A PRIORI ESTIMATES FOR A SUBELLIPTIC SPIN$_{\mathbb{C}}$ DIRAC OPERATOR

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ABSTRACT. Let X be a compact complex manifold with strictly pseudoconvex boundary, Y. In this setting, the Spin$_{\mathbb{C}}$ Dirac operator is canonically identified with $\bar\partial + \bar\partial^* : \mathcal{C}^\infty(X; \Lambda^{0,\text{even}}) \to \mathcal{C}^\infty(X; \Lambda^{0,\text{odd}})$. In this note we prove a priori estimates for a modification of the $\bar\partial$-Neumann boundary condition. This is a step toward obtaining a subelliptic Fredholm Spin$_{\mathbb{C}}$ Dirac operators, whose index equals the holomorphic Euler characteristic of X.

INTRODUCTION

Let X be an even dimensional manifold with an almost complex structure J. It is well known that the almost complex structure defines a Spin$_{\mathbb{C}}$-structure on X. A compatible choice of metric defines a Spin$_{\mathbb{C}}$-Dirac operator, $\eth_{\mathbb{C}}$ which acts on sections of the bundle of complex spinors, $\not{S}$. The metric on X induces a metric on the bundle of spinors. We let $\langle\sigma, \sigma\rangle$ denote the pointwise inner product. This, is turn, defines an inner product of the space of sections of $\not{S}$, that are smooth up to the boundary, by setting:

$$\langle\sigma, \sigma\rangle_{L^2} = \int\limits_X \langle\sigma, \sigma\rangle dV$$

If the complex structure is integrable then the bundle of complex spinors is canonically identified with $\oplus_{q\geq 0}\Lambda^{0,q}$. If the metric is Kähler then the Spin$_{\mathbb{C}}$ Dirac operator is given by

$$\eth_{\mathbb{C}} = \bar\partial + \bar\partial^*.$$

Here $\bar\partial^*$ denotes the formal adjoint of $\bar\partial$. This operator is called the Dolbeault-Dirac operator by Duistermaat, see [2]. If the metric is hermitian, though not Kähler, then

$$\eth_{\mathbb{C}} = \bar\partial + \bar\partial^* + \mathcal{T}_0,$$

here $\mathcal{T}_0$ is a homomorphism carrying $\Lambda^{0,\text{even}}$ to $\Lambda^{0,\text{odd}}$ and vice versa. It vanishes at points where the metric is Kähler. It is customary to write $\eth_{\mathbb{C}} = \eth_{\mathbb{C}}^+ + \eth_{\mathbb{C}}^-$ where

$$\eth_{\mathbb{C}}^+ : \mathcal{C}^\infty(X; \Lambda^{0,\text{even}}) \longrightarrow \mathcal{C}^\infty(X, \Lambda^{0,\text{odd}})$$

Date: December 5, 2003.

Keywords: Spin$_{\mathbb{C}}$ Dirac operator, index, subelliptic boundary value problem, $\bar\partial$-Neumann condition. Research partially supported by NSF grant DMS99-70487. Address: Department of Mathematics, University of Pennsylvania, Philadelphia, PA. E-mail: cle@math.upenn.edu.

and $\eth_{\mathbb{C}}^-$ is the formal adjoint of $\eth_{\mathbb{C}}^+$. If X is a compact manifold then the L^2-closure of $\eth_{\mathbb{C}}^+$ is a Fredholm operator. It has the same principal symbol as $\bar\partial + \bar\partial^*$ and therefore its index is given by

$$\operatorname{Ind}(\eth_{\mathbb{C}}^+) = \sum_{j=0}^{n} (-1)^j \dim H^{0,j}(X). \tag{1}$$

If X is a manifold with boundary then the kernels and cokernels of $\eth_{\mathbb{C}}^{\pm}$ are infinite dimensional. To obtain a Fredholm operator we need to impose boundary conditions. In this instance there are no local boundary conditions for $\eth_{\mathbb{C}}^{\pm}$ which define elliptic problems. In this note we prove the basic *a priori* estimates for a small modification of the classical $\bar\partial$-Neumann condition. In a latter publication we will show that this leads to a Fredholm operator whose index is given by the finite part of the Euler characteristic of the $\bar\partial$-Neumann complex. We restrict our attention to metrics which are Kähler in a neighborhood of bX. With this restriction, and appropriate boundary conditions, the operators $\eth_{\mathbb{C}}^+$ and $\bar\partial + \bar\partial^*$ have the same index. We therefore concentrate on the latter operator.

