ERRATUM FOR "A RELATIVE INDEX ON THE SPACE OF EMBEDDABLE CR-STRUCTURES, I"

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1. Some preliminary remarks

As stated, there is a gap in the proof of Proposition 7.2 in [1]; this is because Lemma 7.1 and formula (7.13) are incorrect. In this note we follow the notation of [1] which in turn uses that of [4]. Let $\Box_t^{p,q}$ denote the $\overline{\partial}$-Neumann operator acting (p,q)-forms defined by the complex structure on X_t. These operators are continuous functions of t in the L^2-strong resolvent sense; however a little care is needed to make this precise as they act on sections of different, though isomorphic bundles. Let $\Lambda_t^{p,q} \to X_t$ denote the (p,q)-form bundle with respect to the complex structure on X_t. There exists a smooth family of bundle isomorphisms $\Phi_t^{p,q} : \Lambda_0^{p,q} \to \Lambda_t^{p,q}$ converging to the identity at $t = 0$. If f is a section of $\Lambda_0^{p,q}$ then $\Phi_t^{p,q} \circ f$ is a section of $\Lambda_t^{p,q}$. These operations are continuous with respect to the H^s-topologies on sections. We denote these operators by $\Phi_t^{p,q}$ as well. As a function of t, $\{\Phi_t^{p,q} : t \in [0,1]\}$ is a strongly continuous family of operators in H^s-topology for any $s \geq 0$.

Proposition 1. *Fix a $q > 0$ and $t_0 \in [0,1]$. If $\lambda_0 \notin \operatorname{spec}(\Box_{t_0}^{p,q})$ then there is an $\epsilon > 0$ such that the family of operators*

$$A_{t,\lambda}^{p,q} = [\Phi_t^{p,q}]^{-1}(\Box_t^{p,q} - \lambda)^{-1}\Phi_t^{p,q}$$

is $L^2(X_0;\Lambda_0^{p,q})$-strongly continuous for $|t - t_0| < \epsilon$ and $|\lambda - \lambda_0| < \epsilon$.

Proof. Using the argument used to prove Proposition 7.1 in [1] one easily shows the existence of an $\epsilon > 0$ such that, for $|t - t_0| < \epsilon$,

$$\operatorname{spec}(\Box_t^{p,q}) \cap D_\epsilon(\lambda_0) = \emptyset$$

where $D_\epsilon(\lambda_0) = \{\lambda : |\lambda - \lambda_0| < \epsilon\}$. For such (t,λ) and $\eta \in \mathcal{C}^\infty(X_0;\Lambda_0^{p,q})$ there is a unique solution to the equation

$$(\Box_t^{p,q} - \lambda)\omega_t = \Phi_t^{p,q}\eta$$

satisfying the the $\overline{\partial}_t$-Neumann boundary conditions. Let $\{t_n\}$ be a sequence in $D_\epsilon(t_0)$ converging to t, also in this disk. Using the estimates in Theorem 3.1.14 in [2] it is easy to show that $\{\omega_{t_n}\}$ is pre-compact in $L^2(X)$, indeed in $H^s(X)$ for any $s \geq 0$. As we have control of the higher Sobolev norms the limit of any convergent subsequence satisfies the $\overline{\partial}_t$-Neumann boundary conditions. Therefore, by uniqueness, any convergent subsequence converges to ω_t. This implies that $\{\omega_{t_n}\}$ itself converges, in the L^2-norm to ω_t. For a fixed $\lambda \in D_\epsilon(\lambda_0)$ and $t \in D_\epsilon(t_0)$ the L^2-norms of the operators $A^{p,q}_{t,\lambda}$ are uniformly bounded. As $\mathcal{C}^\infty(X_0;\Lambda^{p,q}_0)$ is dense in the L^2-topology this completes the proof of the proposition. $\square$

Using this result and Proposition 7.1 in [1] it follows that there is an $\epsilon > 0$ so that $\{A^{p,q}_{t,\lambda} : t \in [0,1]\}$ is a strongly continuous family of operators for $0 < |\lambda| < \epsilon$. Using an argument from section VII.1.1 of [3] we see that this family is uniformly, strongly continuous on compact subsets of $\{\lambda : 0 < |\lambda| < \epsilon\}$. Let $P^t_{p,q}$ denote the orthogonal projection onto the null-space of $\square^{p,q}_t$.

