . \quad (7.6)$$

From this theorem one easily deduces:

Proposition 7.3: If  $f_\Omega$  is a conformal map from a simply connected domain  $\Omega$  onto the unit disk, then:

$$|\mathcal{L}_{f_\Omega}(z)| \leq \frac{3}{2} \mu_\Omega(z) . \quad (7.6')$$

Proof of Proposition 7.3: By expressing  $\mathcal{L}_f$  in terms of  $\mathcal{L}_{f^{-1}}$  and recognizing the right hand side as  $3/2\mu_{D_1}$ , (7.6') follows from (7.6). ■

To apply this estimate we need an expression for the principal curvatures in terms of  $\mathcal{L}_{f_\Omega}$  and  $\mu_\Omega$ . For the remainder of this section we will use the notation  $\mathcal{L}_\Omega$  for  $\mathcal{L}_{f_\Omega}$ .

Proposition 7.4: The principal curvatures of the -1-Weingarten surface,

$\Sigma(\rho)$  are:

$$k_\pm = \left( 1 \pm \left| \frac{\mu_\Omega}{\mathcal{L}_\Omega} \right| \right)^{-1} \quad (7.7)$$

Proof of Proposition 7.4: We begin with the formula derived in §5:

$$KdA = K_m dA_m .$$

Recall that  $K_m = -1$  and  $dA_m = \mu_\Omega dx dy$ . On the other hand, from Proposition 7.2 it follows that:

$$\begin{aligned} dA \Big|_{z=0} &= \sqrt{\det h_{ij}} dx dy \\ &= \mu_\Omega \left( 1 - \left| \frac{\mathcal{L}_\Omega}{\mu_\Omega} \right|^2 \right) dx dy . \end{aligned}$$

At  $z=0$

$$K = \left( \left| \frac{\mathcal{L}_\Omega}{\mu_\Omega} \right|^2 - 1 \right)^{-1} . \quad (7.8)$$

$K$  is given terms of the principal curvatures by

$$K = k_1 k_2 - 1 . \quad (7.9)$$

As  $\Sigma(\rho)$  is Weingarten surface its principal curvatures satisfy the relation:

$$\frac{1 - k_1 k_2}{(1 - k_1)(1 - k_2)} = 1$$

or

$$k_2 = \frac{k_1}{2k_1 - 1} . \quad (7.10)$$

Substituting into (7.9) from (7.10), we easily obtain:

$$k_\pm = K + 1 \pm \sqrt{K(K+1)} ; \quad (7.11)$$

(7.7) follows from (7.11) after substitution from (7.8). ■

The formula was derived for  $z=0$ . However,  $\left| \frac{\mathcal{J}_\Omega}{\mu_\Omega} \right|$  is invariant under conformal changes of parameter and so the formula for  $K$  is valid throughout the domain of the conformal parameter.

The regularity theorem for Weingarten surfaces over simply connected regions is:

**Theorem 7.5:** Suppose  $\Omega \subset S^2$  is a simply connected region,

- a) If  $|\mathcal{J}_\Omega| < (1/2)\mu_\Omega$  everywhere in  $\Omega$  then the  $\alpha$ -Weingarten surfaces with boundary equal to  $\partial\Omega$  are smoothly imbedded for all  $\alpha < 0$ .
- b) If  $|\mathcal{J}_\Omega| < \mu_\Omega$  everywhere in  $\Omega$  then the  $\alpha$ -Weingarten surfaces with boundary equal to  $\partial\Omega$  are immersed for  $\alpha \leq -1$ .
- c) If  $\Omega$  satisfies neither a nor b, then the  $\alpha$ -Weingarten surfaces with boundary equal to  $\partial\Omega$  are immersed for:

$$\alpha < -(\sqrt{6} - 1)^{-1/2}.$$

**Proof of Theorem 7.5:** The proof follows from the formula for the hyperbolic jacobian determinant of  $R_{\rho+t}$ :

$$dA(t) = -dA_\alpha K(t)^{-1}$$

and the formula for the curvature:

$$K(t) = \left( \left| \frac{\mathcal{J}_\Omega}{\mu_\Omega} \right|^2 e^{-2t} - \text{ch}^2 t \right)^{-1}.$$

Since  $dA_\alpha$  is always positive and  $K(t)$  is negative and finite whenever

$$|\mathcal{J}_\Omega|^2 < \mu_\Omega e^{2t} \text{ch}^2 t, \tag{7.12}$$

assertion b follows immediately from (7.12). Assertion c follows from (7.12) and Proposition (7.3).

