A LECTURE ON FILLING PSEUDOCONCAVE HOLES, HAYAMA, JAPAN, 1999

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Introduction

These are notes for a lecture that I presented at the Hayama conference on several complex variables in December of 1999. Most of the work reported here is joint with Gennadi M. Henkin; details and further results can be found in [4].

1. Questions

Let X_- be a compact complex manifold with a strictly pseudoconcave boundary. We consider the following questions:

Question 1: When does there exist a compact complex manifold V such that X_- embeds into V as an open set?

If such a V can be found then we say that X_- is fillable. If $\dim X_- \geq 3$ then using a result of Rossi we know that this can always be done, see [7]. However if $\dim X_- = 2$ then there are many examples which show that this is not always possible, see [3]. This suggests:

Question 2: If $\dim X_- = 2$ and $X_- \hookrightarrow V$ as above then describe the set of small deformations of the complex structures on X_- which can be similarly filled.

(a) Is the set of fillable structures closed in the $\mathcal{C}^\infty$-topology?
(b) Is the set of fillable structures path connected?
(c) Does this set have the structure of a fiber space over a finite dimensional base?

Question 3: Can V be taken to be a projective variety?

2. Projective embeddings

In the early 1960s Andreotti proved that if X_- embeds into projective space then there exists a projective variety V which contains X_- as an open set, see [1]. This is an *extrinsic* result. Andreotti and Tomassini obtained a more intrinsic result: Suppose that X_- is a pseudoconcave space and $F \to X_-$ is a holomorphic line bundle. For each integer d, let $H^0(X_-; F^d)$ denote the space of holomorphic sections of F^d and set

$$\mathcal{A}(X_-; F) = \bigoplus_d H^0(X_-; F^d).$$

Andreotti and Tomassini proved

Theorem 1 (Andreotti-Tomassini [2])**.** *If $\mathcal{A}(X_-; F)$ separates points on X_- and contains local coordinates for a neighborhood of each point on X_- then there exists a projective variety V such that $X_- \hookrightarrow V$ as an open subset.*

The hypotheses easily imply that any relatively compact subset of X_- embeds into projective space and so, by Andreotti's earlier result, is fillable. The difficulty in proving this result is to obtain an embedding of all of X_-. Henkin and I have refined this result in case $\dim X_- = 2$ and F is defined by a smoothly embedded, compact curve $Z \hookrightarrow X_-$ with $N_Z > 0$, see Theorem 4. The basis for this work is the following estimates

Proposition 1. *Suppose that X is a connected, complex surface which contains a positively embedded compact curve Z. Let g denote the genus of Z and k the degree of its normal bundle. Assume moreover that V is compact surface such that $X \hookrightarrow V$. There exists a constant $E(g,k)$ such that for $d \geq 0$ we have the estimates*

$$k\frac{d(d-1)}{2} + (1-g)(d+1) \leq \dim H^0(X;[dZ]) \leq k\frac{d(d-1)}{2} + (1-g)(d+1) + E(g,k). \tag{1}$$

This shows that the $\dim H^0(X;[dZ])$ depends very little on the details of how X is filled, up to a universal constant it depends only on the 1-jet of X along Z.

3. Surface Germs

We now consider an interesting family of examples. Let Σ be a compact Riemann surface and $L \to \Sigma$ a holomorphic line bundle. We say that a surface Y contains the data (Σ, L) if there is an embedding

$$i : \Sigma \longrightarrow Y$$

such that $Ni(\Sigma) = L$. We identify two such surfaces which agree as germs along $i(\Sigma)$ and denote the equivalence classes by $\mathfrak{S}(\Sigma, L)$. We call these the *surface germs* containing (Σ, L). If $L < 0$ then $\mathfrak{S}(\Sigma, L)$ is a finite dimensional space whereas if $L > 0$ it is infinite dimensional, see [5]. If an equivalence class has a representative which is a compact variety then we say that the germ is fillable. If $L > 0$ such a compact variety is necessarily projective.

