

# Subelliptic $\text{Spin}_{\mathbb{C}}$ Dirac operators, II

## Basic Estimates

Charles L. Epstein\*  
Department of Mathematics  
University of Pennsylvania

November 2, 2007: 2nd revised version

*This paper dedicated to Peter D. Lax  
on the occasion of his Abel Prize.*

### Abstract

We assume that the manifold with boundary,  $X$ , has a  $\text{Spin}_{\mathbb{C}}$ -structure with spinor bundle  $\mathcal{S}$ . Along the boundary, this structure agrees with the structure defined by an infinite order integrable almost complex structure and the metric is Kähler. In this case the  $\text{Spin}_{\mathbb{C}}$ -Dirac operator  $\bar{d}$  agrees with  $\bar{\partial} + \bar{\partial}^*$  along the boundary. The induced CR-structure on  $bX$  is integrable and either strictly pseudoconvex or strictly pseudoconcave. We assume that  $E \rightarrow X$  is a complex vector bundle, which has an infinite order integrable complex structure along  $bX$ , compatible with that defined along  $bX$ . In this paper we use boundary layer methods to prove subelliptic estimates for the twisted  $\text{Spin}_{\mathbb{C}}$ -Dirac operator acting on sections on  $\mathcal{S} \otimes E$ . We use boundary conditions that are modifications of the classical  $\bar{\partial}$ -Neumann condition. These results are proved by using the extended Heisenberg calculus.

### Introduction

Let  $X$  be an even dimensional manifold with a  $\text{Spin}_{\mathbb{C}}$ -structure, see [11]. A compatible choice of metric,  $g$ , defines a  $\text{Spin}_{\mathbb{C}}$ -Dirac operator,  $\bar{d}$  which acts on sections of the bundle of complex spinors,  $\mathcal{S}$ . This bundle splits as a direct sum  $\mathcal{S} = \mathcal{S}^e \oplus \mathcal{S}^o$ . The metric on  $TX$  induces a metric on the bundle of spinors. We let  $\langle \sigma, \sigma \rangle_g$  denote

---

\*Keywords:  $\text{Spin}_{\mathbb{C}}$  Dirac operator, index, subelliptic boundary value problem,  $\bar{\partial}$ -Neumann condition, boundary layer, Heisenberg calculus, extended Heisenberg calculus. Research partially supported by NSF grants DMS99-70487 and DMS02-03795 and the Francis J. Carey term chair. E-mail: cle@math.upenn.edu

the pointwise inner product. This, in turn, defines an inner product on the space of sections of  $\mathcal{S}$ , by setting:

$$\langle \sigma, \sigma \rangle_X = \int_X \langle \sigma, \sigma \rangle_g dV_g$$

If  $X$  has an almost complex structure, then this structure defines a  $\text{Spin}_{\mathbb{C}}$ -structure, see [4]. If the complex structure is integrable, then the bundle of complex spinors is canonically identified with  $\bigoplus_{q \geq 0} \Lambda^{0,q}$ . We use the notation

$$\Lambda^e = \bigoplus_{q=0}^{\lfloor \frac{n}{2} \rfloor} \Lambda^{0,2q} \quad \Lambda^o = \bigoplus_{q=0}^{\lfloor \frac{n-1}{2} \rfloor} \Lambda^{0,2q+1}. \quad (1)$$

If the metric is Kähler, then the  $\text{Spin}_{\mathbb{C}}$  Dirac operator is given by

$$\bar{\partial} = \bar{\partial} + \bar{\partial}^*.$$

Here  $\bar{\partial}^*$  denotes the formal adjoint of  $\bar{\partial}$  defined by the metric. This operator is called the Dolbeault-Dirac operator by Duistermaat, see [4]. If the metric is Hermitian, though not Kähler, then

$$\bar{\partial}_{\mathbb{C}} = \bar{\partial} + \bar{\partial}^* + \mathcal{M}_0,$$

with  $\mathcal{M}_0$  a homomorphism carrying  $\Lambda^e$  to  $\Lambda^o$  and vice versa. It vanishes at points where the metric is Kähler. It is customary to write  $\bar{\partial} = \bar{\partial}^e + \bar{\partial}^o$  where

$$\bar{\partial}^e : \mathcal{C}^\infty(X; \mathcal{S}^e) \longrightarrow \mathcal{C}^\infty(X; \mathcal{S}^o),$$

and  $\bar{\partial}^o$  is the formal adjoint of  $\bar{\partial}^e$ .

If  $X$  has a boundary, then the kernels and cokernels of  $\bar{\partial}^{e0}$  are generally infinite dimensional. To obtain a Fredholm operator we need to impose boundary conditions. In this instance, there are no local boundary conditions for  $\bar{\partial}^{e0}$  that define elliptic problems. Starting with the work of Atiyah, Patodi and Singer, the basic boundary value problems for Dirac operators on manifolds with boundary have been defined by classical pseudodifferential projections acting on the sections of the spinor bundle restricted to the boundary. In this paper we analyze *subelliptic* boundary conditions for  $\bar{\partial}^{e0}$  obtained by modifying the classical  $\bar{\partial}$ -Neumann and dual  $\bar{\partial}$ -Neumann conditions. The  $\bar{\partial}$ -Neumann conditions on a strictly pseudoconvex manifold allow for an infinite dimensional null space in degree 0 and, on a strictly pseudoconcave manifold, in degree  $n - 1$ . We modify these boundary conditions by using generalized Szegő projectors, in the appropriate degrees, to eliminate these infinite dimensional spaces.

In this paper we prove the basic analytic results needed to do index theory for these boundary value problems. To that end, we compare the projections defining the subelliptic boundary conditions with the Calderon projector and show that, in a certain sense, these projections are relatively Fredholm. We should emphasize at the outset that these projections are not relatively Fredholm in the usual sense of say Fredholm pairs in a Hilbert space, used in the study of elliptic boundary value problems. Nonetheless, we can use our results to obtain a formula for a parametrix for these subelliptic boundary value problems that is precise enough to prove, among other things, higher norm estimates. This formula is related to earlier work of Greiner and Stein, and Beals and Stanton, see [7, 2]. We use the extended Heisenberg calculus introduced in [6]. Similar classes of operators were also introduced by Greiner and Stein, Beals and Stanton as well as Taylor, see [7, 2, 1, 14]. The results here and their applications in [5] suggest that the theory of Fredholm pairs has an extension to subspaces of  $\mathcal{C}^\infty$  sections where the relative projections satisfy appropriate tame estimates.

In this paper  $X$  is a  $\text{Spin}_\mathbb{C}$ -manifold with boundary. The  $\text{Spin}_\mathbb{C}$  structure along the boundary arises from an almost complex structure that is integrable to infinite order. This means that the induced CR-structure on  $bX$  is integrable and the Nijenhuis tensor vanishes to infinite order along the boundary. We generally assume that this CR-structure is either strictly pseudoconvex (or pseudoconcave). When we say that “ $X$  is a strictly pseudoconvex (or pseudoconcave) manifold,” this is what we mean. We usually treat the pseudoconvex and pseudoconcave cases in tandem. When needed, we use a subscript  $+$  to denote the pseudoconvex case and  $-$ , the pseudoconcave case.

Indeed, as all the important computations in this paper are calculations in Taylor series along the boundary, it suffices to consider the case that the boundary of  $X$  is in fact a hypersurface in a complex manifold, and we often do so. We suppose that the boundary of  $X$  is the zero set of a function  $\rho$  such that

1.  $d\rho \neq 0$  along  $bX$ .
2.  $\partial\bar{\partial}\rho$  is positive definite along  $bX$ . Hence  $\rho < 0$ , if  $X$  is strictly pseudoconvex and  $\rho > 0$ , if  $X$  is strictly pseudoconcave.
3. The length of  $\bar{\partial}\rho$  in the metric with Kähler form  $-i\partial\bar{\partial}\rho$  is  $\sqrt{2}$  along  $bX$ . This implies that the length  $d\rho$  is 2 along  $bX$ .

If  $bX$  is a strictly pseudoconvex or pseudoconcave hypersurface, with respect to the infinite order integrable almost complex structure along  $bX$ , then a defining function  $\rho$  satisfying these conditions can always be found.

The Hermitian metric on  $X$ , near to  $bX$ , is defined by  $\partial\bar{\partial}\rho$ . If the almost complex structure is integrable, then this metric is Kähler. This should be contrasted to

the usual situation when studying boundary value problems of APS type: here one usually assumes that the metric is a product in a neighborhood of the boundary, with the boundary a totally geodesic hypersurface. Since we are interested in using the subelliptic boundary value problems as a tool to study the complex structure of  $X$  and the CR-structure of  $bX$ , this would not be a natural hypothesis. Instead of taking advantage of the simplifications that arise from using a product metric, we use the simplifications that result from using Kähler coordinates.

Let  $\mathcal{P}^{eo}$  denotes the Calderon projectors and  $\mathcal{R}'^{eo}$ , the projectors defining the subelliptic boundary value problems on the even (odd) spinors, respectively. These operators are defined in [5] as well as in Lemmas 4 and 5. The main objects of study in this paper are the operators:

$$\mathcal{T}'^{eo} = \mathcal{R}'^{eo}\mathcal{P}^{eo} + (\text{Id} - \mathcal{R}'^{eo})(\text{Id} - \mathcal{P}^{eo}). \quad (2)$$

These operators are elements of the extended Heisenberg calculus. If  $X$  is strictly pseudoconvex, then  $\mathcal{T}'^{eo}$  is an elliptic operator, in the classical sense, away from the positive contact direction. Along the positive contact direction, most of its principal symbol vanishes. If instead we compute its principal symbol in the Heisenberg sense, we find that this symbol has a natural block structure:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \quad (3)$$

As an element of the Heisenberg calculus  $A_{ij}$  is a symbol of order  $2 - (i + j)$ . The inverse has the identical block structure

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad (4)$$

where the order of  $B_{ij}$  is  $(i + j) - 2$ . The principal technical difficulty that we encounter is that the symbol of  $\mathcal{T}'^{eo}$  along the positive contact direction could, in principle, depend on higher order terms in the symbol of  $\mathcal{P}^{eo}$  as well as the geometry of  $bX$  and its embedding as the boundary of  $X$ . In fact, the Heisenberg symbol of  $\mathcal{T}'^{eo}$  is determined by the principal symbol of  $\mathcal{P}^{eo}$  and depends in a very simple way on the geometry of  $bX \hookrightarrow X$ . It requires some effort to verify this statement and explicitly compute the symbol. Another important result is that the leading order part of  $B_{22}$  vanishes. This allows the deduction of the classical sharp anisotropic estimates for these modifications of the  $\bar{\partial}$ -Neumann problem from our results. Analogous remarks apply to strictly pseudoconcave manifolds with the two changes that the difficulties occur along the negative contact direction, and the block structure depends on the parity of the dimension.

As it entails no additional effort, we work in somewhat greater generality and consider the “twisted”  $\text{Spin}_{\mathbb{C}}$  Dirac operator. To that end, we let  $E \rightarrow X$  denote a complex vector bundle and consider the Dirac operator acting on sections of  $\mathcal{S} \otimes E$ . The bundle  $E$  is assumed to have an almost complex structure near to  $bX$ , that is infinite order integrable along  $bX$ . We assume that this almost complex structure is compatible with that defined on  $X$  along  $bX$ . By this we mean  $E \rightarrow X$  defines, along  $bX$ , an infinite order germ of a holomorphic bundle over the infinite order germ of the holomorphic manifold. We call such a bundle a complex vector bundle *compatible* with  $X$ . When necessary for clarity, we let  $\bar{\partial}_E$  denote the  $\bar{\partial}$ -operator acting on sections of  $\Lambda^{0,q} \otimes E$ . A Hermitian metric is fixed on the fibers of  $E$  and  $\bar{\partial}_E^*$  denotes the adjoint operator. Along  $bX$ ,  $\bar{\partial}_E = \bar{\partial}_E + \bar{\partial}_E^*$ . In most of this paper we simplify the notation by suppressing the dependence on  $E$ .

We first recall the definition of the Calderon projector in this case, which is due to Seeley. We follow the discussion in [3]. We then examine its symbol and the symbol of  $\mathcal{T}_{\pm}^{\text{eo}}$  away from the contact directions. Next we compute the symbol in the appropriate contact direction. We see that  $\mathcal{T}_{\pm}^{\text{eo}}$  is a graded elliptic system in the extended Heisenberg calculus. Using the parametrix for  $\mathcal{T}_{\pm}^{\text{eo}}$  we obtain parametrices for the boundary value problems considered here as well as those introduced in [5]. Using the parametrices we prove subelliptic estimates for solutions of these boundary value problems formally identical to the classical  $\bar{\partial}$ -Neumann estimates of Kohn. We are also able to characterize the adjoints of the graph closures of the various operators as the graph closures of the formal adjoints.

#### Acknowledgments

Boundary conditions similar to those considered in this paper were first suggested to me by Laszlo Lempert. I would like to thank John Roe for some helpful pointers on the  $\text{Spin}_{\mathbb{C}}$  Dirac operator.

## 1 The extended Heisenberg Calculus

The main results in this paper rely on the computation of the symbol of an operator built out of the Calderon projector and a projection operator in the Heisenberg calculus. This operator belongs to the extended Heisenberg calculus, as defined in [6]. While we do not intend to review this construction in detail, we briefly describe the different symbol classes within a single fiber of the cotangent bundle. This suffices for our purposes as all of our symbolic computations are principal symbol computations, which are, in all cases, localized to a single fiber.

Each symbol class is defined by a compactification of the fibers of  $T^*Y$ . In our applications,  $Y$  is a contact manifold of dimension  $2n - 1$ . Let  $L$  denote the contact line within  $T^*Y$ . We assume that  $L$  is oriented and  $\theta$  is a global, positive section of

$L$ . According to Darboux's theorem, there are coordinates  $(y_0, y_1, \dots, y_{2(n-1)})$  for a neighborhood  $U$  of  $p \in Y$ , so that

$$\theta|_U = dy_0 + \frac{1}{2} \sum_{j=1}^{n-1} [y_j dy_{j+n} - y_{j+n} dy_j], \quad (5)$$

Let  $\eta$  denote the local fiber coordinates on  $T^*Y$  defined by the trivialization

$$\{dy_0, \dots, dy_{2(n-1)}\}.$$

We often use the splitting  $\eta = (\eta_0, \eta')$ . In the remainder of this section we do essentially all our calculations at the point  $p$ . As such coordinates can be found in a neighborhood of any point, and in light of the invariance results established in [6], these computations actually cover the general case.

### 1.1 The compactifications of $T^*Y$

We define three compactifications of the fibers of  $T^*Y$ . The first is the standard radial compactification,  ${}^R\overline{T^*Y}$ , defined by adding one point at infinity for each orbit of the standard  $\mathbb{R}_+$ -action,  $(y, \eta) \mapsto (y, \lambda\eta)$ . Along with  $y$ , standard polar coordinates in the  $\eta$ -variables define local coordinates near  $b^R\overline{T^*Y}$ :

$$r_R = \frac{1}{|\eta|}, \quad \omega_j = \frac{\eta_j}{|\eta|}, \quad (6)$$

with  $r_R$  a smooth defining function for  $b^R\overline{T^*Y}$ .

To define the Heisenberg compactification we first need to define a parabolic action of  $\mathbb{R}_+$ . Let  $T$  denote the vector field defined by the conditions  $\theta(T) = 1, i_T d\theta = 0$ . As usual  $i_T$  denotes interior product with the vector field  $T$ . Let  $H^*$  denote the subbundle of  $T^*Y$  consisting of forms that annihilate  $T$ . Clearly  $T^*Y = L \oplus H^*$ , let  $\pi_L \oplus \pi_{H^*}$  denote the bundle projections defined by this splitting. The parabolic action of  $\mathbb{R}_+$  is defined by

$$(y, \eta) \mapsto (y, \lambda\pi_{H^*}(y, \eta) + \lambda^2\pi_T(y, \eta)) \quad (7)$$

In the Heisenberg compactification we add one point at infinity for each orbit under this action. A smooth defining function for the boundary is given by

$$r_H = [|\pi_{H^*}(y, \eta)|^4 + |\pi_T(y, \eta)|^2]^{-\frac{1}{4}}. \quad (8)$$

In [6] it is shown that the smooth structure of  ${}^H\overline{T^*Y}$  depends only on the contact structure, and not the choice of contact form.

In the fiber over  $y = 0$ ,  $r_H = [|\eta'|^4 + |\eta_0|^2]^{-\frac{1}{4}}$ . Coordinates near the boundary in the fiber over  $y = 0$  are given by

$$r_H, \sigma_0 = \frac{\eta_0}{[|\eta'|^4 + |\eta_0|^2]^{\frac{1}{2}}}, \sigma_j = \frac{\eta_j}{[|\eta'|^4 + |\eta_0|^2]^{\frac{1}{4}}}, \quad j = 1, \dots, 2(n-1). \quad (9)$$

The extended Heisenberg compactification can be defined by performing a blowup of either the radial or the Heisenberg compactification. Since we need to lift classical symbols to the extended Heisenberg compactification, we describe the fiber of  ${}^{eH}\overline{T^*Y}$  in terms of a blowup of  ${}^R\overline{T^*Y}$ . In this model we parabolically blowup the boundary of contact line, i.e., the boundary of the closure of  $L$  in  ${}^R\overline{T^*Y}$ . The conormal bundle to the  $b^R\overline{T^*Y}$  defines the parabolic direction. The fiber of the compactified space is a manifold with corners, having three hypersurface boundary components. The two boundary points of  $\overline{L}$  become  $2(n-1)$  dimensional disks. These are called the upper and lower Heisenberg faces. The complement of  $b\overline{L}$  lifts to a cylinder, diffeomorphic to  $(-1, 1) \times S^{2n-3}$ , which was call the ‘‘classical’’ face. Let  $r_{e\pm}$  be defining functions for the upper and lower Heisenberg faces and  $r_c$  a defining function for the classical face. From the definition we see that coordinates near the Heisenberg faces, in the fiber over  $y = 0$ , are given by

$$r_{eH} = [r_R^2 + |\omega'|^4]^{\frac{1}{4}}, \quad \tilde{\sigma}_j = \frac{\omega_j}{r_{eH}}, \quad \text{for } j = 1, \dots, 2n-2, \quad (10)$$

with  $r_{eH}$  a smooth defining function for the Heisenberg faces. In order for an arc within  $T^*Y$  to approach either Heisenberg face it is necessary that, for any  $\epsilon > 0$ ,

$$|\eta'| \leq \epsilon |\eta_0|,$$

as  $|\eta|$  tends to infinity. Indeed, for arcs that terminate on the interior of a Heisenberg face the ratio  $\eta'/\sqrt{|\eta_0|}$  approaches a limit. If  $\eta_0 \rightarrow +\infty (-\infty)$ , then the arc approaches the upper (lower) parabolic face. In the interior of the Heisenberg faces we can use  $[|\eta_0|]^{-\frac{1}{2}}$  as a defining function.

## 1.2 The symbol classes and pseudodifferential operators

The symbols of order zero are defined in all cases as the smooth functions on the compactified cotangent space:

$$S_R^0 = \mathcal{C}^\infty({}^R\overline{T^*Y}), \quad S_H^0 = \mathcal{C}^\infty({}^H\overline{T^*Y}), \quad S_{eH}^0 = \mathcal{C}^\infty({}^{eH}\overline{T^*Y}). \quad (11)$$

In the classical and Heisenberg cases there is a single order parameter for symbols, the symbols of order  $m$  are defined as

$$S_R^m = r_R^{-m} \mathcal{C}^\infty({}^R\overline{T^*Y}), \quad S_H^m = r_H^{-m} \mathcal{C}^\infty({}^H\overline{T^*Y}). \quad (12)$$

In the extended Heisenberg case there are three symbolic orders  $(m_c, m_+, m_-)$ , the symbol classes are defined by

$$S_{eH}^{m_c, m_+, m_-} = r_c^{-m_c} r_{e_+}^{-m_+} r_{e_-}^{-m_-} S_{eH}^0. \quad (13)$$

If  $a$  is a symbol belonging to one of the three classes above, and  $\varphi$  is a smooth function with compact support in  $U$ , then the Weyl quantization rule is used to define the localized operator  $M_\varphi a(X, D)M_\varphi$  :

$$M_\varphi a(X, D)M_\varphi f = \int_{R^{2n-1}} \int_{R^{2n-1}} \varphi(y) a\left(\frac{y+y'}{2}, \eta\right) \varphi(y') f(y') e^{i(\eta, y-y')} \frac{dy' d\eta}{(2\pi)^{2n-1}}. \quad (14)$$

The operator  $M_\varphi$  is multiplication by  $\varphi$ . As usual, the Schwartz kernel of  $a(X, D)$  is assumed to be smooth away from the diagonal.

We denote the classes of pseudodifferential operators defined by the symbol classes  $S_R^m, S_H^m, S_{eH}^{m_c, m_+, m_-}$  by  $\Psi_R^m, \Psi_H^m, \Psi_{eH}^{m_c, m_+, m_-}$ , respectively. As usual, the leading term in the Taylor expansion of a symbol along the boundary can be used to define a principal symbol. Because the defining functions for the boundary components are only determined up to multiplication by a positive function, invariantly, these symbols are sections of line bundles defined on the boundary. We let  ${}^R\sigma_m(A), {}^H\sigma_m(A)$  denote the principal symbols for the classical and Heisenberg pseudodifferential operators of order  $m$ . In each of these cases, the principal symbol uniquely determines a function on the cotangent space, homogeneous with respect to the appropriate  $\mathbb{R}_+$  action. An extended Heisenberg operator has three principal symbols, corresponding to the three boundary hypersurfaces of  ${}^{eH}\overline{T^*Y}$ . For an operator with orders  $(m_c, m_+, m_-)$  they are denoted by  ${}^{eH}\sigma_{m_c}^c(A), {}^{eH}\sigma_{m_+}^+(A), {}^{eH}\sigma_{m_-}^-(A)$ . The classical symbol  ${}^{eH}\sigma_{m_c}^c(A)$  can be represented by a radially homogeneous function defined on  $T^*Y \setminus L$ . The vector field  $T$  defines a splitting to  $T^*Y$  into two half spaces

$$T_\pm^*Y = \{(y, \eta) : \pm\eta(T) > 0\}. \quad (15)$$

The Heisenberg symbols,  ${}^{eH}\sigma_{m_\pm}^\pm(A)$  can be represented by parabolically homogeneous functions defined in the half spaces of  $T_\pm^*Y$ . In most of our computations we use the representations of principal symbols in terms of functions, homogeneous with respect to the appropriate  $\mathbb{R}_+$ -action.