Remark 1. In this paper C is used to denote a variety of *positive* constants which depend only on the geometry of X. We make extensive usage of the boundary adapted $(1, -\frac{1}{2})$-Sobolev space. For a definition see [6]. For our applications, the most important properties of this space are the following facts:

(a) The restriction map $H_{(1,-\frac{1}{2})}(X) \to L^2(bX)$ is continuous.
(b) The Poisson kernel for the Dirac operator is continuous as a map

$$L^2(X; E\restriction_{bX}) \longrightarrow H_{(1,-\frac{1}{2})}(X; E).$$

Here E is an appropriate spinor bundle.

Acknowledgements

The boundary conditions considered in this paper were first suggested to me by Laszlo Lempert. I would like to thank John Roe for some helpful pointers on the $\mathrm{Spin}_{\mathbb{C}}$ Dirac operator.

1. Subelliptic boundary conditions for $\eth_{\mathbb{C}}^{\pm}$

Henceforth X denotes a compact complex manifold with strictly pseudoconvex boundary. The kernels of $\eth_{\mathbb{C}}^{\pm}$ are both infinite dimensional. Let $\mathcal{P}^{\pm}$ denote the operators defined on bX which are the projections onto the boundary values of element in $\ker \eth_{\mathbb{C}}^{\pm}$; these are the Calderon projections. They are classical pseudodifferential operators of order 0; see [1]. The L^2-closure of the operators $\eth_{\mathbb{C}}^{\pm}$, with domains consisting of smooth spinors such that $\mathcal{P}^{\pm}(\sigma\big|_{bX}) = 0$, are elliptic with Fredholm index zero.

Let ρ be a smooth defining function for the boundary of X. If σ is a section of $\Lambda^{p,q}$, smooth up to bX, then the $\bar\partial$-Neumann boundary condition is the requirement that

$$\bar\partial\rho \rfloor \sigma\restriction_{bX} = 0.$$

As all holomorphic functions on X satisfy this condition, using this as a boundary condition for the operator $\eth_{\mathbb{C}}$ again leads to an operator with an infinite dimensional nullspace. Let $\mathcal{S}$ denote an orthogonal projection acting on $\mathscr{C}^\infty(bX)$ with range the boundary values of holomorphic functions on X, or briefly, a *Szegő projection*. For σ, an element of $\mathscr{C}^\infty(\overline{X};\Lambda^{0,\text{even}}\oplus\Lambda^{0,\text{odd}})$, we write

$$\sigma=\sigma_0+\sigma_1+\sigma',$$

with σ' the terms in σ with degrees larger than 1. The augmented $\bar\partial$-Neumann condition, on even degree forms is defined to be

$$\mathcal{R}^+\begin{pmatrix}\sigma_0\\ \sigma'\end{pmatrix}\restriction_{bX}=\begin{pmatrix}\mathcal{S}\sigma_0\\ \bar\partial\rho\rfloor\sigma'\end{pmatrix}\restriction_{bX}=0. \tag{2}$$

The boundary condition on odd degree forms, formally adjoint to (2), is given by

$$\mathcal{R}^-\begin{pmatrix}\sigma_1\\ \sigma'\end{pmatrix}\restriction_{bX}=\begin{pmatrix}(\mathrm{Id}-\mathcal{S})\bar\partial\rho\rfloor\sigma_1\\ \bar\partial\rho\rfloor\sigma'\end{pmatrix}\restriction_{bX}=0. \tag{3}$$

The operations in the lower right impose the $\bar\partial$-Neumann condition in degrees greater than 1; the boundary value of the 0-degree part of σ is orthogonal to the nullspace of $\bar\partial_b$, whereas the $\bar\partial\rho$-component of the $(0,1)$-part has boundary value lying in the nullspace of $\bar\partial_b$. We prove *a priori* estimates for smooth forms satisfying the boundary conditions above.