To prove assertion a we use the formula for the principal curvatures of a  $-1$ -Weingarten surface, (7.7):

$$k_\pm = \left( 1 \pm \left| \frac{\mu_\Omega}{\mathcal{J}_\Omega} \right| \right)^{-1}.$$

If  $|\mathcal{J}_\Omega| < 1/2 |\mu_\Omega|$  then  $|k_\pm| < 1$ .  $\Sigma(\rho)$  is a smooth immersion of  $\Omega$  without boundary points in  $H^3$  and thus is complete. Theorem 3.4 applies to show that the  $-1$ -Weingarten surface is imbedded as well as the family of surfaces parallel to it. ■

**Remark:** If  $\Omega$  satisfies neither a nor b, then the  $-1$ -Weingarten surface with boundary equal to  $\partial\Omega$  has singularities. This could only fail to occur if  $|\mathcal{J}_\Omega| \mu_\Omega^{-1} > 1$  everywhere in  $\Omega$ . If we choose our conformal parameter by stereographically projecting from a point in the interior of  $\Omega$ , then  $\partial\Omega$  will be a compact set and  $\mu_\Omega$  will be bounded from below near  $\partial\Omega$ . Thus  $\mathcal{J}_\Omega(z)^{-1}$  will be holomorphic near  $\partial\Omega$  and satisfy the estimate:

$$|\mathcal{J}_\Omega(z)^{-1}| \leq \delta(z, \partial\Omega),$$

clearly an absurdity for a holomorphic function.

In case a of the theorem the parallel surfaces  $\Sigma_t$  define a foliation of a part of  $H^3$ . To prove this, we need a comparison theorem for surfaces:

Proposition 7.6: Let  $\Omega_1$  and  $\Omega_2$  be two simply connected domains on  $S^2$  and suppose

$$\Omega_1 \subset \subset \Omega_2 .$$

Let  $\rho_1$  and  $\rho_2$  define complete metrics on  $\Omega_1$  and  $\Omega_2$  respectively such that:

$$\rho_2 < \rho_1 \text{ in } \bar{\Omega}_1 .$$

Finally, suppose that the surfaces  $\Sigma(\rho_2)$  and  $\{\Sigma_t(\rho_1) : t \geq 0\}$  are properly imbedded, then

$$\Sigma(\rho_1) \cap \Sigma(\rho_2) = \emptyset .$$

Proof of Proposition 7.6: First we see that there exists a  $t_0 > 0$  such that

$$\Sigma_t(\rho_1) \cap \Sigma(\rho_2) = \emptyset \text{ if } t \geq t_0 .$$

If this were not the case, we could find a sequence of times  $t_n \rightarrow \infty$  and a sequence of points:

$$p_n \in \Sigma_{t_n}(\rho_1) \cap \Sigma(\rho_2) .$$

As  $B^3$  is compact, there is a subsequence of the points  $p_n$  converging to  $p^*$ ; call this sequence  $p_n$  as well. From formula (2.4), it is evident that  $p^*$  lies in  $\bar{\Omega}_1$ . On the other hand,  $\Sigma(\rho_2) \cap \partial B^3 = \partial \Omega_2$  thus we have derived a contradiction as

$$\Omega_1 \subset \subset \Omega_2$$

and  $p^* \in \Sigma(\rho_2) \cap \partial B^3$ .