### 1.3 Symbolic composition formulæ

The quantization rule leads to a different symbolic composition rule for each class of operators. For classical operators, the composition of principal symbols is given



by pointwise multiplication: If  $A \in \Psi_R^m$ ,  $B \in \Psi_R^{m'}$ , then  $A \circ B \in \Psi_R^{m+m'}$  and

$${}^R\sigma_{m+m'}(A \circ B)(p, \eta) = {}^R\sigma_m(A)(p, \eta) {}^R\sigma_{m'}(B)(p, \eta). \quad (16)$$

For Heisenberg operators, the composition rule involves a nonlocal operation in the fiber of the cotangent space. If  $A \in \Psi_H^m$ ,  $B \in \Psi_H^{m'}$ , then  $A \circ B \in \Psi_H^{m+m'}$ . For our purposes it suffices to give a formula for  ${}^H\sigma_{m+m'}(A \circ B)(p, \pm 1, \eta')$ ; the symbol is then extended to  $T_p^*Y \setminus H^*$  as a parabolically homogeneous function of degree  $m + m'$ . It extends to  $H^* \setminus \{0\}$  by continuity. On the hyperplanes  $\eta_0 = \pm 1$  the composite symbol is given by

$$\begin{aligned} & {}^H\sigma_{m+m'}(A \circ B)(p, \pm 1, \eta') = \\ & \frac{1}{\pi^{2(n-1)}} \int_{\mathbb{R}^{2(n-1)}} \int_{\mathbb{R}^{2(n-1)}} a_m(\pm 1, u + \eta') b_{m'}(\pm 1, v + \eta') e^{\pm 2i\omega(u,v)} dudv, \end{aligned} \quad (17)$$

where  $\omega = d\theta'$ , the dual of  $d\theta \upharpoonright_{H^*}$ , and

$$a_m(\eta) = {}^H\sigma_m(A)(p, \eta), \quad b_{m'}(\eta) = {}^H\sigma_{m'}(B)(p, \eta).$$

Note that the composed symbols in each half space are determined by the component symbols in that half space. Indeed the symbols that vanish in a half space define an ideal. These are called the upper and lower Hermite ideals. The right hand side of (17) defines two associative products on appropriate classes of functions defined on  $\mathbb{R}^{2(n-1)}$ , which are sometimes denoted by  $a_m \#_{\pm} b_{m'}$ . An operator in  $\Psi_H^m$  is elliptic if and only if the functions  ${}^H\sigma_m(p, \pm 1, \eta')$  are invertible elements, or units, with respect to these algebra structures.

Using the representations of symbols as homogeneous functions, the compositions for the different types of extended Heisenberg symbols are defined using the appropriate formula above: the classical symbols are composed using (16) and the Heisenberg symbols are composed using (17), with  $+$  for  ${}^{eH}\sigma(+)$  and  $-$  for  ${}^{eH}\sigma(-)$ . These formulæ and their invariance properties are established in [6].

The formula in (17) would be of little use, but for the fact that it has an interpretation as a composition formula for a class of operators acting on  $\mathbb{R}^{n-1}$ . The restrictions of a Heisenberg symbol to the hyperplanes  $\eta_0 = \pm 1$  define *isotropic* symbols on  $\mathbb{R}^{2(n-1)}$ . An isotropic symbol is a smooth function on  $\mathbb{R}^{2(n-1)}$  that satisfies symbolic estimates in all variables, i.e.,  $c(\eta')$  is an isotropic symbol of order  $m$  if, for every  $2(n-1)$ -multi-index  $\alpha$ , there is a constant  $C_\alpha$  so that

$$|\partial_{\eta'}^\alpha c(\eta')| \leq C_\alpha (1 + |\eta'|)^{m-|\alpha|}. \quad (18)$$

We split  $\eta'$  into two parts

$$x = (\eta_1, \dots, \eta_{n-1}), \quad \zeta = (\eta_n, \dots, \eta_{2(n-1)}). \quad (19)$$

If  $c$  is an isotropic symbol, then we define two operators acting on  $\mathcal{S}(\mathbb{R}^{n-1})$  by defining the Schwartz kernels of  $c^\pm(X, D)$  to be

$$k_c^\pm(x, x') = \int_{\mathbb{R}^{n-1}} e^{\pm i\langle \xi, x-x' \rangle} c\left(\frac{x+x'}{2}, \xi\right) d\xi. \quad (20)$$

The utility of the formula in (17) is a consequence of the following proposition:

**Proposition 1.** *If  $c_1$  and  $c_2$  are two isotropic symbols, then the complete symbol of  $c_1^\pm(X, D) \circ c_2^\pm(X, D)$  is  $c_1 \sharp_\pm c_2$ , with  $\omega = \sum dx_j \wedge d\xi_j$ . An isotropic operator  $c^\pm(X, D) : \mathcal{S}(\mathbb{R}^{n-1}) \rightarrow \mathcal{S}(\mathbb{R}^{n-1})$  is invertible if and only if  $c(\eta')$  is a unit with respect to the  $\sharp_\pm$  product.*

*Remark 1.* This result appears in essentially this form in [13]. It is related to an earlier result of Rockland.

If  $A$  is a Heisenberg (or extended Heisenberg operator), then the isotropic symbols  ${}^H\sigma_m(A)(p, \pm 1, \eta')$  ( ${}^{eH}\sigma^\pm(A)(p, \pm 1, \eta')$ ) can be quantized using (20). We denote the corresponding operators by  ${}^H\sigma_m(A)(p, \pm)$ , ( ${}^{eH}\sigma(A)(p, \pm)$ ). We call these “the” model operators defined by  $A$  at  $p$ . Often the point of evaluation,  $p$  is fixed and then it is omitted from the notation. The choice of splitting in (19) cannot in general be done globally. Hence the model operators are not, in general, globally defined. What is important to note is that the invertibility of these operators does *not* depend on the choices made to define them. From the proposition it is clear that  $A$  is elliptic in the Heisenberg calculus if and only if the model operators are everywhere invertible. An operator in the extended Heisenberg calculus is elliptic if and only if these model operators are invertible and the classical principal symbol is nonvanishing.

All these classes of operators are easily extended to act between sections of vector bundles. When necessary we indicate this by using, e.g.  $\Psi_R^m(Y; F_1, F_2)$  to denote classical pseudodifferential operators of order  $m$  acting from sections of the bundle  $F_1$  to sections of the bundle  $F_2$ . In this case the symbols take values in  $P^*(\text{hom}(F_1, F_2))$ , where  $P : T^*Y \rightarrow Y$  is the canonical projection. Unless needed for clarity, the explicit dependence on bundles is suppressed.

#### 1.4 Lifting classical symbols to ${}^{eH}\overline{T^*Y}$

We close our discussion of the extended Heisenberg calculus by considering lifts of classical symbols from  ${}^R\overline{T^*Y}$  to  ${}^{eH}\overline{T^*Y}$ . As above, it suffices to consider what happens on the fiber over  $p$ . This fixed point of evaluation is suppressed to simplify the notation. Let  $a(\eta)$  be a classically homogeneous function of degree  $m$ .

The transition from the radial compactification to the extended Heisenberg compactification involves blowing up the points  $(\pm\infty, 0)$  in the fiber of  ${}^R\overline{T^*Y}$ . We need to understand the behavior of  $a$  near these points. Away from  $\eta = 0$ , we can express  $a(\eta) = r_R^{-m} a_0(\omega)$ , where  $a_0$  is a homogeneous function of degree 0. Using the relations in (6) and (10) we see that

$$r_R = r_{eH}^2 \sqrt{1 - |\tilde{\sigma}'|^4}, \quad \omega' = r_{eH} \tilde{\sigma}'. \quad (21)$$

Near  $b\overline{L}$  we can use  $r_R$  and  $\omega'$  as coordinates, where the function  $a$  has Taylor expansions:

$$a_{\pm}(r_R, \omega') = r_R^{-m} a_0(\pm\sqrt{1 - |\omega'|^2}, \omega') \sim r_R^{-m} \sum_{\alpha} a_{\pm}^{(\alpha)} \omega'^{\alpha}. \quad (22)$$

To find the lift, we substitute from (21) into (22) to obtain

$$a(r_{eH}, \tilde{\sigma}') \sim r_{eH}^{-2m} (1 - |\tilde{\sigma}'|^4)^{-\frac{m}{2}} \sum_{\alpha} a_{\pm}^{(\alpha)} r_{eH}^{|\alpha|} \tilde{\sigma}'^{\alpha}. \quad (23)$$

We summarize these computations in a proposition.

**Proposition 2.** *Let  $a(\eta)$  be a classically homogeneous function of order  $m$  with Taylor expansion given in (22). If  $a_{\pm}^{(\alpha)}$  vanish for  $|\alpha| < k_{\pm}$ , then the symbol  $a \in S_R^m$  lifts to define an element of  $S_{eH}^{m, 2m-k_+, 2m-k_-}$ . The Heisenberg principal symbols (as sections of line bundles on the boundary) are given by*

$${}^{eH}a_{m_{\pm}} = r_{eH}^{k_{\pm}-2m} (1 - |\tilde{\sigma}'|^4)^{-\frac{m}{2}} \sum_{|\alpha|=k_{\pm}} a_{\pm}^{(\alpha)} \tilde{\sigma}'^{\alpha}. \quad (24)$$

*Remark 2.* From this proposition it is clear that the Heisenberg principal symbol of the lift of a classical pseudodifferential operator may not be defined by its classical principal symbol. It may depend on lower order terms in the classical symbol.

To compute with the lifted symbols it is more useful to represent them as Heisenberg homogeneous functions. In the computations that follow we only encounter symbols of the form

$$a(\eta) = \frac{h(\eta)}{|\eta|^k}, \quad (25)$$

with  $h(\eta)$  a polynomial of degree  $l$ . In the fiber over  $p$ , the coordinate  $\eta_0$  is parabolically homogeneous of degree 2 whereas the coordinates in  $\eta'$  are parabolically homogeneous of degree 1. Using this observation, it is straightforward to find the representations, as parabolically homogeneous functions, of the Heisenberg principal symbols defined by  $a(\eta)$ . First observe that  $|\eta'|^2/\eta_0$  is parabolically homogeneous

of degree 0, and therefore, in terms of the *parabolic homogeneities* we have the expansion

$$\begin{aligned} \frac{1}{|\eta|^k} &= \frac{1}{|\eta_0|^k} \frac{1}{\left(1 + \frac{|\eta'|^2}{\eta_0^2}\right)^k} \\ &\sim \frac{1}{|\eta_0|^k} \left[ 1 + \sum_{j=1}^{\infty} \frac{c_{k,j}}{|\eta_0|^j} \left(\frac{|\eta'|^2}{|\eta_0|^2}\right)^j \right]. \end{aligned} \quad (26)$$

Thus  $|\eta|^{-k}$  lifts to define a symbol in  $S_{eH}^{-k, -2k, -2k}$ . Note also that only even parabolic degrees appear in this expansion.

We complete the analysis by expressing  $h(\eta)$  as a polynomial in  $\eta_0$  :

$$h(\eta) = \sum_{j=0}^{l'} \eta_0^j h_j(\eta'), \quad (27)$$

here  $h_j$  is a radially homogeneous polynomial of degree  $l - j$ , and  $l' \leq l$ . We assume that  $h_{l'} \neq 0$ . Evidently  $\eta_0^{l'} h_{l'}(\eta')$  is the term with highest parabolic order, and therefore  $h$  lifts to define a parabolic symbol of order  $l' + l$ . Combining these calculations gives the following result:

**Proposition 3.** *If  $h(\eta)$  is a radially homogeneous polynomial of degree  $l$  with expansion given by (27), then  $h(\eta)|\eta|^{-k}$  lifts to define an element of  $S_{eH}^{l-k, l'+l-2k, l'+l-2k}$ . As parabolically homogeneous functions, the Heisenberg principal symbols are*

$$(\pm 1)^{l'} |\eta_0|^{l'-k} h_{l'}(\eta'). \quad (28)$$

*Proof.* The statement about the orders of the lifted symbols follows immediately from (26) and (27). We observe that  $|\eta_0|^{-\frac{1}{2}}$  is a defining function for the upper and lower Heisenberg faces, and  $\eta'/\sqrt{|\eta_0|}$  is parabolically homogeneous of degree 0. As noted, the term in the expansion of  $h(\eta)|\eta|^{-k}$  with highest parabolic degree is that given in (28). We can express it as the leading term in the Taylor series of the lifted symbol along the Heisenberg face as:

$$(\pm 1)^{l'} |\eta_0|^{l'-k} h_{l'}(\eta') = (\pm 1)^{l'} [\sqrt{|\eta_0|}]^{l'+l-2k} h_{l'} \left( \frac{\eta'}{\sqrt{|\eta_0|}} \right). \quad (29)$$

□

Note that the terms in the parabolic expansions of the lift of  $h(\eta)|\eta|^{-k}$  all have the same parity.

## 2 The symbol of the Dirac Operator and its inverse

Let  $X$  be a manifold with boundary,  $Y$  and suppose that  $X$  has a  $\text{Spin}_\mathbb{C}$ -structure and a compatible metric. Let  $\bar{\partial}_E$  denote the twisted  $\text{Spin}_\mathbb{C}$ -Dirac operator and  $\bar{\partial}_E^{\text{eo}}$  its “even” and “odd” parts. Let  $\rho$  be a defining function for  $bX$ . As noted above,  $E \rightarrow X$  is a complex vector bundle with compatible almost complex structure along  $bX$ . The manifold  $X$  can be included into a larger manifold  $\tilde{X}$  in such a way that its  $\text{Spin}_\mathbb{C}$ -structure and Dirac operator extend smoothly to  $\tilde{X}$  and such that the operators  $\bar{\partial}_E^{\text{eo}}$  are invertible, see Chapter 9 of [3]. Let  $Q_E^{\text{eo}}$  denote the inverses of  $\bar{\partial}_E^{\text{eo}}$ . These are classical pseudodifferential operators of order  $-1$ . The existence of an exact inverse just simplifies the presentation a little, a parametrix suffices for our computations.

Let  $r$  denote the operation of restriction of a section of  $\mathcal{S}^{\text{eo}} \otimes E$ , defined on  $\tilde{X}$  to  $X$ , and  $\gamma_\epsilon$  the operation of restriction of a smooth section of  $\mathcal{S}^{\text{eo}} \otimes E$  to  $Y_\epsilon = \{\rho^{-1}(\epsilon)\}$ . We use the convention used in [5]: if  $X$  is strictly pseudoconvex then  $\rho < 0$  on  $X$  and if  $X$  is strictly pseudoconcave then  $\rho > 0$  on  $X$ . We define the operator

$$\tilde{K}_E^{\text{eo}} \stackrel{d}{=} r Q_E^{\text{eo}} \gamma_0^* : \mathcal{C}^\infty(Y; \mathcal{S}^{\text{oe}} \otimes E \upharpoonright_Y) \longrightarrow \mathcal{C}^\infty(X; \mathcal{S}^{\text{eo}} \otimes E). \quad (30)$$

Here  $\gamma_0^*$  is the formal adjoint of  $\gamma_0$ . We recall that, along  $Y$  the symbol  $\sigma_1(\bar{\partial}_E^{\text{eo}}, d\rho)$  defines an isomorphism

$$\sigma_1(\bar{\partial}_E^{\text{eo}}, d\rho) : \mathcal{S}^{\text{eo}} \otimes E \upharpoonright_Y \longrightarrow \mathcal{S}^{\text{oe}} \otimes E \upharpoonright_Y. \quad (31)$$

Composing, we get the usual Poisson operators

$$\mathcal{K}_{E\pm}^{\text{eo}} = \frac{\mp}{i\sqrt{2}} \tilde{K}_E^{\text{eo}} \circ \sigma_1(\bar{\partial}_E^{\text{eo}}, d\rho) : \mathcal{C}^\infty(Y; \mathcal{S}^{\text{eo}} \otimes E \upharpoonright_Y) \longrightarrow \mathcal{C}^\infty(X; \mathcal{S}^{\text{eo}} \otimes E), \quad (32)$$

which map sections of  $\mathcal{S}^{\text{eo}} \otimes E \upharpoonright_Y$  into the nullspace of  $\bar{\partial}_E^{\text{eo}}$ . The factor  $\mp/\sqrt{2}$  is inserted because  $\rho < 0$  on  $X$ , if  $X$  is strictly pseudoconvex, and  $\|d\rho\| = \sqrt{2}$ .

The Calderon projectors are defined by

$$\mathcal{P}_{E\pm}^{\text{eo}} \stackrel{d}{=} \lim_{\mp\epsilon \rightarrow 0^+} \gamma_\epsilon \mathcal{K}_{E\pm}^{\text{eo}} s \text{ for } s \in \mathcal{C}^\infty(Y; \mathcal{S}^{\text{eo}} \otimes E \upharpoonright_Y). \quad (33)$$

The fundamental result of Seeley is that  $\mathcal{P}_{E\pm}^{\text{eo}}$  are classical pseudodifferential operators of order 0. The ranges of these operators are the boundary values of elements of  $\ker \bar{\partial}_{E\pm}^{\text{eo}}$ . Seeley gave a prescription for computing the symbols of these operators using contour integrals, which we do not repeat, as we shall be computing these symbols in detail in the following sections. See [12]

*Remark 3 (Notational remark).* The notation  $\mathcal{P}_\pm$  used in this paper does *not* follow the usual convention in this field. Usually  $\mathcal{P}_\pm$  would refer to the Calderon projectors defined by approaching a hypersurface in a single invertible double from either side. In this case one would have the identity  $\mathcal{P}_+ + \mathcal{P}_- = \text{Id}$ . In our usage,  $\mathcal{P}_+$  refers to the projector for the pseudoconvex side and  $\mathcal{P}_-$  the projector for the pseudoconcave side. With our convention it is not usually true that  $\mathcal{P}_+ + \mathcal{P}_- = \text{Id}$ .

As we need to compute the symbol of  $Q_E^{\text{eo}}$  in some detail, we now consider how to find it. We start with the formally self adjoint operators  $D_E^{\text{eo}} = \bar{\partial}_E^{\text{eo}} \partial_E^{\text{oe}}$ . If  $Q_{E(2)}^{\text{eo}}$  is the inverse of  $D_E^{\text{eo}}$  then

$$Q_E^{\text{eo}} = \bar{\partial}_E^{\text{oe}} Q_{E(2)}^{\text{eo}}. \quad (34)$$

In carefully chosen coordinates, it is a simple matter to get a precise description of symbols of  $\bar{\partial}_E^{\text{eo}}$  and  $Q_{E(2)}^{\text{eo}}$  and thereby the symbols of  $Q_E^{\text{eo}}$ . Throughout this and the following section we repeatedly use the fact that the principal symbol of a classical, Heisenberg or extended Heisenberg pseudodifferential operator is well defined as a (collection of) homogeneous functions on the cotangent bundle. To make these computations tractable it is crucial to carefully normalize the coordinates. At the boundary, there is a complex interplay between the Kähler geometry of  $X$  and the CR-geometry of  $bX$ . For this reason the initial computations are done in a Kähler coordinate system about a fixed point  $p \in bX$ . In order to compute the symbol of the Calderon projector we need to switch to a boundary adapted coordinate system. Finally, to analyze the Heisenberg symbols of  $\mathcal{T}_{E\pm}^{\text{eo}}$  we need to use Darboux coordinates at  $p$ . Since the boundary is assumed to be strictly pseudoconvex (pseudoconcave), the relevant geometry is the same at every boundary point, hence there is no loss in generality in doing the computations at a fixed point.

We now suppose that, in a neighborhood of the boundary,  $X$  is a complex manifold and the Kähler form of the metric is given by  $\omega_g = -i\partial\bar{\partial}\rho$ . We are implicitly assuming that  $bX$  is either strictly pseudoconvex or strictly pseudoconcave. Our convention on the sign of  $\rho$  implies that, in either case,  $\omega_g$  is positive definite near to  $bX$ . As noted above it is really sufficient to assume that  $X$  has an almost complex structure along  $bX$  that is integrable to infinite order, however, to simplify the exposition we assume that there is a genuine complex structure in a neighborhood of  $bX$ . We fix an Hermitian metric  $h$  on sections of  $E$ .

Fix a point  $p$  on the boundary of  $X$  and let  $(z_1, \dots, z_n)$  denote Kähler coordinates centered at  $p$ . This means that

1.  $p \leftrightarrow (0, \dots, 0)$
2. The Hermitian metric tensor  $g_{i\bar{j}}$  in these coordinates satisfies

$$g_{i\bar{j}} = \frac{1}{2}\delta_{i\bar{j}} + O(|z|^2). \quad (35)$$

As a consequence of Lemma 2.3 in [15], we can choose a local holomorphic frame  $(e_1(z), \dots, e_r(z))$  for  $E$  such that

$$h(e_j(z), e_k(z)) = \delta_{jk} + O(|z|^2). \quad (36)$$

Equation (35) implies that, after a linear change of coordinates, we can arrange to have

$$\rho(z) = -2 \operatorname{Re} z_1 + |z|^2 + \operatorname{Re}(bz, z) + O(|z|^3). \quad (37)$$

In this equation  $b$  is an  $n \times n$  complex matrix and

$$(w, z) = \sum_{j=1}^n w_j z_j. \quad (38)$$

We use the conventions for Kähler geometry laid out in Section IX.5 of [10]. The underlying real coordinates are denoted by  $(x_1, \dots, x_{2n})$ , with  $z_j = x_j + ix_{j+n}$ , and  $(\zeta_1, \dots, \zeta_{2n})$  denote the linear coordinates defined on the fibers of  $T^*X$  by the local coframe field  $\{dx_1, \dots, dx_{2n}\}$ .

In this coordinate system we now compute the symbols of  $\bar{\partial}_E = \bar{\partial}_E + \bar{\partial}_E^*$ ,  $D_E^{\text{eo}}$ ,  $Q_{E(2)}^{\text{eo}}$  and  $Q_E^{\text{eo}}$ . For these calculations the following notation proves very useful: a term which is a symbol of order at most  $k$  vanishing, at  $p$ , to order  $l$  is denoted by  $\mathfrak{D}_k(|z|^l)$ . As we work with a variety of operator calculi, it is sometimes necessary to be specific as to the sense in which the order should be taken. The notation  $\mathfrak{D}_j^C$  refers to terms of order at most  $j$  in the sense of the class  $C$ . If  $C = eH$  we sometimes use an appropriate multi-order. If no symbol class is specified, then the order is with respect to the classical, radial scaling. If no rate of vanishing is specified, it should be understood to be  $O(1)$ .

Recall that, with respect to the standard Euclidean metric

$$\langle \partial_{\bar{z}_j}, \partial_{\bar{z}_k} \rangle_{\text{eucl}} = \frac{1}{2} \text{ and } \langle d\bar{z}_j, d\bar{z}_k \rangle_{\text{eucl}} = 2. \quad (39)$$

Orthonormal bases for  $T^{1,0}X$  and  $\Lambda^{1,0}X$ , near to  $p$ , take the form

$$Z_j = \sqrt{2}(\partial_{z_j} + e_{jk}(z)\partial_{z_k}), \quad \omega_j = \frac{1}{\sqrt{2}}(dz_j + f_{jk}(z)dz_k), \quad (40)$$

with  $e_{jk}$  and  $f_{jk}$  both  $O(|z|^2)$ . With respect to the trivialization of  $E$  given above, the symbol of  $\bar{\partial}_E$  is a polynomial in  $\zeta$  of the form

$$\sigma(\bar{\partial}_E)(z, \zeta) = d(z, \zeta) = d_1(z, \zeta) + d_0(z), \quad (41)$$

with  $d_j(z, \cdot)$  a polynomial of degree  $j$  such that

$$d_1(z, \zeta) = d_1(0, \zeta) + \mathfrak{D}_1(|z|^2), \quad d_0(z) = \mathfrak{D}_0(|z|). \quad (42)$$

The linear polynomial  $d_1(0, \xi)$  is the symbol of  $\bar{\partial}_E + \bar{\partial}_E^*$  on  $\mathbb{C}^n$  with respect to the flat metric. These formulæ imply that

$$\sigma(D_E^{\text{eo}}) = \Delta_2(z, \xi) + \Delta_1(z, \xi) + \Delta_0(z, \xi), \quad (43)$$

with  $\Delta_j(z, \cdot)$  a polynomial of degree  $j$  such that

$$\begin{aligned} \Delta_2(z, \xi) &= \Delta_2(0, \xi) + \mathfrak{D}_2(|z|^2) \\ \Delta_1(z, \xi) &= \mathfrak{D}_1(|z|), \quad \Delta_0(z, \xi) = \mathfrak{D}_0(1). \end{aligned} \quad (44)$$

As the metric is Kähler,  $D_E^{\text{eo}}$  is half the Riemannian Laplacian, hence the principal symbol at zero is

$$\Delta_2(0, \xi) = \frac{1}{2}|\xi|^2 \otimes \text{Id}. \quad (45)$$

Here  $\text{Id}$  is the identity homomorphism on the appropriate bundle. As it has no significant effect on our subsequence computations, or results, we heretofore suppress the explicit dependence on the bundle  $E$ , except where necessary.