2. A PRIORI ESTIMATES

The $\bar\partial$-Neumann conditions leads to basic integration-by-parts formulæ for $\eth^{\pm}_{\mathbb{C}}$.

Lemma 1. *If $\sigma\in\mathscr{C}^\infty(\overline{X};\Lambda^{0,\text{even}})$ (or $\sigma\in\mathscr{C}^\infty(\overline{X};\Lambda^{0,\text{odd}})$ satisfies* (2) *((3)), then*

$$\langle\eth^{\pm}_{\mathbb{C}}\sigma,\eth^{\pm}_{\mathbb{C}}\sigma\rangle_{L^2}=\langle\bar\partial\sigma,\bar\partial\sigma\rangle_{L^2}+\langle\bar\partial^*\sigma,\bar\partial^*\sigma\rangle_{L^2} \tag{4}$$

Proof of the lemma. The proof of the lemma is a simple consequence of the facts that

(a) $\bar\partial^2=0$
(b) If η is a $(0,j)$-form satisfying $\bar\partial\rho\rfloor\eta\restriction_{bX}=0$, then, for β any smooth $(0,j-1)$-form we have

$$\langle\beta,\bar\partial^*\eta\rangle_{L^2}=\langle\bar\partial\beta,\eta\rangle_{L^2}. \tag{5}$$

If $\sigma=\sigma_0+\sigma_2+\cdots+\sigma_{2k}$, then we need to show that

$$\langle\bar\partial\sigma_{2j},\bar\partial^*\sigma_{2(j+1)}\rangle_{L^2}=0.$$

This follows immediately from (a), (b), and the fact that $\sigma_{2(j+1)}$ satisfies (2). A similar proof applies in the odd case. □

The lemma implies that

(6)
$$\langle\eth^{+}_{\mathbb{C}}\sigma,\eth^{+}_{\mathbb{C}}\sigma\rangle_{L^2}=\langle\bar\partial\sigma',\bar\partial\sigma'\rangle_{L^2}+\langle\bar\partial^*\sigma',\bar\partial^*\sigma'\rangle_{L^2}+\langle\bar\partial\sigma_0,\bar\partial\sigma_0\rangle_{L^2},$$
$$\langle\eth^{-}_{\mathbb{C}}\sigma,\eth^{-}_{\mathbb{C}}\sigma\rangle_{L^2}=\langle\bar\partial\sigma',\bar\partial\sigma'\rangle_{L^2}+\langle\bar\partial^*\sigma',\bar\partial^*\sigma'\rangle_{L^2}+\langle\bar\partial\sigma_1,\bar\partial\sigma_1\rangle_{L^2}+\langle\bar\partial^*\sigma_1,\bar\partial^*\sigma_1\rangle_{L^2}.$$

The "basic estimate" for the $\bar\partial$-Neumann problem therefore implies (in the odd or even case) that there is a positive constant C so that

$$\|\sigma'\|^2_{(1,-\frac{1}{2})} \leq C[\|\eth^{\pm}_{\mathbb{C}}\sigma'\|_{L^2} + \|\sigma'\|^2_{L^2}]; \tag{7}$$

see [4].

To prove an *a priori* estimate in the even case we need to consider σ_0. Let $\mathcal{B}$ be the Bergmann projector on X. We also have the classical estimate

$$\|(\mathrm{Id} - \mathcal{B})\sigma_0\|_{(1,-\frac{1}{2})} \leq C[\|\bar\partial\sigma_0\|^2_{L^2} + \|\sigma_0\|^2_{L^2}]. \tag{8}$$

To handle $\mathcal{B}\sigma_0$ we use the boundary condition, which implies that

$$0 = \mathcal{S}(\sigma_0 \restriction_{bX}) = \mathcal{S}([(\mathrm{Id} - \mathcal{B})\sigma_0] \restriction_{bX}) + \mathcal{S}([\mathcal{B}\sigma_0] \restriction_{bX}).$$

