Tempering the Polylogarithm

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1 The polylogarithm

The power series

$$\mathrm{Li}_s(z) = \sum_{n\geq 1} \frac{z^n}{n^s}$$

converges when $|z| < 1$, defining the classical polylogarithm function, equal to $-\log(1-z)$ when $s = 1$. In general, the behavior of these functions at $z = 1$ is complicated: it is known, for example, that Li_s has an analytic continuation to the cut plane $\mathbb{C} - 1, \infty)$. The fact that the 'modified' polylogarithm of Bloch, Ramakrishnan, Wigner, Wojtkowiak, Zagier, ..., defined as the real, or imaginary part of

$$\sum_{n-1\geq k\geq 0} \frac{(-2)^k B_k}{k} \log^k |z| \cdot \mathrm{Li}_{n-k}(z),$$

according to whether $n \in \mathbb{Z}$ is even or odd, see [1], extends to define a smooth function on the whole complex plane gives some idea of the complexity of its branch-point behavior.

The closely related series

$$\mathrm{li}_s(x) = \sum_{n\geq 1} \frac{e^{nx}}{n^s} = \mathrm{Li}_s(e^x)$$

converges when $\mathrm{Re}\, s \geq 0$ and $\mathrm{Re}\, x < 0$. This note is concerned with the function, or rather the tempered distribution, it defines upon restriction to the real axis. In [5],

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the second author considered the polylogarithm on the unit circle; our recent interest in its behavior on the real line comes from problems in statistical mechanics, see [6].

We would like to thank Don Zagier for pointing out that, when $\mathrm{Re}\, s < 1$, the methods of [7] can be used to establish that

$$\mathrm{li}_s(x) = \Gamma(1-s)(-x)^{s-1} + \sum_{k \geq 0} \zeta(s-k) \frac{x^k}{k}$$

(if $|x| < 2\pi$ as well, which ensures convergence of the series on the right). Our starting point is the simpler observation that, when $\mathrm{Re}\, s \geq 1$, and $x < 0$,

$$\partial_x \mathrm{li}_s(x) = \mathrm{li}_{s-1}(x).$$

It is useful to establish some notation. Recall that convolution of a smooth function supported on the positive real axis with the distribution-valued divided power

$$\gamma_+^s(x) = \frac{x_+^{s-1}}{\Gamma(s)}$$

defines an entire function of the complex variable s, see [2][Ch. I §3.5], representing fractional differentiation of order $-s$, see [2][Ch. I §5.5].

Proposition 1. *If* $\mathrm{Re}\, s > 0$ *and* $0 > x \in \mathbb{R}$*, then*

$$\mathrm{li}_s(x) = (\gamma_+^s * \mathrm{li}_0)(x)$$

as functions holomorphic in the right half s*-plane, with values in smooth functions of* $x < 0$*; where*

$$\mathrm{li}_0(x) = (e^{-x} - 1)^{-1} = x^{-1} \sum_{n \geq 0} (-1)^{n-1} B_n \frac{x^n}{n} .$$

Proof. The classical integral representation

$$\sum_{n \geq 0} \frac{z^n}{(n+a)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - ze^{-t}}\, dt$$

for the 'Lerch transcendent', valid for $\mathrm{Re}\, a > 0$, $\mathrm{Re}\, s > 0$, and $|z| < 1$, is obtained by expanding the denominator in the integral as a power series. Take $a = 1$, $z = e^x$ with $x < 0$, and multiply both sides by z to get

$$\mathrm{li}_s(x) = \int_{-\infty}^{\infty} \frac{t_+^{s-1}}{\Gamma(s)} \frac{e^{-(t-x)}}{1 - e^{-(t-x)}}\, dt = (\gamma_+^s * \mathrm{li}_0)(x).$$

□

In our case γ_+^s has at worst polynomial growth on the right half-line, $\mathrm{li}_0(x) \to 0$ exponentially as $x \to -\infty$, and $\mathrm{li}_0(x) + 1 \to 0$ exponentially as $x \to \infty$. Near $x = 0$, however,

$$\mathrm{li}_0 \equiv -x^{-1} \text{ mod smooth functions}$$

is singular, and the behavior of the convolution there is quite interesting.