Theorem 2. *If $L > 0$ then the set of fillable germs in $\mathfrak{S}(\Sigma, L)$ is finite dimensional. Let g denote the genus of Σ and k the degree of L, there are constants $N(g,k)$ and $D(g,k)$ such that any fillable germ in $\mathfrak{S}(\Sigma, L)$ has a representative as surface of degree at most $D(g,k)$ in a projective space of dimension at most $N(g,k)$. The image of Σ is a hyperplane section.*

This result is a rather straightforward consequence of (1). The case $\Sigma = \mathbb{P}^1$ and $L = \mathcal{O}(1)$ was considered by Morrow and Rossi. They show that the only fillable germ is the standard embedding of $\mathbb{P}^1$ as a linear subspace of $\mathbb{P}^2$.

4. Sufficient conditions for the fillable structure to be closed

Let X_- be a smooth complex surface with strictly pseudoconcave boundary. We suppose that X_- embeds into $\mathbb{P}^N$ and that $Z = X_- \cap \mathbb{P}^{N-1}$ is a smooth compact hyperplane section.

Theorem 3. *If $H^1(Z; N_Z) = 0$ then the set of small deformations of the complex structure on X_- which are fillable is closed in the $\mathcal{C}^\infty$-topology.*

The condition $H^1(Z; N_Z) = 0$ is Kodaira's stability condition for the curve Z. It implies that under small deformations of the complex structure on X_- one can find holomorphic curves which are small deformations of Z. The proof of this result requires a refinement of the Andreotti-Tomassini result. We start with a definition:

Definition 1. Suppose that (X_-, Z) is as above and that there exists a d such that

(a). $H^0(X_-; [dZ])$ embeds a neighborhood of Z into projective space.

(b). There is a proper analytic subset $G \subset X_- \setminus Z$ such that $X_- \setminus G$ is also embedded by this space of sections.

We then say that (X_-, Z) is **almost embeddable.** We call a map with these properties a **generically one-to-one map.**

Theorem 4. *If (X_-, Z) is almost embeddable then it can be embedded into projective space.*

Proof. This result is proved in 3 steps:

(a). Let ρ be a non-negative defining function for bX_- and set

$$X_\epsilon = \{x \,:\, \rho(x) > \epsilon\}.$$

Using the notion of the *relative index* defined in [3] we show that it suffices to prove that there exists a sequence $\{\epsilon_n\}$ converging to zero such that each X_{ϵ_n} is embeddable.

(b). Let $\psi : X_- \to \mathbb{P}^N$ be a generically one-to-one map. There exists a projective variety V such that $\psi(X_-) \subset V$. It is easy to see that we can assume that V is a normal variety. Let $\pi : \widehat{V} \to V$ denote the resolution of the singularities of V, then there exists a meromorphic map $\widehat{\psi} : X_- \to \widehat{V}$ such that $\psi = \pi \circ \widehat{\psi}$. Because the $\dim X_- = 2$ the indeterminancy locus of $\widehat{\psi}$ is a **discrete** set.

(c). To complete the proof we need a local Castelnuovo theorem:

Lemma 1. *If $\varphi : (D, 0) \to (\mathbb{C}^2, 0)$ is a germ of a holomorphic map such that*

(a) $\varphi(x, 0) = (0, 0)$,

(b) $\varphi\big|_{D \setminus \{y=0\}}$ *is an embedding*

then φ is holomorphically conjugate to a composition of standard blowdowns.

□

The complete proof of this result can be found in [4].

5. CR-cobordisms

The final topic we consider is whether fillability is a cobordism invariant.

Definition 2. A compact CR-manifold X_0 is CR-cobordant to another compact CR-manifold X_1 if there is a compact complex manifold with boundary Y so that $bY = X_0 \sqcup X_1$.