As $\mathcal{S}([\mathcal{B}\sigma_0] \restriction_{bX}) = [\mathcal{B}\sigma_0] \restriction_{bX}$ this implies that

$$[\mathcal{B}\sigma_0] \restriction_{bX} = \mathcal{S}([(\mathrm{Id} - \mathcal{B})\sigma_0] \restriction_{bX}).$$

If $\mathcal{K}$ is the Poisson kernel for $\eth^+$ then

$$\mathcal{B}\sigma_0 = \mathcal{K}\mathcal{S}([(\mathrm{Id} - \mathcal{B})\sigma_0] \restriction_{bX}),$$

which, in turn, shows that

$$\|\mathcal{B}\sigma_0\|_{(1,-\frac{1}{2})} \leq C\|(\mathrm{Id} - \mathcal{B})\sigma_0\|_{(1,-\frac{1}{2})}. \tag{9}$$

Here we use the fact that $\mathcal{K}$ is a continuous map from $L^2(bX; \Lambda^{0,\mathrm{even}} \restriction_{bX})$ to $H_{(1,-\frac{1}{2})}(X; \Lambda^{0,\mathrm{even}})$; see [1].

Combining this estimate with (8) and (6) we obtain the basic *a priori* estimate for the even case:

Lemma 2. *There is a positive constant, C such that if* $\sigma \in \mathcal{C}^\infty(\overline{X}; \Lambda^{0,\mathrm{even}})$, *satisfies* (2)*, then*

$$\|\sigma\|^2_{(1,-\frac{1}{2})} \leq C[\|\eth^{+}_{\mathbb{C}}\sigma\|^2_{L^2} + \|\sigma\|^2_{L^2}]. \tag{10}$$

To obtain an analogous result for the odd case we need to estimate σ_1 in terms of $\|\eth^-\sigma_1\|_{L^2}$. We use the method employed in [5].

Lemma 3. *Suppose that* $\sigma_1 \in \mathcal{C}^\infty(\overline{X}; \Lambda^{0,1})$ *satisfies* (3)*; write* $\sigma_1 = \sigma_{10} + f\bar\partial\rho$, *where* $f = \bar\partial\rho\rfloor\sigma_1$. *There is a positive constant* C, *independent of* σ_1 *so that*

$$\|\sigma_{10}\|^2_{(1,-\frac{1}{2})} + \|f\|^2_{H^1} \leq C[\|\sigma_1\|^2_{L^2} + \|\eth^-\sigma_1\|^2_{L^2}]. \tag{11}$$

Proof. As σ_{10} satisfies the $\bar\partial$-Neumann condition, it satisfies the classical $\frac{1}{2}$-estimate,

$$\|\sigma_{10}\|^2_{(1,-\frac{1}{2})} \leq C[\|\eth^{-}_{\mathbb{C}}\sigma_{10}\|^2_{L^2} + \|\sigma_{10}\|^2_{L^2}]. \tag{12}$$

To prove (11) we first show that $\|\eth^-_{\mathbb{C}}(f\bar\partial\rho)\|_{L^2}$ bounds the H^1-norm of f and then handle the cross terms which arise in the computation of $\|\eth^-_{\mathbb{C}}(\sigma_{10} + f\bar\partial\rho)\|^2_{L^2}$.

Let W denote the unique $(1,0)$-vector field, defined in a neighborhood of bX such that W annihilates the orthocomplement of $\partial\rho$ and satisfies

$$\partial\rho(W) = 1.$$

We cover bX by neighborhoods $\{U_j\}$ such that in each U_j there is an orthogonal basis for $T^{1,0}X$ of the form $\{W, Z_1, \dots, Z_n\}$. Let $\{\partial\rho, \omega_1, \dots, \omega_n\}$ denote the dual basis of $(1,0)$-forms.