2 Extension of li_s as a tempered distribution

Much of the material in this section is standard and can be found in [3]. We let $\mathcal{S}(\mathbb{R})$ denote the Schwartz class functions on $\mathbb{R}$ with topology defined by the seminorms

$$\| f \|_N = \max_{0 \le j \le N} \sup_{x \in \mathbb{R}} (1 + |x|)^N |\partial_x^j f(x)|, \; N \in \mathbb{N}. \tag{1}$$

The dual space $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions, with the usual weak topology. This implies that $l \in \mathcal{S}'(\mathbb{R})$ if and only if there is an N so that

$$|l(f)| \le C_N \| f \|_N. \tag{2}$$

Hence a given tempered distribution is always of finite order, and growth at infinity, and therefore can be continuously extended to a much larger space. Schwartz space is identified with a subset of $\mathcal{S}'(\mathbb{R})$ by $f \mapsto l_f$

$$l_f(\varphi) = \int \varphi(x) f(x) dx. \tag{3}$$

It is a very useful and important fact that $\mathcal{S}(\mathbb{R})$ is a *dense* subspace.

Recall that if φ is a tempered distribution, then the Fourier transform $\hat{\varphi}$ is the tempered distribution *defined* by duality, with:

$$\langle \hat{\varphi}, f \rangle = \langle \varphi, \hat{f} \rangle \text{ for all } f \in \mathcal{S}(\mathbb{R}). \tag{4}$$

This defines a tempered distribution, because the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R})$ to itself. In general the product of two distributions is not defined, and because $\widehat{\varphi * \psi} = \hat{\varphi} \hat{\psi}$, this implies that the convolution of two tempered distribution is not always defined. Of course it may be defined for a given pair.

When it is defined, similar considerations are used to define the convolution of two distributions. We start with the case $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ and observe that, for $f \in \mathcal{S}(\mathbb{R})$ we have:

$$\begin{aligned} \langle \varphi * \psi, f \rangle &= \int \left[\int \varphi(x) \psi(t - x) dx \right] f(t) dt \\ &= \int \varphi(x) \left[\int \psi(y) f(x + y) dy \right] dx \end{aligned} \tag{5}$$

We changed the order of integrations and variables, with $t - x = y$ to go from the first line to the second. Because $\mathcal{S}(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ is dense, this relation defines $\psi * \varphi$, whenever this convolution defines a tempered distribution. For $x \in \mathbb{R}$ we define the operation $\tau_x \ \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by

$$\tau_x f(y) = f(x + y). \tag{6}$$

The convolution of φ and ψ defines a tempered distribution precisely when

1. $x \mapsto \langle \varphi, \tau_x f \rangle$ is in the domain of ψ.

2. There is an N and a $C_N >$ so that

$$|\langle \psi, \langle \varphi, \tau_x f \rangle \rangle| \leq C_N \| f \|_N. \tag{7}$$

In this case the convolution $\varphi * \psi$ is defined by

$$\langle \varphi * \psi, \ f \rangle = \langle \varphi(x), \langle \psi, \tau_x f \rangle \rangle. \tag{8}$$

We can now prove that the tempered distributions

$$\gamma_+^s(x) = \frac{x_+^{s-1}}{\Gamma(s)}, \text{ and } \mathrm{li}_0 = \mathrm{PV} \frac{1}{e^{-x} - 1}, \tag{9}$$

can be convolved to give an entire family of tempered distributions. Observe that, the discussion above implies that if $\gamma_+^s * \mathrm{li}_0$ makes sense as a distribution, then for all Schwartz class functions f, we must have

$$\langle \gamma_+^s * \mathrm{li}_0, \ f \rangle = \mathrm{PV} \int_{-\infty}^{\infty} \frac{1}{e^{-t} - 1} \langle \gamma_+^s(\cdot), \ f(\cdot + t) \rangle dt. \tag{10}$$

The crux of the matter is therefore to analyze

$$F_s(t) = \langle \gamma_+^s(\cdot), \ f(\cdot + t) \rangle, \tag{11}$$

as a function of (t, s). If $\mathrm{Re}\, s > 1$, then $F_s(t)$ is given by an absolutely convergent integral and for any $k \in \mathbb{N}$, we can integrate by parts to obtain:

$$F_s(t) = \frac{(-1)^k}{\Gamma(s + k)} \int_0^{\infty} x^{s+k-1} f^k(x + t) dx. \tag{12}$$

