

# Subelliptic $\text{Spin}_{\mathbb{C}}$ Dirac operators, III

## The Atiyah-Weinstein conjecture

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*This paper is dedicated to my wife Jane  
for her enduring love and support.*

### Abstract

In this paper we extend the results obtained in [9, 10] to manifolds with  $\text{Spin}_{\mathbb{C}}$ -structures defined, near the boundary, by an almost complex structure. We show that on such a manifold with a strictly pseudoconvex boundary, there are modified  $\bar{\partial}$ -Neumann boundary conditions defined by projection operators,  $\mathcal{R}_+^{\text{eo}}$ , which give subelliptic Fredholm problems for the  $\text{Spin}_{\mathbb{C}}$ -Dirac operator,  $\bar{\partial}_+^{\text{eo}}$ . We introduce a generalization of Fredholm pairs to the “tame” category. In this context, we show that the index of the graph closure of  $(\bar{\partial}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}})$  equals the relative index, on the boundary, between  $\mathcal{R}_+^{\text{eo}}$  and the Calderon projector,  $\mathcal{P}_+^{\text{eo}}$ . Using the relative index formalism, and in particular, the comparison operator,  $\mathcal{F}_+^{\text{eo}}$ , introduced in [9, 10], we prove a trace formula for the relative index that generalizes the classical formula for the index of an elliptic operator. Let  $(X_0, J_0)$  and  $(X_1, J_1)$  be strictly pseudoconvex, almost complex manifolds, with  $\phi : bX_1 \rightarrow bX_0$ , a contact diffeomorphism. Let  $\mathcal{S}_0, \mathcal{S}_1$  denote generalized Szegő projectors on  $bX_0, bX_1$ , respectively, and  $\mathcal{R}_0^{\text{eo}}, \mathcal{R}_1^{\text{eo}}$ , the subelliptic boundary conditions they define. If  $\bar{X}_1$  is the manifold  $X_1$  with its orientation reversed, then the glued manifold  $X = X_0 \amalg_{\phi} \bar{X}_1$  has a canonical  $\text{Spin}_{\mathbb{C}}$ -structure and Dirac operator,  $\bar{\partial}_X^{\text{eo}}$ .

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Applying these results and those of our previous papers we obtain a formula for the relative index,  $\text{R-Ind}(\mathcal{S}_0, \phi^*\mathcal{S}_1)$ ,

$$\text{R-Ind}(\mathcal{S}_0, \phi^*\mathcal{S}_1) = \text{Ind}(\bar{\partial}_X^c) - \text{Ind}(\bar{\partial}_{X_0}^c, \mathcal{R}_0^c) + \text{Ind}(\bar{\partial}_{X_1}^c, \mathcal{R}_1^c). \quad (1)$$

For the special case that  $X_0$  and  $X_1$  are strictly pseudoconvex complex manifolds and  $\mathcal{S}_0$  and  $\mathcal{S}_1$  are the classical Szegő projectors defined by the complex structures this formula implies that

$$\text{R-Ind}(\mathcal{S}_0, \phi^*\mathcal{S}_1) = \text{Ind}(\bar{\partial}_X^c) - \chi'_0(X_0) + \chi'_0(X_1), \quad (2)$$

which is essentially the formula conjectured by Atiyah and Weinstein, see [37]. We show that, for the case of embeddable CR-structures on a compact, contact 3-manifold, this formula specializes to show that the boundedness conjecture for relative indices from [7] reduces to a conjecture of Stipsicz concerning the Euler numbers and signatures of Stein surfaces with a given contact boundary, see [35].

## Introduction

Let  $X$  be an even dimensional manifold with a  $\text{Spin}_{\mathbb{C}}$ -structure, see [21]. A compatible choice of metric,  $g$ , and connection  $\nabla^{\mathcal{S}}$ , define a  $\text{Spin}_{\mathbb{C}}$ -Dirac operator,  $\bar{\partial}$  which acts on sections of the bundle of complex spinors,  $\mathcal{S}$ . This bundle splits as a direct sum  $\mathcal{S} = \mathcal{S}^e \oplus \mathcal{S}^o$ . If  $X$  has a boundary, then the kernels and cokernels of  $\bar{\partial}^{e,o}$  are generally infinite dimensional. To obtain a Fredholm operator we need to impose boundary conditions. In this instance, there are no local boundary conditions for  $\bar{\partial}^{e,o}$  that define elliptic problems. In our earlier papers, [9, 10], we analyzed *subelliptic* boundary conditions for  $\bar{\partial}^{e,o}$  obtained by modifying the classical  $\bar{\partial}$ -Neumann and dual  $\bar{\partial}$ -Neumann conditions for  $X$ , under the assumption that the  $\text{Spin}_{\mathbb{C}}$ -structure near to the boundary of  $X$  is that defined by an integrable almost complex structure, with the boundary of  $X$  either strictly pseudoconvex or pseudoconcave. The boundary conditions considered in our previous papers have natural generalizations to almost complex manifolds with strictly pseudoconvex or pseudoconcave boundary.

A notable feature of our analysis is that, properly understood, we show that the natural generality for Kohn's classic analysis of the  $\bar{\partial}$ -Neumann problem is that of an almost complex manifold with a strictly pseudoconvex contact boundary. Indeed it is quite clear that analogous results hold true for almost complex manifolds with contact boundary satisfying the obvious generalizations of the conditions  $Z(q)$ , for a  $q$  between 0 and  $n$ , see [14]. The principal difference between the integrable and non-integrable cases is that in the latter case one must consider all form degrees at once because, in general,  $\bar{\partial}^2$  does not preserve form degree.

Before going into the details of the geometric setup we briefly describe the philosophy behind our analysis. There are three principles:

1. On an almost complex manifold the  $\text{Spin}_{\mathbb{C}}$ -Dirac operator,  $\bar{\partial}$ , is the proper replacement for  $\bar{\partial} + \bar{\partial}^*$ .
2. Indices can be computed using trace formulæ.
3. The index of a boundary value problem should be expressed as a relative index between projectors on the boundary.

The first item is a well known principle that I learned from reading [6]. Technically, the main point here is that  $\bar{\partial}^2$  differs from a metric Laplacian by an operator of order zero. As to the second item, this is a basic principle in the analysis of elliptic operators as well. It allows one to take advantage of the remarkable invariance properties of the trace. The last item is not entirely new, but our applications require a substantial generalization of the notion of Fredholm pairs. In an appendix we define *tame Fredholm pairs* and prove generalizations of many standard results. Using this approach we reduce the Atiyah-Weinstein conjecture to Bojarski's theorem, which expresses the index of a Dirac operator on a compact manifold as a relative index of a pair of Calderon projectors defined on a separating hypersurface. That Bojarski's theorem would be central to the proof of formula (1) was suggested by Weinstein in [37].

The Atiyah-Weinstein conjecture, first enunciated in the 1970s, was a conjectured formula for the index of a class of elliptic Fourier integral operators defined by contact transformations between co-sphere bundles of compact manifolds. We close this introduction with a short summary of the evolution of this conjecture and the prior results. In the original conjecture one began with a contact diffeomorphism between co-sphere bundles:  $\phi : S^*M_1 \rightarrow S^*M_0$ . This contact transformation defines a class of elliptic Fourier integral operators. There are a variety of ways to describe an operator from this class; we use an approach that makes the closest contact with the analysis in this paper.

Let  $(M, g)$  be a smooth Riemannian manifold; it is possible to define complex structures on a neighborhood of the zero section in  $T^*M$  so that the zero section and fibers of  $\pi : T^*M \rightarrow M$  are totally real, see [24], [16, 17]. For each  $\epsilon > 0$ , let  $B_\epsilon^*M$  denote the co-ball bundle of radius  $\epsilon$ , and let  $\Omega^{n,0}B_\epsilon^*M$  denote the space of holomorphic  $(n, 0)$ -forms on  $B_\epsilon^*M$  with tempered growth at the boundary. For small enough  $\epsilon > 0$ , the push-forward defines maps

$$G_\epsilon : \Omega^{n,0}B_\epsilon^*M \longrightarrow \mathcal{C}^{-\infty}(M), \quad (3)$$

such that forms smooth up to the boundary map to  $\mathcal{C}^\infty(M)$ . Boutet de Monvel and Guillemin conjectured, and Epstein and Melrose proved that there is an  $\epsilon_0 > 0$  so

that, if  $\epsilon < \epsilon_0$ , then  $G_\epsilon$  is an isomorphism, see [11]. With  $S_\epsilon^*M = bB_\epsilon^*M$ , we let  $\Omega_b^{n,0}S_\epsilon^*M$  denote the distributional boundary values of elements of  $\Omega^{n,0}B_\epsilon^*M$ . One can again define a push-forward map

$$G_{b\epsilon} : \Omega_b^{n,0}S_\epsilon^*M \longrightarrow \mathcal{C}^{-\infty}(M). \quad (4)$$

In his thesis, Raul Tataru showed that, for small enough  $\epsilon$ , this map is also an isomorphism, see [36]. As the canonical bundle is holomorphically trivial for  $\epsilon$  sufficiently small, it suffices to work with holomorphic functions (instead of  $(n, 0)$ -forms).