The symbol  $\sigma(Q_{(2)}^{\text{eo}}) = \tilde{q} = \tilde{q}_{-2} + \tilde{q}_{-3} + \dots$  is determined by the usual symbolic relations:

$$\begin{aligned} \tilde{q}_{-2} &= \Delta_2^{-1} \\ \tilde{q}_{-3} &= -\tilde{q}_{-2}[\Delta_1 \tilde{q}_{-2} + i D_{\xi_j} \Delta_2 D_{x_j} \tilde{q}_{-2}], \end{aligned} \quad (46)$$

*etc.* Using the expressions in (44) we obtain that

$$\begin{aligned} \tilde{q}_{-2} &= \frac{2}{|\xi|^2}(\text{Id} + \mathfrak{D}_0(|z|^2)) \\ \tilde{q}_{-3} &= \frac{\mathfrak{D}_1(|z|)}{|\xi|^4}, \end{aligned} \quad (47)$$

and generally for  $k \geq 2$  we have

$$\begin{aligned} \tilde{q}_{-2k} &= \sum_{j=0}^{l_k} \frac{\mathfrak{D}_{2j}(1)}{|\xi|^{2(k+j)}} \\ \tilde{q}_{-(2k+1)} &= \sum_{j=0}^{l'_k} \frac{\mathfrak{D}_{1+2j}(1)}{|\xi|^{2(k+j+1)}}. \end{aligned} \quad (48)$$

The exact form of denominator is important in the computation of the symbol of Calderon projectors. The numerators are polynomials in  $\xi$  of the indicated degrees.

Set

$$\sigma(Q^{\text{eo}}) = q = q_{-1} + q_{-2} + \dots \quad (49)$$



As it has no bearing on the calculation, for the moment we do not keep track of whether to use the even or odd part of the operator. Note that the symbol of  $Q_{(2)}^{\text{eo}}$  is the same for both parities. From the standard composition formula, we obtain that

$$\begin{aligned} q_{-1} &= d_1 \tilde{q}_{-2} \\ q_{-2} &= d_1 \tilde{q}_{-3} + d_0 \tilde{q}_{-2} + i \sum_{j=1}^{2n} D_{\xi_j} d_1 D_{x_j} \tilde{q}_{-2}. \end{aligned} \quad (50)$$

Generally, we have

$$\begin{aligned} q_{-(2+k)}(x, \xi) &= d_0(x) \tilde{q}_{-(2+k)}(x, \xi) + d_1(x, \xi) \tilde{q}_{-(3+k)}(x, \xi) + \\ & i \sum_{|\alpha|=1} D_{\xi}^{\alpha} d_1(x, \xi) D_x^{\alpha} \tilde{q}_{-(2+k)}(x, \xi). \end{aligned} \quad (51)$$

Combining (42) and (47) shows that

$$q_{-2} = \mathfrak{D}_{-2}(|z|). \quad (52)$$

Using the expressions in (48) we see that for  $k \geq 2$  we have

$$q_{-2k} = \sum_{j=0}^{l_k} \frac{\mathfrak{D}_{2j}(1)}{|\xi|^{2(k+j)}}, \quad q_{-(2k-1)} = \sum_{j=0}^{l'_k} \frac{\mathfrak{D}_{2j+1}(1)}{|\xi|^{2(k+j)}}. \quad (53)$$

In order to compute the symbol of the Calderon projector, we introduce boundary adapted coordinates,  $(t, x_2, \dots, x_{2n})$  where

$$t = -\frac{1}{2}\rho(z) = x_1 + O(|x|^2). \quad (54)$$

Note that  $t$  is positive on a pseudoconvex manifold and negative on a pseudoconcave manifold.

We need to use the change of coordinates formula to express the symbol in the new variables. From [8] we obtain the following prescription: Let  $w = \phi(x)$  be a diffeomorphism and  $a(x, \xi)$  the symbol of a classical pseudodifferential operator  $A$ . Let  $(w, \eta)$  be linear coordinates in the cotangent space, then  $a_{\phi}(w, \eta)$ , the symbol of  $A$  in the new coordinates, is given by

$$a_{\phi}(\phi(x), \eta) \sim \sum_{k=0}^{\infty} \sum_{\alpha \in \mathcal{F}_k} \frac{(-i)^k \partial_{\xi}^{\alpha} a(x, d\phi(x)^t \eta) \partial_x^{\alpha} e^{i(\Phi_x(\tilde{x}), \eta)}}{\alpha!} \Big|_{x=\tilde{x}}, \quad (55)$$

where

$$\Phi_x(\tilde{x}) = \phi(\tilde{x}) - \phi(x) - d\phi(x)(\tilde{x} - x). \quad (56)$$

Here  $\mathcal{J}_k$  are multi-indices of length  $k$ . Our symbols are matrix valued, e.g.  $q_{-2}$  is really  $(q_{-2})_{pq}$ . As the change of variables applies component by component, we suppress these indices in the computations that follow.

In the case at hand, we are interested in evaluating this expression at  $z = x = 0$ , where we have  $d\phi(0) = \text{Id}$  and

$$\Phi_0(\tilde{x}) = \left(-\frac{1}{2}[|\tilde{z}|^2 + \text{Re}(b\tilde{z}, \tilde{z}) + O(|\tilde{z}|^3)], \dots, 0\right).$$

Note also that, in (55), the symbol  $a$  is only differentiated in the fiber variables and, therefore, any term of the symbol that vanishes at  $z = 0$ , in the Kähler coordinates, does not contribute to the symbol at 0 in the boundary adapted coordinates. Of particular importance is the fact that the term  $q_{-2}$  vanishes at  $z = 0$  and therefore does not contribute to the final result. Indeed we shall see that only the principal symbol  $q_{-1}$  contributes to the Heisenberg principal symbol along the positive (or negative) contact direction.

The  $k = 1$  term from (55) vanishes, the  $k = 2$  term is given by

$$-\frac{i\zeta_1}{2} \text{tr}[\partial_{\xi_j \xi_k}^2 q(0, \zeta) \partial_{x_j x_k}^2 \phi(0)]. \quad (57)$$

For  $k > 2$ , the terms have the form

$$\sum_{\alpha \in \mathcal{J}_k} \partial_{\xi}^{\alpha} q(0, \zeta) p^{\alpha}(\zeta_1). \quad (58)$$

Here  $p^{\alpha}$  is a polynomial of degree at most  $\lfloor \frac{|\alpha|}{2} \rfloor$ . As we shall see, the terms for  $k > 2$  do not contribute to the final result.

To compute the  $k = 2$  term we need to compute the Hessians of  $q_{-1}$  and  $\phi(x)$  at  $x = 0$ . We define the  $2n \times 2n$  matrix  $B$  so that

$$\text{Re}(bz, z) = \langle Bx, x \rangle; \quad (59)$$

if  $b = b^0 + ib^1$ , then

$$B = \begin{pmatrix} b^0 & -b^1 \\ -b^1 & -b^0 \end{pmatrix}. \quad (60)$$

With these definitions we see that

$$\partial_{x_j x_k}^2 \phi(0) = -(\text{Id} + B). \quad (61)$$

We further simplify the notation by letting  $d_1(\zeta) \stackrel{d}{=} d_1(0, \zeta)$ , then

$$q_{-1}(0, \zeta) = \frac{2d_1(\zeta)}{|\zeta|^2}. \quad (62)$$

Differentiating gives

$$\frac{\partial q_{-1}}{\partial \xi_j} = \frac{2\partial_{\xi_j} d_1}{|\xi|^2} - \frac{2\xi_j d_1}{|\xi|^4} \quad (63)$$

and

$$\frac{\partial^2 q_{-1}}{\partial \xi_k \partial \xi_j} = -4 \frac{d_1 \text{Id} + \xi \otimes \partial_{\xi} d_1^t + \partial_{\xi} d_1 \otimes \xi^t}{|\xi|^4} + 16d_1 \frac{\xi \otimes \xi^t}{|\xi|^6}. \quad (64)$$

Here  $\xi$  and  $\partial_{\xi} d_1$  are regarded as column vectors.

We now compute the principal part of the  $k = 2$  term

$$\begin{aligned} q_{-2}^c(\xi) &= i\xi_1 \text{tr} \left[ (\text{Id} + B) \left( -2 \frac{d_1 \text{Id} + \xi \otimes \partial_{\xi} d_1^t + \partial_{\xi} d_1 \otimes \xi^t}{|\xi|^4} + 8d_1 \frac{\xi \otimes \xi^t}{|\xi|^6} \right) \right] \\ &= 4i\xi_1 \left[ (1-n) \frac{d_1}{|\xi|^4} + \frac{2d_1 \langle B\xi, \xi \rangle}{|\xi|^6} - \frac{\langle B\xi, \partial_{\xi} d_1 \rangle}{|\xi|^4} \right]. \end{aligned} \quad (65)$$

Because  $q_{-2}$  vanishes at 0 and because the order of a symbol is preserved under a change of variables we see that the symbol of  $Q^{\text{eo}}$  at  $p$  is therefore

$$q(0, \xi) = \frac{2d_1(\xi)}{|\xi|^2} + q_{-2}^c(\xi) + \mathfrak{D}_{-3}(1). \quad (66)$$

For the computation of the Calderon projector it is useful to be a little more precise about the error term. The terms of highest symbolic order are multiples of terms of the form  $\xi_1^k \partial_{\xi}^{\alpha} q_{-j}$  where  $|\alpha| = 2k$ . We describe, in a proposition, the types of terms that arise as error terms in (66)

**Proposition 4.** *The  $\mathfrak{D}_{-3}(1)$ -term in (66) is a sum of terms of the form appearing in (53) along with terms of the forms*

$$\begin{aligned} &\frac{\xi_1^l h_{2m}(\xi)}{|\xi|^{2(k+l'+m)}} \text{ with either } k = 1 \text{ and } l \geq 2 \text{ or } k \geq 2 \\ &\frac{\xi_1^l h_{2m+1}(\xi)}{|\xi|^{2(k+l'+m)}} \text{ with } k \geq 2. \end{aligned} \quad (67)$$

Here  $l' \geq l$ ,  $m$  is a nonnegative integer and  $h_j(\xi)$  is a radially homogeneous polynomial of degree  $j$ .

*Proof.* This statement is an immediate consequence of (53), (55) and the fact that  $\Phi_0(\tilde{x})$  vanishes quadratically at  $\tilde{x} = 0$ .  $\square$

### 3 The symbol of the Calderon projector

We are now prepared to compute the symbol of the Calderon projector; it is expressed as 1-variable contour integral in the symbol of  $Q^{\text{eo}}$ . If  $q(t, x', \zeta_1, \zeta')$  is the symbol of  $Q^{\text{eo}}$  in the boundary adapted coordinates, then the symbol of the Calderon projector is

$$\mathfrak{p}_{\pm}(x', \zeta') = \frac{1}{2\pi} \int_{\Gamma_{\pm}(\zeta_1)} q(0, x', \zeta_1, \zeta') d\zeta_1 \circ \sigma_1(\bar{\partial}^{\text{eo}}, \mp i dt). \quad (68)$$

Here we recall that  $q(0, x', \zeta_1, \zeta')$  is a meromorphic function of  $\zeta_1$ . For each fixed  $\zeta'$ , the poles of  $q$  lie on the imaginary axis. If  $X$  is strictly pseudoconvex, then  $t > 0$  on  $X$  and we take  $\Gamma_+(\zeta_1)$  to be a contour enclosing the poles of  $q(0, x', \cdot, \zeta')$  in the upper half plane. If  $X$  is strictly pseudoconcave, then  $t < 0$  on  $X$  and  $\Gamma_-(\zeta_1)$  is a contour enclosing the poles of  $q(0, x', \cdot, \zeta')$  in the lower half plane. In a moment we use a residue computation to evaluate these integrals. For this purpose we note that the contour  $\Gamma_+(\zeta_1)$  is positively oriented, while  $\Gamma_-(\zeta_1)$  is negatively oriented.

The Calderon projector is a classical pseudodifferential operator of order 0 and therefore its symbol has an asymptotic expansion of the form

$$\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_{-1} + \dots \quad (69)$$

The contact line,  $L_p$ , is defined in  $T_p^*Y$  by the equations

$$\zeta_2 = \dots = x_n = x_{n+2} = \dots = \zeta_{2n} = 0, \quad (70)$$

and  $\zeta_{n+1}$  is a coordinate along the contact line. Because  $t = -\frac{1}{2}\rho$ , the positive contact direction is given by  $\zeta_{n+1} < 0$ . If  $X$  is pseudoconvex then, for  $\zeta' \notin L_p^+$ , it suffices to compute  $\mathfrak{p}_0$ , whereas if  $X$  is pseudoconcave, then for  $\zeta' \notin L_p^-$  it suffices to compute  $\mathfrak{p}_0$ . We begin our computations with the principal symbol

**Proposition 5.** *If  $X$  is strictly pseudoconvex (pseudoconcave) and  $p \in bX$  with coordinates normalized at  $p$  as above, then*

$$\mathfrak{p}_0^{\text{eo}}(0, \zeta') = \frac{d_1^{\text{oe}}(\pm i|\zeta'|, \zeta')}{|\zeta'|} \circ \sigma_1(\bar{\partial}^{\text{eo}}, \mp i dt). \quad (71)$$

*Proof.* The leading term in the symbol of the Calderon projector comes from

$$q_{-1}(0, \zeta) = \frac{2d_1(\zeta)}{|\zeta|^2} = \frac{2d_1(\zeta_1, \zeta')}{(\zeta_1 + i|\zeta'|)(\zeta_1 - i|\zeta'|)}. \quad (72)$$

Evaluating the contour integral in (68) gives (71).  $\square$

Along the contact directions we need to evaluate higher order terms. We begin by showing that the error terms in (66) contribute terms that lift to have Heisenberg order less than  $-2$

**Proposition 6.** *The error terms in (66) contribute terms to the symbol of the Calderon projector that lift to have Heisenberg orders at most  $-4$ .*

*Proof.* We first check the terms that come from the lower order terms in the symbol of  $Q^{\text{eo}}$  before changing variables. These are of the forms given in (53) with  $k \geq 2$ . It suffices to consider a term of the form

$$\frac{h_{2j+1}(\zeta)}{|\zeta|^{2(k+j)}} \quad (73)$$

for  $k \geq 2$  and  $j \geq 0$ . Applying the contour integration to such a term gives a multiple of

$$\partial_{\zeta_1}^{k+j-1} \left[ \frac{h_{2j+1}(\zeta)}{(\zeta_1 \pm i|\zeta'|)^{k+j}} \right]_{\zeta_1 = \pm i|\zeta'|}. \quad (74)$$

As  $\zeta_{n+1}$  has Heisenberg order 2, it is not difficult to see that the highest parabolic order term results if  $h_{2j+1}(\zeta) = \zeta_{n+1}^{2j+1}$ . Differentiating gives a term of the form

$$\frac{\zeta_{n+1}^{2j+1}}{|\zeta'|^{2k+2j-1}}. \quad (75)$$

Proposition 3 implies that this term lifts to have Heisenberg order  $4 - 4k$ . As  $k \geq 2$  the proposition follows in this case.

Among the terms that come from the change of variables formula, there are two cases to consider: those coming from  $q_{-1}$  and those coming from  $q_{-k}$  for  $k \geq 3$ . Recall that  $q_{-2}$  does not contribute anything to the symbol at  $p$ . The terms in (55) coming from the principal symbol are of the form

$$\frac{\zeta_1^l h_{1+2j}(\zeta)}{|\zeta|^{2(1+j+l')}} \text{ where } 2 \leq l \leq l' \text{ and } j \geq 0. \quad (76)$$

Clearly the worst case is when  $l = l'$  and  $h_{2j+1} = \zeta_{n+1}^{2j+1}$ . The contour integral applied to such a term produces a multiple of

$$\frac{\zeta_{n+1}^{2j+1}}{|\zeta'|^{l+2j+1}}. \quad (77)$$

This lifts to have Heisenberg order  $-2l$ . As  $l \geq 2$ , this completes the analysis of the contribution of the principal symbol.

Finally we need to consider terms of the forms given in (67) with  $k \geq 2$  and  $l \geq 1$ . As before, the worse case is with  $l = l'$  and  $h_{2j+1}(\xi) = \frac{\xi^{2j+1}}{\xi_{n+1}}$ . The contour integral gives a term of the form

$$\frac{\xi_{n+1}^{2j+1}}{|\xi'|^{2j+1}} \frac{1}{|\xi'|^{2k+l-2}}. \quad (78)$$

As  $2k+l \geq 5$ , these terms lift to have Heisenberg order at most  $-6$ . This completes the proof of the proposition.  $\square$

To finish our discussion of the symbol of the Calderon projector we need to compute the symbol along the contact direction. This entails computing the contribution from  $q_{-2}^c$ . As we now show, terms arising from the holomorphic Hessian of  $\rho$  do not contribute anything to the symbol of the Calderon projector. To do these computations we need to have an explicit formula for the principal symbol  $d_1(\xi)$  of  $\bar{\partial}$  at  $p$ . For the purposes of these and our subsequent computations, it is useful to use the chiral operators  $\bar{\partial}^{\text{eo}}$ . As we are working in a Kähler coordinate system, we only need to find the symbols of  $\bar{\partial}^{\text{eo}}$  for  $\mathbb{C}^n$  with the flat metric. Let  $\sigma$  denote a section of  $\Lambda^{\text{eo}} \otimes E$ . We split  $\sigma$  into its normal and tangential parts at  $p$ :

$$\sigma = \sigma^t + \frac{d\bar{z}_1}{\sqrt{2}} \wedge \sigma^n, \quad i_{\partial_{\bar{z}_1}} \sigma^t = 0, \quad i_{\partial_{\bar{z}_1}} \sigma^n = 0. \quad (79)$$

With this splitting we see that

$$\begin{aligned} \bar{\partial}^e \sigma &= \sqrt{2} \begin{pmatrix} \partial_{\bar{z}_1} \otimes \text{Id}_{E,n} & \mathfrak{D}_t \\ -\mathfrak{D}_t & -\partial_{z_1} \otimes \text{Id}_{E,n} \end{pmatrix} \begin{pmatrix} \sigma^t \\ \sigma^n \end{pmatrix} \\ \bar{\partial}^o \sigma &= \sqrt{2} \begin{pmatrix} -\partial_{z_1} \otimes \text{Id}_{E,n} & -\mathfrak{D}_t \\ \mathfrak{D}_t & \partial_{\bar{z}_1} \otimes \text{Id}_{E,n} \end{pmatrix} \begin{pmatrix} \sigma^n \\ \sigma^t \end{pmatrix}, \end{aligned} \quad (80)$$

where  $\text{Id}_{E,n}$  is the identity matrix acting on the normal, or tangential parts of  $\Lambda^{\text{eo}} \otimes E \downarrow_{bX}$  and

$$\mathfrak{D}_t = \sum_{j=2}^n [\partial_{z_j} e_j - \partial_{\bar{z}_j} \epsilon_j] \text{ with } e_j = i\sqrt{2}\partial_{\bar{z}_j} \text{ and } \epsilon_j = \frac{d\bar{z}_j}{\sqrt{2}} \wedge. \quad (81)$$

These symbols are expressed in the block matrix structure shown in (3). It is now a simple matter to compute  $d_1^{\text{eo}}(\xi)$ :

$$\begin{aligned} d_1^e(\xi) &= \frac{1}{\sqrt{2}} \begin{pmatrix} (i\xi_1 - \xi_{n+1}) \otimes \text{Id}_{E,n} & \mathfrak{d}(\xi'') \\ -\mathfrak{d}(\xi'') & -(i\xi_1 + \xi_{n+1}) \otimes \text{Id}_{E,n} \end{pmatrix} \\ d_1^o(\xi) &= \frac{1}{\sqrt{2}} \begin{pmatrix} -(i\xi_1 + \xi_{n+1}) \otimes \text{Id}_{E,n} & -\mathfrak{d}(\xi'') \\ \mathfrak{d}(\xi'') & (i\xi_1 - \xi_{n+1}) \otimes \text{Id}_{E,n} \end{pmatrix} \end{aligned} \quad (82)$$

where  $\zeta'' = (\zeta_2, \dots, \zeta_n, \zeta_{n+2}, \dots, \zeta_{2n})$  and

$$\mathfrak{d}(\zeta'') = \sum_{j=2}^n [(i\zeta_j + \zeta_{n+j})e_j - (i\zeta_j - \zeta_{n+j})\epsilon_j]. \quad (83)$$

As  $\epsilon_j^* = e_j$  we see that  $\mathfrak{d}(\zeta'')$  is a self adjoint symbol.

In the next section we show that, in the block structure shown in equation (3), the (1, 1) block of the symbol of  $\mathcal{T}^{\text{eo}}$  has Heisenberg order 0, the (1, 2) and the (2, 1) blocks have Heisenberg order  $-1$ . The symbol  $q_{-2}^c$  produces a term that lifts to have Heisenberg order  $-2$  and therefore we only need to compute the (2, 2) block arising from this term.

We start with the nontrivial term of order  $-1$ .

**Lemma 1.** *If  $X$  is either pseudoconvex or pseudoconcave we have that*

$$\frac{1}{2\pi} \int_{\Gamma_{\pm}(\zeta')} \frac{4i\zeta_1(1-n)d_1(\zeta_1, \zeta')d\zeta_1}{|\zeta|^4} = -\frac{i(n-1)\partial_{\zeta_1}d_1}{|\zeta'|} \quad (84)$$

*Remark 4.* As  $d_1$  is a linear polynomial,  $\partial_{\zeta_1}d_1$  is a constant matrix.

*Proof.* The residue theorem implies that

$$\frac{1}{2\pi} \int_{\Gamma_{\pm}(\zeta')} \frac{4i\zeta_1(1-n)d_1(\zeta_1, \zeta')d\zeta_1}{|\zeta|^4} = \pm 4(n-1)\partial_{\zeta_1} \left[ \frac{\zeta_1 d_1}{(\zeta_1 \pm i|\zeta'|)^2} \right]_{\zeta_1 = \pm i|\zeta'|}. \quad (85)$$

The lemma follows from this equation by an elementary computation.  $\square$

We complete the computation by evaluating the contribution from the other terms in  $q_{-2}^c$  along the contact line.

**Proposition 7.** *For  $\zeta'$  along the positive (negative) contact line we have, for  $j = 1, 2$ , that*

$$\int_{\Gamma_{\pm}(\zeta')} \left[ \frac{2d_1(\zeta)\langle B\zeta, \zeta \rangle - |\zeta|^2 \langle B\zeta, \partial_{\zeta}d_1 \rangle}{|\zeta|^6} \right]_{jj} \zeta_1 d\zeta_1 = 0. \quad (86)$$

*The subscript 11 refers to the upper left block and 22 the lower right block of the matrix. If  $\zeta_{n+1} < 0$ , then we use  $\Gamma_+(\zeta')$ , whereas if  $\zeta_{n+1} > 0$ , then we use  $\Gamma_-(\zeta')$ .*

*Proof.* To prove this result we need to evaluate the contour integral with

$$\zeta' = \zeta'_c = (0, \dots, 0, \zeta_{n+1}, 0, \dots, 0),$$

recalling that the positive contact line corresponds to  $\zeta_{n+1} < 0$ . Hence, along the positive contact line  $|\zeta'| = -\zeta_{n+1}$ . Because

$$[d_1^e]_{11} = [d_1^o]_{22} \text{ and } [d_1^e]_{22} = [d_1^o]_{11}, \quad (87)$$

it suffices to prove the result for the (2, 2) block in both the even and odd cases. We first compute the integrand along  $\zeta'_c$ .