Definition 3. A compact CR-manifold X_0 is strictly CR-cobordant to another compact CR-manifold X_1 if there is a compact, complex manifold Y with $bY = X_0 \sqcup X_1$ and there is a strictly plurisubharmonic function ρ, defined on Y so that the boundary of Y is a union of non-critical level sets of ρ. In this case, the boundaries of Y are necessarily pseudoconvex or pseudoconcave.

Usually we suppose that X_0 is a connected, strictly pseudoconvex CR-manifold whereas X_1 may be a finite union of connected, strictly pseudoconcave CR-manifolds. If the components of X_1 are all fillable then clearly X_0 is fillable as well. On the other hand, it is not so clear that the fillability of X_0 implies that the components of X_1 can also be filled. If Y defines a CR-cobordism between X_0 and X_1 it is not even clear that holomorphic functions defined in a neighborhood of X_0 extend to all of Y. If we also have a plurisubharmonic exhaustion ρ then we can use Lewy extension and induction over the level sets of ρ to extend holomorphic functions to all of Y.

Theorem 5. *Suppose that X_0 is a compact connected, strictly pseudoconvex CR-manifold which is the boundary of a fillable, pseudoconcave surface V_-. Suppose that V_- has an embedding, ψ into projective space such that*

(a). *There exists a normal sub-variety W with $\psi(V_-) \subset W$,*
(b). *$\psi(X_0) = \psi(bV_-)$ lies in an affine chart.*

If X_1 is strictly CR-cobordant to X_0 with Y a complex manifold defining the cobordism then $V_- \sqcup Y$ is also projectively fillable.

Proof. The proof of this result is done in two steps. First we must extend the embedding ψ to $V_- \sqcup Y$. Let ρ denote the strictly plurisubharmonic exhaustion defined on Y. Without loss of generality we can assume that ρ has only Morse-type singularities.

Since $\psi(X_0)$ lies in an affine chart we can extend ψ to Y by extending the coordinate functions of ψ with respect to this affine structure. The extension of the coordinate functions is effected by using induction over the level sets of ρ and a small extension of the Lewy extension theorem. The extension we require is to handle the critical level sets of ρ. Since the critical points are isolated, this is actually an elementary result. One shows that the "power series" expansion at a critical point converges in an open polydisk and represents the given function in a pseudoconcave open set with the critical point on its boundary.

Let $\widehat{\psi}$ denote the extension of ψ to $Y' = V_- \sqcup Y$. There is clearly a neighborhood of $\overline{V_-}$ in Y' in which $\widehat{\psi}$ is a embedding. To show that $\widehat{\psi}$ embeds all of Y' we again use induction over the level sets of ρ. Let

$$E = \{y \in Y' \,:\, \operatorname{rank} d\rho(y) < 2\}.$$

The proof is completed by showing that $E = \emptyset$ and using the fact that $\widehat{\psi}(Y') \subset W$, a normal variety. The fact that $E = \emptyset$ is a consequence of a lemma:

Lemma 2. *Let D be the open unit ball in $\mathbb{C}^2$ and (W, p) be the germ of a normal surface. Let $F : (D, 0) \to (W, p)$ be the germ of a holomorphic map such that $\operatorname{rank} dF(x) = 2$ for the $x \neq 0$ and φ be a strictly plurisubharmonic, Morse function defined in a neighborhood of 0 with $\varphi(0) = 0$. If $F \restriction_{\varphi(x)>0}$ is a one-to-one map then W is smooth at p and F is the germ of an embedding.*

Proof. The hypotheses of the theorem imply that F is the germ of a finite branched holomorphic cover. Using results of H. Cartan and D. Prill it can be shown that this germ is equivalent to a quotient map $\mathbb{C}^2 \to \mathbb{C}^2/G$ where G is a finite subgroup of $U(2)$ which acts freely on the unit sphere, see [6]. The lemma follows by considering the possible groups which can arise. □

□

References

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