As $f\restriction_{bX}$ satisfies a global boundary condition, care is required in the use of partitions of unity. In a fixed neighborhood U_l we can write

$$\sigma_{10} = \sum_{j=1}^{n} a_j \bar{\omega}_j,$$

we then have the formulæ

$$\begin{aligned}\bar{\partial}\sigma_{10} &= \sum_{j\neq k} \overline{Z}_k a_j \bar{\omega}_k \wedge \bar{\omega}_j + \sum_j [a_j \bar{\partial}\bar{\omega}_j + \overline{W} a_j \bar{\partial}\rho \wedge \bar{\omega}_j] \\ \bar{\partial}^*\sigma_{10} &= -\sum_{j\neq k} [Z_j a_j + c_j a_j].\end{aligned} \tag{13}$$

Here $\{c_j\}$ are smooth functions. We also have

$$\begin{aligned}\bar{\partial}(f\bar{\partial}\rho) &= \sum_{j=1}^{n} \overline{Z}_j f \bar{\omega}_j \wedge \bar{\partial}\rho \\ \bar{\partial}^*(f\bar{\partial}\rho) &= -Wf + cf.\end{aligned} \tag{14}$$

Here c is a smooth function; the second formula holds in a neighborhood of bX.

It suffices to assume that f is supported in a small neighborhood of bX. If ψ is a smooth function with support in U_l then a simple integration by parts shows that

$$\begin{aligned}\|\bar{\partial}(\psi f\bar{\partial}\rho)\|^2_{L^2} &= \sum_{j=1}^{n} \int_X |\overline{Z}_j(\psi f)|^2 dV \\ &= \sum_{j=1}^{n} \int_X |Z_j(\psi f)|^2 dV + \operatorname{Re}\int_X L_0(\psi f)\overline{(\psi f)} dV,\end{aligned} \tag{15}$$

where L_0 is a smooth vector field defined on $\overline{U}_l$. Thus we see that

$$\|\bar{\partial}(\psi f\bar{\partial}\rho)\|^2_{L^2} = \frac{1}{2}\left[\sum_{j=1}^{n} \int_X [|\overline{Z}_j(\psi f)|^2 + |Z_j(\psi f)|^2 dV] + \operatorname{Re}\int_X L_0\psi f\overline{(\psi f)} dV\right]. \tag{16}$$

On the other hand

$$\|\bar{\partial}^*(f\bar{\partial}\rho)\|^2_{L^2} = \int_X |Wf|^2 dV + \operatorname{Re}\int_X L_0' f\bar{f} dV,$$

where L_0' is another smooth first order differential operator. Integrating by parts in this formula gives

$$\int_X |Wf|^2 dV = \int_X |\overline{W} f|^2 dV + \operatorname{Re}\int_X L_0'' f \bar{f} dV + \int_{bX} (\sigma(W, d\rho) f \overline{W} \bar{f} - \sigma(\overline{W}, d\rho) f W \bar{f}) dS, \tag{17}$$

again L_0'' is a smooth first order operator. The vector field $W = N - iT$ where N is a outward pointing normal vector and T is tangent to bX. Since $\sigma(d\rho, W) = \sigma(\overline{W}, d\rho) = 1$, the boundary term in (17) can be rewritten as

$$2i \int_{bX} f T \bar{f} dS.$$

The Toeplitz operator $\mathcal{S} - iT\mathcal{S}$ has a positive definite symbol of positive order in the Heisenberg calculus; see [3]. In light of the fact that $\mathcal{S} f = f$, this implies that there is a positive constant C so that

$$-C\| f \restriction_{bX} \|_{L^2}^2 \leq 2i \int_{bX} f T \bar{f} dS. \tag{18}$$

Combining this with (17) shows that there is a positive constant C so that

$$\frac{1}{2}\int_X [|Wf|^2 + |\overline{W} f|^2 + \operatorname{Re}(f L_0'' \bar{f})] dV - \|f\|_{(1,-\frac{1}{2})} \leq C\|\bar{\partial}^*(f\bar{\partial}\rho)\|_{L^2}^2. \tag{19}$$

To prove this estimate we did not use a partition of unity. Combining (19) with (16), summed over a partition of unity, gives the estimate

$$\|f\|_{H^1}^2 + \operatorname{Re}\int_X f L_0 \bar{f} - \|f\|_{(1,-\frac{1}{2})} \leq C[\|\bar{\partial}(f\bar{\partial}\rho)\|_{L^2}^2 + \|\bar{\partial}^*(f\bar{\partial}\rho)\|_{L^2}^2 + \|f\bar{\partial}\rho\|_{L^2}^2], \tag{20}$$

for a positive constant C. Applying the Cauchy-Schwarz inequality and standard interpolation inequalities, shows that there is a positive constant C so that

$$\|f\|_{H^1}^2 \leq C[\|\bar{\partial}(f\bar{\partial}\rho)\|_{L^2}^2 + \|\bar{\partial}^*(f\bar{\partial}\rho)\|_{L^2}^2 + \|f\bar{\partial}\rho\|_{L^2}^2], \tag{21}$$