**Lemma 2.** *For  $\zeta'$  along the contact line we have*

$$\left[ \frac{2d_1^e(\zeta) \langle B\zeta, \zeta \rangle - |\zeta|^2 \langle B\zeta, \partial_\zeta d_1^e \rangle}{|\zeta|^6} \right]_{22} = \frac{(\zeta_1 b_{11}^1 + \zeta_{n+1} b_{11}^0) + i(\zeta_1 b_{11}^0 - \zeta_{n+1} b_{11}^1)}{(\zeta_{n+1} + i\zeta_1)(\zeta_{n+1} - i\zeta_1)^3} \otimes \text{Id}_{E,n} \quad (88)$$

$$\left[ \frac{2d_1^o(\zeta) \langle B\zeta, \zeta \rangle - |\zeta|^2 \langle B\zeta, \partial_\zeta d_1^o \rangle}{|\zeta|^6} \right]_{22} = \frac{(\zeta_1 b_{11}^1 + \zeta_{n+1} b_{11}^0) - i(\zeta_1 b_{11}^0 - \zeta_{n+1} b_{11}^1)}{(\zeta_{n+1} - i\zeta_1)(\zeta_{n+1} + i\zeta_1)^3} \otimes \text{Id}_{E,n}. \quad (89)$$

The subscript 22 refers to the lower right block of the matrix.

*Proof.* Observe that along the contact line

$$\langle B\zeta, \zeta \rangle = b_{11}^0(\zeta_1^2 - \zeta_{n+1}^2) - 2b_{11}^1 \zeta_1 \zeta_{n+1}.$$

We outline the proof for the even case. The lower right block of  $d_1^e(\zeta)$  equals  $-(i\zeta_1 + \zeta_{n+1}) \otimes \text{Id}_{E,n}$ , thus

$$[\partial_\zeta d_1^e]_{22} = (-i, 0, \dots, 0, -1, 0, \dots, 0) \otimes \text{Id}_{E,n}.$$

Putting these expressions into the formula on the left hand side of (88) gives  $\text{Id}_{E,n}$  times

$$\frac{-2(i\zeta_1 + \zeta_{n+1})(b_{11}^0(\zeta_1^2 - \zeta_{n+1}^2) - 2b_{11}^1 \zeta_1 \zeta_{n+1})}{|\zeta|^3} - \frac{(\zeta_1 b_{11}^1 + \zeta_{n+1} b_{11}^0) - i(\zeta_1 b_{11}^0 - \zeta_{n+1} b_{11}^1)}{|\zeta|^4}. \quad (90)$$

To complete the calculation we express  $|\zeta|^2 = (\zeta_{n+1} + i\zeta_1)(\zeta_{n+1} - i\zeta_1)$ , cancel and place the result over a common denominator. This leads to the cancellation of a second factor of  $\zeta_{n+1} + i\zeta_1$ . The odd case follows, *mutatis mutandis*, using

$$d_1^o(\zeta) = (i\zeta_1 - \zeta_{n+1}) \otimes \text{Id}_{E,n}.$$

The details are left to the reader.  $\square$



To complete the proof of the proposition we need to compute the contour integrals of the expressions in (88) and (89) times  $\zeta_1$ , along the appropriate end of the contact line. We state these computations as lemmas.

**Lemma 3.** *If  $\zeta_{n+1} < 0$ , then*

$$\begin{aligned} \text{even} \quad & \int_{\Gamma_+(\zeta'_c)} \frac{(\zeta_1 b_{11}^1 + \zeta_{n+1} b_{11}^0) + i(\zeta_1 b_{11}^0 - \zeta_{n+1} b_{11}^1)}{(\zeta_1 - i\zeta_{n+1})(\zeta_1 + i\zeta_{n+1})^3} \zeta_1 d\zeta_1 = 0 \\ \text{odd} \quad & \int_{\Gamma_+(\zeta'_c)} \frac{(\zeta_1 b_{11}^1 + \zeta_{n+1} b_{11}^0) - i(\zeta_1 b_{11}^0 - \zeta_{n+1} b_{11}^1)}{(\zeta_1 + i\zeta_{n+1})(\zeta_1 - i\zeta_{n+1})^3} \zeta_1 d\zeta_1 = 0 \end{aligned} \quad (91)$$

*Note that this implies that, if  $\zeta_{n+1} > 0$ , then the same integrals vanish if  $\Gamma_+(\zeta'_c)$  is replaced by  $\Gamma_-(\zeta'_c)$ .*

*Proof.* The second statement follows by observing that the singular terms in the integrand in the upper half plane are those coming from  $(\zeta_1 + i\zeta_{n+1})$ . If  $\zeta_{n+1} > 0$ , then these become the singular terms in the lower half plane. Using a residue computation we see that the even case gives

$$\begin{aligned} & (\pi i) \partial_{\zeta_1}^2 \left[ \frac{(\zeta_1 b_{11}^1 + \zeta_{n+1} b_{11}^0) + i(\zeta_1 b_{11}^0 - \zeta_{n+1} b_{11}^1)}{(\zeta_1 - i\zeta_{n+1})} \right]_{\zeta_1 = -i\zeta_{n+1}} \\ &= \frac{2\pi}{(-2i\zeta_{n+1})^2} \left[ b_{11}^1 + i b_{11}^0 - \frac{(\zeta_1 b_{11}^1 + \zeta_{n+1} b_{11}^0) + i(\zeta_1 b_{11}^0 - \zeta_{n+1} b_{11}^1)}{\zeta_1 - i\zeta_{n+1}} \right]_{\zeta_1 = -i\zeta_{n+1}}. \end{aligned} \quad (92)$$

The quantity in the brackets is easily seen to vanish. The odd case follows easily from the observation that

$$\left[ (\zeta_1 b_{11}^1 + \zeta_{n+1} b_{11}^0) - i(\zeta_1 b_{11}^0 - \zeta_{n+1} b_{11}^1) \right]_{\zeta_1 = -i\zeta_{n+1}} = 0. \quad (93)$$

□

The two lemmas prove the proposition. □

As a corollary, we have a formula for the  $-1$  order term in the symbol of the Calderon projector

**Corollary 1.** *If  $X$  is strictly pseudoconvex (pseudoconcave), then, in the normalizations defined above, for  $j = 1, 2$ , we have*

$$[\mathfrak{p}_{-1}^{\text{eo}}(0, \zeta')]_{jj} = -\frac{i(n-1)\partial_{\zeta_1} d_1^{\text{oe}}}{|\zeta'|} \circ \sigma_1(\bar{\partial}^{\text{eo}}, \mp i dt). \quad (94)$$

We have shown that the order  $-1$  term in the symbol of the Calderon projector, along the appropriate half of the contact line, is given by the right hand side of equation (84). It is determined by the principal symbol of  $Q^{\text{eo}}$  and does not depend on the higher order geometry of  $bX$ . As we have shown that all other terms in the symbol of  $Q^{\text{eo}}$  contribute terms that lift to have Heisenberg order less than  $-2$ , these computations allow us to find the principal symbols of  $\mathcal{T}_{\pm}^{\text{eo}}$  and deduce the main results of the paper. As noted above, the off diagonal blocks have Heisenberg order  $-1$ , so the classical terms of order less than zero cannot contribute to their principal parts.

## 4 The subelliptic boundary conditions

We now give formulæ for the chiral forms of the subelliptic boundary conditions defined in [5] as well as the isomorphisms  $\sigma_1(\partial^{\text{eo}}, \mp idt)$ . We begin by recalling the basic properties of compatible almost complex structures defined on a contact field and of the symbol of a generalized Szegő projector. Let  $\theta$  denote a positive contact form defining  $H$ . An almost complex structure on  $H$  is compatible if

1.  $X \mapsto d\theta(JX, X)$  defines an inner product on  $H$ .
2.  $d\theta(JX, JY) = d\theta(X, Y)$  for sections of  $H$ .

Let  $\omega'$  be the dual symplectic form on  $H^*$  and  $J'$  the dual almost complex structure. The symbol of a field of harmonic oscillators is defined by

$$h_J(\eta) = \omega'(J'\pi_{H^*}(\eta), \pi_{H^*}(\eta)). \quad (95)$$

The model operator defined by the symbol  $h_J$  is a harmonic oscillator, as such its minimum eigenstate or vacuum state is one dimensional. The projector onto the vacuum state has symbol  $s_{J0} = 2^{1-n} e^{-h_J}$ . An operator  $\mathcal{G}'$  in the Heisenberg calculus with principal symbol  $s_{J0}$ , for a compatible almost complex structure  $J$ , such that

$$[\mathcal{G}']^2 = \mathcal{G}' \text{ and } [\mathcal{G}']^* = \mathcal{G}' \quad (96)$$

is called a generalized Szegő projector. Generalized conjugate Szegő projectors are analogously defined, with the symbol supported on the lower half space. A generalized Szegő projector acting on sections of a complex vector bundle  $F \rightarrow bX$  is an operator in  $\Psi_H^0(Y; F)$ , which satisfies the conditions in (96) and its principal symbol is  $s_{J0} \otimes \text{Id}_F$ .

**Lemma 4.** According to the splittings of sections of  $\Lambda^{\text{eo}} \otimes E$  given in (79), the subelliptic boundary conditions, defined by the generalized Szegő projector  $\mathcal{P}'$ , on even (odd) forms are given by  $\mathcal{R}'_{\pm} \sigma \upharpoonright_{bX} = 0$  where

$$\mathcal{R}'_{+} \sigma \upharpoonright_{bX} = \begin{pmatrix} \mathcal{P}' & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Id} & \mathbf{0} \end{pmatrix} \begin{bmatrix} \sigma^t \\ \sigma^n \end{bmatrix}_{bX} \quad \mathcal{R}'_{+} \sigma \upharpoonright_{bX} = \begin{pmatrix} 1 - \mathcal{P}' & 0 & \mathbf{0} \\ 0 & \text{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{bmatrix} \sigma^n \\ \sigma^t \end{bmatrix}_{bX} \quad (97)$$

**Lemma 5.** According to the splittings of sections of  $\Lambda^{\text{eo}} \otimes E$  given in (79), the subelliptic boundary conditions, defined by the generalized conjugate Szegő projector  $\bar{\mathcal{P}}'$ , on even (odd) forms are given by  $\mathcal{R}'_{\pm} \sigma \upharpoonright_{bX} = 0$  where, if  $n$  is even, then

$$\mathcal{R}'_{-} \sigma \upharpoonright_{bX} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Id} & 0 \\ \mathbf{0} & 0 & 1 - \bar{\mathcal{P}}' \end{pmatrix} \begin{bmatrix} \sigma^t \\ \sigma^n \end{bmatrix}_{bX} \quad \mathcal{R}'_{-} \sigma \upharpoonright_{bX} = \begin{pmatrix} \text{Id} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & 0 & \bar{\mathcal{P}}' \end{pmatrix} \begin{bmatrix} \sigma^n \\ \sigma^t \end{bmatrix}_{bX} . \quad (98)$$

If  $n$  is odd, then

$$\mathcal{R}'_{-} \sigma \upharpoonright_{bX} = \begin{pmatrix} \mathbf{0} & 0 & \mathbf{0} \\ 0 & \bar{\mathcal{P}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \text{Id} \end{pmatrix} \begin{bmatrix} \sigma^t \\ \sigma^n \end{bmatrix}_{bX} \quad \mathcal{R}'_{-} \sigma \upharpoonright_{bX} = \begin{pmatrix} \text{Id} & 0 & \mathbf{0} \\ 0 & 1 - \bar{\mathcal{P}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{bmatrix} \sigma^n \\ \sigma^t \end{bmatrix}_{bX} . \quad (99)$$

*Remark 5.* These boundary conditions are introduced in [5]. For the purposes of this paper, these formulæ can be taken as the definitions of the projections  $\mathcal{R}'_{\pm}$ , which, in turn, define the boundary conditions.

**Lemma 6.** The isomorphisms at the boundary between  $\Lambda^{\text{eo}} \otimes E$  and  $\Lambda^{\text{oe}} \otimes E$  are given by

$$\sigma_1(\bar{\partial}_{\pm}^{\text{eo}}, \mp idt) \sigma^t = \frac{\pm}{\sqrt{2}} \sigma^t, \quad \sigma_1(\bar{\partial}_{\pm}^{\text{eo}}, \mp idt) \sigma^n = \frac{\mp}{\sqrt{2}} \sigma^n. \quad (100)$$

We have thus far succeeded in computing the symbols of the Calderon projectors to high enough order to compute the principal symbols of  $\mathcal{T}_{\pm}^{\text{eo}}$  as elements of the extended Heisenberg calculus. The computations have been carried out in a coordinate system adapted to the boundary. This suffices to examine the classical parts of the symbols. In the next section we further normalize the coordinates, in order to analyze the Heisenberg symbols.

We close this section by computing the classical parts of the symbols of  $\mathcal{T}_{\pm}^{\text{eo}}$  and showing that they are invertible on the complement of the appropriate half of the contact line. Recall that the positive contact ray,  $L^+$ , is given at  $p$  by  $\zeta'' = 0, \zeta_{n+1} < 0$ .

**Proposition 8.** *If  $X$  is strictly pseudoconvex, then, on the complement of the positive contact direction, the classical symbols  ${}^R\sigma_0(\mathcal{T}_+^{\text{eo}})$  are given by*

$$\begin{aligned} {}^R\sigma_0(\mathcal{T}_+^{\text{e}})(0, \zeta') &= \frac{1}{2|\zeta'|} \begin{pmatrix} (|\zeta'| + \zeta_{n+1}) \text{Id} & -\mathfrak{d}(\zeta'') \\ \mathfrak{d}(\zeta'') & (|\zeta'| + \zeta_{n+1}) \text{Id} \end{pmatrix} \\ {}^R\sigma_0(\mathcal{T}_+^{\text{o}})(0, \zeta') &= \frac{1}{2|\zeta'|} \begin{pmatrix} (|\zeta'| + \zeta_{n+1}) \text{Id} & \mathfrak{d}(\zeta'') \\ -\mathfrak{d}(\zeta'') & (|\zeta'| + \zeta_{n+1}) \text{Id} \end{pmatrix} \end{aligned} \quad (101)$$

*These symbols are invertible on the complement of  $L^+$ .*

*Proof.* Away from the positive contact direction  $\mathcal{R}_+^{\text{eo}}$  are classical pseudodifferential operators with

$${}^R\sigma_0(\mathcal{R}_+^{\text{e}}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Id} \end{pmatrix}, \quad {}^R\sigma_0(\mathcal{R}_+^{\text{o}}) = \begin{pmatrix} \text{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (102)$$

The formulæ in (101) follow easily from these relations, along with (71), (82), and (100). To show that these symbols are invertible away from the positive contact direction, it suffices to show that their determinants do not vanish. Up to the factor of  $(2|\zeta'|)^{-1}$ , these symbols are of the form  $\lambda \text{Id} + B$  where  $\lambda$  is real (and nonnegative) and  $B$  is skew-adjoint. As a skew-adjoint matrix has purely imaginary spectrum, the determinants of these symbols vanish if and only if  $\mathfrak{d}(\zeta'') = 0$  and  $|\zeta'| + \zeta_{n+1} = 0$ . The first condition implies that  $|\zeta'| = |\zeta_{n+1}|$ , hence these determinant vanish if and only if  $\zeta'$  belongs to the positive contact ray.  $\square$

An essentially identical argument, taking into account the fact that  $\mathcal{R}_-^{\text{eo}}$  are classical pseudodifferential operators on the complement of  $L^-$ , suffices to treat the pseudoconcave case.

**Proposition 9.** *If  $X$  is strictly pseudoconcave, then, on the complement of the negative contact direction, the classical symbols  ${}^R\sigma_0(\mathcal{T}_-^{\text{eo}})$  are given by*

$$\begin{aligned} {}^R\sigma_0(\mathcal{T}_-^{\text{e}})(0, \zeta') &= \frac{1}{2|\zeta'|} \begin{pmatrix} (|\zeta'| - \zeta_{n+1}) \text{Id} & \mathfrak{d}(\zeta'') \\ -\mathfrak{d}(\zeta'') & (|\zeta'| - \zeta_{n+1}) \text{Id} \end{pmatrix} \\ {}^R\sigma_0(\mathcal{T}_-^{\text{o}})(0, \zeta') &= \frac{1}{2|\zeta'|} \begin{pmatrix} (|\zeta'| - \zeta_{n+1}) \text{Id} & -\mathfrak{d}(\zeta'') \\ \mathfrak{d}(\zeta'') & (|\zeta'| - \zeta_{n+1}) \text{Id} \end{pmatrix} \end{aligned} \quad (103)$$

*These symbols are invertible on the complement of  $L^-$ .*

*Remark 6.* Propositions 8 and 9 are classical and implicitly stated, for example, in the work of Greiner and Stein, and Beals and Stanton, see [2, 7].

## 5 The Heisenberg symbols of $\mathcal{T}_\pm^{\text{eo}}$

To compute the Heisenberg symbols of  $\mathcal{T}_\pm^{\text{eo}}$  we change coordinates, one last time, to get Darboux coordinates at  $p$ . Up to this point we have used the coordinates  $(\xi_2, \dots, \xi_{2n})$  for  $T_p^*bX$ , which are defined by the coframe  $dx_2, \dots, dx_{2n}$ , with  $dx_{n+1}$  the contact direction. Recall that the contact form  $\theta$ , defined by the complex structure and defining function  $\rho/2$ , is given by  $\theta = \frac{i}{2}\bar{\partial}\rho$ . The symplectic form on  $H$  is defined by  $d\theta$ . At  $p$  we have

$$\theta_p = -\frac{1}{2}dx_{n+1}, \quad d\theta_p = \sum_{j=2}^n dx_j \wedge dx_{j+n}. \quad (104)$$

By comparison with (5), we see that properly normalized coordinates for  $T_p^*bX$  are obtained by setting

$$\eta_0 = -2\xi_{n+1}, \quad \eta_j = \xi_{j+1}, \quad \eta_{j+n-1} = \xi_{j+n+1} \text{ for } j = 1, \dots, n-1. \quad (105)$$

As usual we let  $\eta' = (\eta_1, \dots, \eta_{2(n-1)})$ ; whence  $\xi'' = \eta'$ .

As a first step in lifting the symbols of the Calderon projectors to the extended Heisenberg compactification, we re-express them, through order  $-1$  in the  $\xi$ -coordinates:

$$\mathfrak{p}_+^e(\xi') = \frac{1}{2|\xi'|} \left[ \begin{pmatrix} (|\xi'| - \xi_{n+1}) \text{Id} & \mathfrak{d}(\xi'') \\ \mathfrak{d}(\xi'') & (|\xi'| + \xi_{n+1}) \text{Id} \end{pmatrix} - (n-1) \begin{pmatrix} \text{Id} & \mathbf{0} \\ \mathbf{0} & \text{Id} \end{pmatrix} \right] \quad (106)$$

$$\mathfrak{p}_+^o(\xi') = \frac{1}{2|\xi'|} \left[ \begin{pmatrix} (|\xi'| + \xi_{n+1}) \text{Id} & \mathfrak{d}(\xi'') \\ \mathfrak{d}(\xi'') & (|\xi'| - \xi_{n+1}) \text{Id} \end{pmatrix} - (n-1) \begin{pmatrix} \text{Id} & \mathbf{0} \\ \mathbf{0} & \text{Id} \end{pmatrix} \right] \quad (107)$$

$$\mathfrak{p}_-^e(\xi') = \frac{1}{2|\xi'|} \left[ \begin{pmatrix} (|\xi'| + \xi_{n+1}) \text{Id} & -\mathfrak{d}(\xi'') \\ -\mathfrak{d}(\xi'') & (|\xi'| - \xi_{n+1}) \text{Id} \end{pmatrix} + (n-1) \begin{pmatrix} \text{Id} & \mathbf{0} \\ \mathbf{0} & \text{Id} \end{pmatrix} \right] \quad (108)$$

$$\mathfrak{p}_-^o(\xi') = \frac{1}{2|\xi'|} \left[ \begin{pmatrix} (|\xi'| - \xi_{n+1}) \text{Id} & -\mathfrak{d}(\xi'') \\ -\mathfrak{d}(\xi'') & (|\xi'| + \xi_{n+1}) \text{Id} \end{pmatrix} + (n-1) \begin{pmatrix} \text{Id} & \mathbf{0} \\ \mathbf{0} & \text{Id} \end{pmatrix} \right] \quad (109)$$

Various identity and zero matrices appear in these symbolic computations. Precisely which matrix is needed depends on the dimension, the bundle  $E$ , the parity, etc. We do not encumber our notation with these distinctions.

In order to compute  ${}^H\sigma(\mathcal{T}_{\pm}^{\text{eo}})$ , we represent the Heisenberg symbols as model operators and use operator composition. To that end we need to quantize  $\mathfrak{d}(\eta')$  as well as the terms coming from the diagonals in (106)–(109). We first treat the pseudoconvex side. In this case we need to consider the symbols on positive Heisenberg face, where the function  $|\zeta'| + \zeta_{n+1}$  vanishes.

We express the various terms in  $\mathfrak{p}_+^{\text{eo}}$ , near the positive contact line as sums of Heisenberg homogeneous terms

$$\begin{aligned} |\zeta'| &= \frac{\eta_0}{2}(1 + \mathfrak{D}_{-2}^H) \\ |\zeta'| - \zeta_{n+1} &= \eta_0(1 + \mathfrak{D}_{-2}^H), \quad |\zeta'| + \zeta_{n+1} = \frac{|\eta'|^2}{\eta_0}(1 + \mathfrak{D}_{-2}^H) \\ \mathfrak{d}(\zeta'') &= \sum_{j=1}^{n-1} [(i\eta_j + \eta_{n+j-1})e_j - (i\eta_j - \eta_{n+j-1})\epsilon_j]. \end{aligned} \quad (110)$$

Recall that the notation  $\mathfrak{D}_j^H$  denotes a term of Heisenberg order at most  $j$ . To find the model operators, we split  $\eta' = (w, \varphi)$ . Using the quantization rule in (20) (with the  $+$  sign) we see that

$$\begin{aligned} \eta_j - i\eta_{n+j-1} &\leftrightarrow C_j \stackrel{d}{=} (w_j - \partial_{w_j}) \\ \eta_j + i\eta_{n+j-1} &\leftrightarrow C_j^* \stackrel{d}{=} (w_j + \partial_{w_j}) \\ |\eta'|^2 &\leftrightarrow \mathfrak{H} \stackrel{d}{=} \sum_{j=1}^{n-1} w_j^2 - \partial_{w_j}^2. \end{aligned} \quad (111)$$

The following standard identities are useful

$$\sum_{j=1}^{n-1} C_j^* C_j - (n-1) = \mathfrak{H} = \sum_{j=1}^{n-1} C_j C_j^* + (n-1) \quad (112)$$

We let  $\mathfrak{D}_+$  denote the model operator defined, using the  $+$  quantization, by  $\mathfrak{d}(\zeta'')$ , it is given by

$$\mathfrak{D}_+ = i \sum_{j=1}^{n-1} [C_j e_j - C_j^* \epsilon_j]. \quad (113)$$

This is the model operator defined by  $\bar{\partial}_b + \bar{\partial}_b^*$  acting on  $\oplus_q \Lambda_b^{0,q} \otimes E$ . This operator can be split into even and odd parts,  $\mathfrak{D}_+^{\text{eo}}$  and these chiral forms of the operator are what appear in the model operators below. To keep the notation from becoming too complicated we suppress this dependence.