As  $\Sigma(\rho_2)$  is properly embedded it divides  $B^3$  into two connected components,  $D_1$  and  $D_2$ . We label them so that  $\Omega_1$  lies in  $\bar{D}_1$ . From the above argument, it follows that

$$\Sigma_t(\rho_1) \subset \subset D_1 \tag{7.13}$$

for  $t$  large enough. As  $\partial \Omega_1$  is disjoint from  $\partial \Omega_2$ , there is a first time  $t_1$  such that

$$\Sigma_{t_1}(\rho_1) \cap \Sigma(\rho_2) \neq \emptyset .$$

These two surfaces are tangent at some finite point,  $q$ . Since both  $\Sigma(\rho_2)$  is imbedded and (7.13) holds, it follows that the inward pointing normals of  $\Sigma_{t_1}(\rho_1)$  and  $\Sigma(\rho_2)$  agree at  $q$ . Thus the Gauss maps of the two surfaces agree at  $q$ ; call the common value  $\theta$ . From the definitions of these surfaces as envelopes of horospheres, it follows that

$$H(\theta, \rho_1(\theta) + t_1) = H(\theta, \rho_2(\theta)) .$$

From this we conclude that

$$t_1 = \rho_2(\theta) - \rho_1(\theta) < 0 .$$

Therefore,  $\Sigma(\rho_1) \cap \Sigma(\rho_2) = \emptyset$ . ■

Corollary 7.7: The  $\alpha$ -Weingarten surfaces for  $\alpha < 0$  and  $\Omega$  satisfying assertion a of Theorem 7.5 foliate  $\bigcup_{t \in \mathbb{R}} \Sigma_t(\rho)$ .

Proof of Corollary 7.7: Proposition 7.6 does not apply immediately as all the surfaces  $\Sigma_t(\rho)$  share a common boundary. To remedy this, we consider the subregion  $\Omega_r$  of  $\Omega$  given by:

$$\Omega_r = f_{\Omega}^{-1}\{|z| \leq r\} .$$

The conformal map for  $\Omega_r$  is

$$f_r = r^{-1}f \Big|_{\Omega_r} .$$

An easy calculation shows:

$$\mathcal{H}_r = \mathcal{H}_{f_r} = \mathcal{H}_\Omega$$

and the comparison principle for hyperbolic metrics [Ah12] states:

$$\mu_\Omega < \mu_{\Omega_r} \text{ on } \Omega_r .$$

Thus:

$$\begin{aligned} \left| \frac{\mathcal{H}_r}{\mu_{\Omega_r}} \right| &= \left| \frac{\mathcal{H}_\Omega}{\mu_{\Omega_r}} \right| \\ &< \left| \frac{\mathcal{H}_\Omega}{\mu_\Omega} \right| \\ &\leq \alpha \end{aligned}$$

for  $\alpha = \sup_\Omega \left| \frac{\mathcal{H}_\Omega}{\mu_\Omega} \right| < 1/2$ . Thus, for all  $r \leq 1$ :

$$\left| \frac{\mathcal{H}_r}{\mu_{\Omega_r}} \right| < 1/2 \text{ in } \Omega_r .$$

Letting  $2\rho_r = \log \mu_{\Omega_r}^{-2}$  it follows from Theorem 7.5 a that  $\Sigma_t(\rho_r)$  is embedded for every  $t$ . Proposition 7.6 applies and we conclude that  $\Sigma_t(\rho_r)$  and  $\Sigma_s(\rho)$  are disjoint for  $t > s$  and  $r$  less than 1. Letting  $r \rightarrow 1$ , it follows that  $\Sigma_t(\rho) \cap \Sigma_s(\rho)$  consists of points of tangency; at the common image of such points under the respective Gauss maps,  $\theta$ :