Let  $M_0$  and  $M_1$  be compact manifolds and  $\phi : S^*M_1 \rightarrow S^*M_0$  a contact diffeomorphism. Such a transformation canonically defines a contact diffeomorphism  $\phi_\epsilon : S_\epsilon^*M_1 \rightarrow S_\epsilon^*M_0$  for all  $\epsilon > 0$ . For sufficiently small positive  $\epsilon$ , we define the operator:

$$F_\epsilon^\phi f = G_{b\epsilon}^1 \phi_\epsilon^* [G_{b\epsilon}^0]^{-1} f. \quad (5)$$

This is an elliptic Fourier integral operator, with canonical relation essentially the graph of  $\phi$ . The original Atiyah-Weinstein conjecture (circa 1975) was a formula for the index of this operator as the index of the  $\text{Spin}_\mathbb{C}$ -Dirac operator on the compact  $\text{Spin}_\mathbb{C}$ -manifold  $B_\epsilon^*M_0 \amalg_\phi \overline{B_\epsilon^*M_1}$ . Here  $\overline{X}$  denotes a reversal of the orientation of the oriented manifold  $X$ . If we let  $\mathcal{S}_\epsilon^j$  denote the Szegő projectors onto the boundary values of holomorphic functions on  $B_\epsilon^*M_j$ ,  $j = 0, 1$ , then, using the Epstein-Melrose-Tataru result, Zelditch observed that the index of  $F_\epsilon^\phi$  could be computed as the relative index between the Szegő projectors,  $\mathcal{S}_\epsilon^0$ , and  $[\phi^{-1}]^* \mathcal{S}_\epsilon^1 \phi^*$ , defined on  $S_\epsilon^*M_0$ , i.e.

$$\text{Ind}(F_\epsilon^\phi) = \text{R-Ind}(\mathcal{S}_\epsilon^0, [\phi^{-1}]^* \mathcal{S}_\epsilon^1 \phi^*). \quad (6)$$

Weinstein subsequently generalized the conjecture to allow for contact transforms  $\phi : bX_1 \rightarrow bX_0$ , where  $X_0, X_1$  are strictly pseudoconvex complex manifolds with boundary, see [37]. In this paper Weinstein suggests a variety of possible formulæ depending upon whether or not the  $X_j$  are Stein manifolds.

Several earlier papers treat special cases of this conjecture (including the original conjectured formula). In [12], Epstein and Melrose consider operators defined by contact transformations  $\phi : Y \rightarrow Y$ , for  $Y$  an arbitrary compact, contact manifold. If  $\mathcal{S}$  is any generalized Szegő projector defined on  $Y$ , then they show that  $\text{R-Ind}(\mathcal{S}, [\phi^{-1}]^* \mathcal{S} \phi^*)$  depends only on the contact isotopy class of  $\phi$ . In light of its topological character, Epstein and Melrose call this relative index the *contact degree* of  $\phi$ , denoted  $\text{c-deg}(\phi)$ . It equals the index of the  $\text{Spin}_\mathbb{C}$ -Dirac operator on the mapping torus  $Z_\phi = Y \times [0, 1]/(y, 0) \sim (\phi(y), 1)$ . Generalized Szegő projectors were originally introduced by Boutet de Monvel and Guillemin, in the context

of the Hermite calculus, see [5]. A discussion of generalized Szegő projectors and their relative indices, in the Heisenberg calculus, can be found in [12].

Leichtnam, Nest and Tsygan consider the case of contact transformations  $\phi : S^*M_1 \rightarrow S^*M_0$  and obtain a cohomological formula for the index of  $F_\epsilon^\phi$ , see [23]. The approaches of these two papers are quite different: Epstein and Melrose express the relative index as a spectral flow, which they compute by using the extended Heisenberg calculus to deform, through Fredholm operators, to the  $\text{Spin}_\mathbb{C}$ -Dirac operator on  $Z_\phi$ . Leichtnam, Nest and Tsygan use the deformation theory of Lie algebroids and the general algebraic index theorem from [27] to obtain their formula for the index of  $F_\epsilon^\phi$ . In this paper we also make extensive usage of the extended Heisenberg calculus, but the outline of the argument here is quite different here from that in [12].

One of our primary motivations for studying this problem was to find a formula for the relative index between pairs of Szegő projectors,  $\mathcal{S}_0, \mathcal{S}_1$ , defined by embeddable, strictly pseudoconvex CR-structures on a compact, 3-dimensional contact manifold  $(Y, H)$ . In [7] we conjectured that, among small embeddable deformations, the relative index,  $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)$  should assume finitely many distinct values. It is shown that the relative index conjecture implies that the set of small embeddable perturbations of an embeddable CR-structure on  $(Y, H)$  is closed in the  $\mathcal{C}^\infty$ -topology.

Suppose that  $j_0, j_1$  are embeddable CR-structures on  $(Y, H)$ , which bound the strictly pseudoconvex, complex surfaces  $(X_0, J_0), (X_1, J_1)$ , respectively. In this situation our general formula, (2) takes a very explicit form:

$$\begin{aligned} \text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = & \dim H^{0,1}(X_0, J_0) - \dim H^{0,1}(X_1, J_1) + \\ & \frac{\text{sig}[X_0] - \text{sig}[X_1] + \chi[X_0] - \chi[X_1]}{4}. \end{aligned} \quad (7)$$

Here  $\text{sig}[M]$  is the signature of the oriented 4-manifold  $M$  and  $\chi(M)$  is its Euler characteristic. In [35], Stipsicz conjectures that, among Stein manifolds  $(X, J)$  with  $(Y, H)$  as boundary, the characteristic numbers  $\text{sig}[X], \chi[X]$  assume only finitely many values. Whenever Stipsicz's conjecture is true it implies a strengthened form of the relative index conjecture: the function  $\mathcal{S}_1 \mapsto \text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)$  is bounded from above throughout the entire deformation space of embeddable CR-structures on  $(Y, H)$ . Many cases of Stipsicz's conjecture are proved in [30, 35]. As a second consequence of (7) we show that, if  $\dim M_j = 2$ , then  $\text{Ind}(F_\epsilon^\phi) = 0$ .

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## 1 Outline of Results

Let  $X$  be an even dimensional manifold and let  $\mathcal{S} \rightarrow X$  denote the bundle of complex spinors defined by a  $\text{Spin}_{\mathbb{C}}$ -structure on  $X$ . A choice of metric on  $X$  and compatible connection,  $\nabla^{\mathcal{S}}$ , on the bundle  $\mathcal{S}$  define the  $\text{Spin}_{\mathbb{C}}$ -Dirac operator,  $\bar{D}$  :

$$\bar{D}\sigma = \sum_{j=0}^{\dim X} \mathbf{c}(\omega_j) \cdot \nabla_{V_j}^{\mathcal{S}} \sigma, \quad (8)$$

with  $\{V_j\}$  a local framing for the tangent bundle and  $\{\omega_j\}$  the dual coframe. Here  $\mathbf{c}(\omega) \cdot$  denotes the Clifford action of  $T^*X$  on  $\mathcal{S}$ . It is customary to split  $\bar{D}$  into its chiral parts:  $\bar{D} = \bar{D}^e + \bar{D}^o$ , where

$$\bar{D}^{\text{eo}} : \mathcal{C}^\infty(X; \mathcal{S}^{\text{eo}}) \longrightarrow \mathcal{C}^\infty(X; \mathcal{S}^{\text{eo}}).$$

The operators  $\bar{D}^o$  and  $\bar{D}^e$  are formal adjoints.

An almost complex structure on  $X$  defines a  $\text{Spin}_{\mathbb{C}}$ -structure, and bundle of complex spinors  $\mathcal{S}$ , see [6]. The bundle of complex spinors is canonically identified with  $\bigoplus_{q \geq 0} \Lambda^{0,q}$ . We use the notation

$$\Lambda^e = \bigoplus_{q=0}^{\lfloor \frac{n}{2} \rfloor} \Lambda^{0,2q} \quad \Lambda^o = \bigoplus_{q=0}^{\lfloor \frac{n-1}{2} \rfloor} \Lambda^{0,2q+1}. \quad (9)$$

These bundles are in turn canonically identified with the bundles of even and odd spinors,  $\mathcal{S}^{\text{eo}}$ , which are defined as the  $\pm 1$ -eigenspaces of the orientation class. A metric  $g$  on  $X$  is compatible with the almost complex structure, if for every  $x \in X$  and  $V, W \in T_x X$ , we have:

$$g_x(J_x V, J_x W) = g_x(V, W). \quad (10)$$

Let  $X$  be a compact manifold with a co-oriented contact structure  $H \subset T_b X$ , on its boundary. Let  $\theta$  denote a globally defined contact form in the given co-orientation class. An almost complex structure  $J$  defined in a neighborhood of  $bX$  is compatible with the contact structure if, for every  $x \in bX$ , we have

$$\begin{aligned} J_x H_x &\subset H_x, \text{ and for all } V, W \in H_x \\ d\theta_x(J_x V, W) + d\theta_x(V, J_x W) &= 0 \\ d\theta_x(V, J_x V) &> 0, \text{ if } V \neq 0. \end{aligned} \quad (11)$$

We usually assume that  $g \upharpoonright_{H \times H} = d\theta(\cdot, J\cdot)$ . If the almost complex is not integrable, then  $\bar{\partial}^2$  does not preserve the grading of  $\mathcal{S}$  defined by the  $(0, q)$ -types.