The right hand is an analytic function in $-k < \mathrm{Re}(s) < k$ with values in the space of functions, $\mathcal{F}_k^+$, which we now define:

Definition 1. A function $f \in \mathscr{C}^\infty(\mathbb{R})$ belongs to $\mathscr{F}_k^+$ if

1. $f(x)$ and all its derivatives are rapidly decreasing as x tends to $+\infty$.
2. For all j
$$\limsup_{x\to-\infty}\left|\frac{\partial_x^j f(x)}{(1+|x|)^k}\right| < \infty. \tag{13}$$

Briefly, $f \in \mathscr{F}_k^+$ if f is smooth, in Schwartz class at $+\infty$ and, of tempered growth at $-\infty$. The topology on $\mathscr{F}_k^+$ is defined by the semi-norms:

$$|F|_{k,l} = \sup_{x>0}(1+|x|)^l \max_{0\le j\le l}|\partial_x^j F(x)| + \sup_{x\le 0}(1+|x|)^{-k}\max_{0\le j\le l}|\partial_x^j F(x)|. \tag{14}$$

The statement that F_s is an analytic function from $-k < \operatorname{Re} s < k$ to $\mathscr{F}_k^+$ is now a simple consequence of formula (12). Moreover, $f \mapsto F_s$ is continuous as a mapping from $\mathscr{S}(\mathbb{R})$ to $\mathscr{F}_k^+$, that is for each l there is an N_l and a $C_{s,l}$ so that

$$|F_s|_{k,l} \le C_{s,l}\|f\|_{N_l}. \tag{15}$$

The constants $C_{s,l}$ are locally uniformly bounded in $|\operatorname{Re} s| < k$.

To complete our discussion of $\gamma_+^s * \mathrm{li}_0$ it remains only to show that $\mathscr{F}_k^+$ is in the domain of li_0 for all $k \in \mathbb{N}$. To that end we choose an even function $\chi \in \mathscr{C}_c^\infty((-1,1))$, which equals 1 in the interval $-\frac{1}{2}, \frac{1}{2}$. For any such function we have

$$\langle \mathrm{li}_0, f\rangle = \int_{-\infty}^{\infty}\frac{(1-\chi(t))f(t)dt}{e^{-t}-1} + \lim_{\epsilon\to 0^+}\int_{\epsilon<|t|}\frac{\chi(t)f(t)dt}{e^{-t}-1}. \tag{16}$$

Observe that
$$\frac{1}{e^{-t}-1} = \begin{cases} -1+O(e^{-t}) & \text{as } t\to+\infty \\ O(e^{-|t|}) & \text{as } t\to-\infty \end{cases} \tag{17}$$

From these estimates it follows that the first term on the r.h.s. of (16) is clearly a continuous functional on $\mathscr{F}_k^+$, for every k.

We consider the principal value part. An elementary calculation shows that, for a differentiable f we have:

$$\lim_{\epsilon\to 0^+}\int_{\epsilon<|t|}\frac{\chi(t)f(t)dt}{e^{-t}-1} = \int \chi(t)f(t)\left[\frac{1}{e^{-t}-1}+\frac{1}{t}\right]dt - \int \chi(t)\frac{f(t)-f(0)}{t}dt. \tag{18}$$

The sum $(e^{-t}-1)^{-1}+t^{-1}$ is a smooth function, hence the right hand side again clearly defines a continuous functional on $\mathscr{F}_k^+$, for every k. This proves the theorem

Theorem 1. *The family $s \mapsto \gamma_+^s * \mathrm{li}_0$ is an entire family of tempered distributions.*

In the sequel we use li_s to denote the distribution $\gamma_+^s * \mathrm{li}_0$. While it is not immediately clear, we will show that the two, a priori different, definitions of li_0 do coincide.

The theorem and the well known functional equation, $\partial_x \gamma_+^s = \gamma_+^{s-1}$, show that the functional equation satisfied by $\mathrm{li}_s(x)$ in $x < 0$ extends to the whole real line.