With these preliminaries, we can compute the model operators for  $\mathcal{P}_+^e$  and  $\text{Id} - \mathcal{P}_+^e$  in the positive contact direction. They are:

$${}^{eH}\sigma(\mathcal{P}_+^e)(+) = \begin{pmatrix} \text{Id} & \frac{\mathcal{D}_+}{\eta_0} \\ \frac{\mathcal{D}_+}{\eta_0} & \mathfrak{H}_{-(n-1)} \end{pmatrix} \quad {}^{eH}\sigma(\text{Id} - \mathcal{P}_+^e)(+) = \begin{pmatrix} \frac{\mathfrak{H}_{+n-1}}{\eta_0^2} & -\frac{\mathcal{D}_+}{\eta_0} \\ -\frac{\mathcal{D}_+}{\eta_0} & \text{Id} \end{pmatrix}. \quad (114)$$

The denominators involving  $\eta_0$  are meant to remind the reader of the Heisenberg orders of the various blocks:  $\eta_0^{-1}$  indicates a term of Heisenberg order  $-1$  and  $\eta_0^{-2}$  a term of order  $-2$ . Similar computations give the model operators in the odd case:

$${}^{eH}\sigma(\mathcal{P}_+^o)(+) = \begin{pmatrix} \frac{\mathfrak{H}_{-(n-1)}}{\eta_0^2} & \frac{\mathcal{D}_+}{\eta_0} \\ \frac{\mathcal{D}_+}{\eta_0} & \text{Id} \end{pmatrix} \quad {}^{eH}\sigma(\text{Id} - \mathcal{P}_+^o)(+) = \begin{pmatrix} \text{Id} & -\frac{\mathcal{D}_+}{\eta_0} \\ -\frac{\mathcal{D}_+}{\eta_0} & \frac{\mathfrak{H}_{+n-1}}{\eta_0^2} \end{pmatrix}. \quad (115)$$

Let  $\pi'_0 = {}^{eH}\sigma(+)(\mathcal{P}')$ ; this is a self adjoint rank one projection defined by a compatible almost complex structure on  $H$ , then

$${}^{eH}\sigma(\mathcal{R}'^e)(+) = \begin{pmatrix} \pi'_0 & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Id} & \mathbf{0} \end{pmatrix}, \quad {}^{eH}\sigma(\mathcal{R}'^o)(+) = \begin{pmatrix} 1 - \pi'_0 & 0 & \mathbf{0} \\ 0 & \text{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \quad (116)$$

We can now compute the model operators for  $\mathcal{T}_+^{\text{eo}}$  on the upper Heisenberg face.

**Proposition 10.** *If  $X$  is strictly pseudoconvex, then, at  $p \in bX$ , the model operators for  $\mathcal{T}_+^{\text{eo}}$ , in the positive contact direction, are given by*

$${}^{eH}\sigma(\mathcal{T}_+^e)(+) = \begin{pmatrix} \pi'_0 & 0 & -\left[1 - 2\pi'_0 & 0\right] \frac{\mathcal{D}_+}{\eta_0} \\ 0 & \mathbf{0} & \left[0 & \text{Id}\right] \frac{\mathcal{D}_+}{\eta_0} \\ \frac{\mathcal{D}_+}{\eta_0} & \frac{\mathfrak{H}_{-(n-1)}}{\eta_0^2} & \end{pmatrix} \quad (117)$$

$${}^{eH}\sigma(\mathcal{T}_+^o)(+) = \begin{pmatrix} \pi'_0 & 0 & \left[1 - 2\pi'_0 & 0\right] \frac{\mathcal{D}_+}{\eta_0} \\ 0 & \mathbf{0} & \left[0 & \text{Id}\right] \frac{\mathcal{D}_+}{\eta_0} \\ -\frac{\mathcal{D}_+}{\eta_0} & \frac{\mathfrak{H}_{+(n-1)}}{\eta_0^2} & \end{pmatrix}. \quad (118)$$

*Proof.* Observe that the Heisenberg orders of the blocks in (117) and (118) are

$$\begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}. \quad (119)$$

Proposition 6 shows that all other terms in the symbol of the Calderon projector lead to diagonal terms of Heisenberg order at most  $-4$ , and off diagonal terms of order at most  $-2$ . This, along with the computations above, completes the proof of the proposition.  $\square$

A similar analysis applies for the pseudoconcave case. Here we use that, near the negative contact line, we have

$$\begin{aligned} |\zeta'| &= -\frac{\eta_0}{2}(1 + \mathfrak{D}_{-2}^H) \\ |\zeta'| + \zeta_{n+1} &= -\eta_0(1 + \mathfrak{D}_{-2}^H), \quad |\zeta'| - \zeta_{n+1} = -\frac{|\eta'|^2}{\eta_0}(1 + \mathfrak{D}_{-2}^H) \end{aligned} \quad (120)$$

The formula for  $\mathfrak{d}(\zeta'')$  is the same, however, the quantization rule is slightly different, note the  $\pm$  in equation (20). Using the  $-$  sign we get the following quantizations:

$$\begin{aligned} \eta_j - i\eta_{n+j-1} &\leftrightarrow C_j^* \stackrel{d}{=} (w_j + \partial_{w_j}) \\ \eta_j + i\eta_{n+j-1} &\leftrightarrow C_j \stackrel{d}{=} (w_j - \partial_{w_j}) \\ |\eta'|^2 &\leftrightarrow \mathfrak{H} \stackrel{d}{=} \sum_{j=1}^{n-1} w_j^2 - \partial_{w_j}^2. \end{aligned} \quad (121)$$

With the  $-$  sign we therefore obtain that the model operator defined by  $\mathfrak{d}(\zeta'')$  is

$$\mathfrak{D}_- = i \sum_{j=1}^{n-1} [C_j^* e_j - C_j \epsilon_j]. \quad (122)$$

Using computations identical to those above, we find that the model operators for  $\mathcal{P}_-^{\text{eo}}$  and  $\text{Id} - \mathcal{P}_-^{\text{eo}}$ , along the negative contact direction are:

$$\begin{aligned} {}^{eH}\sigma(\mathcal{P}_-^{\text{e}})(-) &= \begin{pmatrix} \text{Id} & \frac{\mathfrak{D}_-}{|\eta_0|} \\ \frac{\mathfrak{D}_-}{|\eta_0|} & \frac{\mathfrak{H} + (n-1)}{|\eta_0|^2} \end{pmatrix} & {}^{eH}\sigma(\text{Id} - \mathcal{P}_-^{\text{e}})(-) &= \begin{pmatrix} \frac{\mathfrak{H} - (n-1)}{|\eta_0|^2} & -\frac{\mathfrak{D}_-}{|\eta_0|} \\ -\frac{\mathfrak{D}_-}{|\eta_0|} & \text{Id} \end{pmatrix} \\ {}^{eH}\sigma(\mathcal{P}_-^{\text{o}})(-) &= \begin{pmatrix} \frac{\mathfrak{H} + (n-1)}{|\eta_0|^2} & \frac{\mathfrak{D}_-}{|\eta_0|} \\ \frac{\mathfrak{D}_-}{|\eta_0|} & \text{Id} \end{pmatrix} & {}^{eH}\sigma(\text{Id} - \mathcal{P}_-^{\text{o}})(-) &= \begin{pmatrix} \text{Id} & -\frac{\mathfrak{D}_-}{|\eta_0|} \\ -\frac{\mathfrak{D}_-}{|\eta_0|} & \frac{\mathfrak{H} - (n-1)}{|\eta_0|^2} \end{pmatrix}. \end{aligned} \quad (123)$$

The Heisenberg orders of the various blocks are indicated by powers of  $|\eta_0|$ , as we evaluate the symbols along the hyperplane  $\eta_0 = -1$  to obtain the model operators. Let  $\bar{\pi}'_0$  denote the rank one projection, which is the principal symbol of  $\bar{\mathcal{P}}'$ . If  $n$  is even, then

$${}^{eH}\sigma(\mathcal{R}'^{\text{e}})(-) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Id} & 0 \\ \mathbf{0} & 0 & 1 - \bar{\pi}'_0 \end{pmatrix} \quad {}^{eH}\sigma(\mathcal{R}'^{\text{o}})(-) = \begin{pmatrix} \text{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & 0 & \bar{\pi}'_0 \end{pmatrix}. \quad (124)$$



If  $n$  is odd, then

$${}^{eH}\sigma(\mathcal{R}'^e)(-) = \begin{pmatrix} \mathbf{0} & 0 & \mathbf{0} \\ 0 & \bar{\pi}'_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \text{Id} \end{pmatrix} \quad {}^{eH}\sigma(\mathcal{R}'^o)(-) = \begin{pmatrix} \text{Id} & 0 & \mathbf{0} \\ 0 & 1 - \bar{\pi}'_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \quad (125)$$

**Proposition 11.** *If  $X$  is strictly pseudoconcave, then at  $p \in bX$ , the model operators for  $\mathcal{T}_-^{\text{eo}}$ , in the negative contact direction, are given, for  $n$  even by,*

$${}^{eH}\sigma(\mathcal{T}_-^e)(-) = \begin{pmatrix} \frac{\mathfrak{H}_-(n-1)}{|\eta_0|^2} & -\frac{\mathfrak{D}_-}{|\eta_0|} \\ \left[ \text{Id} & 0 \right] \frac{\mathfrak{D}_-}{|\eta_0|} & \mathbf{0} & 0 \\ 0 & 1 - 2\bar{\pi}'_0 & 0 & \bar{\pi}'_0 \end{pmatrix} \quad (126)$$

$${}^{eH}\sigma(\mathcal{T}_-^o)(-) = \begin{pmatrix} \frac{\mathfrak{H}_+(n-1)}{|\eta_0|^2} & \frac{\mathfrak{D}_-}{|\eta_0|} \\ -\left[ \text{Id} & 0 \right] \frac{\mathfrak{D}_-}{|\eta_0|} & \mathbf{0} & 0 \\ 0 & 1 - 2\bar{\pi}'_0 & 0 & \bar{\pi}'_0 \end{pmatrix} \quad (127)$$

If  $n$  is odd, then

$${}^{eH}\sigma(\mathcal{T}_-^e)(-) = \begin{pmatrix} \mathbf{0} & 0 & -\left[ \text{Id} & 0 \right] \frac{\mathfrak{D}_-}{|\eta_0|} \\ 0 & \bar{\pi}'_0 & \frac{\mathfrak{H}_+(n-1)}{|\eta_0|^2} \\ \frac{\mathfrak{D}_-}{|\eta_0|} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (128)$$

$${}^{eH}\sigma(\mathcal{T}_-^o)(-) = \begin{pmatrix} \mathbf{0} & 0 & \left[ \text{Id} & 0 \right] \frac{\mathfrak{D}_-}{|\eta_0|} \\ 0 & \bar{\pi}'_0 & \frac{\mathfrak{H}_-(n-1)}{|\eta_0|^2} \\ -\frac{\mathfrak{D}_-}{|\eta_0|} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (129)$$

*Proof.* In the even case the Heisenberg orders of the blocks are

$$\begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}, \quad (130)$$

while in the odd case they are

$$\begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}. \quad (131)$$

As before, the proposition follows from this observation, the computations above, and Proposition 6.  $\square$

This brings us to the main technical result in this paper.

**Theorem 1.** *If  $X$  is strictly pseudoconvex (pseudoconcave),  $E \rightarrow X$  a compatible complex vector bundle and  $\mathcal{S}$  ( $\bar{\mathcal{S}}$ ) a generalized (conjugate) Szegő projector, defined by a compatible deformation of the almost complex structure on  $H$  induced by the embedding of  $bX$  as the boundary of  $X$ , then the operators  $\mathcal{T}_{E+}^{\text{eo}}$  ( $\mathcal{T}_{E-}^{\text{eo}}$ ) are graded elliptic elements of the extended Heisenberg calculus. If  $X$  is pseudoconvex or  $X$  is pseudoconcave and  $n$  is odd, then, as block matrices, the parametrices for  $\mathcal{T}_{E\pm}^{\text{eo}}$  have Heisenberg orders*

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \quad (132)$$

*If  $X$  is pseudoconcave and  $n$  is even, then, as block matrices, the parametrices for  $\mathcal{T}_{E-}^{\text{eo}}$  have Heisenberg orders*

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \quad (133)$$

*Proof.* Using standard symbolic arguments, to prove the theorem it suffices to construct operators  $\mathcal{U}_{\pm}^{\text{eo}}, \mathcal{V}_{\pm}^{\text{eo}}$ , in the extended Heisenberg calculus, so that

$$\begin{aligned} \mathcal{U}_{\pm}^{\text{eo}} \mathcal{T}_{\pm}^{\text{eo}} &= \text{Id} + \mathcal{O}_{-1,-1}^{eH} \\ \mathcal{T}_{\pm}^{\text{eo}} \mathcal{V}_{\pm}^{\text{eo}} &= \text{Id} + \mathcal{O}_{-1,-1}^{eH}. \end{aligned} \quad (134)$$

As usual, this just amounts to the invertibility of the principal symbols. Away from the positive (negative) Heisenberg face this is clear, as the operator is classically elliptic of order 0. Along the Heisenberg face, the operator is graded so a little discussion is required. For a graded Heisenberg operator  $\mathcal{A}$ , denote the matrix of model operators by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

The blocks of  $A$  have orders either  $i + j - 4$  or  $2 - (i + j)$ . Suppose the model operators are invertible with inverses given by

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

The orders of the blocks of  $B$  are either  $4 - (i + j)$  or  $(i + j) - 2$ . Let  $\mathcal{B}$  denote an extended Heisenberg operator with principal symbol given by  $B$ . Then, in the first case, we have

$$\mathcal{A}\mathcal{B} = \begin{pmatrix} \text{Id} + \mathcal{E}_{-1} & \mathcal{E}_{-1} \\ \mathcal{E}_0 & \text{Id} + \mathcal{E}_{-1} \end{pmatrix} \quad \mathcal{B}\mathcal{A} = \begin{pmatrix} \text{Id} + \mathcal{F}_{-1} & \mathcal{F}_0 \\ \mathcal{F}_{-1} & \text{Id} + \mathcal{F}_{-1} \end{pmatrix}. \quad (135)$$

Here  $\mathcal{E}_j, \mathcal{F}_j$  denote operators with the indicated Heisenberg orders. Setting

$$\mathcal{B}_r = \mathcal{B} \begin{pmatrix} \text{Id} & 0 \\ -\mathcal{E}_0 & \text{Id} \end{pmatrix}, \quad \mathcal{B}_l = \begin{pmatrix} \text{Id} & -\mathcal{F}_0 \\ 0 & \text{Id} \end{pmatrix} \mathcal{B} \quad (136)$$

gives the right and left parametrices called for in equation (134). A similar argument works if the orders of  $A$  are  $(i + j) - 2$ . Thus it suffices to show that the model operators  ${}^{eH}\sigma(\mathcal{T}_{\pm}^{\text{co}})(\pm)$  are invertible, in the graded sense used above. This is done in the next two sections.  $\square$

*Remark 7.* In the analysis below we show that the order 2 block in the parametrix is absent, hence it is not necessary to correct  $\mathcal{B}$  with a triangular matrix.

## 6 Invertibility of the model operators with classical Szegő projectors

In this section we prove Theorem 1 with the additional assumption that the principal symbol of  $\mathcal{S}'$  ( $\bar{\mathcal{S}}'$ ) agrees with the principal symbol,  $\pi_0, (\bar{\pi}_0)$  of the classical Szegő projector (conjugate Szegő projector) defined by the CR-structure on  $bX$ . In this case the structure of the model operators is a little simpler. It is not necessary to assume that the CR-structure on  $bX$  is embeddable, as all that we require are the symbolic identities

$$\sigma_1(\bar{\partial}_b \mathcal{S}) = 0, \quad \text{and} \quad \sigma_1(\bar{\partial}_b^* \bar{\mathcal{S}}) = 0. \quad (137)$$

Note that  $\mathcal{S}_E$  (or  $\bar{\mathcal{S}}_E$ ) are projectors onto sections of  $\Lambda_b^{\text{co}} \otimes E \downarrow_{bX}$ . Because the complex structure of  $E$  is compatible with that of  $X$ , using the holomorphic frame introduced in (36), we see that

$$\sigma(\mathcal{S}_E) = \sigma(\mathcal{S}) \otimes \text{Id}_E, \quad \sigma(\bar{\mathcal{S}}_E) = \sigma(\bar{\mathcal{S}}) \otimes \text{Id}_E. \quad (138)$$

Thus we may continue to suppress the explicit dependence on  $E$ .

The operators  $\{C_j\}$  are called the creation operators and the operators  $\{C_j^*\}$  the annihilation operators. They satisfy the commutation relations

$$[C_j, C_k] = [C_j^*, C_k^*] = 0, \quad [C_j, C_k^*] = -2\delta_{jk} \quad (139)$$

The operators  $\mathcal{D}_{\pm}$  act on sums of the form

$$\omega = \sum_{k=0}^{n-1} \sum_{I \in \mathcal{J}'_k} f_I \bar{\omega}^I, \quad (140)$$

here  $\mathcal{J}'_k$  are increasing multi-indices of length  $k$ . The coefficients,  $\{f_I\}$  are sections of the appropriate holomorphic bundle, assumed trivialized near  $p$ , as described in Section 2, with vanishing connection coefficients. We refer to the terms with  $|I| = k$  as the terms of degree  $k$ . For an increasing  $k$ -multi-index  $I = 1 \leq i_1 < i_2 < \cdots < i_k \leq n-1$ ,  $\bar{\omega}^I$  is defined by

$$\bar{\omega}^I = \frac{1}{2^{\frac{k}{2}}} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_k}. \quad (141)$$

We first describe the relationships among the operators  $\pi_0$ ,  $\mathcal{D}_+$ ,  $\bar{\pi}_0$  and  $\mathcal{D}_-$ .

**Lemma 7.** *Let  $\pi_0$  and  $\bar{\pi}_0$  be the symbols of the classical Szegő projector and conjugate Szegő projector respectively, then*

$$\begin{bmatrix} \pi_0 & 0 \\ 0 & \mathbf{0} \end{bmatrix} \mathcal{D}_+ = 0 \text{ and } \begin{bmatrix} \mathbf{0} & 0 \\ 0 & \bar{\pi}_0 \end{bmatrix} \mathcal{D}_- = 0 \quad (142)$$

*Proof.* The range of  $\mathcal{D}_+$  in degree 0, where  $\pi_0$  acts, is spanned by expressions of the form

$$\sum_{j=1}^{n-1} C_j f_I e_j \bar{\omega}^I, \text{ with } |I| = 1. \quad (143)$$

Taking the adjoint, the first identity in (137) is equivalent to  $\pi_0 C_j = 0$  for all  $j$ , and the lemma follows in this case. The range of  $\mathcal{D}_-$  in degree  $n-1$ , where  $\bar{\pi}_0$  acts, is spanned by expressions of the form

$$\sum_{j=1}^{n-1} C_j f_I e_j \bar{\omega}^I, \text{ with } |I| = n-2. \quad (144)$$

Once again, (137) implies that  $\bar{\pi}_0 C_j = 0$ ; the proof of the lemma is complete.  $\square$

This lemma simplifies the analysis of the model operators for  $\mathcal{T}_\pm^{\text{eo}}$ . The following lemma is useful in finding their inverses.

**Lemma 8.** *Let  $\Pi_q$  denote projection onto the terms of degree  $q$ ,*

$$\Pi_q \omega = \sum_{I \in \mathcal{J}'_q} f_I \bar{\omega}^I. \quad (145)$$

*The operators  $\mathcal{D}_\pm$  satisfy the identities*

$$\mathcal{D}_+^2 = \sum_{j=1}^{n-1} C_j C_j^* \otimes \text{Id} + \sum_{q=0}^{n-1} 2q \Pi_q, \quad \mathcal{D}_-^2 = \sum_{j=1}^{n-1} C_j C_j^* \otimes \text{Id} + \sum_{q=0}^{n-1} 2(n-1-q) \Pi_q. \quad (146)$$

*Proof.* In the proof of this lemma we make extensive usage of the following classical identities, whose verification we leave to the reader.

**Lemma 9.** *The operators  $\{e_j, \epsilon_j\}$  satisfy the following relations*

$$\begin{aligned} e_j e_k &= -e_k e_j, & \epsilon_j \epsilon_k &= -\epsilon_k \epsilon_j \text{ for all } j, k, \\ \epsilon_j e_k &= -e_j \epsilon_k \text{ if } j \neq k. \end{aligned} \quad (147)$$

For  $j = k$  we have

$$\epsilon_j e_j \bar{\omega}^I = \begin{cases} \bar{\omega}^I & \text{if } j \in I \\ 0 & \text{if } j \notin I \end{cases} \quad e_j \epsilon_j \bar{\omega}^I = \begin{cases} \bar{\omega}^I & \text{if } j \notin I \\ 0 & \text{if } j \in I \end{cases} \quad (148)$$

We start with  $\mathcal{D}_+$ , using the lemma we obtain that

$$\begin{aligned} \mathcal{D}_+^2 &= - \sum_{j \neq k} \left( \frac{1}{2} [C_j, C_k] e_j e_k + \frac{1}{2} [C_j^*, C_k^*] \epsilon_j \epsilon_k - [C_j, C_k^*] e_j \epsilon_k \right) + \\ &\quad \sum_{j=1}^{n-1} [C_j C_j^* e_j \epsilon_j + C_j^* C_j \epsilon_j e_j]. \end{aligned} \quad (149)$$

It follows from the commutation relations that the sum over  $j \neq k$  vanishes. Using (139) we rewrite the second sum as

$$\sum_{j=1}^{n-1} [C_j C_j^* e_j \epsilon_j + (C_j C_j^* + 2) \epsilon_j e_j]. \quad (150)$$

The statement of the lemma follows easily from (150), and the fact that

$$\sum_{j=1}^{n-1} \epsilon_j e_j \bar{\omega}^I = |I| \bar{\omega}^I. \quad (151)$$

The argument for  $\mathcal{D}_-$  is quite similar. The analogous sum over  $j \neq k$  vanishes and we see that

$$\begin{aligned} \mathcal{D}_-^2 &= \sum_{j=1}^{n-1} [C_j^* C_j e_j \epsilon_j + C_j C_j^* \epsilon_j e_j] \\ &= \sum_{j=1}^{n-1} [(C_j C_j^* + 2) e_j \epsilon_j + C_j C_j^* \epsilon_j e_j]. \end{aligned} \quad (152)$$

The proof is completed as before using

$$\sum_{j=1}^{n-1} e_j \epsilon_j \bar{\omega}^I = (n-1-|I|)\bar{\omega}^I. \quad (153)$$

instead of (151).  $\square$

Before we construct the explicit inverses, we show that  ${}^{eH}\sigma(\mathcal{T}_{\pm}^{\text{eo}})(\pm)$  are Fredholm elements (in the graded sense), in the isotropic algebra. Notice that this is a purely symbolic statement in the isotropic algebra. The isotropic blocks have orders

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad (154)$$

on the pseudoconvex side and on the pseudoconcave side if  $n$  is odd, and orders

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad (155)$$

on the pseudoconcave side if  $n$  is even. The leading order part in the isotropic algebra is independent of the choice of generalized (conjugate) Szegő projector. In the former case we can think of the operator as defining a map from  $H^1(\mathbb{R}^{n-1}; E_1) \oplus H^2(\mathbb{R}^{n-1}; E_2)$  to  $H^1(\mathbb{R}^{n-1}; F_1) \oplus H^0(\mathbb{R}^{n-1}; F_2)$  for appropriate vector bundles  $E_1, E_2, F_1, F_2$ . In the later case the map is from  $H^2(\mathbb{R}^{n-1}; E_1) \oplus H^1(\mathbb{R}^{n-1}; E_2)$  to  $H^0(\mathbb{R}^{n-1}; F_1) \oplus H^1(\mathbb{R}^{n-1}; F_2)$ . It is as maps between these spaces that the model operators are Fredholm.