To finish we need to show that the cross terms are of lower order. Supposing that σ_{10} is supported in a chart U_l we have the formula

$$\langle \bar{\partial}\sigma_{10}, \bar{\partial}(f\bar{\partial}\rho)\rangle + \langle \bar{\partial}^*\sigma_{10}, \bar{\partial}^*(f\bar{\partial}\rho)\rangle = \operatorname{Re}\left[\int_X \left(\sum_{j=1}^n [Z_j a_j \overline{Wf} - \overline{W} a_j Z_j \bar{f}] + \sum a_j L_j' \bar{f}\right) dV\right], \tag{22}$$

with $\{L'_j\}$ smooth first order operators. Integrating by parts in the first sum on the right-hand side gives,

$$\operatorname{Re}\left[\int_X \sum_{j=1}^{n}[Z_j a_j \overline{Wf} - \overline{W} a_j Z_j \bar{f}]dV\right] = \operatorname{Re}\sum_{j=1}^{n}\Bigg[\int_X a_j L''_j \bar{f} dV + \int_{bX} a_j Z_j \bar{f} dS\Bigg]. \tag{23}$$

In light of the fact that $\bar\partial_b f \restriction_{bX}= 0$, the boundary term in (23) is zero. Applying the Cauchy-Schwarz inequality we easily combine (12), (21) and (23) to complete the proof of (11). □

The *a priori* estimate for the odd Dirac operator with boundary condition defined by $\mathcal{R}^-$ is summarized in the following lemma.

Lemma 4. *Suppose that* $\sigma \in \mathcal{C}^\infty(\overline{X}; \Lambda^{0,\text{odd}})$ *satisfies* (3) *and* $\sigma = \sigma_{10} + f\bar\partial\rho + \sigma'$, *as in Lemma 3. Then there is a positive constant C so that*

$$\|\sigma'\|^2_{(1,-\frac{1}{2})} + \|\sigma_{10}\|^2_{(1,-\frac{1}{2})} + \|f\|^2_{H^1} \leq C[\|\eth^-_{\mathbb{C}}\sigma\|^2_{L^2} + \|\sigma\|^2_{L^2}] \tag{24}$$

3. The kernels of $\eth^\pm_{\mathbb{C}}$

The estimates (24) and (10) imply that the closures of $\eth^\pm_{\mathbb{C}}$ on the domains defined by $\mathcal{R}^\pm$ have compact resolvent and therefore closed ranges and finite dimensional kernels. Higher norm estimates can be derived exactly as for the usual $\bar\partial$-Neumann problem. We will return to this in a later publication. Note that such estimates imply that the kernels of $\eth^\pm_{\mathbb{C}}$ are contained in $\mathcal{C}^\infty(\overline{X}; \oplus\Lambda^{0,q})$. For $q > 0$ we let $\mathcal{H}^{0,q}_{\bar\partial}(X)$ denote the finite dimensional vector space of $\bar\partial$-Neumann harmonic $(0, q)$-forms:

$$\mathcal{H}^{0,q}_{\bar\partial}(X) = \{\omega \in \mathcal{C}^\infty(\overline{X}; \Lambda^{0,q}) : \bar\partial\omega = 0, \quad \bar\partial^*\omega = 0, \quad \bar\partial\rho\rfloor\omega \restriction_{bX}= 0\}.$$

It follows easily from (4) that

$$\ker \eth^+_{\mathbb{C}} = \bigoplus_{j=1}^{k} \mathcal{H}^{0,2j}_{\bar\partial}(X), \tag{25}$$

with k the greatest integer in $\frac{\dim X}{2}$. Away from degree 1 a similar result is immediate for $\eth^-_{\mathbb{C}}$. The result also holds in degree 1.