As noted, the almost complex structure defines the bundles  $T^{1,0}X, T^{0,1}X$  as well as the form bundles  $\Lambda^{0,q}X$ . This in turn defines the  $\bar{\partial}$ -operator. The bundles  $\Lambda^{0,q}$  have a splitting at the boundary into almost complex normal and tangential parts, so that a section  $s$  satisfies:

$$s \upharpoonright_{bX} = s^t + \bar{\partial}\rho \wedge s^n, \text{ where } \bar{\partial}\rho \lrcorner s^t = \bar{\partial}\rho \lrcorner s^n = 0. \quad (12)$$

Here  $\rho$  is defining function for  $bX$ . The  $\bar{\partial}$ -Neumann condition for sections  $s \in \mathcal{C}^\infty(X; \Lambda^{0,q})$  is the requirement that

$$\bar{\partial}\rho \lrcorner [s]_{bX} = 0, \quad (13)$$

i.e.,  $s^n = 0$ . As before this does not impose any requirement on forms of degree  $(0, 0)$ .

The contact structure on  $bX$  defines the class of generalized Szegő projectors acting on scalar functions, see [10, 12] for the definition. Using the identifications of  $\mathcal{S}^{\text{eo}}$  with  $\Lambda^{0,\text{eo}}$ , a generalized Szegő projector,  $\mathcal{S}$ , defines a modified (strictly pseudoconvex)  $\bar{\partial}$ -Neumann condition as follows:

$$\begin{aligned} \mathcal{R}\sigma^{00} &\stackrel{d}{=} \mathcal{S}[\sigma^{00}]_{bX} = 0 \\ \mathcal{R}\sigma^{01} &\stackrel{d}{=} (\text{Id} - \mathcal{S})[\bar{\partial}\rho \lrcorner \sigma^{01}]_{bX} = 0 \\ \mathcal{R}\sigma^{0q} &\stackrel{d}{=} [\bar{\partial}\rho \lrcorner \sigma^{0q}]_{bX} = 0, \text{ for } q > 1. \end{aligned} \quad (14)$$

We choose the defining function so that  $s^t$  and  $\bar{\partial}\rho \wedge s^n$  are orthogonal, hence the mapping  $\sigma \rightarrow \mathcal{R}\sigma$  is a self adjoint projection operator. Following the practice in [9, 10] we use  $\mathcal{R}^{\text{eo}}$  to denote the restrictions of this projector to the subbundles of even and odd spinors.

We follow the conventions for the  $\text{Spin}_{\mathbb{C}}$ -structure and Dirac operator on an almost complex manifold given in [6]. Lemma 5.5 in [6] states that the principal symbol of  $\bar{\partial}_X$  agrees with that of the Dolbeault-Dirac operator  $\bar{\partial} + \bar{\partial}^*$ , which implies that  $(\bar{\partial}_X^{\text{eo}}, \mathcal{R}^{\text{eo}})$  are formally adjoint operators. It is a consequence of our analysis that, as unbounded operators on  $L^2$ ,

$$(\bar{\partial}_X^{\text{eo}}, \mathcal{R}^{\text{eo}})^* = \overline{(\bar{\partial}_X^{\text{oe}}, \mathcal{R}^{\text{oe}})}. \quad (15)$$

The almost complex structure is only needed to define the boundary condition. Hence we assume that  $X$  is a  $\text{Spin}_{\mathbb{C}}$ -manifold, where the  $\text{Spin}_{\mathbb{C}}$ -structure is defined by an almost complex structure  $J$  defined in a neighborhood of the boundary.

In this paper we begin by showing that the analytic results obtained in our earlier papers remain true in the almost complex case. As noted above, this shows that integrability is not needed for the validity of Kohn's estimates for the  $\bar{\partial}$ -Neumann problem. By working with  $\text{Spin}_{\mathbb{C}}$ -structures we are able to fashion a much more flexible framework for studying index problems than that presented in [9, 10]. As before, we compare the projector  $\mathcal{R}$  defining the subelliptic boundary conditions with the Calderon projector for  $\bar{\partial}$ , and show that these projectors are, in a certain sense, relatively Fredholm. These projectors are not relatively Fredholm in the usual sense of say Fredholm pairs in a Hilbert space, used in the study of elliptic boundary value problems. We circumvent this problem by extending the theory of Fredholm pairs to that of *tame Fredholm pairs*. We then use our analytic results to obtain a formula for a parametrix for these subelliptic boundary value problems that is precise enough to prove, among other things, higher norm estimates. The extended Heisenberg calculus introduced in [13] remains at the center of our work. The basics of this calculus are outlined in [10].

If  $\mathcal{R}^{\text{eo}}$  are projectors defining modified  $\bar{\partial}$ -Neumann conditions and  $\mathcal{P}^{\text{eo}}$  are the Calderon projectors, then we show that the comparison operators,

$$\mathcal{T}^{\text{eo}} = \mathcal{R}^{\text{eo}}\mathcal{P}^{\text{eo}} + (\text{Id} - \mathcal{R}^{\text{eo}})(\text{Id} - \mathcal{P}^{\text{eo}}) \quad (16)$$

are graded elliptic elements of the extended Heisenberg calculus. As such there are parametrices  $\mathcal{U}^{\text{eo}}$  that satisfy

$$\mathcal{T}^{\text{eo}}\mathcal{U}^{\text{eo}} = \text{Id} - K_1^{\text{eo}}, \quad \mathcal{U}^{\text{eo}}\mathcal{T}^{\text{eo}} = \text{Id} - K_2^{\text{eo}}, \quad (17)$$

where  $K_1^{\text{eo}}, K_2^{\text{eo}}$  are smoothing operators. We define Hilbert spaces,  $\mathcal{H}_{\mathcal{U}^{\text{eo}}}$  to be the closures of  $\mathcal{C}^{\infty}(bX; \mathcal{F}^{\text{eo}}|_{bX})$  with respect to the inner products

$$\langle \sigma, \sigma \rangle_{\mathcal{H}_{\mathcal{U}^{\text{eo}}}} = \langle \sigma, \sigma \rangle_{L^2} + \langle \mathcal{U}^{\text{eo}}\sigma, \mathcal{U}^{\text{eo}}\sigma \rangle_{L^2}. \quad (18)$$

The operators  $\mathcal{R}^{\text{eo}}\mathcal{P}^{\text{eo}}$  are Fredholm from range  $\mathcal{P}^{\text{eo}} \cap L^2$  to range  $\mathcal{R}^{\text{eo}} \cap \mathcal{H}_{\mathcal{U}^{\text{eo}}}$ . As usual, we let  $\text{R-Ind}(\mathcal{P}^{\text{eo}}, \mathcal{R}^{\text{eo}})$  denote the indices of these restrictions; we show that

$$\text{Ind}(\bar{\partial}^{\text{eo}}, \mathcal{R}^{\text{eo}}) = \text{R-Ind}(\mathcal{P}^{\text{eo}}, \mathcal{R}^{\text{eo}}). \quad (19)$$

Using the standard formalism for computing indices we show that

$$\text{R-Ind}(\mathcal{P}^{\text{eo}}, \mathcal{R}^{\text{eo}}) = \text{tr } \mathcal{R}^{\text{eo}} K_1^{\text{eo}} \mathcal{R}^{\text{eo}} - \text{tr } \mathcal{P}^{\text{eo}} K_2^{\text{eo}} \mathcal{P}^{\text{eo}}. \quad (20)$$

There is some subtlety in the interpretation of this formula in that  $\mathcal{R}^{\text{eo}} K_1^{\text{eo}} \mathcal{R}^{\text{eo}}$  act on  $\mathcal{H}_{\mathcal{U}^{\text{eo}}}$ . But, as is also used implicitly in the elliptic case, we show that the computation of the trace does not depend on the topology of the underlying Hilbert space.



Among other things, this formula allows us to prove that the indices of the boundary problems  $(\bar{\partial}^{\text{eo}}, \mathcal{R}^{\text{eo}})$  depend continuously on the data defining the boundary condition and the  $\text{Spin}_{\mathbb{C}}$ -structure, allowing us to employ deformation arguments.

To obtain the gluing formula we use the invertible double construction introduced in [3]. Using this construction, we are able to express the relative index between two generalized Szegő projectors as the index of the  $\text{Spin}_{\mathbb{C}}$ -Dirac operators on a compact manifold with corrections coming from boundary value problems on the ends. Let  $X_0, X_1$  be  $\text{Spin}_{\mathbb{C}}$ -manifolds with contact boundaries. Assume that the  $\text{Spin}_{\mathbb{C}}$ -structures are defined in neighborhoods of the boundaries by compatible almost complex structures, such that  $bX_0$  is contact isomorphic to  $bX_1$ , let  $\phi : bX_1 \rightarrow bX_0$  denote a contact diffeomorphism. If  $\overline{X_1}$  denotes  $X_1$  with its orientation reversed, then  $\tilde{X}_{01} = X_0 \amalg_{\phi} \overline{X_1}$  is a compact manifold with a canonical  $\text{Spin}_{\mathbb{C}}$ -structure and Dirac operator,  $\bar{\partial}_{\tilde{X}_{01}}^{\text{eo}}$ . Even if  $X_0$  and  $X_1$  have globally defined almost complex structures, the manifold  $X$ , in general, does not. In case  $X_0 = X_1$ , as  $\text{Spin}_{\mathbb{C}}$ -manifolds, then this is the invertible double introduced in [3], where they show that  $\bar{\partial}_{\tilde{X}_{01}}$  is an invertible operator.