Corollary 1. *Under the hypotheses of Proposition 7.1 in* [1] *the conjugated projectors* $\{[\Phi^{p,q}_t]^{-1}P^t_{p,q}\Phi^{p,q}_t : t \in [0,1]\}$ *are strongly continuous in the* H^s*-operator topology for any* $s \geq 0$.

Proof. These operators are given by the Cauchy integral

$$[\Phi^{p,q}_t]^{-1}P^t_{p,q}\Phi^{p,q}_t = \int\limits_{|\lambda|=\frac{\epsilon}{2}} [\Phi^{p,q}_t]^{-1}(\square^{p,q}_t - \lambda)^{-1}\Phi^{p,q}_t \frac{d\lambda}{2\pi i}. \tag{1}$$

The strong continuity of the projectors in the L^2-norm follows easily from the uniform strong continuity of the integrand on the path of integration. It follows from (7.4) in [1] that for each $s \geq 0$ there is a constant C_s, independent of t so that

$$\|P^t_{p,q}\omega\|_{H^s} \leq C_s\|\omega\|_{L^2}. \tag{2}$$

The interpolation inequalities for Sobolev norms imply that for each $s > 0$ there is a constant C'_s so that

$$\|f\|_{H^s} \leq C'_s\|f\|^{\frac{1}{2}}_{L^2}\|f\|^{\frac{1}{2}}_{H^{2s}}.$$

Using this observation and (2) we obtain estimates

$$\begin{aligned} \|P^{t_1}_{p,q}\omega - P^{t_2}_{p,q}\omega\|^2_{H^s} &\leq C'_s\|P^{t_1}_{p,q}\omega - P^{t_2}_{p,q}\omega\|_{L^2}\|P^{t_1}_{p,q}\omega - P^{t_2}_{p,q}\omega\|_{H^{2s}} \\ &\leq 2C_sC'_s\|P^{t_1}_{p,q}\omega - P^{t_2}_{p,q}\omega\|_{L^2}\|\omega\|_{L^2}, \end{aligned} \tag{3}$$

where $t_1, t_2 \in [0,1]$. This shows that the projectors are also strongly continuous in the H^s-topology. $\square$

2. Correction to the proof of Proposition 7.2

We return now to the proof of Proposition 7.2 in [1]. The proof is correct up to equation (7.12). Recall equations (7.10) and (7.11)

$$\begin{aligned} \bar\partial_t v_t &= \alpha'_t - \beta_t + P^t_{0,1}(\alpha'_t - \beta_t), \\ \bar\partial^t_b v_t &= \bar\partial^t_b u + [P^t_{0,1}(\alpha'_t - \beta_t)]_b. \end{aligned} \tag{4}$$

The second equation in (4) shows that the cohomology class of $P^t_{0,1}(\alpha'_t - \beta_t)$ is in the null-space, $K^{0,1}_t$ of the restriction map $H^{0,1}(X_t) \to H^{0,1}_b(bX_t)$. To complete the proof of Proposition 7.2 in [1] we need to find functions g_t so that

$$\begin{aligned}\bar{\partial}_b^t g_t &= [P_{0,1}^t(\alpha_t' - \beta_t)]_b \\ \|g_t\|_{L^2(bX_t)} &\leq C\|\bar{\partial}_b^t u\|_{L^2(bX_t)},\end{aligned} \tag{5}$$

where of course the constant C is independent of u and t.