Lemma 5. *If* σ_1 *is a smooth* $(0, 1)$*-form which satisfies* (3) *and* $\eth^-_{\mathbb{C}}\sigma_1 = 0$, *then* σ_1 *satisfies the* $\bar\partial$*-Neumann condition.*

Proof. We use the Hodge decomposition defined by the $\bar\partial$-Neumann operator to write

$$\sigma_1 = \bar\partial\bar\partial^*\mathcal{G}^{0,1}_{\bar\partial}\sigma_1 + \mathcal{P}^{0,1}\sigma_1.$$

The $\bar{\partial}^*\bar{\partial}\mathcal{G}_{\bar{\partial}}^{0,1}\sigma_1$ term is absent because $\bar{\partial}^*\bar{\partial}\mathcal{G}_{\bar{\partial}}^{0,1}\sigma_1 = \bar{\partial}^*\mathcal{G}_{\bar{\partial}}^{0,1}\bar{\partial}\sigma_1$, and $\bar{\partial}\sigma_1 = 0$. The second term on the right-hand side satisfies the $\bar{\partial}$-Neumann condition, so it suffices to show that the first term is zero.

Let $a = \bar{\partial}^*\mathcal{G}_{\bar{\partial}}^{0,1}\sigma_1$, and write

$$a \restriction_{bX} = \alpha_0 + \alpha_1,$$

where $\alpha_0 = \mathcal{S}a \restriction_{bX}$. Let a_0 be the homomorphic extension of α_0 and $a_1 = a - a_0$. We integrate by parts to obtain

$$\begin{aligned} \langle\bar{\partial}a, \bar{\partial}a\rangle_{L^2} &= \langle\bar{\partial}a_1, \bar{\partial}a\rangle_{L^2} \\ &= \langle a_1, \bar{\partial}^*\bar{\partial}a\rangle_{L^2} + \langle\sigma(\bar{\partial}, dr)a_1, \bar{\partial}a\rangle_{L^2(bX)}. \end{aligned} \tag{26}$$

Recall that $\bar{\partial}^*\bar{\partial}a = 0$, and, therefore, the first term on the right-hand side in (26) vanishes. As

$$\langle\sigma(\bar{\partial}, dr)a_1, \bar{\partial}a\rangle_{L^2(bX)} = \langle a_1, \sigma(\bar{\partial}^*, dr)\bar{\partial}a\rangle_{L^2(bX)},$$

and $\sigma(\bar{\partial}^*, dr)\bar{\partial}a \restriction_{bX}$ belongs to the range of $\mathcal{S}$, whereas $a_1 \restriction_{bX}$ is perpindicular to the range of $\mathcal{S}$, the second term also vanishes, and therefore $\bar{\partial}a = 0$. □

Using this lemma and (4) we obtain (modulo the smoothness of $\ker\eth_{\mathbb{C}}^-$) that if $\dim X$ is even then

$$\ker\eth_{\mathbb{C}}^- = \bigoplus_{j=1}^{k}\mathcal{H}_{\bar{\partial}}^{0,2j-1}(X). \tag{27}$$

and, if $\dim X$ is odd, then

$$\ker\eth_{\mathbb{C}}^- = \bigoplus_{j=0}^{k}\mathcal{H}_{\bar{\partial}}^{0,2j+1}(X). \tag{28}$$

To show that the closure of $\eth_{\mathbb{C}}^+$ is a Fredholm operator and compute its index, it remains to show that the adjoint of the closure of $\eth_{\mathbb{C}}^+$ is the closure of $\eth_{\mathbb{C}}^-$. Once the higher norm estimates for $\eth_{\mathbb{C}}^-$ are established, then this result follows exactly as for the $\bar{\partial}$-Neumann problem. One shows that the range of $(\eth_{\mathbb{C}}^-)^*\eth_{\mathbb{C}}^- + \mathrm{Id}$, restricted to the smooth elements in its domain, is dense in L^2. This implies that the domain $(\eth_{\mathbb{C}}^+)^*$ equals the closure of the domain of $\eth_{\mathbb{C}}^-$.

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