**Proposition 12.** *The model operators,  ${}^{eH}\sigma(\mathcal{T}_{\pm}^{\text{eo}})(\pm)$ , are graded Fredholm elements in the isotropic algebra.*

*Proof.* As noted above this is a purely symbolic statement in the isotropic algebra. It suffices to show that the model operators are invertible, by appropriately graded elements of the isotropic algebra, up to an error of lower order. Equation (149) shows that

$$[\mathfrak{H}^{-1}\mathcal{D}_{\pm}]\mathcal{D}_{\pm} = \mathcal{D}_{\pm}[\mathfrak{H}^{-1}\mathcal{D}_{\pm}] = \text{Id} + \mathfrak{D}_{-1}^{\text{iso}}. \quad (156)$$

Here  $\mathfrak{D}_j^{\text{iso}}$  is a term of order at most  $j$  in the isotropic algebra. Up to lower order terms, the model operators are

$$\begin{aligned} {}^{eH}\sigma(\mathcal{T}_+^{\text{eo}})(+) &= \begin{pmatrix} \mathbf{0} & \mp\mathcal{D}_+ \\ \pm\mathcal{D}_+ & \mathfrak{H} \end{pmatrix} \\ n \text{ odd} \quad {}^{eH}\sigma(\mathcal{T}_-^{\text{eo}})(-) &= \begin{pmatrix} \mathbf{0} & \mp\mathcal{D}_- \\ \pm\mathcal{D}_- & \mathfrak{H} \end{pmatrix} \\ n \text{ even} \quad {}^{eH}\sigma(\mathcal{T}_-^{\text{eo}})(-) &= \begin{pmatrix} \mathfrak{H} & \mp\mathcal{D}_- \\ \pm\mathcal{D}_- & \mathbf{0} \end{pmatrix} \end{aligned} \quad (157)$$

The isotropic principal symbol of  $\mathfrak{H}$  is  $|\eta'|^2$ . For these computations, we let  $\mathfrak{H}^{-1}$  denote a model operator with isotropic principal symbol  $|\eta'|^{-2}$ . Using (156), we see that the operators in (157) have right parametrices:

$$\begin{aligned}
& \begin{pmatrix} \mathbf{0} & \mp \mathcal{D}_+ \\ \pm \mathcal{D}_+ & \mathfrak{H} \end{pmatrix} \begin{pmatrix} \text{Id} & \pm \mathfrak{H}^{-1} \mathcal{D}_+ \\ \mp \mathfrak{H}^{-1} \mathcal{D}_+ & \mathbf{0} \end{pmatrix} = \text{Id} + \mathcal{D}_{-1}^{\text{iso}} \\
n \text{ odd} \quad & \begin{pmatrix} \mathbf{0} & \mp \mathcal{D}_- \\ \pm \mathcal{D}_- & \mathfrak{H} \end{pmatrix} \begin{pmatrix} \text{Id} & \pm \mathfrak{H}^{-1} \mathcal{D}_- \\ \mp \mathfrak{H}^{-1} \mathcal{D}_- & \mathbf{0} \end{pmatrix} = \text{Id} + \mathcal{D}_{-1}^{\text{iso}} \\
n \text{ even} \quad & \begin{pmatrix} \mathfrak{H} & \mp \mathcal{D}_- \\ \pm \mathcal{D}_- & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \pm \mathfrak{H}^{-1} \mathcal{D}_- \\ \mp \mathfrak{H}^{-1} \mathcal{D}_- & \text{Id} \end{pmatrix} = \text{Id} + \mathcal{D}_{-1}^{\text{iso}}
\end{aligned} \tag{158}$$

The same model operators provide left parametrices as well. This proves the proposition.  $\square$

*Remark 8.* Note that the block of the principal symbols of the parametrices, expected to have order 2, actually vanishes. As a result, the inverses of the model operators have Heisenberg order at most 1, which in turn allows us to deduce the standard subelliptic  $\frac{1}{2}$ -estimates for these boundary value problems.

The operators  $\mathcal{D}_\pm^e$  and  $\mathcal{D}_\pm^o$  are adjoint to one another. From (146) and the well known properties of the harmonic oscillator, it is clear that  $\mathcal{D}_+^e \mathcal{D}_+^o$  is invertible. As  $\mathcal{D}_+^e$  has a one dimensional null space this easily implies that  $\mathcal{D}_+^o$  is injective with image orthogonal to the range of  $\pi_0$ , while  $\mathcal{D}_+^e$  is surjective. The analogous statements for  $\mathcal{D}_-^e$  depend on the parity of  $n$ , as  $\mathcal{D}_-^e$  has a null space of dimension one spanned by the forms of degree  $n - 1$  in the image of  $\bar{\pi}_0$ . If  $n$  is even, then  $\mathcal{D}_-^e$  is injective and  $\mathcal{D}_-^o$  is surjective, with a one dimensional null space spanned by the range of  $\bar{\pi}_0$ . If  $n$  is odd, then  $\mathcal{D}_-^o$  is injective and  $\mathcal{D}_-^e$  is surjective. With these observations we easily invert the model operators.

We begin with the  $+$  side. Let  $[\mathcal{D}_+^e]^{-1}u$  denote the unique solution to the equation

$$\mathcal{D}_+^e v = u,$$

orthogonal to the null space of  $\mathcal{D}_+^e$ . We let

$$\overset{\square}{u} = \begin{pmatrix} 1 - \pi_0 & 0 \\ 0 & \text{Id} \end{pmatrix} u; \tag{159}$$

this is the projection onto the range of  $\mathcal{D}_+^o$  and

$$u_0 = \begin{pmatrix} \pi_0 & 0 \\ 0 & \mathbf{0} \end{pmatrix} u, \tag{160}$$

denotes the projection onto the nullspace of  $\mathcal{D}_+^e$ . We let  $[\mathcal{D}_+^o]^{-1}$  denote the unique solution to

$$\mathcal{D}_+^o v = \overset{\square}{u}.$$

Proposition 12 shows that these partial inverses are isotropic operators of order  $-1$ .

With this notation we find the inverse of  ${}^{eH}\sigma(\mathcal{T}_+^e)(+)$ . The vector  $[u, v]$  satisfies

$${}^{eH}\sigma(\mathcal{T}_+^e)(+) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad (161)$$

if and only if

$$\begin{aligned} u &= a_0 + [\mathcal{D}_+^e]^{-1}(\mathfrak{H} - (n-1))[\mathcal{D}_+^o]^{-1}\overset{\square}{a} + [\mathcal{D}_+^e]^{-1}b \\ v &= -[\mathcal{D}_+^o]^{-1}\overset{\square}{a}. \end{aligned} \quad (162)$$

Writing out the inverse as a block matrix of operators, with appropriate factors of  $\eta_0$  included, gives:

$$[{}^{eH}\sigma(\mathcal{T}_+^e)(+)]^{-1} = \begin{bmatrix} \begin{pmatrix} \pi_0 & 0 \\ 0 & \mathbf{0} \end{pmatrix} + [\mathcal{D}_+^e]^{-1}(\mathfrak{H} - (n-1))[\mathcal{D}_+^o]^{-1} \begin{pmatrix} 1 - \pi_0 & 0 \\ 0 & \text{Id} \end{pmatrix} & \eta_0[\mathcal{D}_+^e]^{-1} \\ -\eta_0[\mathcal{D}_+^o]^{-1} \begin{pmatrix} 1 - \pi_0 & 0 \\ 0 & \text{Id} \end{pmatrix} & \mathbf{0} \end{bmatrix} \quad (163)$$

The isotropic operators  $[\mathcal{D}_+^{eo}]^{-1}$  are of order  $-1$ , whereas  $[\mathcal{D}_+^e]^{-1}(\mathfrak{H} - (n-1))[\mathcal{D}_+^o]^{-1}$  is of order zero. The Schwartz kernel of  $\pi_0$  is rapidly decreasing. From this we conclude that the Heisenberg orders, as a block matrix, of the parametrix for  $[{}^{eH}\sigma(\mathcal{T}_+^e)(+)]$  are

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \quad (164)$$

We get a 1 in the lower right corner because the principal symbol, a priori of order 2, of this entry vanishes. The solution for the odd case is given by

$$\begin{aligned} u &= a_0 + [\mathcal{D}_+^e]^{-1}(\mathfrak{H} + (n-1))[\mathcal{D}_+^o]^{-1}\overset{\square}{a} - [\mathcal{D}_+^e]^{-1}b \\ v &= [\mathcal{D}_+^o]^{-1}\overset{\square}{a}. \end{aligned} \quad (165)$$

Once again the 2, 2 block of  $[{}^{eH}\sigma(\mathcal{T}_+^o)(+)]^{-1}$  vanishes, and the principal symbol has the Heisenberg orders indicated in (164).



We complete this analysis by writing the solutions to

$${}^{eH}\sigma(\mathcal{T}_-^{\text{eo}})(-)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad (166)$$

in the various cases. For  $n$  even, the operator  $\mathcal{D}_-^e$  is injective and  $\mathcal{D}_-^o$  has a one dimensional null space. We let  $u_0$  denote the projection of  $u$  onto the null space and  $\overset{\square}{u}$  the projection onto its complement. With the notation for the partial inverses of  $\mathcal{D}_-^{\text{eo}}$  analogous to that used in the  $+$  case, we have the solution operators:

$$\begin{aligned} \text{even} \quad u &= [\mathcal{D}_-^e]^{-1}\overset{\square}{b} \\ v &= b_0 + [\mathcal{D}_-^o]^{-1}(\mathfrak{H} - (n-1))[\mathcal{D}_-^e]^{-1}\overset{\square}{b} - [\mathcal{D}_-^o]^{-1}a \\ \text{odd} \quad u &= -[\mathcal{D}_-^e]^{-1}\overset{\square}{b} \\ v &= b_0 + [\mathcal{D}_-^o]^{-1}(\mathfrak{H} - (n-1))[\mathcal{D}_-^e]^{-1}\overset{\square}{b} + [\mathcal{D}_-^o]^{-1}a \end{aligned} \quad (167)$$

Here and in (168), “even” and “odd” refer to the parity of the spinor. For  $n$  even, the operator  $\mathcal{D}_-^o$  is injective and  $\mathcal{D}_-^e$  has a one dimensional null space. We let  $u_0$  denote the projection of  $u$  onto the null space and  $\overset{\square}{u}$  the projection onto its complement.

$$\begin{aligned} \text{even} \quad u &= a_0 + [\mathcal{D}_-^e]^{-1}(\mathfrak{H} + (n-1))[\mathcal{D}_-^o]^{-1}\overset{\square}{a} + [\mathcal{D}_-^e]^{-1}b \\ v &= -[\mathcal{D}_-^o]^{-1}\overset{\square}{a}, \\ \text{odd} \quad u &= a_0 + [\mathcal{D}_-^e]^{-1}(\mathfrak{H} + (n-1))[\mathcal{D}_-^o]^{-1}\overset{\square}{a} - [\mathcal{D}_-^e]^{-1}b \\ v &= [\mathcal{D}_-^o]^{-1}\overset{\square}{a}. \end{aligned} \quad (168)$$

If  $n$  is even, then the  $(1, 1)$  block of the principal symbols of  $[{}^{eH}\sigma(\mathcal{T}_-^{\text{eo}})(-)]^{-1}$  vanishes and therefore the Heisenberg orders of the blocks of the parametrices are

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \quad (169)$$

If  $n$  is odd, then the  $(2, 2)$  block the principal symbols of  $[{}^{eH}\sigma(\mathcal{T}_-^{\text{eo}})(-)]^{-1}$  vanishes and therefore the Heisenberg orders of the blocks of the parametrices are

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \quad (170)$$

For the case of classical Szegő projectors, Lemma 7 implies that the model operators satisfy

$$[{}^{eH}\sigma(\mathcal{T}_\pm^{\text{eo}})(\pm)]^* = {}^{eH}\sigma(\mathcal{T}_\pm^{\text{oe}})(\pm). \quad (171)$$

From Proposition 12 we know that these are Fredholm operators. Since we have shown that all the operators  ${}^{eH}\sigma(\mathcal{T}_{\pm}^{\text{eo}})(\pm)$  are surjective, i.e., have a left inverse, it follows that all are in fact injective and therefore invertible. In all cases this completes the proof of Theorem 1 in the special case that the principal symbols of  $\mathcal{P}'$  or  $\bar{\mathcal{P}}'$  agree with those of the classical Szegő projector or conjugate Szegő projector.

## 7 Invertibility of the model operators with generalized Szegő projectors

The proof of Theorem 1, with generalized Szegő projectors, is not much different from that covered in the previous section. We show here that the parametrices for  ${}^{eH}\sigma(\mathcal{T}_{-}^{\text{eo}})(-)$  differ from those with classical Szegő projectors (or conjugate Szegő projectors) by operators of finite rank. The Schwartz kernels of the correction terms are in the Hermite ideal, and so do not affect the Heisenberg orders of the blocks in the parametrix. As before the principal symbol in the (2, 2) block (or (1, 1) block, where appropriate) vanishes.

In [6] we characterize the set of compatible almost complex structures in the following way:

**Lemma 10.** *Let  $J_1$  and  $J_2$  be compatible almost complex structures on the co-oriented contact manifold  $Y$ . For each  $p \in Y$  there is a Darboux coordinate system centered at  $p$ , so that, if  $(\eta_0, \eta')$  are the linear coordinates on  $T_p^*Y$ , then*

$$h_{J_1}(\eta) = \sum_{j=1}^{2(n-1)} \eta_j^2 \text{ and } h_{J_2}(\eta) = \sum_{j=1}^{n-1} [\mu_j \eta_j^2 + \mu_j^{-1} \eta_{j+n-1}^2] \quad (172)$$

for positive numbers  $(\mu_1, \dots, \mu_{n-1})$ .

We split the coordinates  $\eta'$  into  $(w_1, \dots, w_{n-1}; \varphi_1, \dots, \varphi_{n-1})$ . Let  $\mathfrak{H}_1$  and  $\mathfrak{H}_2$  denote the harmonic oscillators obtained by quantizing these symbols with respect to this splitting, then the ground states for these operators are spanned by

$$\begin{aligned} v_0^1 &= m^1 \exp \left[ -\frac{1}{2} \sum_{j=1}^{n-1} w_j^2 \right] \\ v_0^2 &= m^2 \exp \left[ -\frac{1}{2} \sum_{j=1}^{n-1} \left( \frac{w_j}{\mu_j} \right)^2 \right], \end{aligned} \quad (173)$$

with  $m^j$  chosen so that  $\|v_0^j\|_{L^2} = 1$ . From these expressions we easily deduce the following result.

**Lemma 11.** *If  $J_1$  and  $J_2$  are compatible almost complex structures, then, with respect to the  $L^2$ -inner product on  $\mathbb{R}^{n-1}$  defined by a choice of splitting of  $H_p$ , we have*

$$\langle v_0^1, v_0^2 \rangle > 0. \quad (174)$$

*On a compact manifold, this inner product is a smooth function, bounded below by a positive constant. If  $\pi_0^j$  denote the projections onto the respective vacuum states, then*

$$\langle v_0^1, v_0^2 \rangle^2 = \text{tr } \pi_0^1 \pi_0^2, \quad (175)$$

*is therefore well defined independent of the choice of quantization.*

*Proof.* Only the second statement requires a proof. In terms of any Darboux coordinate system, the projection onto the vacuum state has Schwartz kernel

$$v_0^j \otimes v_0^{jt}. \quad (176)$$

This shows that (175) is correct. It is shown in [6] that the trace is independent of the choice of quantization.  $\square$

For our applications, the following corollary is very useful.

**Corollary 2.** *Let  $J_1$  and  $J_2$  be compatible almost complex structures, In a choice of quantization we define the model operator*

$$P_{21} = \frac{\pi_0^2 \pi_0^1}{\text{tr } \pi_0^2 \pi_0^1}. \quad (177)$$

*This operator is globally defined, belongs to the Hermite ideal, and satisfies*

$$\pi_0^1 P_{21} = \pi_0^1. \quad (178)$$

*Proof.* The first statement follows from Lemma 11 and the fact that the symbols of the projectors are globally defined. The relation in (177) is easily proved using the representations of  $\pi_0^j$  given in (176). The fact that  $P_{21}$  belongs to the Hermite ideal is again immediate from the fact that its Schwartz kernel belongs to  $\mathcal{S}(\mathbb{R}^{2(n-1)})$ .  $\square$

*Remark 9.* The relation (178) implies that

$$\pi_0^1 (P_{21} \pi_0^1 - \pi_0^1) = 0. \quad (179)$$

An analogous result, which we use in the sequel, holds for generalized conjugate Szegő projectors.

With these preliminaries, we can now complete the proof of Theorem 1. For clarity, we use  ${}^{eH}\sigma(\mathcal{T}'_{\pm})(\pm)$  to denote the model operators with the classical (conjugate) Szegő projection, and  ${}^{eH}\sigma(\mathcal{T}^{\text{eo}}_{\pm})(\pm)$  with a generalized Szegő projection (or generalized conjugate Szegő projection).

**Proposition 13.** *If  $\pi'_0(\bar{\pi}'_0)$  is a generalized (conjugate) Szegő projection, which is a deformation of  $\pi_0, (\bar{\pi}_0)$ , then  ${}^{eH}\sigma(\mathcal{T}'_{\pm})(\pm)$  are invertible elements of the isotropic algebra. The inverses satisfy*

$$[{}^{eH}\sigma(\mathcal{T}'_{+}(\text{+}))^{-1}]^{-1} = [{}^{eH}\sigma(\mathcal{T}^{\text{eo}}_{+}(\text{+}))^{-1}]^{-1} + \begin{pmatrix} c_1 & c_2 \\ c_3 & 0 \end{pmatrix}, \quad (180)$$

if  $n$  is even, then

$$[{}^{eH}\sigma(\mathcal{T}'_{-}(\text{-}))^{-1}]^{-1} = [{}^{eH}\sigma(\mathcal{T}^{\text{eo}}_{-}(\text{-}))^{-1}]^{-1} + \begin{pmatrix} 0 & c_2 \\ c_3 & c_1 \end{pmatrix}, \quad (181)$$

and if  $n$  is odd, then

$$[{}^{eH}\sigma(\mathcal{T}'_{-}(\text{-}))^{-1}]^{-1} = [{}^{eH}\sigma(\mathcal{T}^{\text{eo}}_{-}(\text{-}))^{-1}]^{-1} + \begin{pmatrix} c_1 & c_2 \\ c_3 & 0 \end{pmatrix}. \quad (182)$$

Here  $c_1, c_2, c_3$  are finite rank operators in the Hermite ideal.

*Proof.* The arguments for the different cases are very similar. We give the details for one + case and one – case and formulæ for the answers in representative cases. In these formulæ we let  $z_0$  denote the unit vector spanning the range of  $\pi_0$  and  $z'_0$ , the unit vector spanning the range of  $\pi'_0$ .

Proposition 12 implies that  ${}^{eH}\sigma(\mathcal{T}'_{\pm})(\pm)$  are Fredholm operators. Since the differences

$${}^{eH}\sigma(\mathcal{T}'_{\pm})(\pm) - {}^{eH}\sigma(\mathcal{T}^{\text{eo}}_{\pm})(\pm)$$

are finite rank operators, it follows that  ${}^{eH}\sigma(\mathcal{T}'_{\pm})(\pm)$  have index zero. It therefore suffices to construct a left inverse.

We begin with the + even case by rewriting the equation

$${}^{eH}\sigma(\mathcal{T}'_{+}(\text{+})) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad (183)$$

as

$$\begin{aligned} \begin{bmatrix} \pi'_0 & 0 \\ 0 & \mathbf{0} \end{bmatrix} [u + \mathcal{D}_+^{\circ} v] &= \begin{bmatrix} \pi'_0 & 0 \\ 0 & \mathbf{0} \end{bmatrix} a \\ \begin{bmatrix} 1 - \pi'_0 & 0 \\ 0 & \text{Id} \end{bmatrix} \mathcal{D}_+^{\circ} v &= - \begin{bmatrix} 1 - \pi'_0 & 0 \\ 0 & \text{Id} \end{bmatrix} a \\ \mathcal{D}_+^{\circ} u + (\mathfrak{H} - (n - 1))v &= b. \end{aligned} \quad (184)$$

We solve the middle equation in (184) first. Let

$$\alpha_1 = \left( \frac{z'_0 \otimes z'_0}{\langle z'_0, z_0 \rangle} - \pi_0 \right) \Pi_0 a, \quad (185)$$

and note that  $\pi_0 \alpha_1 = 0$ . Corollary 2 shows that this model operator provides a globally defined symbol. The section  $v$  is determined as the unique solution to

$$\mathcal{D}_+^{\circ} v = -(\overset{\square}{a} - \alpha_1). \quad (186)$$

By construction  $(1 - \pi'_0)(a_0 + \alpha_1) = 0$  and therefore the second equation is solved. The section  $\overset{\square}{u}$  is now uniquely determined by the last equation in (184):

$$\overset{\square}{u} = [\mathcal{D}_+^e]^{-1}(b + (\mathfrak{H} - (n - 1))) [\mathcal{D}_+^{\circ}]^{-1}(\overset{\square}{a} - \alpha_1). \quad (187)$$

This leaves only the first equation, which we rewrite as

$$\begin{bmatrix} \pi'_0 & 0 \\ 0 & \mathbf{0} \end{bmatrix} u_0 = \begin{bmatrix} \pi'_0 & 0 \\ 0 & \mathbf{0} \end{bmatrix} (a - \mathcal{D}_+^{\circ} v - \overset{\square}{u}). \quad (188)$$

It is immediate that

$$u_0 = \frac{z_0 \otimes z_0''}{\langle z_0, z'_0 \rangle} \Pi_0 (a - \mathcal{D}_+^{\circ} v - \overset{\square}{u}). \quad (189)$$

By comparing these equations to those in (162) we see that  $[{}^e H \sigma(\mathcal{T}'_+^e)(+)]^{-1}$  has the required form. The finite rank operators are finite sums of terms involving  $\pi_0$ ,  $z_0 \otimes z_0''$  and  $z_0'' \otimes z_0$ , and are therefore in the Hermite ideal.

The solution in the  $+$  odd case is given by

$$\begin{aligned} v &= [\mathcal{D}_+^{\circ}]^{-1}(\overset{\square}{a} - \alpha_1) \\ \overset{\square}{u} &= [\mathcal{D}_+^e]^{-1}[(\mathfrak{H} + (n - 1))v - b] \\ u_0 &= \frac{z_0 \otimes z_0''}{\langle z_0, z'_0 \rangle} \Pi_0 (a + \mathcal{D}_+^{\circ} v - \overset{\square}{u}) \end{aligned} \quad (190)$$

As before  $\alpha_1$  is given by (185). Again the inverse of  ${}^e H \sigma(\mathcal{T}'_+^{\circ})(+)$  has the desired form.