Let  $\mathcal{S}_0, \mathcal{S}_1$  be generalized Szegő projectors on  $bX_0, bX_1$ , respectively. If  $\mathcal{R}_0^{\text{e}}, \mathcal{R}_1^{\text{e}}$  are the subelliptic boundary conditions they define, then the main result of this paper is the following formula:

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = \text{Ind}(\bar{\partial}_{\tilde{X}_{01}}^{\text{e}}) - \text{Ind}(\bar{\partial}_{X_0}^{\text{e}}, \mathcal{R}_0^{\text{e}}) + \text{Ind}(\bar{\partial}_{X_1}^{\text{e}}, \mathcal{R}_1^{\text{e}}). \quad (21)$$

As detailed in the introduction, such a formula was conjectured, in a more restricted case, by Atiyah and Weinstein, see [37]. Our approach differs a little from that conjectured by Weinstein, in that  $\tilde{X}_{01}$  is constructed using the extended invertible double construction rather than the stabilization of the almost complex structure on the glued space described in [37]. A result of Cannas da Silva implies that the stable almost complex structure on  $\tilde{X}_{01}$  defines a  $\text{Spin}_{\mathbb{C}}$ -structure, which very likely agrees with that used here, see [15]. Our formula is very much in the spirit suggested by Atiyah and Weinstein, though we have not found it necessary to restrict to  $X_0, X_1$  to be Stein manifolds (or even complex manifolds), nor have we required the use of “pseudoconcave caps” in the non-Stein case. It is quite likely that there are other formulæ involving the pseudoconcave caps and they will be considered in a subsequent publication.

In the case that  $X_0$  is isotopic to  $X_1$  through  $\text{Spin}_{\mathbb{C}}$ -structures compatible with the contact structure on  $Y$ , then  $\tilde{X}_{01}$ , with its canonical  $\text{Spin}_{\mathbb{C}}$ -structure, is isotopic to the invertible double of  $X_0 \simeq X_1$ . In [3] it is shown that in this case,  $\bar{\partial}_{\tilde{X}_{01}}^{\text{eo}}$  are invertible operators and hence  $\text{Ind}(\bar{\partial}_{\tilde{X}_{01}}^{\text{e}}) = 0$ . Thus (21) states that

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = \text{Ind}(\bar{\partial}_{X_1}^{\text{e}}, \mathcal{R}_1^{\text{e}}) - \text{Ind}(\bar{\partial}_{X_0}^{\text{e}}, \mathcal{R}_0^{\text{e}}). \quad (22)$$

If  $X_0 \simeq X_1$  are complex manifolds with strictly pseudoconvex boundaries, and the complex structures are isotopic as above (through compatible almost complex structures), and the Szegő projectors are those defined by the complex structure, then formula (77) in [9] implies that  $\text{Ind}(\partial_{X_j}^c, \mathcal{R}_j^c) = \chi'_0(X_j)$  and therefore:

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = \chi'_0(X_1) - \chi'_0(X_0). \quad (23)$$

When  $\dim_{\mathbb{C}} X_j = 2$ , this formula becomes:

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = \dim H^{0,1}(X_0) - \dim H^{0,1}(X_1), \quad (24)$$

which has applications to the relative index conjecture in [7]. In the case that  $\dim_{\mathbb{C}} X_j = 1$ , a very similar formula was obtained by Segal and Wilson, see [33, 19]. A detailed analysis of the complex 2-dimensional case is given in Section 12, where we prove (7).

In Section 11 we show how these results can be extended to allow for vector bundle coefficients. An interesting consequence of this analysis is a proof, which makes no mention of K-theory, that the index of a classically elliptic operator on a compact manifold  $M$  equals that of a  $\text{Spin}_{\mathbb{C}}$ -Dirac operator on the glued space  $B^*M \amalg_{S^*M} \overline{B^*M}$ . Hence, using relative indices and the extended Heisenberg calculus, along with Getzler's rescaling argument we obtain an entirely analytic proof of the Atiyah-Singer formula.

*Remark 1.* In this paper we restrict our attention to the pseudoconvex case. There are analogous results for other cases with non-degenerate  $d\theta(\cdot, J\cdot)$ . We will return to these in a later publication. The subscript  $+$  sometimes refers to the fact that the underlying manifold is pseudoconvex. Sometimes, however, we use  $\pm$  to designate the two sides of a separating hypersurface.

## 2 The symbol of the Dirac Operator and its inverse

In this section we show that, under appropriate geometric hypotheses, the results of Sections 2–5 of [10] remain valid, with small modifications, for the  $\text{Spin}_{\mathbb{C}}$ -Dirac operator on an almost complex manifold, with strictly pseudoconvex boundary. As noted above the  $\text{Spin}_{\mathbb{C}}$ -structure only need be defined by an almost complex structure near the boundary. This easily implies that the operators  $\mathcal{T}_{\pm}^{\text{eo}}$  are elliptic elements of the extended Heisenberg calculus. To simplify the exposition we treat only the pseudoconvex case. The results in the pseudoconcave case are entirely analogous. For simplicity we also omit vector bundle coefficients. There is no essential difference if they are included; the modifications necessary to treat this case are outlined in Section 11.

Let  $X$  be a manifold with boundary,  $Y$ . We suppose that  $(Y, H)$  is a contact manifold and  $X$  has an almost complex structure  $J$ , defined near the boundary, compatible with the contact structure, with respect to which the boundary is strictly pseudoconvex, see [2]. We let  $g$  denote a metric on  $X$  compatible with the almost complex structure: for every  $x \in X$ ,  $V, W \in T_x X$  we have

$$g_x(J_x V, J_x W) = g_x(V, W). \quad (25)$$

We suppose that  $\rho$  is a defining function for the boundary of  $X$  that is negative on  $X$ . Let  $\bar{\partial}$  denote the (possibly non-integrable)  $\bar{\partial}$ -operator defined by  $J$ . We assume that  $JH \subset H$ , and that the one form,

$$\theta = \frac{i}{2} \bar{\partial} \rho \upharpoonright_{TbX}, \quad (26)$$

is a contact form for  $H$ . The quadratic form defined on  $H \times H$  by

$$\mathcal{L}(V, W) = d\theta(V, JW) \quad (27)$$

is assumed to be positive definite. In the almost complex category this is the statement that  $bX$  is strictly pseudoconvex.

Let  $T$  denote the Reeb vector field:  $\theta(T) = 1, i_T d\theta = 0$ . For simplicity we assume that

$$g \upharpoonright_{H \times H} = \mathcal{L} \text{ and } g(T, V) = 0, \quad \forall V \in H. \quad (28)$$

Note that (25) and (28) imply that  $J$  is compatible with  $d\theta$  in that, for all  $V, W \in H$  we have

$$d\theta(JV, JW) = d\theta(V, W) \text{ and } d\theta(V, JV) > 0 \text{ if } V \neq 0. \quad (29)$$

**Definition 1.** Let  $X$  be a  $\text{Spin}_{\mathbb{C}}$ -manifold with almost complex structure  $J$ , defined near  $bX$ . If the  $\text{Spin}_{\mathbb{C}}$ -structure near  $bX$  is that specified by  $J$ , then the quadruple  $(X, J, g, \rho)$  satisfying (25)–(28) defines a *normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold*.

On an almost complex manifold with compatible metric there is a  $\text{Spin}_{\mathbb{C}}$ -structure so that the bundle of complex spinors  $\mathcal{S} \rightarrow X$  is a complex Clifford module. As noted above, if the  $\text{Spin}_{\mathbb{C}}$ -structure is defined by an almost complex structure, then  $\mathcal{S} \simeq \oplus \Lambda^{0,q}$ . Under this isomorphism, the Clifford action of a real one-form  $\xi$  is given by

$$c(\xi) \cdot \sigma \stackrel{d}{=} (\xi - iJ\xi) \wedge \sigma - \xi \lrcorner \sigma. \quad (30)$$

It is extended to the complexified Clifford algebra complex linearly. We largely follow the treatment of  $\text{Spin}_{\mathbb{C}}$ -geometry given in [6], though with some modifications to make easier comparisons with the results of our earlier papers.

There is a compatible connection  $\nabla^{\mathcal{F}}$  on  $\mathcal{F}$  and a formally self adjoint  $\text{Spin}_{\mathbb{C}}$ -Dirac operator defined on sections of  $\mathcal{F}$  by

$$\bar{\partial}\sigma = \frac{1}{2} \sum_{j=1}^{2n} \mathbf{e}(\omega_j) \cdot \nabla_{V_j}^{\mathcal{F}} \sigma, \quad (31)$$

with  $\{V_j\}$  a local framing for the tangent bundle and  $\{\omega_j\}$  the dual coframe. Here we differ slightly from [6] by including the factor  $\frac{1}{2}$  in the definition of  $\bar{\partial}$ . This is so that, in the case that  $J$  is integrable, the leading order part of  $\bar{\partial}$  is  $\bar{\partial} + \bar{\partial}^*$  (rather than  $2(\bar{\partial} + \bar{\partial}^*)$ ), which makes for a more direct comparison with results in [9, 10].

The spinor bundle splits into even and odd components  $\mathcal{F}^{\text{eo}}$ , and the Dirac operator splits into even and odd parts,  $\bar{\partial}^{\text{eo}}$ , where

$$\bar{\partial}^{\text{eo}} : \mathcal{C}^\infty(X; \mathcal{F}^{\text{eo}}) \longrightarrow \mathcal{C}^\infty(X; \mathcal{F}^{\text{oe}}). \quad (32)$$

Note that, in each fiber, Clifford multiplication by a non-zero co-vector gives an isomorphism  $\mathcal{F}^{\text{eo}} \leftrightarrow \mathcal{F}^{\text{oe}}$ .