Corollary 1. *In the sense of distributions, $\partial_x \mathrm{li}_s = \mathrm{li}_{s-1}$, for all $s \in \mathbb{C}$.*

3 The singularities of $\mathrm{li}_s(x)$.

We now turn to a consideration of the singularities of the distribution $\mathrm{li}_s(x)$, as a function of x. In the previous section we obtained the formula:

$$\langle \gamma_+^s * \mathrm{li}_0, f\rangle = \int_{-\infty}^{\infty} \frac{(1-\chi(t))F_s(t)dt}{e^{-t}-1} + \lim_{\epsilon\to 0^+} \int_{\epsilon<|t|} \frac{\chi(t)F_s(t)dt}{e^{-t}-1}. \tag{19}$$

Using formula (12) it is straightforward to show that

$$\int_{-\infty}^{\infty} \frac{(1-\chi(t))F_s(t)dt}{e^{-t}-1} = \frac{1}{\Gamma(s+k)} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{(1-\chi(x-y))y^{s+k-1}dy}{e^{y-x}-1} f^k(x)dx. \tag{20}$$

Hence there is an analytic family of smooth, tempered *functions* $G_s(x)$ so that the first term on the right hand side of (19) has a representation as

$$\int_{-\infty}^{\infty} \frac{(1-\chi(t))F_s(t)dt}{e^{-t}-1} = \langle G_s, f\rangle. \tag{21}$$

Thus the singularities of $\mathrm{li}_s(x)$ are in the second term in (19).

Using (18) we can show that the first term contributes a smooth term and therefore the distribution, g_s defined by

$$\begin{aligned}\langle g_s, f\rangle &= -\int \chi(t)\frac{F_s(t)-F_s(0)}{t}dt \\ &= -\,\mathrm{PV}\int \chi(t)\frac{F_s(t)dt}{t}.\end{aligned} \tag{22}$$

has the same singularity as li_s.

Given a choice of smooth, even cutoff function χ we define the operator on Schwartz class functions:

$$\mathcal{H}_\chi f = -\,\mathrm{PV}\int \chi(t)\frac{f(x-t)dt}{t}. \tag{23}$$

The distribution $h_0 = \mathrm{PV}\left[\frac{\chi(t)}{t}\right]$ is compactly supported and therefore has a smooth Fourier transform, on the other hand $h_1 = \left[\frac{1-\chi(t)}{t}\right]$ is smooth and belongs to L^2, hence its Fourier transform is rapidly decreasing. The Fourier transform of PV $\left[\frac{1}{t}\right]$ is well known to be $-\pi i \operatorname{sgn}\xi$. This shows that the Fourier transform of h_0 is smooth and rapidly approaches $-\pi i(\pm 1)$, as $\xi \to \pm\infty$. Thus $\mathcal{H}_\chi$ maps $\mathcal{S}$ to itself and therefore $\mathcal{H}_\chi$ extends as a map from $\mathcal{S}'$ to itself. An elementary calculation shows that, for $\operatorname{Re} s > -k$ we have

$$\langle g_s,\, f\rangle = \langle \mathcal{H}_\chi \gamma_+^{s+k}, (-1)^k f^k\rangle. \tag{24}$$

The singularity of $\mathrm{li}_s(x)$ at $x = 0$ agrees with that of

$$g_s = \partial_x^k \mathcal{H}_\chi \gamma_+^{s+k} = \mathcal{H}_\chi \gamma_+^s. \tag{25}$$

It would be tempting to say that this agrees with the singularity of the Hilbert transform of γ_+^s, but for the fact that the Hilbert transform does not preserve Schwartz space, and hence does not have an extension to tempered distributions. Some care is required to compute the singular part of of g_s. We make extensive usage of the fact that the Fourier transform of a compactly supported distribution is smooth.

Notice that in (24) the power $s+k$ is positive, this, coupled with the functional equation

$$\partial_x \gamma_+^s = \gamma_+^{s-1}. \tag{26}$$

facilitate the computations which follow. We first compute the Fourier transform of γ_+^s

Proposition 2. *The Fourier transform of the tempered distribution γ_+^s is $e^{-i\frac{\pi s}{2}}\eta_-^{-s}$, where*

$$\eta_\pm^s(\xi) = \lim_{\epsilon\downarrow 0}(\xi \pm i\epsilon)^s. \tag{27}$$

The complex power $z \mapsto z^s$ is taken to be real on the positive real axis and analytic in $\mathbb{C} \setminus (-\infty, 0$.