In [5] it is shown that $K_t^{0,1} = [H_t^{n,n-1}]'$, thus the hypothesis of the proposition implies that

$$m = \dim K_t^{0,1} = \dim H_t^{n,n-1}$$

is independent of t. If * denotes the Hodge star operator (defined by the complex structure on X_t) then results in section 6 of [5] imply that $K_t^{0,1}$ is spanned by $^\star \operatorname{ran} P_{n,n-1}^t$. These are, however not harmonic representatives. Let $\{\theta_1, \ldots, \theta_m\}$ be a basis for $\operatorname{ran} P_{n,n-1}^0$ and set

$$\eta_j^t = {}^\star P_{n,n-1}^t \Phi_t^{n,n-1} \theta_j.$$

Since $\overline{\partial}_t \eta_j^t = 0$ the $\overline{\partial}$-Neumann-Hodge decomposition gives

$$\begin{aligned}\eta_j^t &= P_{0,1}^t \eta_j^t + \overline{\partial}_t[\overline{\partial}_t^* G_{0,1}^t \eta_j^t] \\ &= \xi_j^t + \overline{\partial}_t g_j^t.\end{aligned} \tag{6}$$

Note that $[\eta_j^t]_b = 0$ and therefore

$$[\xi_j^t]_b = -\bar{\partial}_b^t g_j^t. \tag{7}$$

It follows from (7.4) in [1] and (2) that for each $s \geq 0$ there is a constant C_s'', independent of t so that

$$\|g_j^t\|_{H^s} \leq C_s'' \|\eta_j^t\|_{L^2}. \tag{8}$$

By replacing the basis $\{\theta_j\}$ with a different basis we can arrange to have $\{\xi_j^0\}$ an orthonormal basis of harmonic representatives for $K_0^{0,1}$. It follows from Corollary 1 that there is an $\epsilon > 0$ such that, if $0 \leq t \leq \epsilon$ then the forms $\{\xi_j^t\}$ are a harmonic basis for $K_t^{0,1}$ and

$$< \xi_j^t, \xi_k^t >= \delta_{jk} + e_{jk}(t)$$

where $\lim_{t\to 0} \|e(t)\| = 0$.

For sufficiently small t, the error term in (4) has the representation

$$P_{0,1}^t(\alpha_t' - \beta_t) = \sum_{j=1}^m c_j^t \xi_j^t, \tag{9}$$

where the coefficients $\{c_j^t\}$ satisfy the estimates

$$\sum_{j=1}^m |c_j^t|^2 \leq 2 \sum_{j=1}^m | < (\alpha_t' - \beta_t), \xi_j^t > |^2. \tag{10}$$

Because "$(\alpha'_t - \beta_t) \in H^{-\frac{1}{2}}(X)$," the estimates, (2) satisfied by the $\{\eta^t_j\}$ and $\{\xi^t_j\}$ imply that there is a constant C', independent of u and t so that

$$\sum_{j=1}^{m} | < (\alpha'_t - \beta_t), \xi^t_j > |^2 \leq C' \|\bar{\partial}^t_b u\|^2_{L^2}. \tag{11}$$

Combining equations (4), (7) and (9) we see that, for sufficiently small t

$$\bar{\partial}^t_b u = \bar{\partial}^t_b [v_t + \sum_{j=1}^{m} c^t_j g^t_j].$$

Using the estimate (7.12) in [1] along with (8), (10) and (11) we see that for such t there is a constant C'' so that

$$\|v_t + \sum_{j=1}^{m} c^t_j g^t_j\|_{L^2(bX)} \leq C'' \|\bar{\partial}^t_b u\|_{L^2(bX)}.$$

After replacing the family $\Phi^{p,q}_t$ with $\Phi^{p,q}_t \circ [\Phi^{p,q}_{t_0}]^{-1}$, the argument can be repeated to obtain a neighborhood of each $t_0 \in [0,1]$ where such an estimate holds. By compactness there is a single constant which works for all $t \in [0,1]$. This completes the proof of Proposition 7.2 in [1].

References

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