In the  $-$  case, the computations are nearly identical for  $n$  odd. We leave the details to the reader, and conclude by providing the solution for  $n$  even. We let  $\bar{z}_0$  and  $\bar{z}'_0$  denote unit vectors spanning the ranges of  $\bar{\pi}_0$  and  $\bar{\pi}'_0$  respectively. We let

$$\beta_1 = \left( \frac{\bar{z}'_0 \otimes \bar{z}'_0}{\langle \bar{z}'_0, \bar{z}_0 \rangle} - \bar{\pi}_0 \right) \Pi_{n-1} b \quad (191)$$

The solution to

$$[{}^{eH}\sigma(\mathcal{T}'_{-}{}^e)(-)]^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad (192)$$

is given by

$$\begin{aligned} u &= [\mathcal{D}'_{-}{}^e]^{-1}(\bar{b} - \beta_1) \\ \bar{v} &= [\mathcal{D}'_{-}{}^o]^{-1}((\mathfrak{H} - (n-1))u - a) \\ v_0 &= \frac{\bar{z}_0 \otimes \bar{z}_0''}{\langle \bar{z}_0, \bar{z}_0' \rangle} \Pi_{n-1}(b - \mathcal{D}'_{-}{}^e u - \bar{v}). \end{aligned} \quad (193)$$

The result for  $\mathcal{T}'_{-}{}^o$  is

$$\begin{aligned} u &= -[\mathcal{D}'_{-}{}^e]^{-1}(\bar{b} - \beta_1) \\ \bar{v} &= [\mathcal{D}'_{-}{}^o]^{-1}(a - (\mathfrak{H} - (n-1))u) \\ v_0 &= \frac{\bar{z}_0 \otimes \bar{z}_0''}{\langle \bar{z}_0, \bar{z}_0' \rangle} \Pi_{n-1}(b + \mathcal{D}'_{-}{}^e u - \bar{v}). \end{aligned} \quad (194)$$

We leave the computations in the case of  $n$  odd to the reader. In all cases we see that the parametrices have the desired grading and this completes the proof of the proposition.  $\square$

As noted above, the operators  ${}^{eH}\sigma(\mathcal{T}'_{\pm}{}^{eo})(\pm)$  are Fredholm operators of index zero. Hence, Solvability of the equations

$${}^{eH}\sigma(\mathcal{T}'_{\pm}{}^{eo})(\pm) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad (195)$$

for all  $[a, b]$  implies the uniqueness and therefore the invertibility of the model operators. This completes the proof of Theorem 1. We now turn to applications of these results.

## 8 The Fredholm property

Let  $\mathcal{D}$  be a (pseudo)differential operator acting on smooth sections of  $F \rightarrow X$ , and  $\mathcal{B}$  a (pseudodifferential) boundary operator acting on sections of  $F|_{bX}$ . The pair  $(\mathcal{D}, \mathcal{B})$  is the densely defined operator,  $\sigma \mapsto \mathcal{D}\sigma$ , acting on sections of  $F$ , smooth on  $\bar{X}$ , that satisfy

$$\mathcal{B}[\sigma]_{bX} = 0. \quad (196)$$

The notation  $\overline{(\mathcal{D}, \mathcal{B})}$  is the closure of  $(\mathcal{D}, \mathcal{B})$  in the graph norm

$$\|\sigma\|_{\mathcal{G}}^2 = \|\mathcal{D}\sigma\|_{L^2}^2 + \|\sigma\|_{L^2}^2. \quad (197)$$

We let  $H_{\mathcal{G}}$  denote the domain of the closure, with norm defined by  $\|\cdot\|_{\mathcal{G}}$ . The following general result about Dirac operators, proved in [3], is useful for our analysis:

**Proposition 14.** *Let  $X$  be a compact manifold with boundary and  $\mathcal{D}$  an operator of Dirac type acting on sections of  $F \rightarrow X$ . The trace map from smooth sections of  $F$  to sections of  $F|_{bX}$ ,*

$$\sigma \mapsto \sigma|_{bX},$$

*extends to define a continuous map from  $H_{\mathcal{G}}$  to  $H^{-\frac{1}{2}}(bX; F|_{bX})$ .*

The results of the previous sections show that the operators  $\mathcal{T}'_{\pm}{}^{\text{eo}}$  are elliptic elements in the extended Heisenberg calculus. We now let  $\mathcal{Q}'_{\pm}{}^{\text{eo}}$  denote a left and right parametrix defined so that

$$\begin{aligned} \mathcal{Q}'_{\pm}{}^{\text{eo}}\mathcal{T}'_{\pm}{}^{\text{eo}} &= \text{Id} + K_1 \\ \mathcal{T}'_{\pm}{}^{\text{eo}}\mathcal{Q}'_{\pm}{}^{\text{eo}} &= \text{Id} + K_2, \end{aligned} \quad (198)$$

with  $K_1, K_2$  finite rank smoothing operators. The principal symbol computations show that  $\mathcal{Q}'_{\pm}{}^{\text{eo}}$  has classical order 0 and Heisenberg order at most 1. Such an operator defines a bounded map from  $H^{\frac{1}{2}}(bX)$  to  $L^2(bX)$ . This follows because such operators are contained in  $\Psi_{eH}^{\frac{1}{2}, 1, 1}$ . If  $\Delta$  is a positive (elliptic) Laplace operator, then  $\mathcal{L} = (\Delta + 1)^{\frac{1}{4}}$  lifts to define an invertible elliptic element of this operator class. An operator  $A \in \Psi_{eH}^{\frac{1}{2}, 1, 1}$  can be expressed in the form

$$A = A'\mathcal{L} \text{ where } A' \in \Psi_{eH}^{0, 0, 0}. \quad (199)$$

It is shown in [6], that operators in  $\Psi_{eH}^{0, 0, 0}$  act boundedly on  $H^s$ , for all real  $s$ . This proves the following result:

**Proposition 15.** *The operators  $\mathcal{Q}'_{\pm}{}^{\text{eo}}$  define bounded maps from  $H^s(bX; F)$  to  $H^{s-\frac{1}{2}}(bX; F)$  for  $s \in \mathbb{R}$ . Here  $F$  is an appropriate vector bundle over  $bX$ .*

*Remark 10.* Various similar results appear in the literature, for example in [7] and [2]. While the simple result in the proposition is adequate for our purposes, much more precise, anisotropic estimates can also be deduced.

The mapping properties of the boundary parametrices allow us to show that the graph closures of the operators  $(\check{\mathcal{D}}_{\pm}{}^{\text{eo}}, \mathcal{R}'_{\pm}{}^{\text{eo}})$  are Fredholm. As usual  $E \rightarrow X$  is a compatible complex vector bundle. Except when needed for clarity, the explicit dependence on  $E$  is suppressed.

**Theorem 2.** *Let  $X$  be a strictly pseudoconvex (pseudoconcave) manifold. The graph closures of  $(\bar{\partial}_{E+}^{\text{co}}, \mathcal{R}'_{E+})$ ,  $((\bar{\partial}_{E-}^{\text{co}}, \mathcal{R}'_{E-}))$ , respectively, are Fredholm operators.*

*Proof.* The argument is formally identical for all the different cases, so we do just the case of  $(\bar{\partial}_+^{\text{e}}, \mathcal{R}'_+)$ . As before  $Q^{\text{e}}$  is a fundamental solution for  $\bar{\partial}_+^{\text{e}}$  and  $\mathcal{K}$  is the Poisson kernel mapping the range of  $\mathcal{P}_+^{\text{e}}$  into the null space of  $\bar{\partial}_+^{\text{e}}$ . We need to show that the range of the closure is closed, of finite codimension, and that the null space is finite dimensional.

Let  $f$  be an  $L^2$ -section of  $\Lambda^{\circ} \otimes E$ ; with

$$u_1 = Q^{\text{e}} f \text{ and } u_0 = -\mathcal{K}\mathcal{U}'_{+^{\text{e}}}\mathcal{R}'_{+^{\text{e}}}[u_1]_{bX}, \quad (200)$$

we let  $u = u_0 + u_1$ . Proposition 15 and standard estimates imply that, for  $s \geq 0$ , there are constants  $C_{s1}, C_{s2}$ , independent of  $f$ , so that

$$\|u_1\|_{H^{s+1}} \leq C_{s1}\|f\|_{H^s}, \quad \|u_0\|_{H^{s+\frac{1}{2}}} \leq C_{s2}\|f\|_{H^s}. \quad (201)$$

The crux of the matter is to show that  $\mathcal{R}'_{+^{\text{e}}}[u_0 + u_1]_{bX} = 0$ . For data satisfying finitely many linear conditions, this is a consequence of the following lemma.

**Lemma 12.** *If  $\mathcal{T}'_{+^{\text{e}}}v \in \text{Im } \mathcal{R}'_{+^{\text{e}}}$ , then*

$$\mathcal{T}'_{+^{\text{e}}}\mathcal{P}_+^{\text{e}}v = \mathcal{T}'_{+^{\text{e}}}v. \quad (202)$$

*Proof of the lemma.* As  $(\text{Id} - \mathcal{R}'_{+^{\text{e}}})\mathcal{T}'_{+^{\text{e}}} = \mathcal{T}'_{+^{\text{e}}}(\text{Id} - \mathcal{P}_+^{\text{e}})$  we see that the hypothesis of the lemma implies that

$$\mathcal{T}'_{+^{\text{e}}}(\text{Id} - \mathcal{P}_+^{\text{e}})v = (\text{Id} - \mathcal{R}'_{+^{\text{e}}})\mathcal{T}'_{+^{\text{e}}}v = 0. \quad (203)$$

The conclusion follows from this relation.  $\square$

Since  $u_0 \in \ker \bar{\partial}_+^{\text{e}}$  it follows that  $(\text{Id} - \mathcal{P}_+^{\text{e}})[u_0]_{bX} = 0$ , and therefore the definition of  $u_0$  implies that:

$$\begin{aligned} \mathcal{R}'_{+^{\text{e}}}[u_0 + u_1]_{bX} &= \mathcal{T}'_{+^{\text{e}}}[u_0]_{bX} + \mathcal{R}'_{+^{\text{e}}}[u_1]_{bX} \\ &= -\mathcal{T}'_{+^{\text{e}}}\mathcal{P}_+^{\text{e}}\mathcal{U}'_{+^{\text{e}}}\mathcal{R}'_{+^{\text{e}}}[u_1]_{bX} + \mathcal{R}'_{+^{\text{e}}}[u_1]_{bX}. \end{aligned} \quad (204)$$

If

$$K_2\mathcal{R}'_{+^{\text{e}}}[u_1]_{bX} = K_2\mathcal{R}'_{+^{\text{e}}}[Q^{\text{e}}f]_{bX} = 0, \quad (205)$$

then

$$\mathcal{T}'_{+^{\text{e}}}\mathcal{U}'_{+^{\text{e}}}\mathcal{R}'_{+^{\text{e}}}[u_1]_{bX} = \mathcal{R}'_{+^{\text{e}}}[u_1]_{bX} \in \text{Im } \mathcal{R}'_{+^{\text{e}}}. \quad (206)$$



Hence, applying Lemma 12, we see that

$$\begin{aligned}\mathcal{T}'_+ \mathcal{P}'_+ \mathcal{U}'_+ \mathcal{R}'_+[u_1]_{bX} &= \mathcal{T}'_+ \mathcal{U}'_+ \mathcal{R}'_+[u_1]_{bX} \\ &= \mathcal{R}'_+[u_1]_{bX}\end{aligned}\quad (207)$$

Combining (204) and (207) gives the desired result:

$$\mathcal{R}'_+[u_0 + u_1]_{bX} = 0. \quad (208)$$

It is also clear that, if  $f \in H^s$ , then  $u \in H^{s+\frac{1}{2}}$ . In particular, if  $f$  is smooth, then so is  $u$ . Hence  $u$  belongs to the domain of  $(\bar{\partial}'_+, \mathcal{R}'_+)$ .

The operator  $K_2$  is a finite rank smoothing operator, and therefore the composition

$$f \mapsto K_2 \mathcal{R}'_+[Q^e f]_{bX} \quad (209)$$

has a kernel of the form

$$\sum_{j=1}^M u_j(x) v_j(y) \text{ for } (x, y) \in bX \times X, \quad (210)$$

with

$$u_j \in \mathcal{C}^\infty(bX) \text{ and } v_j \in \mathcal{C}^\infty(\bar{X}).$$

Hence, an  $L^2$ -section,  $f$  satisfying (205) can be obtained as the limit of a sequence of smooth sections  $\langle f^n \rangle$  that also satisfy this condition. Let  $\langle u^n \rangle$  be the smooth solutions to

$$\bar{\partial}'_+ u^n = f^n, \quad \mathcal{R}'_+[u^n]_{bX} = 0, \quad (211)$$

constructed above. The estimates in (201) show that  $\langle u^n \rangle$  converges to a limit  $u$  in  $H^{\frac{1}{2}}$ . It is also clear that  $\bar{\partial}'_+ u^n$  converges weakly to  $\bar{\partial}'_+ u$ , and in  $L^2$  to  $f$ . Therefore  $\langle u^n \rangle$  converges to  $u$  in the graph norm. This shows that  $u$  is in the domain of the closure and satisfies  $\bar{\partial}'_+ u = f$ . As the composition

$$f \mapsto K_2 \mathcal{R}'_+[Q^e f]_{bX},$$

is bounded, it follows that the range of  $\overline{(\bar{\partial}'_+, \mathcal{R}'_+)}$  contains a closed subspace of finite codimension and is therefore also a closed subspace of finite codimension.

To complete the proof of the theorem we need to show that the null space is finite dimensional. Suppose that  $u$  belongs to the null space of  $\overline{(\bar{\partial}'_+, \mathcal{R}'_+)}$ . This implies that there is a sequence of smooth sections  $\langle u^n \rangle$  in the domain of the operator, converging to  $u$  in the graph norm, such that  $\|\bar{\partial}'_+ u^n\|_{L^2}$  converges to zero.

Hence  $\bar{\partial}_+^e u = 0$  in the weak sense. Proposition 14 shows that  $u$  has boundary values in  $H^{-\frac{1}{2}}(bX)$  and that, in the sense of distributions,

$$\mathcal{R}'_+[u]_{bX} = \lim_{n \rightarrow \infty} \mathcal{R}'_+[u^n]_{bX} = 0.$$

Since  $u$  is in the null space of  $\bar{\partial}_+^e$ , it is also the case that  $\mathcal{P}'_+[u]_{bX} = [u]_{bX}$ . These two facts imply that  $\mathcal{F}'_+[u]_{bX} = 0$ . Composing on the left with  $\mathcal{U}'_+$  shows that

$$(\text{Id} + K_1)[u]_{bX} = 0. \quad (212)$$

As  $K_1$  is a finite rank smoothing operator, we conclude that  $[u]_{bX}$  and therefore  $u$  are smooth. By the unique continuation property for Dirac operators, the dimension of the null space of  $(\bar{\partial}_+^e, \mathcal{R}'_+)$  is bounded by the dimension of the null space of  $(\text{Id} + K_1)$ . This completes the proof of the assertion that  $(\bar{\partial}_+^e, \mathcal{R}'_+)$  is a Fredholm operator. The proofs in the other cases, are up to minor changes in notation, identical.  $\square$

*Remark 11.* In the proof of the theorem we have constructed right parametrices  $\mathcal{Q}'_{\pm}{}^{\text{eo}}$  for the boundary value problems  $(\bar{\partial}_{\pm}^{\text{eo}}, \mathcal{R}'_{\pm}{}^{\text{eo}})$ , which gain a half a derivative.

We close this section with Sobolev space estimates for the operators  $(\bar{\partial}_{\pm}^{\text{eo}}, \mathcal{R}'_{\pm}{}^{\text{eo}})$ .

**Theorem 3.** *Let  $X$  be a strictly pseudoconvex (pseudoconcave) manifold, and  $E \rightarrow X$  a compatible complex vector bundle. For each  $s \geq 0$ , there is a positive constant  $C_s$  such that if  $u$  is an  $L^2$ -solution to*

$$\bar{\partial}_{E\pm}^{\text{eo}} u = f \in H^s(X) \text{ and } \mathcal{R}'_{E\pm}{}^{\text{eo}}[u]_{bX} = 0$$

*in the sense of distributions, then*

$$\|u\|_{H^{s+\frac{1}{2}}} \leq C_s [\|\bar{\partial}_{E\pm}^{\text{eo}} u\|_{H^s} + \|u\|_{L^2}]. \quad (213)$$

*Proof.* With  $u_1 = Q^{\text{eo}} f$ , we see that  $u_1 \in H^{s+1}(X)$  and

$$\bar{\partial}_{\pm}^{\text{eo}}(u - u_1) = 0 \text{ with } \mathcal{R}'_{\pm}{}^{\text{eo}}[u - u_1]_{bX} = -\mathcal{R}'_{\pm}{}^{\text{eo}}[u_1]_{bX}.$$

These relations imply that  $\mathcal{P}'_{\pm}{}^{\text{eo}}[u - u_1]_{bX} = [u - u_1]_{bX}$  and therefore

$$-\mathcal{R}'_{\pm}{}^{\text{eo}}[u_1]_{bX} = \mathcal{R}'_{\pm}{}^{\text{eo}}[u - u_1]_{bX} = \mathcal{F}'_{\pm}{}^{\text{eo}}[u - u_1]_{bX}. \quad (214)$$

We apply  $\mathcal{U}'_{\pm}{}^{\text{eo}}$  to this equation to deduce that

$$(\text{Id} + K_1)[u - u_1]_{bX} = -\mathcal{U}'_{\pm}{}^{\text{eo}} \mathcal{R}'_{\pm}{}^{\text{eo}}[u_1]_{bX}. \quad (215)$$

Because  $K_1$  is a smoothing operator, Proposition 15 implies that there is a constant  $C'_s$  so that

$$\|[u - u_1]_{bX}\|_{H^s(bX)} \leq C'_s[\|u_1\|_{H^{s+\frac{1}{2}}(bX)} + \|[u - u_1]\|_{H^{-\frac{1}{2}}(bX)}]. \quad (216)$$

As the Poisson kernel carries  $H^s(bX)$  to  $H^{s+\frac{1}{2}}(X)$ , boundedly, this estimate shows that  $u = u - u_1 + u_1$  belongs to  $H^{s+\frac{1}{2}}(X)$  and that there is a constant  $C_s$  so that

$$\|u\|_{H^{s+\frac{1}{2}}} \leq C_s[\|f\|_{H^s} + \|u\|_{L^2}] \quad (217)$$

This proves the theorem.  $\square$

*Remark 12.* In the case  $s = 0$ , this proof gives a slightly better result: the Poisson kernel actually maps  $L^2(bX)$  into  $H_{(1, -\frac{1}{2})}(X)$  and therefore the argument shows that there is a constant  $C_0$  such that if  $u \in L^2$ ,  $\bar{\partial}_\pm^{\text{eo}} u \in L^2$  and  $\mathcal{R}'_{\pm \text{eo}}[u]_{bX} = 0$ , then

$$\|u\|_{(1, -\frac{1}{2})} \leq C_0[\|f\|_{L^2} + \|u\|_{L^2}] \quad (218)$$

This is just the standard  $\frac{1}{2}$ -estimate for the operators  $\overline{(\bar{\partial}_\pm^{\text{eo}}, \mathcal{R}'_{\pm \text{eo}})}$

It is also possible to prove localized versions of these results. The higher norm estimates have the same consequences as for the  $\bar{\partial}$ -Neumann problem. Indeed, under certain hypotheses these estimates imply higher norm estimates for the second order operators considered in [5]. We prove these in the next section after showing that the closures of the formal adjoints of  $(\bar{\partial}_\pm^{\text{eo}}, \mathcal{R}'_{\pm \text{eo}})$  are the  $L^2$ -adjoints of these operators.

## 9 Adjoints of the $\text{Spin}_{\mathbb{C}}$ Dirac operators

In the previous section we proved that the operators  $\overline{(\bar{\partial}_\pm^{\text{eo}}, \mathcal{R}'_{\pm \text{eo}})}$  are Fredholm operators, as well as estimates that they satisfy. In this section we show that the  $L^2$ -adjoints of these operators are the closures of the formal adjoints.

**Theorem 4.** *If  $X$  is strictly pseudoconvex (pseudoconcave),  $E \rightarrow X$  a compatible complex vector bundle, then we have the following relations:*

$$(\bar{\partial}_{E\pm}^{\text{eo}}, \mathcal{R}'_{E\pm \text{eo}})^* = \overline{(\bar{\partial}_{E\pm}^{\text{oe}}, \mathcal{R}'_{E\pm \text{oe}})}. \quad (219)$$

*We take  $+$  if  $X$  is pseudoconvex and  $-$  if  $X$  is pseudoconcave.*

*Proof.* The argument follows a standard outline. It is clear that

$$\overline{(\partial_{\pm}^{\text{oe}}, \mathcal{R}'_{\pm}{}^{\text{oe}})} \subset (\partial_{\pm}^{\text{eo}}, \mathcal{R}'_{\pm}{}^{\text{eo}})^* \quad (220)$$

Suppose that the containment is proper. This would imply that, for any nonzero, real  $\mu$  there exists a nonzero section  $v \in \text{Dom}_{L^2}((\partial_{\pm}^{\text{eo}}, \mathcal{R}'_{\pm}{}^{\text{eo}})^*)$ , such that, for all  $w \in \text{Dom}((\partial_{\pm}^{\text{oe}}, \mathcal{R}'_{\pm}{}^{\text{oe}}))$ ,

$$\langle [\partial_{\pm}^{\text{eo}}]^* v, \partial_{\pm}^{\text{oe}} w \rangle + \mu^2 \langle v, w \rangle = 0. \quad (221)$$

Suppose that  $\mathcal{R}'_{\pm}{}^{\text{eo}} \partial_{\pm}^{\text{oe}} w \upharpoonright_{bX} = 0$ . Since  $v$  belongs to  $\text{Dom}_{L^2}((\partial_{\pm}^{\text{eo}}, \mathcal{R}'_{\pm}{}^{\text{eo}})^*)$ , we can integrate by parts to obtain that

$$\langle v, (\partial_{\pm}^{\text{eo}} \partial_{\pm}^{\text{oe}} + \mu^2) w \rangle = 0. \quad (222)$$

This reduces the proof of the theorem to the following proposition.

**Proposition 16.** *For any nonzero real number  $\mu$ , and*

$$f \in \mathcal{C}^{\infty}(\overline{X}; \mathcal{S}^{\text{oe}} \otimes E),$$

*there is a section  $w \in \mathcal{C}^{\infty}(\overline{X}; \mathcal{S}^{\text{oe}} \otimes E)$ , which satisfies*

$$\begin{aligned} (\partial_{\pm}^{\text{eo}} \partial_{\pm}^{\text{oe}} + \mu^2) w &= f \\ \mathcal{R}'_{\pm}{}^{\text{oe}} w \upharpoonright_{bX} &= 0 \quad \text{and} \quad \mathcal{R}'_{\pm}{}^{\text{eo}} \partial_{\pm}^{\text{oe}} w \upharpoonright_{bX} = 0. \end{aligned} \quad (223)$$

Before proving the proposition, we show how it implies the theorem. Let  $w, f$  be as in (223). The boundary conditions satisfied by  $\partial_{\pm}^{\text{oe}} w$  and (222) imply that we have

$$\langle v, f \rangle = 0. \quad (224)$$

As  $f \in \mathcal{C}^{\infty}(\overline{X}; \mathcal{S}^{\text{oe}} \otimes E)$  is arbitrary, this shows that  $v = 0$  as well and thereby completes the proof of the theorem.  $\square$

The proposition is a consequence of Theorems 1 and 3.