Proof. If $s > 0$, then we can compute the Fourier transform using

$$\begin{aligned}\widehat{\gamma_+^s}(\xi) &= \lim_{\epsilon\downarrow 0} \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-\epsilon x} e^{-ix\xi} dx \\ &= \lim_{\epsilon\downarrow 0} \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-x(\epsilon+i\xi)} dx.\end{aligned} \tag{28}$$

Given our choice of branch for z^{s-1}, the last integral can be regarded as a contour integral along the ray $\{x(\epsilon + i\xi) \quad x > 0\}$, which lies in the right half plane. The conclusion follows from an elementary contour deformation argument. For $s \leq 0$ we use the functional equation to conclude that

$$\widehat{\gamma_+^{s-k}} = (i\xi)^k \widehat{\gamma_+^s}, \tag{29}$$

which is easily seen to extend our formula for $\widehat{\gamma_+^s}$ to the whole complex plane. ☐

The Fourier transform of $\mathscr{H}_\chi \gamma_+^s$ is $-h_0(\xi) e^{-i\frac{\pi s}{2}} \eta_-^{-s}$. Let $\psi(\xi)$ be a smooth, even, non-negative function which vanishes in a neighborhood of 0, and equals 1 for $|\xi| > 1$, then

$$\mathscr{H}_\chi \gamma_+^s = -e^{-i\frac{\pi s}{2}} \left[\mathscr{F}^{-1}((1-\psi)h_0(\xi)\eta_-^{-s}) + \mathscr{F}^{-1}(\psi h_0(\xi)\eta_-^{-s})\right] \tag{30}$$

The distribution $(1-\psi)h_0(\xi)\eta_-^{-s}$ is compactly supported, hence its inverse Fourier transform is a smooth function. The singularity of g_s is therefore equal to that of

$$g_{s0} = -e^{-i\frac{\pi s}{2}} \mathscr{F}^{-1}(\psi h_0(\xi)\eta_-^{-s}). \tag{31}$$

From the remarks above, it is clear that the difference $\psi h_0(\xi) - \psi(-i\pi \operatorname{sgn}\xi)$ is a smooth rapidly decreasing function, and therefore the singularity of g_{s0} equals that of

$$g_{s1} = i\pi e^{-i\frac{\pi s}{2}} \mathscr{F}^{-1}(\psi \operatorname{sgn}\xi \, \eta_-^{-s}). \tag{32}$$

If we let $\psi_+(\xi) = \chi_{0,\infty)}(\xi)\psi(\xi)$, then we can write g_{s1} as

$$g_{s1} = i\pi \mathscr{F}^{-1}\left[\frac{e^{-i\frac{\pi s}{2}}\psi_+(\xi)}{\xi^s} - \frac{e^{i\frac{\pi s}{2}}\psi_+(-\xi)}{|\xi|^s}\right] \tag{33}$$

For $\operatorname{Re} s > 1$, we see that

$$g_{s1} = \frac{i}{2}\left[e^{-i\frac{\pi s}{2}} \int_0^\infty \frac{\psi_+(\xi)e^{ix\xi}d\xi}{\xi^s} - e^{i\frac{\pi s}{2}} \int_0^\infty \frac{\psi_+(\xi)e^{-ix\xi}d\xi}{\xi^s}\right]. \tag{34}$$

Finally, we see that, for all $s \in \mathbb{C}$, the distributions $\Gamma(1-s)\gamma_+^{1-s} - \frac{\psi_+(\xi)}{\xi^s}$ are compactly supported and therefore g_{s1} has the same singularity as

$$g_{s2} = i\pi\,\Gamma(1-s)\left[e^{-i\frac{\pi s}{2}}\mathscr{F}^{-1}(\gamma_+^{1-s})(x) - e^{i\frac{\pi s}{2}}\mathscr{F}^{-1}(\gamma_+^{1-s})(-x)\right]. \tag{35}$$