Fix a point  $p$  on the boundary of  $X$  and let  $(x_1, \dots, x_{2n})$  denote normal coordinates centered at  $p$ . This means that

1.  $p \leftrightarrow (0, \dots, 0)$
2. The Hermitian metric tensor  $g_{i\bar{j}}$  in these coordinates satisfies

$$g_{i\bar{j}} = \frac{1}{2} \delta_{i\bar{j}} + O(|x|^2). \quad (33)$$

If  $V \in T_p X$  is a unit vector, then  $V^{0,1} = \frac{1}{2}(V + iJV)$ , and

$$\langle V^{0,1}, V^{0,1} \rangle_g = \frac{1}{2}. \quad (34)$$

Without loss of generality we may also assume that the coordinates are ‘‘almost complex’’ and adapted to the contact geometry *at*  $p$ : that is the vectors  $\{\partial_{x_j}\} \subset T_p X$  satisfy

$$\begin{aligned} J_p \partial_{x_j} &= \partial_{x_{j+n}} \text{ for } j = 1, \dots, n \\ \{\partial_{x_2}, \dots, \partial_{x_{2n}}\} &\in T_p bX \\ \{\partial_{x_2}, \dots, \partial_{x_n}, \partial_{x_{n+2}}, \dots, \partial_{x_{2n}}\} &\in H_p. \end{aligned} \quad (35)$$

We let

$$z_j = x_j + ix_{j+n}.$$

As  $d\rho|_{bX} = 0$ , equation (35) implies that

$$\rho(z) = -\frac{2}{\alpha} \operatorname{Re} z_1 + \langle az, z \rangle + \operatorname{Re}(bz, z) + O(|z|^3). \quad (36)$$

In this equation  $\alpha > 0$ ,  $a$  and  $b$  are  $n \times n$  complex matrices, with  $a = a^*$ ,  $b = b^t$ , and

$$\langle w, z \rangle = \sum_{j=1}^n w_j \bar{z}_j \text{ and } (w, z) = \sum_{j=1}^n w_j z_j. \quad (37)$$

With these normalizations we have the following formulæ for the contact form at  $p$  :

**Lemma 1.** *Under the assumptions above*

$$\theta_p = -\frac{1}{2\alpha} dx_{n+1} \text{ and } d\theta_p = \sum_{j=2}^n dx_j \wedge dx_{j+n}. \quad (38)$$

*Proof.* The formula for  $\theta_p$  follows from (36). The normality of the coordinates, (28) and (35) imply that, for a one-form  $\phi_p$  we have

$$d\theta_p = \sum_{j=2}^n dx_j \wedge dx_{j+n} + \theta_p \wedge \phi_p. \quad (39)$$

The assumption that the Reeb vector field is orthogonal to  $H_p$  and (35) imply that  $\partial_{x_{n+1}}$  is a multiple of the Reeb vector field. Hence  $\phi_p = 0$ .  $\square$

For symbolic calculations the following notation proves very useful: a term which is a symbol of order at most  $k$  vanishing, at  $p$ , to order  $l$  is denoted by  $\mathfrak{D}_k(|x|^l)$ . As we work with a variety of operator calculi, it is sometimes necessary to be specific as to the sense in which the order should be taken. The notation  $\mathfrak{D}_j^C$  refers to terms of order at most  $j$  in the sense of the symbol class  $C$ . If no symbol class is specified, then the order is with respect to the classical, radial scaling. If no rate of vanishing is specified, it should be understood to be  $O(1)$ .

If  $\{f_j\}$  is an orthonormal frame for  $TX$ , then the Laplace operator on the spinor bundle is defined by

$$\Delta = \sum_{j=1}^{2n} \nabla_{f_j}^{\mathfrak{s}} \circ \nabla_{f_j}^{\mathfrak{s}} - \nabla_{\nabla_{f_j}^{\mathfrak{s}} f_j}^{\mathfrak{s}}. \quad (40)$$

Here  $\nabla^{\mathfrak{s}}$  is the Levi-Civita connection on  $TX$ . As explained in [6], the reason for using the  $\operatorname{Spin}_C$ -Dirac operator as a replacement for  $\bar{\partial} + \bar{\partial}^*$  is because of its very close connection to the Laplace operator.

**Proposition 1.** *Let  $(X, g, J)$  be a Hermitian, almost complex manifold and  $\bar{\partial}$  the  $\text{Spin}_{\mathbb{C}}$ -Dirac operator defined by this data. Then*

$$\bar{\partial}^2 = \frac{1}{2}\Delta + R, \quad (41)$$

where  $R : \mathcal{S} \rightarrow \mathcal{S}$  is an endomorphism.

After changing to the normalizations used here, e.g.  $\langle V^{0,1}, V^{0,1} \rangle_g = \frac{1}{2}$ , this is Theorem 6.1 in [6]. Using this result we can compute the symbols of  $\bar{\partial}$  and  $\bar{\partial}^2$  at  $p$ . Recall that

$$\nabla^g \partial_{x_k} = O(|x|). \quad (42)$$

We can choose a local orthonormal framing for  $\mathcal{S}$ ,  $\{\sigma_J\}$  ( $J = (j_1, \dots, j_q)$  with  $1 \leq j_1 < \dots < j_q \leq n$ ) so that

$$\sigma_J - d\bar{z}^J = O(|x|) \text{ and } \nabla^{\mathcal{S}} \sigma_J = O(|x|) \quad (43)$$

as well.

With respect to this choice of frame, the symbol of  $\bar{\partial}$ , in a geodesic normal coordinate system, is

$$\sigma(\bar{\partial})(x, \xi) = d_1(x, \xi) + d_0(x). \quad (44)$$

Because the connection coefficients vanish at  $p$  we obtain:

$$d_1(x, \xi) = d_1(0, \xi) + \mathfrak{D}_1(|x|), \quad d_0(z) = \mathfrak{D}_0(|x|). \quad (45)$$

The linear polynomial  $d_1(0, \xi)$  is the symbol of  $\bar{\partial} + \bar{\partial}^*$  on  $\mathbb{C}^n$  with respect to the flat metric. This is slightly different from the Kähler case where  $d_1(x, \xi) - d_1(0, \xi) = \mathfrak{D}_1(|x|^2)$ . First order vanishing is sufficient for our applications, we only needed the quadratic vanishing to obtain the formula for the symbol of  $\bar{\partial}^2$ , obtained here from Proposition 1.

Proposition 1 implies that

$$\sigma(\bar{\partial}^2)(x, \xi) = \sigma(\Delta - R)(x, \xi) = \Delta_2(x, \xi) + \Delta_1(x, \xi) + \Delta_0(x), \quad (46)$$

where  $\Delta_j$  is a polynomial in  $\xi$  of degree  $j$  and

$$\begin{aligned} \Delta_2(x, \xi) &= \Delta_2(0, \xi) + \mathfrak{D}_2(|x|^2) \\ \Delta_1(x, \xi) &= \mathfrak{D}_1(|x|), \quad \Delta_0(x, \xi) = \mathfrak{D}_0(1). \end{aligned} \quad (47)$$

Because we are working in geodesic normal coordinates, the principal symbol at  $p$  is

$$\Delta_2(0, \xi) = \frac{1}{2}|\xi|^2 \otimes \text{Id}. \quad (48)$$

Here  $\text{Id}$  is the identity homomorphism on the appropriate bundle. These formulæ are justified in Section 11, where we explain the modifications needed to include vector bundle coefficients.

The manifold  $X$  can be included into a larger manifold  $\tilde{X}$  (the invertible double) in such a way that its  $\text{Spin}_\mathbb{C}$ -structure and Dirac operator extend smoothly to  $\tilde{X}$  and such that the extended operators  $\tilde{\partial}^{\text{eo}}$  are invertible. We return to this construction in Section 7. Let  $Q^{\text{eo}}$  denote the inverses of  $\tilde{\partial}^{\text{eo}}$  extended to  $\tilde{X}$ . These are classical pseudodifferential operators of order  $-1$ .

We set  $\tilde{X} \setminus Y = \tilde{X}_+ \sqcup \tilde{X}_-$ , where  $\tilde{X}_+ = X$ ; note that  $\rho < 0$  on  $\tilde{X}_+$ , and  $\rho > 0$  on  $\tilde{X}_-$ . Let  $r_\pm$  denote the operations of restriction of a section of  $\mathcal{F}^{\text{eo}}$ , defined on  $\tilde{X}$  to  $\tilde{X}_\pm$ , and  $\gamma_\epsilon$  the operation of restriction of a smooth section of  $\mathcal{F}^{\text{eo}}$  to  $Y_\epsilon = \{\rho^{-1}(\epsilon)\}$ . Define the operators

$$\tilde{K}_\pm^{\text{eo}} \stackrel{d}{=} r_\pm Q^{\text{eo}} \gamma_0^* : \mathcal{C}^\infty(Y; \mathcal{F}^{\text{eo}} \upharpoonright_Y) \longrightarrow \mathcal{C}^\infty(\tilde{X}_\pm; \mathcal{F}^{\text{eo}}). \quad (49)$$

Here  $\gamma_0^*$  is the formal adjoint of  $\gamma_0$ . We recall that, along  $Y$ , the symbol  $\sigma_1(\tilde{\partial}^{\text{eo}}, d\rho)$  defines an isomorphism

$$\sigma_1(\tilde{\partial}^{\text{eo}}, d\rho) : \mathcal{F}^{\text{eo}} \upharpoonright_Y \longrightarrow \mathcal{F}^{\text{eo}} \upharpoonright_Y. \quad (50)$$

Composing, we get the usual Poisson operators

$$\mathcal{H}_\pm^{\text{eo}} = \frac{\mp}{i \|d\rho\|} \tilde{K}_\pm^{\text{eo}} \circ \sigma_1(\tilde{\partial}^{\text{eo}}, d\rho) : \mathcal{C}^\infty(Y; \mathcal{F}^{\text{eo}} \upharpoonright_Y) \longrightarrow \mathcal{C}^\infty(\tilde{X}_\pm; \mathcal{F}^{\text{eo}}), \quad (51)$$

which map sections of  $\mathcal{F}^{\text{eo}} \upharpoonright_Y$  into the nullspaces of  $\tilde{\partial}_\pm^{\text{eo}}$ . The factor  $\mp$  is inserted because  $\rho < 0$  on  $X$ .