*Proof of Proposition 16.* The first step is to show that (223) has a weak solution for any non-zero real number  $\mu$ , after which, we use a small extension of Theorem 3 to show that this solution is actually in  $\mathcal{C}^{\infty}(\overline{X}; \mathcal{S}^{\text{oe}} \otimes E)$ .

**Lemma 13.** *Let  $Q(w) = \langle \partial_{\pm}^{\text{oe}} w, \partial_{\pm}^{\text{oe}} w \rangle$ , denote the non-negative, symmetric quadratic form with domain:*

$$\text{Dom}(Q) = \{w \in L^2(X) : \partial_{\pm}^{\text{oe}} w \in L^2(X) \text{ and } \mathcal{R}'_{\pm}{}^{\text{oe}} w \upharpoonright_{bX} = 0\}. \quad (225)$$

*The form  $Q$  is closed and densely defined. Let  $L$  denote the self adjoint operator defined by  $Q$ . If  $w \in \text{Dom}(L)$ , then*

$$\partial_{\pm}^{\text{eo}} \partial_{\pm}^{\text{oe}} w \in L^2 \text{ and } \mathcal{R}'_{\pm}{}^{\text{eo}} \partial_{\pm}^{\text{oe}} w \upharpoonright_{bX} = 0. \quad (226)$$

*Remark 13.* That a densely defined, closed, symmetric, non-negative quadratic form defines a self adjoint operator is the content of Theorem VI.2.6 in [9]. For the remainder of this section we let  $\rho$  denote a defining function for  $bX$ , with  $d\rho$  of unit length along the sets  $\{\rho = \epsilon\}$ , for  $\epsilon$  sufficiently small.

*Proof of Lemma.* It is clear that  $Q$  is densely defined. That the form is closed is an immediate consequence of Proposition 14. By definition, the domain of  $L$  consists of sections  $w \in \text{Dom}(Q)$ , such that there exists a  $g \in L^2$ , for which

$$Q(w, v) = \langle g, v \rangle \quad (227)$$

for all  $v \in \text{Dom}(Q)$ . Since all smooth sections with compact support lie in  $\text{Dom}(Q)$ , it follows from (227) that

$$\bar{\partial}_\pm^{\text{eo}} \bar{\partial}_\pm^{\text{oe}} w = g \in L^2, \quad (228)$$

where the operator,  $\bar{\partial}_\pm^{\text{eo}} \bar{\partial}_\pm^{\text{oe}}$ , is applied in the distributional sense. This in turn implies that  $w \in H_{\text{loc}}^2(X)$ , and that  $\bar{\partial}_\pm^{\text{oe}} w$  has restrictions to the sets  $\{\rho = \epsilon\}$ , which depend continuously on  $\epsilon$  in the  $H^{-\frac{1}{2}}(bX)$ -topology.

Now let  $v$  be a section, smooth in the closure of  $X$ , though not necessarily in  $\text{Dom}(Q)$ . The regularity properties of  $w$  imply that

$$Q(w, v) = \langle \bar{\partial}_\pm^{\text{eo}} \bar{\partial}_\pm^{\text{oe}} w, v \rangle + \langle \bar{\partial}_\pm^{\text{oe}} w, \sigma(\bar{\partial}_\pm^{\text{oe}}, -id\rho)v \rangle_{bX}. \quad (229)$$

If  $v \in \text{Dom}(Q)$ , then (228) shows that the boundary term in (229) must vanish. If  $h$  is any smooth even (odd) section defined on  $bX$ , then by smoothly extending  $(\text{Id} - \mathcal{R}'_{\pm^{\text{oe}}})\sigma(\bar{\partial}_\pm^{\text{eo}}, id\rho)h$  to  $\bar{X}$  we obtain a smooth section  $v_h \in \text{Dom}(Q)$ , with

$$v_h \upharpoonright_{bX} = (\text{Id} - \mathcal{R}'_{\pm^{\text{oe}}})\sigma(\bar{\partial}_\pm^{\text{eo}}, id\rho)h. \quad (230)$$

The identity

$$\mathcal{R}'_{\pm^{\text{eo}}} = \sigma(\bar{\partial}_\pm^{\text{oe}}, -id\rho)(\text{Id} - \mathcal{R}'_{\pm^{\text{oe}}})\sigma(\bar{\partial}_\pm^{\text{eo}}, id\rho) \quad (231)$$

is easily established; it is equivalent to the symmetry of the non-chiral operator  $(\bar{\partial}_\pm, \mathcal{R}'_{\pm})$ . Hence, if  $w \in \text{Dom}(L)$ , then, for any smooth section  $h$ , we have

$$\begin{aligned} 0 &= \langle \bar{\partial}_\pm^{\text{oe}} w, \sigma(\bar{\partial}_\pm^{\text{oe}}, -id\rho)v_h \rangle_{bX} \\ &= \langle \bar{\partial}_\pm^{\text{oe}} w, \sigma(\bar{\partial}_\pm^{\text{oe}}, -id\rho)(\text{Id} - \mathcal{R}'_{\pm^{\text{oe}}})\sigma(\bar{\partial}_\pm^{\text{eo}}, id\rho)h \rangle_{bX} \\ &= \langle \mathcal{R}'_{\pm^{\text{eo}}} \bar{\partial}_\pm^{\text{oe}} w, h \rangle_{bX}. \end{aligned} \quad (232)$$

As  $h$  is an arbitrary smooth section, this verifies the final assertion of the lemma.  $\square$

The operator  $L$  is non-negative and self adjoint. Hence for any real  $\mu \neq 0$ , and  $f \in \mathcal{C}^\infty(\overline{X}; \mathcal{S}^{\text{oe}} \otimes E)$ , there is a unique  $w \in \text{Dom}(L)$  satisfying (223) in the sense of distributions. To complete the proof of the proposition we need to show that this solution is smooth.

We rewrite this in terms of the system of first order equations:

$$\begin{aligned} \mathcal{D}_\pm^\mu \begin{pmatrix} u \\ v \end{pmatrix} &\stackrel{d}{=} \begin{pmatrix} \tilde{\partial}_\pm^{\text{oe}} & -\mu \\ \mu & \tilde{\partial}_\pm^{\text{eo}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \\ \mathcal{R}_\pm \begin{pmatrix} u \\ v \end{pmatrix} &\stackrel{d}{=} \begin{pmatrix} \mathcal{R}'_\pm{}^{\text{oe}} & 0 \\ 0 & \mathcal{R}'_\pm{}^{\text{eo}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \Big|_{bX} = 0. \end{aligned} \quad (233)$$

Clearly the solution constructed above satisfies

$$\mathcal{D}_\pm^\mu \begin{pmatrix} w \\ \frac{1}{\mu} \tilde{\partial}_\pm^{\text{oe}} w \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \text{ and } \mathcal{R}_\pm \begin{pmatrix} w \\ \frac{1}{\mu} \tilde{\partial}_\pm^{\text{oe}} w \end{pmatrix} \Big|_{bX} = 0, \quad (234)$$

in the sense of distributions. To complete the proof of the proposition it suffices to establish a regularity result for  $(\mathcal{D}_\pm^\mu, \mathcal{R}_\pm)$  analogous to Theorem 3. Indeed essentially the same argument applies to this case.

Let  $\mathcal{P}_\pm^\mu$  denote the Calderon projector for the operator  $\mathcal{D}_\pm^\mu$ , and set

$$\mathcal{T}_\pm^\mu = \mathcal{R}_\pm \mathcal{P}_\pm^\mu + (\text{Id} - \mathcal{R}_\pm)(\text{Id} - \mathcal{P}_\pm^\mu). \quad (235)$$

Theorem 1 implies that  $\mathcal{T}_\pm^0$  is a graded elliptic element of the extended Heisenberg calculus. Let  $\mathcal{U}_\pm^0$  denote a parametrix for  $\mathcal{T}_\pm^0$ . We now show that

$$\mathcal{T}_\pm^\mu = \mathcal{T}_\pm^0 + \mathfrak{D}_{-1,-2}^{eH} \quad (236)$$

Here  $\mathfrak{D}_{-1,-2}^{eH}$  is an extended Heisenberg operator, having Heisenberg order  $-2$  on the appropriate parabolic face and classical order  $-1$ . As the extended Heisenberg order of  $\mathcal{U}_\pm^0$  is  $(0, 1)$  we see that this operator is also a parametrix for  $\mathcal{T}_\pm^\mu$ . We now verify (236).

The operator  $\mathcal{D}_\pm^\mu [\mathcal{D}_\pm^\mu]^*$  is given by

$$\mathcal{D}_\pm^\mu [\mathcal{D}_\pm^\mu]^* = \begin{pmatrix} \tilde{\partial}_\pm^{\text{oe}} \tilde{\partial}_\pm^{\text{eo}} + \mu^2 & 0 \\ 0 & \tilde{\partial}_\pm^{\text{eo}} \tilde{\partial}_\pm^{\text{oe}} + \mu^2 \end{pmatrix}. \quad (237)$$

The fundamental solution  $\mathcal{Q}_\pm^{\mu(2)}$  has the form

$$\mathcal{Q}_\pm^{\mu(2)} = \begin{pmatrix} \mathcal{Q}_\pm^{\text{eo}(2)\mu} & 0 \\ 0 & \mathcal{Q}_\pm^{\text{oe}(2)\mu} \end{pmatrix}, \quad (238)$$

where  $Q_{\pm}^{\text{eo}(2)\mu} = (\partial_{\pm}^{\text{oe}} \partial_{\pm}^{\text{eo}} + \mu^2)^{-1}$ . A fundamental solution for  $\mathcal{D}_{\pm}^{\mu}$  is then given by

$$\mathcal{Q}_{\pm}^{\mu} = [\mathcal{D}_{\pm}^{\mu}]^* \mathcal{Q}_{\pm}^{\mu(2)} = \begin{pmatrix} \partial_{\pm}^{\text{eo}} Q_{\pm}^{\text{eo}(2)\mu} & \mu Q_{\pm}^{\text{oe}(2)\mu} \\ -\mu Q_{\pm}^{\text{eo}(2)\mu} & \partial_{\pm}^{\text{oe}} Q_{\pm}^{\text{oe}(2)\mu} \end{pmatrix}. \quad (239)$$

The claim in (236) follows from the observation that

$$Q_{\pm}^{\text{eo}(2)\mu} - Q_{\pm}^{\text{eo}(2)0} \in \mathfrak{D}_{-4}, \quad (240)$$

which is a consequence of the resolvent identity

$$(\partial_{\pm}^{\text{oe}} \partial_{\pm}^{\text{eo}} + \mu^2)^{-1} - (\partial_{\pm}^{\text{oe}} \partial_{\pm}^{\text{eo}})^{-1} = -\mu^2 (\partial_{\pm}^{\text{oe}} \partial_{\pm}^{\text{eo}} + \mu^2)^{-1} (\partial_{\pm}^{\text{oe}} \partial_{\pm}^{\text{eo}})^{-1}, \quad (241)$$

and the fact that  $\partial_{\pm}^{\text{oe}} \partial_{\pm}^{\text{eo}}$  is elliptic of order 2. Using (240) in (239) shows that

$$\mathcal{Q}_{\pm}^{\mu} = \mathcal{Q}_{\pm}^0 + \begin{pmatrix} \mathfrak{D}_{-3} & \mathfrak{D}_{-2} \\ \mathfrak{D}_{-2} & \mathfrak{D}_{-3} \end{pmatrix}. \quad (242)$$

We can now apply Proposition 6 to conclude that the  $\mathfrak{D}_{-3}$  terms along the diagonal in  $\mathcal{Q}_{\pm}^{\mu}$  can only change the symbol of  $\mathcal{P}_{\pm}^0$  by terms with Heisenberg order  $-4$ . The residue computations in Section 3 show that the  $\mathfrak{D}_{-2}$  off diagonal terms can only contribute terms to  $\mathcal{P}_{\pm}^{\mu}$  at Heisenberg order  $-2$ , hence

$$\mathcal{P}_{\pm}^{\mu} = \mathcal{P}_{\pm}^0 + \begin{pmatrix} \mathfrak{S}_{-2,-4}^{eH} & \mathfrak{S}_{-1,-2}^{eH} \\ \mathfrak{S}_{-1,-2}^{eH} & \mathfrak{S}_{-2,-4}^{eH} \end{pmatrix}. \quad (243)$$

The truth of (236) is an immediate consequence of (243) and the fact that  $\mathcal{U}_{\pm}^0$  has extended Heisenberg orders  $(0, 1)$ .

As noted above, this shows that the leading order part of the parametrix for  $\mathcal{T}_{\pm}^{\mu}$  has the form

$$\begin{pmatrix} \mathcal{U}_{\pm}^{\text{oe}} & \mathbf{0} \\ \mathbf{0} & \mathcal{U}_{\pm}^{\text{eo}} \end{pmatrix}. \quad (244)$$

We let  $\mathcal{U}_{\pm}^{\mu}$  denote a parametrix chosen so that

$$\mathcal{U}_{\pm}^{\mu} \mathcal{T}_{\pm}^{\mu} = \text{Id} + K_1^{\mu} \quad \mathcal{T}_{\pm}^{\mu} \mathcal{U}_{\pm}^{\mu} = \text{Id} + K_2^{\mu} \quad (245)$$

with  $K_1^{\mu}, K_2^{\mu}$  smoothing operators of finite rank. Arguing as in Theorem 3, one easily proves the desired regularity:

**Lemma 14.** *Let  $\mu \in \mathbb{C}$  and  $s \geq 0$ , if  $(f, g)$  belongs to  $L^2$ , and satisfies*

$$\mathcal{D}_{\pm}^{\mu} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \in H^s \text{ and } \mathcal{R}_{\pm} \begin{pmatrix} f \\ g \end{pmatrix} \upharpoonright_{bX} = 0, \quad (246)$$

in the sense of distributions, then  $f, g \in H^{s+\frac{1}{2}}$ . There is a constant  $C_{s,\mu}$ , independent of  $(f, g)$  so that

$$\|(f, g)\|_{H^{s+\frac{1}{2}}} \leq C_{s,\mu} \left[ \|\mathcal{D}_{\pm}^{\mu} \begin{pmatrix} f \\ g \end{pmatrix}\|_{H^s} + \left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{L^2} \right]. \quad (247)$$

*Proof.* Let  $U_1 = \mathcal{Q}_{\pm}^{\mu}(a, b)$ , so that  $U_1 \in H^{s+1}$ , and

$$\mathcal{D}_{\pm}^{\mu}(U - U_1) = 0. \quad (248)$$

On the one hand  $\mathcal{R}_{\pm}([U - U_1]_{bX}) = -\mathcal{R}_{\pm}([U_1]_{bX}) \in H^{s+\frac{1}{2}}(bX)$ . On the other hand  $[U - U_1]_{bX} \in \text{Im } \mathcal{P}_{\pm}^{\mu}$  and therefore

$$-\mathcal{R}_{\pm}([U_1]_{bX}) = \mathcal{R}_{\pm}([U - U_1]_{bX}) = \mathcal{F}_{\pm}^{\mu}([U - U_1]_{bX}).$$

We apply  $\mathcal{U}_{\pm}^{\mu}$  to this relation to obtain

$$-\mathcal{U}_{\pm}^{\mu} \mathcal{R}_{\pm}([U_1]_{bX}) = (\text{Id} + K_1^{\mu})([U - U_1]_{bX}) \quad (249)$$

Rewriting this result gives

$$[U]_{bX} = -\mathcal{U}_{\pm}^{\mu} \mathcal{R}_{\pm}([U_1]_{bX}) + (\text{Id} + K_1^{\mu})([U_1]_{bX}) - K_1^{\mu}[U]_{bX}. \quad (250)$$

All terms on the right hand side of (250), but the last are, by construction, in  $H^s(bX)$ . Proposition 14 implies that  $[U]_{bX} \in H^{-\frac{1}{2}}$ , as  $K_1^{\mu}$  is a smoothing operator, the last term,  $K_1^{\mu}[U]_{bX}$ , is smooth. Thus  $[U - U_1]_{bX}$  is in  $H^s(bX)$ , and  $U - U_1$  therefore belongs to  $H^{s+\frac{1}{2}}(X)$ ; hence  $U = U_1 + U - U_1$  does as well. The estimate (247) follows easily from the definition of  $U_1$  and (250).  $\square$

Thus the solution  $w$  constructed above is smooth on  $\overline{X}$ ; this completes the proofs of the proposition and Theorem 4  $\square$

Using Theorem 4 we can describe the domains of  $(\overline{\partial_{\pm}^{\text{eo}}}, \overline{\mathcal{R}'_{\pm}{}^{\text{eo}}})$ .

**Corollary 3.** *The domains of the closures,  $(\overline{\partial_{\pm}^{\text{eo}}}, \overline{\mathcal{R}'_{\pm}{}^{\text{eo}}})$ , are given by*

$$\text{Dom}(\overline{\partial_{\pm}^{\text{eo}}}, \overline{\mathcal{R}'_{\pm}{}^{\text{eo}}}) = \{u \in L^2(X; F) : \partial_{\pm}^{\text{eo}} u \in L^2(X; F'), \mathcal{R}'_{\pm}{}^{\text{eo}} u \upharpoonright_{bX} = 0\} \quad (251)$$

*Remark 14.* Note that Proposition 14 implies that  $u \upharpoonright_{bX} \in H^{-\frac{1}{2}}(bX)$ . It is in this sense that the boundary condition in (251) should be understood.



*Proof.* By Theorem 4, we need only show that  $u$  satisfying the conditions in (251) belong to  $\text{Dom}((\bar{\partial}_{\pm}^{\text{oe}}, \mathcal{R}'_{\pm}{}^{\text{oe}})^*)$ . To show this we need only show that for *smooth* sections,  $v$ , with  $\mathcal{R}'_{\pm}{}^{\text{oe}}v \upharpoonright_{bX} = 0$ , we have

$$\langle \bar{\partial}_{\pm}^{\text{oe}}v, u \rangle = \langle v, \bar{\partial}_{\pm}^{\text{eo}}u \rangle. \quad (252)$$

This follows by a simple limiting argument, because the map  $\epsilon \mapsto u \upharpoonright_{\rho=\epsilon}$  is continuous in the  $H^{-\frac{1}{2}}$ -topology and  $v$  is smooth.  $\square$

As a corollary of Lemma 14 we get estimates for the second order operators  $\bar{\partial}_{\pm}^{\text{oe}}\bar{\partial}_{\pm}^{\text{eo}}$ , with subelliptic boundary conditions.

**Corollary 4.** *Let  $X$  be a strictly pseudoconvex (pseudoconcave) manifold,  $E \rightarrow X$  a compatible complex vector bundle. For  $s \geq 0$  there exist constants  $C_s$  such that if  $u \in L^2$ ,  $\bar{\partial}_{E\pm}^{\text{eo}}u \in L^2$ ,  $\bar{\partial}_{E\pm}^{\text{oe}}\bar{\partial}_{E\pm}^{\text{eo}}u \in H^s$  and  $\mathcal{R}'_{E\pm}{}^{\text{eo}}[u]_{bX} = 0$ ,  $\mathcal{R}'_{E\pm}{}^{\text{oe}}[\bar{\partial}_{E\pm}^{\text{eo}}u] = 0$  in the sense of distributions, then*

$$\|u\|_{H^{s+1}} \leq C_s[\|\bar{\partial}_{E\pm}^{\text{oe}}\bar{\partial}_{E\pm}^{\text{eo}}u\|_{H^s} + \|u\|_{L^2}]. \quad (253)$$

*Proof.* We apply Lemma 14 to  $U = (u, \bar{\partial}_{\pm}^{\text{eo}}u)$ . Initially we see that  $\mathcal{D}_{\pm}^0U \in L^2$ . The lemma shows that  $\bar{\partial}_{\pm}^{\text{eo}}u \in H^{\frac{1}{2}}$ , and therefore  $\mathcal{D}_{\pm}^0U \in H^{\frac{1}{2}}$ . Applying the lemma recursively, we eventually deduce that  $\mathcal{D}_{\pm}^0U \in H^s$  and that there is constant  $C'_s$  so that

$$\|u\|_{H^{s+\frac{1}{2}}} + \|\bar{\partial}_{\pm}^{\text{eo}}u\|_{H^{s+\frac{1}{2}}} \leq C'_s[\|\bar{\partial}_{\pm}^{\text{oe}}\bar{\partial}_{\pm}^{\text{eo}}u\|_{H^s} + \|u\|_{L^2}]. \quad (254)$$

It follows from Theorem 3 that, for a constant  $C''_s$ , we have

$$\|u\|_{H^{s+1}} \leq C''_s[\|u\|_{H^{s+\frac{1}{2}}} + \|\bar{\partial}_{\pm}^{\text{eo}}u\|_{H^{s+\frac{1}{2}}}] \quad (255)$$

Combining the two estimates gives (253).  $\square$

In the case that  $X$  is a complex manifold with boundary, these estimates imply analogous results for the modified  $\bar{\partial}$ -Neumann problem acting on individual form degrees. These results are stated and deduced from Corollary 4 in [5].

## References

- [1] R. BEALS AND P. GREINER, *Calculus on Heisenberg Manifolds*, vol. 119 of Annals of Mathematics Studies, Princeton University Press, 1988.
- [2] R. BEALS AND N. STANTON, *The heat equation for the  $\bar{\partial}$ -Neumann problem*, I, Comm. PDE, 12 (1987), pp. 351–413.

- [3] B. BOOSS-BAVNBK AND K. P. WOJCIECHOWSI, *Elliptic Boundary Problems for the Dirac Operator*, Birkhäuser, Boston, 1996.
- [4] J. DUISTERMAAT, *The heat kernel Lefschetz fixed point theorem formula for the Spin-c Dirac Operator*, Birkhäuser, Boston, 1996.
- [5] C. L. EPSTEIN, *Subelliptic Spin<sub>C</sub> Dirac operators, I*, *Annals of Math.*, 166 (2007), pp. 183-214.
- [6] C. L. EPSTEIN AND R. MELROSE, *The Heisenberg algebra, index theory and homology*, preprint, 2004.
- [7] P. C. GREINER AND E. M. STEIN, *Estimates for the  $\bar{\partial}$ -Neumann problem*, *Mathematical Notes*, Princeton University Press, 1977.
- [8] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators*, vol. 3, Springer Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.
- [9] T. KATO, *Perturbation Theory for Linear Operators, corrected 2nd printing* Springer Verlag, Berlin Heidelberg, 1980.
- [10] S. KOBAYASHI AND K. NOMIZU, *Foundations of Differential Geometry, Vol. 2*, John Wiley and Sons, New York, 1969.
- [11] H. B. LAWSON JR. AND M.-L. MICHELSON, *Spin Geometry*, vol. 38 of *Princeton Mathematical Series*, Princeton University Press, 1989.
- [12] R. SEELEY, *Singular integrals and boundary value problems*, *Amer. Jour. of Math.*, 88 (1966), pp. 781–809.
- [13] M. E. TAYLOR, *Noncommutative microlocal analysis, part I*, vol. 313 of *Mem. Amer. Math. Soc.*, AMS, 1984.
- [14] ———, *Partial Differential Equations, Vol. 2*, vol. 116 of *Applied Mathematical Sciences*, Springer Verlag, New York, 1996.
- [15] R. WELLS, *Differential Analysis on Complex Manifolds*, vol. 165 of *Graduate Texts in Mathematics*, Springer Verlag, New York, 1980.