Using calculations similar to the proof of Proposition to evaluate the right hand side of (35) we obtain:

Proposition 3. *The function* li_s, *taking values in smooth functions on the negative real line when* $\mathrm{Re}\, s \geq 0$, *extends to an entire function of* s, *taking values in tempered distributions on the whole real line, satisfying the congruence*

$$\begin{aligned}\mathrm{li}_s(x) &\equiv -\frac{\Gamma(1-s)}{2}\left[e^{-i\pi s}\eta_+^{s-1} + e^{i\pi s}\eta_-^{s-1}\right] \\ &= -\frac{\Gamma(1-s)}{2}\left[e^{-i\pi s}(x+i0)^{s-1} + e^{i\pi s}(x-i0)^{s-1}\right]\end{aligned} \tag{36}$$

(modulo meromorphic functions with smooth coefficients).

Note that when $x < 0$ this accounts precisely for Zagier's correction.

As follows easily from the computations used in the proof of Corollary 2, the value of this distribution at $s = 1$ is

$$\mathrm{li}_1(x) = -\log|1-e^x|.$$

Note, this is an identity, not a congruence. Using the functional equation from Corollary 1 and this equation, we can show, as asserted above, that our notation is consistent. For $x \neq 0$,

$$\partial_x \mathrm{li}_1(x) = \frac{1}{e^{-x}-1} = \mathrm{li}_0(x), \tag{37}$$

which implies that, the distribution $\mathrm{li}_s\big|_{s=0}$, given by analytic continuation, agrees with $\mathrm{PV}(e^{-x}-1)^{-1}$.

The families of distributions $\eta_\pm^s$ are entire, hence the expression on the right hand side of equation (36) is holomorphic for $\mathrm{Re}\, s < 0$, but has simple poles at positive integers; its residue at $s = n$, however, is (up to sign) the integral divided power $x^{n-1}/(n-1)$, which is smooth. This shows that, while li_s is itself an entire family of distributions, its separation into regular and singular parts cannot be done holomorphically. Near $s = 1$, for example, this implies that

$$\mathrm{li}_s(x) \equiv \frac{1}{s-1} + \dots\ ,$$

which is related to a similar property of $\zeta(s)$. More generally,

Corollary 2. *For a non-negative integer n,*

$$\mathrm{li}_{-n}(x) \equiv (-1)^{n-1} n x^{-n-1}$$

modulo smooth functions, while for positive integers n,

$$\mathrm{li}_n(x) \equiv \frac{-\log|x| x^{n-1}}{(n-1)} \ .$$

Proof. The first assertion is a consequence of [2][Ch. I §3.7], but we give elementary proofs of both formulæ. These statements follow from the equation (22) for the singular part of g_s. For a positive integer, we see that

$$\begin{aligned} \langle g_n, f \rangle &= -\frac{1}{\Gamma(n)} \sum_{j=0}^{n-1} \binom{n-1}{j} \mathrm{PV} \int_{-\infty}^{\infty} \frac{\chi(t)(-t)^j dt}{t} \int_t^{\infty} y^{n-1-j} f(y) dy \\ &\equiv \frac{-1}{\Gamma(n)} \mathrm{PV} \int_{-\infty}^{\infty} \frac{\chi(t) dt}{t} \int_t^{\infty} y^{n-1} f(y) dy \end{aligned} \tag{38}$$

For $t > 0$ define

$$w(t) = -\int_t^{\infty} \frac{\chi(s) ds}{s} \equiv \chi(t) \log t. \tag{39}$$

Integrating by parts shows that

$$\begin{aligned} \langle g_n, f \rangle &\equiv -\frac{1}{\Gamma(n)} \lim_{\epsilon \downarrow 0} \left[\int_{-\infty}^{\infty} w(|t|) t^{n-1} f(t) dt + w(\epsilon) \int_{-\epsilon}^{\epsilon} y^{n-1} f(y) dy \right] \\ &\equiv \int_{-\infty}^{\infty} \frac{-\log|t| t^{n-1}}{(n-1)} f(t) dt \end{aligned} \tag{40}$$

as asserted.