The Calderon projectors are defined by

$$\mathcal{P}_\pm^{\text{eo}} \stackrel{d}{=} \lim_{\mp\epsilon \rightarrow 0^+} \gamma_\epsilon \mathcal{H}_\pm^{\text{eo}} s \text{ for } s \in \mathcal{C}^\infty(Y; \mathcal{F}^{\text{eo}} \upharpoonright_Y). \quad (52)$$

The fundamental result of Seeley is that  $\mathcal{P}_\pm^{\text{eo}}$  are classical pseudodifferential operators of order 0. The ranges of these operators are the boundary values of elements of  $\ker \tilde{\partial}_\pm^{\text{eo}}$ . Seeley gave a prescription for computing the symbols of these operators using contour integrals, which we do not repeat here, as we shall be computing these symbols in detail in the following sections, see [32].

*Remark 2 (Notational remark).* Unlike in [9, 10], the notation  $\mathcal{P}_+^{\text{eo}}$  and  $\mathcal{P}_-^{\text{eo}}$  refers to the Calderon projectors defined on the two sides of a separating hypersurface in a single manifold  $\tilde{X}$ , with an invertible  $\text{Spin}_\mathbb{C}$ -Dirac operator. This is the more standard usage; in this case we have the identities  $\mathcal{P}_+^{\text{eo}} + \mathcal{P}_-^{\text{eo}} = \text{Id}$ . In our earlier papers  $\mathcal{P}_+^{\text{eo}}$  are the Calderon projectors on a pseudoconvex manifold, and  $\mathcal{P}_-^{\text{eo}}$ , the Calderon projectors on a pseudoconcave manifold.

Given the formulæ above for  $\sigma(\bar{\partial})$  and  $\sigma(\bar{\partial}^2)$  the computation of the symbol of  $Q^{\text{eo}}$  proceeds exactly as in the Kähler case. As we only need the principal symbol, it suffices to do the computations in the fiber over a fixed point  $p \in bX$ . Set

$$\sigma(Q^{\text{eo}}) = q = q_{-1} + q_{-2} + \dots \quad (53)$$

We summarize the results of these calculations in the following proposition:

**Proposition 2.** *Let  $(X, J, g, \rho)$  define a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold. For  $p \in bX$ , let  $(x_1, \dots, x_{2n})$  denote boundary adapted, geodesic normal coordinates centered at  $p$ . The symbols of  $Q^{\text{eo}}$  at  $p$  are given by*

$$\begin{aligned} q_{-1}(\xi) &= \frac{2d_1(\xi)}{|\xi|^2} \\ q_{-2} &= \mathfrak{D}_{-2}(|z|) \end{aligned} \quad (54)$$

Here  $\xi$  are the coordinates on  $T_p^*X$  defined by  $\{dx_j\}$ ,  $|\xi|$  is the standard Euclidean norm, and  $d_1(\xi)$  is the symbol of  $\bar{\partial} + \bar{\partial}^*$  on  $\mathbb{C}^n$  with respect to the flat metric. For  $k \geq 2$  we have:

$$q_{-2k} = \sum_{j=0}^{l_k} \frac{\mathfrak{D}_{2j}(1)}{|\xi|^{2(k+j)}}, \quad q_{-(2k-1)} = \sum_{j=0}^{l'_k} \frac{\mathfrak{D}_{2j+1}(1)}{|\xi|^{2(k+j)}}. \quad (55)$$

The terms in the numerators of (55) are polynomials in  $\xi$  of the indicated degrees.

In order to compute the symbol of the Calderon projector, we introduce boundary adapted coordinates,  $(t, x_2, \dots, x_{2n})$  where

$$t = -\frac{\alpha}{2}\rho(z) = x_1 + O(|x|^2). \quad (56)$$

Note that  $t$  is positive on a pseudoconvex manifold and  $dt$  is an inward pointing, unit co-vector.

We need to use the change of coordinates formula to express the symbol in the new variables. From [18] we obtain the following prescription: Let  $w = \phi(x)$  be a diffeomorphism and  $c(x, \xi)$  the symbol of a classical pseudodifferential operator  $C$ . Let  $(w, \eta)$  be linear coordinates in the cotangent space, then  $c_\phi(w, \eta)$ , the symbol of  $C$  in the new coordinates, is given by

$$c_\phi(\phi(x), \eta) \sim \sum_{k=0}^{\infty} \sum_{\theta \in \mathcal{F}_k} \frac{(-i)^k \partial_\xi^\theta c(x, d\phi(x)^t \eta) \partial_{\tilde{x}}^\theta e^{i(\Phi_x(\tilde{x}), \eta)}}{\theta!} \Big|_{x=\tilde{x}}, \quad (57)$$



where

$$\Phi_x(\tilde{x}) = \phi(\tilde{x}) - \phi(x) - d\phi(x)(\tilde{x} - x). \quad (58)$$

Here  $\mathcal{J}_k$  are multi-indices of length  $k$ . Our symbols are matrix valued, e.g.  $q_{-2}$  is really  $(q_{-2})_{pq}$ . As the change of variables applies component by component, we suppress these indices in the computations that follow.

In the case at hand, we are interested in evaluating this expression at  $z = x = 0$ , where we have  $d\phi(0) = \text{Id}$  and

$$\Phi_0(\tilde{x}) = \left(-\frac{\alpha}{2}[\langle a\tilde{z}, \tilde{z} \rangle + \text{Re}(b\tilde{z}, \tilde{z}) + O(|\tilde{z}|^3)], 0, \dots, 0\right).$$

This is exactly as in the Kähler case, but for two small modifications: In the Kähler case  $\alpha = 1$  and  $a = \text{Id}$ . These differences slightly modify the symbolic result, but not the invertibility of the symbols of  $\mathcal{T}_+^{\text{eo}}$ . As before, only the  $k = 2$  term is of importance. It is given by

$$-\frac{i\xi_1}{2} \text{tr}[\partial_{\xi_j \xi_k}^2 q(0, \xi) \partial_{x_j x_k}^2 \phi(0)]. \quad (59)$$

To compute this term we need to compute the Hessians of  $q_{-1}$  and  $\phi(x)$  at  $x = 0$ . We define the  $2n \times 2n$  real matrices  $A, B$  so that

$$\langle az, z \rangle = \langle Ax, x \rangle \text{ and } \text{Re}(bz, z) = \langle Bx, x \rangle; \quad (60)$$

if  $a = a^0 + ia^1$  and  $b = b^0 + ib^1$ , then

$$A = \begin{pmatrix} a^0 & -a^1 \\ a^1 & a^0 \end{pmatrix} \quad B = \begin{pmatrix} b^0 & -b^1 \\ -b^1 & -b^0 \end{pmatrix}. \quad (61)$$

Here  $a^{0t} = a^0 a^{1t} = -a^1$ , and  $b^{0t} = b^0$ ,  $b^{1t} = b^1$ . With these definitions we see that

$$\partial_{x_j x_k}^2 \phi(0) = -\alpha(A + B). \quad (62)$$

As before we compute:

$$\frac{\partial^2 q_{-1}}{\partial \xi_k \partial \xi_j} = -4 \frac{d_1 \text{Id} + \xi \otimes \partial_\xi d_1^t + \partial_\xi d_1 \otimes \xi^t}{|\xi|^4} + 16d_1 \frac{\xi \otimes \xi^t}{|\xi|^6}. \quad (63)$$

Here  $\xi$  and  $\partial_\xi d_1$  are regarded as column vectors. The principal part of the  $k = 2$  term is

$$q_{-2}^c(\xi) = i\xi_1 \alpha \text{tr} \left[ (A + B) \left( -2 \frac{\text{Id} d_1 + \xi \otimes \partial_\xi d_1^t + \partial_\xi d_1 \otimes \xi^t}{|\xi|^4} + 8d_1 \frac{\xi \otimes \xi^t}{|\xi|^6} \right) \right] \quad (64)$$

Observe that  $q_{-2}^c$  depends linearly on  $A$  and  $B$ . It is shown in Proposition 6 of [10] that the contribution, along the contact direction, of a matrix with the symmetries of  $B$  vanishes. Because  $q_{-2}$  vanishes at 0 and because the order of a symbol is preserved under a change of variables we see that the symbol of  $Q^{\text{eo}}$  at  $p$  is

$$q(0, \xi) = \frac{2d_1(\xi)}{|\xi|^2} + q_{-2}^c(\xi) + \mathfrak{D}_{-3}(1). \quad (65)$$

As before the  $\mathfrak{D}_{-3}$ -term contributes nothing to the extended Heisenberg principal symbol of the Calderon projector. Only the term

$$q_{-2}^{cA}(\xi) = 2i\xi_1\alpha \left[ -\frac{\text{tr} Ad_1}{|\xi|^4} + 4\frac{d_1\langle A\xi, \xi \rangle}{|\xi|^6} - 2\frac{\langle A\xi, \partial_\xi d_1 \rangle}{|\xi|^4} \right]. \quad (66)$$

makes a contribution. To find the contribution of  $q_{-2}^{cA}$  to the symbol of the Calderon projector, we need to compute the contour integral

$$\mathfrak{p}_{-2\pm}^c(p, \xi') = \frac{1}{2\pi} \int_{\Gamma_\pm(\xi')} q_{-2}^{cA}(\xi) d\xi_1. \quad (67)$$

Let  $\xi = (\xi_1, \xi')$ . As this term is lower order, in the classical sense, we only need to compute it for  $\xi'$  along the contact line. We do this computation in the next section.