$$\rho(\theta) + t = \rho(\theta) + s ,$$

an obvious contradiction to the choice of  $t$  and  $s$ . ■

Remarks: 1. For general regions  $\Omega$  with  $|\mathcal{H}_\Omega| \mu_\Omega^{-1} < 1/2$  we do not yet know if  $\bigcup_t \Sigma_t$  is all of  $\mathbb{H}^3$ . This will be the case if  $|\nabla \rho| \rightarrow \infty$  on  $\partial\Omega$ . The map  $F(\theta) = \lim_{t \rightarrow -\infty} R_{\rho+t}(\theta)$  then defines a homeomorphism of  $\Omega$  onto  $\Omega^c$  which fixes  $\partial\Omega$  pointwise. A simple degree argument then shows that  $\bigcup_t \Sigma_t = \mathbb{H}^3$ . It seems likely that this is true in full generality, but we have not yet obtained uniform lower estimates for  $|\nabla \rho|$  near  $\partial\Omega$ .

2. The estimate  $|\mathcal{H}_\Omega| \mu_\Omega^{-1} < 1/2$  holds whenever  $\Omega$  is the stereographic image of a convex region in  $\mathbb{E}$ , [Ne]. Estimates of the form  $|\mathcal{H}_\Omega| \mu_\Omega^{-1} < m$  for an  $m < 3/2$  follow if  $f$  has a quasiconformal extension to all of  $S^2$ ; [Ah1, We] and [Pom].

The final result in this section describes the lines of curvature on  $\alpha$ -Weingarten surfaces.

Theorem 7.8: The lines of curvature on an  $\alpha$ -Weingarten surface project under the Gauss map to the positive and negative trajectories of the holomorphic quadratic differential:  $q_\Omega(z) = \mathcal{H}_\Omega(z) dz^2$ .

Remark: This theorem holds for all values of  $\alpha$ . For  $\alpha > 0$  a function  $\rho$  defining the surface is locally expressible in terms of a holomorphic function  $f$  and  $\mathcal{H}_\Omega = \mathcal{H}_f$  locally. We will only treat the case  $\alpha < 0$ . Note  $q_\Omega$  is holomorphic on  $\Sigma(\rho)$  if and only if  $\alpha = 1$ .

Proof of Theorem 7.8: The positive and negative trajectories are the integral curves of the line fields defined by:

$$\operatorname{Im} q_{\Omega} = 0 .$$

If  $\mathcal{A}_{\Omega}(z) = a + ib$  then line fields at  $z$  are given by:

$(a - \sqrt{a^2 + b^2}, -b)$  and  $(a + \sqrt{a^2 + b^2}, -b)$ . The principal directions are determined at  $z=0$  by the matrix:

$$\begin{pmatrix} \operatorname{Re} \mathcal{A}_{\Omega} - \operatorname{Im} \mathcal{A}_{\Omega} \\ -\operatorname{Im} \mathcal{A}_{\Omega} - \operatorname{Re} \mathcal{A}_{\Omega} \end{pmatrix} = \begin{pmatrix} a & -b \\ -b & -a \end{pmatrix} .$$

The eigenvectors are easily seen to coincide with the vectors determining the trajectories of  $q_{\Omega}$ . ■

### 8. Higher Connectivity

The solution of the Dirichlet problem for a Weingarten surface over a simply connected region,  $\Omega$  was reduced to the construction of a complete hyperbolic metric on  $\Omega$ . The regularity depended upon an estimate for  $|\mathcal{A}_{\Omega}|^{\mu_{\Omega}^{-1}}$ .

An essentially arbitrary domain,  $\Omega$  on  $S^2$  has a complete hyperbolic metric. This is a consequence of the general uniformization theorem. If  $S^2 \setminus \Omega$  consists of more than two points, then there is a conformal covering map  $g$  from the unit disk to  $\Omega$ .  $g$  is locally univalent so we can define a local inverse  $f: \Omega \rightarrow D_1$ . If  $f_1$  is a different local inverse, then there exists  $\alpha$  and  $\beta$  such that:

$$f_1 = (\alpha f + \beta)(\bar{\beta} f + \bar{\alpha})^{-1}$$

$$|\alpha|^2 - |\beta|^2 = 1$$

Thus the hyperbolic metric for  $\Omega$  can be expressed without ambiguity by:

$$e^{2\rho} d\sigma^2 = \frac{4|f'|^2}{(1-|f|^2)^2} |dz|^2 .$$

The quadratic differential

$$q_{\Omega}(z) = \mathcal{A}_f(z) dz^2$$

is also well defined because the Schwarzian derivative is invariant under Mobius transformations. Moreover, the formula for the curvature:

$$K = \left( \left| \frac{\mathcal{A}_f}{\mu_{\Omega}} \right|^2 - 1 \right)^{-1}$$

is still valid. Thus the smoothness of the  $\alpha$ -Weingarten surfaces over  $\Omega$  is still determined by the ratio  $|\mathcal{A}_f|^{\mu_{\Omega}^{-1}}$ .

In [Ge] the following lemma is proved for arbitrary domains  $\Omega$ :



Lemma 8.1: If  $f$  is univalent in  $\Omega$  then

$$|\mathcal{K}_f(z)| \leq 6 \delta(z, \partial\Omega)^{-2} . \quad (8.1)$$

(Recall  $\delta(\cdot, \cdot)$  is Euclidean distance in the conformal parameter  $z$ )  
As the proof only requires an estimate of  $\mathcal{K}_f$  in a disk contained in  $\Omega$ , it is clear that  $f$  need not be single valued. Gehring's argument actually proves:

Lemma 8.1': If  $f$  is holomorphic and locally univalent though not necessarily single valued in  $\Omega$  then:

$$|\mathcal{K}_f(z)| \leq 6 \delta(z, \partial\Omega)^{-2} . \quad (8.1)$$

To prove regularity for the  $\alpha$ -Weingarten surfaces, we need a lower bound for  $\mu_\Omega$  in terms of  $\delta(z, \partial\Omega)$ . For a general domain no such estimate is true. However, if  $\Omega$  has the property that

$$\partial\Omega = \cup Y_i$$

where each  $Y_i$  has more than two points and the component of  $S^2 \setminus Y_i$  which contains  $\Omega$  is simply connected, then such an estimate holds. Let the component of  $S^2 \setminus Y_i$  described above be denoted by  $\Omega_i$ . Estimate (7.2) applies to the domains  $\Omega_i$  to give:

$$1/4 \delta(z, \partial\Omega_i)^{-2} \leq \mu_{\Omega_i} . \quad (8.2)$$

We use the comparison principle for hyperbolic metrics to conclude:

Proposition 8.2: Let  $\Omega$  be as described above, then

$$1/4 \delta(z, \partial\Omega)^{-2} \leq \mu_\Omega(z) . \quad (8.3)$$

Proof of Proposition 8.2: The comparison principle for hyperbolic metrics [Ah12] states that if  $\Omega_1 \subset \Omega_2$  then

$$\mu_{\Omega_1} \geq \mu_{\Omega_2} .$$

$\Omega \subset \Omega_i$  for every  $i$  and therefore

$$\mu_{\Omega_i} \leq \mu_\Omega \quad \text{for every } i .$$

We apply (8.2) and take the supremum over  $i$  to obtain:

$$\frac{1}{4} \sup_i \delta(z, \partial\Omega_i)^{-2} \leq \mu_\Omega(z) .$$

The estimate, (8.3) follows as  $\cup Y_i = \partial\Omega$ . ■

Corollary 8.3: For  $\Omega$  as described above:

$$|\mathcal{K}_\Omega| \mu_\Omega^{-1} \leq 24 . \quad (8.4)$$

As a corollary of the corollary, we have:

Corollary 8.4: If  $\Omega$  is as described above, then the  $\alpha$ -Weingarten surfaces with boundary equal to  $\partial\Omega$  are smoothly immersed if

$$\alpha < - (4\sqrt{6} - 1)^{-1} .$$

The argument is identical to that used in the proof of Theorem 7.5.

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