For $-n \leq 0$ we use equation (12), with $k = 1 + n$, and (22) to conclude that

$$\langle g_{-n}, f \rangle = -\mathrm{PV} \int \frac{\chi(t)(-1)^n f^n(t) dt}{t}. \tag{41}$$

Thus

$$g_{-n} \equiv -\partial_t^n \mathrm{PV} \frac{\chi(t)}{t}. \tag{42}$$

The right hand side is equivalent to a homogeneous extension of $(-1)^n n t^{-1-n}$, as asserted. $\square$

Note that if $n > 2$,

$$(\gamma_+^{n-1}(x)\log|x|)' \equiv \gamma_+^{n-2}\log|x|$$

modulo smooth functions, and that

$$\mathrm{li}_1(x) = -\log|1-e^x| \equiv -\log|x|.$$

The Taylor coefficients at zero of the smooth difference can be read off from the power series formula

$$\log(\frac{x}{1-e^x}) = \sum_{k\geq 1}\frac{1}{k}\Big(\sum_{n\geq 1} B_n\frac{(-x)^n}{n}\Big)^k .$$

The modified polylogarithm, as defined in the introduction, is (essentially) the real or imaginary part, depending on whether n is even or odd, of a certain combination of products of classical and polylogarithms, with coefficients as above.

Corollary 3. *When n is even, the distribution analog*

$$\lambda i_n(x) = \sum_{n-1\geq k\geq 0} B_k\frac{(-2x)^k}{k}\cdot \mathrm{li}_{n-k}(x)$$

of the 'modified' polylogarithm is a smooth function of x.

Remark 1. The corollary asserts that if we regard the polylogarithm as a distribution, then this combination is in fact smooth when n is even (which recovers the smoothness of the classical 'modified' polylogarithm). When n is odd, the argument below shows that the singular part of the corresponding combination is real, so the imaginary part is again smooth. Thus our analysis of the singularities of the distribution extension of the polylogarithm is precise enough to recover the smoothness of the classical modified polylogarithm in all cases.

Proof. We claim that Corollary 2 implies that

$$\lambda i_n(x) \equiv -(-2)^{n-1}B^{n-1}(-\tfrac{1}{2})\cdot\gamma_+^{n-1}(x)\log|x| ,$$

where the Bernoulli polynomials are defined by the generating function

$$\sum_{n\geq 0} B^n(q)\frac{t^n}{n} = \frac{te^{(q+1)t}}{e^t-1}$$

see [4][§2]. They satisfy

$$B^n(q) = \sum_{n\geq k\geq 0}\binom{n}{k} B_k q^{n-k} .$$

Substituting the expression for the singular part of li_{n-k} implies that

$$\lambda i_n(x) \equiv - \sum_{n-1 \geq k \geq 0} \frac{(-2)^k B_k}{k(n-k-1)} \cdot x^{n-1} \log|x| \ ,$$

and rewriting $(-2)^k$ as $(-\frac{1}{2})^{1-n}(-\frac{1}{2})^{n-k-1}$ yields the assertion. It follows from the definition that

$$\sum_{n \geq 0} B^n(-\tfrac{1}{2}) \frac{t^n}{n} = \frac{t}{2 \sinh \frac{1}{2} t}$$

is an even function, so $B^{\mathrm{odd}}(-\frac{1}{2}) = 0$. □

It is obvious, on the other hand, that $B^{\mathrm{even}}(-\frac{1}{2}) \in \mathbb{R}$.

As a final corollary we observe that:

Corollary 4. *The function* $s \mapsto \mathrm{Li}_s$ *extends to an entire function of* s*, taking values in distributions on the positive real axis, defined on compactly-supported test functions by*

$$f \mapsto \int_0^\infty \mathrm{Li}_s(t) f(t) \ dt = \int_{-\infty}^\infty \mathrm{li}_s(x) \cdot e^x f(e^x) \ dx \ .$$

Remark 2. The domain of Li_s contains the set of functions such that $e^x f(e^x) \in \mathcal{S}(\mathbb{R})$. For example a smooth function on $0, \infty)$ such that, for every $k \in \mathbb{N}$, there is a C_k satisfying

$$\sup_{y \geq 0} |(1+y^2)\partial_y^k f(y)| \leq C_k, \tag{43}$$

belongs to the domain of Li_s.

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