### 3 The symbol of the Calderon projector

We are now prepared to compute the symbol of the Calderon projector; it is expressed as 1-variable contour integral in the symbol of  $Q^{\text{eo}}$ . If  $q(t, x', \xi_1, \xi')$  is the symbol of  $Q^{\text{eo}}$  in the boundary adapted coordinates, then the symbol of the Calderon projector is

$$\mathfrak{p}_\pm(x', \xi') = \frac{1}{2\pi} \int_{\Gamma_\pm(\xi_1)} q(0, x', \xi_1, \xi') d\xi_1 \circ \sigma_1(\partial^{\text{eo}}, \mp idt). \quad (68)$$

Here we recall that  $q(0, x', \xi_1, \xi')$  is a meromorphic function of  $\xi_1$ . For each fixed  $\xi'$ , the poles of  $q$  lie on the imaginary axis. For  $t > 0$ , we take  $\Gamma_+(\xi_1)$  to be a contour enclosing the poles of  $q(0, x', \cdot, \xi')$  in the upper half plane, for  $t < 0$ ,  $\Gamma_-(\xi_1)$  is a contour enclosing the poles of  $q(0, x', \cdot, \xi')$  in the lower half plane. In a moment we use a residue computation to evaluate these integrals. For this purpose we note that the contour  $\Gamma_+(\xi_1)$  is positively oriented, while  $\Gamma_-(\xi_1)$  is negatively oriented.

The Calderon projector is a classical pseudodifferential operator of order 0 and therefore its symbol has an asymptotic expansion of the form

$$\mathfrak{p}_\pm = \mathfrak{p}_{0\pm} + \mathfrak{p}_{-1\pm} + \dots \quad (69)$$

The contact line,  $L_p$ , is defined in  $T_p^*Y$  by the equations

$$\xi_2 = \dots = \xi_n = \xi_{n+2} = \dots = \xi_{2n} = 0, \quad (70)$$

and  $\xi_{n+1}$  is a coordinate along the contact line. Because  $t = -\frac{\alpha}{2}\rho$ , the positive contact direction is given by  $\xi_{n+1} < 0$ . As before we have the principal symbols of  $\mathcal{P}_\pm^{\text{eo}}$  away from the contact line:

**Proposition 3.** *If  $\tilde{X}$  is an invertible double, containing  $X$  as an open set, and  $p \in bX$  with coordinates normalized at  $p$  as above, then*

$$\mathfrak{p}_{0\pm}^{\text{eo}}(0, \xi') = \frac{d_1^{\text{eo}}(\pm i|\xi'|, \xi')}{|\xi'|} \circ \sigma_1(\tilde{\mathfrak{d}}^{\text{eo}}, \mp i dt). \quad (71)$$

Along the contact directions we need to evaluate higher order terms; as shown in [10], the error terms in (65) contribute terms that lift to have Heisenberg order less than  $-2$ . To finish our discussion of the symbol of the Calderon projector we need to compute the symbol along the contact direction. This entails computing the contribution from  $q_{-2}^{cA}$ . As before, the terms arising from the holomorphic Hessian of  $\rho$  do not contribute anything to the symbol of the Calderon projector. However, the terms arising from  $\partial_{z_j \bar{z}_k}^2$  still need to be computed. To do these computations, we need to have an explicit formula for the principal symbol  $d_1(\xi)$  of  $\tilde{\mathfrak{d}}$  at  $p$ . For the purposes of these and our subsequent computations, it is useful to use the chiral operators  $\tilde{\mathfrak{d}}^{\text{eo}}$ . As we are working in a geodesic normal coordinate system, we only need to find the symbols of  $\tilde{\mathfrak{d}}^{\text{eo}}$  for  $\mathbb{C}^n$  with the flat metric. Let  $\sigma$  denote a section of  $\Lambda^{\text{eo}}$ . We split  $\sigma$  into its normal and tangential parts at  $p$ :

$$\sigma = \sigma^t + \frac{d\bar{z}_1}{\sqrt{2}} \wedge \sigma^n, \quad i_{\partial_{z_1}} \sigma^t = i_{\partial_{z_1}} \sigma^n = 0. \quad (72)$$

With this splitting we see that

$$\begin{aligned} \tilde{\mathfrak{d}}^{\text{e}} \sigma &= \sqrt{2} \begin{pmatrix} \partial_{\bar{z}_1} \otimes \text{Id}_n & \mathfrak{D}_t \\ -\mathfrak{D}_t & -\partial_{z_1} \otimes \text{Id}_n \end{pmatrix} \begin{pmatrix} \sigma^t \\ \sigma^n \end{pmatrix} \\ \tilde{\mathfrak{d}}^{\text{o}} \sigma &= \sqrt{2} \begin{pmatrix} -\partial_{z_1} \otimes \text{Id}_n & -\mathfrak{D}_t \\ \mathfrak{D}_t & \partial_{\bar{z}_1} \otimes \text{Id}_n \end{pmatrix} \begin{pmatrix} \sigma^n \\ \sigma^t \end{pmatrix}, \end{aligned} \quad (73)$$

where  $\text{Id}_n$  is the identity matrix acting on the normal, or tangential parts of  $\Lambda^{\text{eo}} \upharpoonright_{bX}$  and

$$\mathfrak{D}_t = \sum_{j=2}^n [\partial_{z_j} e_j - \partial_{\bar{z}_j} \epsilon_j] \text{ with } e_j = i\sqrt{2}\partial_{z_j} \text{ and } \epsilon_j = \frac{d\bar{z}_j}{\sqrt{2}} \wedge. \quad (74)$$

It is now a simple matter to compute  $d_1^{\text{eo}}(\xi)$  :

$$\begin{aligned} d_1^{\text{e}}(\xi) &= \frac{1}{\sqrt{2}} \begin{pmatrix} (i\xi_1 - \xi_{n+1}) \otimes \text{Id}_n & \mathfrak{d}(\xi'') \\ -\mathfrak{d}(\xi'') & -(i\xi_1 + \xi_{n+1}) \otimes \text{Id}_n \end{pmatrix} \\ d_1^{\text{o}}(\xi) &= \frac{1}{\sqrt{2}} \begin{pmatrix} -(i\xi_1 + \xi_{n+1}) \otimes \text{Id}_n & -\mathfrak{d}(\xi'') \\ \mathfrak{d}(\xi'') & (i\xi_1 - \xi_{n+1}) \otimes \text{Id}_n \end{pmatrix} \end{aligned} \quad (75)$$

where  $\xi'' = (\xi_2, \dots, \xi_n, \xi_{n+2}, \dots, \xi_{2n})$  and

$$\mathfrak{d}(\xi'') = \sum_{j=2}^n [(i\xi_j + \xi_{n+j})e_j - (i\xi_j - \xi_{n+j})\epsilon_j]. \quad (76)$$

As  $\epsilon_j^* = e_j$  we see that  $\mathfrak{d}(\xi'')$  is a self adjoint symbol.

The principal symbols of  $\mathcal{T}_+^{\text{eo}}$  have the same block structure as in the Kähler case. The symbol  $q_{-2}^c$  produces a term that lifts to have Heisenberg order  $-2$  and therefore, in the pseudoconvex case, we only need to compute the  $(2, 2)$  block arising from this term.

We start with the nontrivial term of order  $-1$ .

**Lemma 2.** *If  $X$  is either pseudoconvex or pseudoconcave we have that*

$$\frac{1}{2\pi} \int_{\Gamma_{\pm}(\xi')} \frac{2i\xi_1 \alpha \text{tr} A d_1(\xi_1, \xi') d\xi_1}{|\xi|^4} = -\frac{i\alpha \text{tr} A \partial_{\xi_1} d_1}{2|\xi'|} \quad (77)$$

*Remark 3.* As  $d_1$  is a linear polynomial,  $\partial_{\xi_1} d_1$  is a constant matrix.

*Proof.* See Lemma 1 in [10]. □

We complete the computation by evaluating the contribution from the other terms in  $q_{-2}^{cA}$  along the contact line.

**Proposition 4.** *For  $\xi'$  along the positive (negative) contact line we have*

$$\frac{1}{2\pi} \int_{\Gamma_{\pm}(\xi')} [q_{-2}^{cA}(p, \xi)] d\xi_1 = -\frac{\alpha(a_{11}^0 - \frac{1}{2} \text{tr} A)}{|\xi'|} \partial_{\xi_1} d_1. \quad (78)$$

*If  $\xi_{n+1} < 0$ , then we use  $\Gamma_+(\xi')$ , whereas if  $\xi_{n+1} > 0$ , then we use  $\Gamma_-(\xi')$ .*

*Proof.* To prove this result we need to evaluate the contour integral with

$$\xi' = \xi'_c = (0, \dots, 0, \xi_{n+1}, 0, \dots, 0),$$

recalling that the positive contact line corresponds to  $\xi_{n+1} < 0$ . Hence, along the positive contact line  $|\xi'| = -\xi_{n+1}$ . We first compute the integrand along  $\xi'_c$ .

**Lemma 3.** *For  $\xi'$  along the contact line we have*

$$\left[ \frac{2d_1^c(\xi) \langle A\xi, \xi \rangle - |\xi|^2 \langle A\xi, \partial_\xi d_1^c \rangle}{|\xi|^6} \right] = \frac{a_{11}^0}{|\xi|^4} d_1^c(\xi) \quad (79)$$

$$\left[ \frac{2d_1^o(\xi) \langle A\xi, \xi \rangle - |\xi|^2 \langle A\xi, \partial_\xi d_1^o \rangle}{|\xi|^6} \right] = \frac{a_{11}^0}{|\xi|^4} d_1^o(\xi) \quad (80)$$

*Proof.* As  $a_{11}^1 = 0$  we observe that along the contact line

$$\begin{aligned} \langle A\xi, \xi \rangle &= a_{11}^0 (\xi_1^2 + \xi_{n+1}^2) \\ \langle A\xi, \partial_\xi d_1^c \rangle &= a_{11}^0 \begin{pmatrix} (i\xi_1 - \xi_{n+1}) \otimes \text{Id} & 0 \\ 0 & -(i\xi_1 + \xi_{n+1}) \otimes \text{Id} \end{pmatrix} = a_{11}^0 d_1^c \\ \langle A\xi, \partial_\xi d_1^o \rangle &= a_{11}^0 \begin{pmatrix} -(i\xi_1 + \xi_{n+1}) \otimes \text{Id} & 0 \\ 0 & (i\xi_1 - \xi_{n+1}) \otimes \text{Id} \end{pmatrix} = a_{11}^0 d_1^o \end{aligned} \quad (81)$$

As  $\xi_1^2 + \xi_{n+1}^2 = |\xi|^2$  for  $\xi'$  along the contact line these formulæ easily imply (79) and (80).  $\square$

The proposition is an easy consequence of these formulæ.  $\square$

For subsequent calculations we set

$$\beta \stackrel{d}{=} \frac{1}{2} \text{tr} A - a_{11}^0 = \sum_{j=2}^n [\partial_{x_j}^2 \rho + \partial_{y_j}^2 \rho]_{x=p}. \quad (82)$$

As a corollary, we have a formula for the  $-1$  order term in the symbol of the Calderon projector

**Corollary 1.** *In the normalizations defined above, along the contact directions, we have*

$$\mathfrak{p}_{-1}^{\text{co}}(0, \xi') = -\frac{i\alpha\beta\partial_{\xi_1} d_1^{\text{co}}}{|\xi'|} \circ \sigma_1(\partial^{\text{co}}, \mp idt). \quad (83)$$

*Remark 4.* In the Kähler case  $\alpha = 1$  and  $\beta = n - 1$ . The values of these numbers turn out to be unimportant. It is only important that  $\alpha > 0$  and that they depend smoothly on local geometric data, which they obviously do.

We have shown that the order  $-1$  term in the symbol of the Calderon projector, along the appropriate half of the contact line, is given by the right hand side of equation (83). It is determined by the principal symbol of  $Q^{\text{eo}}$  and does not depend on the higher order geometry of  $bX$ . As all other terms in the symbol of  $Q^{\text{eo}}$  contribute terms that lift to have Heisenberg order less than  $-2$ , these computations allow us to find the principal symbols of  $\mathcal{T}_+^{\text{eo}}$  and extend the main results of [10] to the pseudoconvex almost complex category. As noted above, the off diagonal blocks have Heisenberg order  $-1$ , so the classical terms of order less than zero cannot contribute to their principal parts.

We now give formulæ for the chiral forms of the subelliptic boundary conditions defined in [9] as well as the isomorphisms  $\sigma_1(\bar{\partial}^{\text{eo}}, \mp idt)$ . Let  $\mathcal{S}$  be a generalized Szegő projector.

**Lemma 4.** *According to the splittings of sections of  $\Lambda^{\text{eo}}$  given in (72), the subelliptic boundary conditions, defined by the generalized Szegő projector  $\mathcal{S}$ , on even (odd) forms are given by  $\mathcal{R}_+^{\text{eo}}\sigma \upharpoonright_{bX} = 0$  where*

$$\mathcal{R}_+^{\text{e}}\sigma \upharpoonright_{bX} = \begin{pmatrix} \mathcal{S} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \text{Id} \end{pmatrix} \begin{bmatrix} \sigma^t \\ \sigma^n \end{bmatrix}_{bX}, \quad \mathcal{R}_+^{\text{o}}\sigma \upharpoonright_{bX} = \begin{pmatrix} 1 - \mathcal{S} & 0 & \mathbf{0} \\ 0 & \text{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{bmatrix} \sigma^n \\ \sigma^t \end{bmatrix}_{bX} \quad (84)$$

**Lemma 5.** *The isomorphisms at the boundary between  $\Lambda^{\text{eo}}$  and  $\Lambda^{\text{oe}}$  are given by*

$$\sigma_1(\bar{\partial}_\pm^{\text{eo}}, \mp idt)\sigma^t = \frac{\pm}{\sqrt{2}}\sigma^t, \quad \sigma_1(\bar{\partial}_\pm^{\text{eo}}, \mp idt)\sigma^n = \frac{\mp}{\sqrt{2}}\sigma^n. \quad (85)$$

Thus far, we have succeeded in computing the symbols of the Calderon projectors to high enough order to compute the principal symbols of  $\mathcal{T}_+^{\text{eo}}$  as elements of the extended Heisenberg calculus. The computations have been carried out in a coordinate system adapted to the boundary. This suffices to examine the classical parts of the symbols. Recall that the positive contact direction  $L^+$ , is given at  $p$  by  $\xi'' = 0, \xi_{n+1} < 0$ . As before we obtain:

**Proposition 5.** *If  $(X, J, g, \rho)$  is a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold, then, on the complement of the positive contact direction, the classical symbols  ${}^R\sigma_0(\mathcal{T}_+^{\text{eo}})$  are given by*

$$\begin{aligned} {}^R\sigma_0(\mathcal{T}_+^{\text{e}})(0, \xi') &= \frac{1}{2|\xi'|} \begin{pmatrix} (|\xi'| + \xi_{n+1}) \text{Id} & -\mathfrak{d}(\xi'') \\ \mathfrak{d}(\xi'') & (|\xi'| + \xi_{n+1}) \text{Id} \end{pmatrix} \\ {}^R\sigma_0(\mathcal{T}_+^{\text{o}})(0, \xi') &= \frac{1}{2|\xi'|} \begin{pmatrix} (|\xi'| + \xi_{n+1}) \text{Id} & \mathfrak{d}(\xi'') \\ -\mathfrak{d}(\xi'') & (|\xi'| + \xi_{n+1}) \text{Id} \end{pmatrix} \end{aligned} \quad (86)$$

*These symbols are invertible on the complement of  $L^+$ .*

*Proof.* See Proposition 8 in [10]. □

## 4 The Heisenberg symbols of $\mathcal{T}_+^{\text{eo}}$

To compute the Heisenberg symbols of  $\mathcal{T}_+^{\text{eo}}$  we change coordinates, one last time, to get Darboux coordinates at  $p$ . Up to this point we have used the coordinates  $(\xi_2, \dots, \xi_{2n})$  for  $T_p^*bX$ , which are defined by the coframe  $dx_2, \dots, dx_{2n}$ , with  $dx_{n+1}$  the contact direction. Recall that the contact form  $\theta$ , defined by the complex structure and defining function  $\rho$ , is given by  $\theta = \frac{i}{2}\bar{\partial}\rho$ . The symplectic form on  $H$  is defined by  $d\theta$ . At  $p$  we have

$$\theta_p = -\frac{1}{2\alpha}dx_{n+1}, \quad d\theta_p = \sum_{j=2}^n dx_j \wedge dx_{j+n}. \quad (87)$$

By comparison with equation (5) in [10], we see that properly normalized coordinates for  $T_p^*bX$  (i.e., Darboux coordinates) are obtained by setting

$$\eta_0 = -2\alpha\xi_{n+1}, \quad \eta_j = \xi_{j+1}, \quad \eta_{j+n-1} = \xi_{j+n+1} \text{ for } j = 1, \dots, n-1. \quad (88)$$

As usual we let  $\eta' = (\eta_1, \dots, \eta_{2(n-1)})$ ; whence  $\xi'' = \eta'$ .

As a first step in lifting the symbols of the Calderon projectors to the extended Heisenberg compactification, we re-express them, through order  $-1$  in the  $\xi$ -coordinates:

$$\mathfrak{p}_+^e(\xi') = \frac{1}{2|\xi'|} \left[ \begin{pmatrix} (|\xi'| - \xi_{n+1}) \text{Id} & \mathfrak{d}(\xi'') \\ \mathfrak{d}(\xi'') & (|\xi'| + \xi_{n+1}) \text{Id} \end{pmatrix} - \alpha\beta \begin{pmatrix} \text{Id} & \mathbf{0} \\ \mathbf{0} & \text{Id} \end{pmatrix} \right] \quad (89)$$

$$\mathfrak{p}_+^o(\xi') = \frac{1}{2|\xi'|} \left[ \begin{pmatrix} (|\xi'| + \xi_{n+1}) \text{Id} & \mathfrak{d}(\xi'') \\ \mathfrak{d}(\xi'') & (|\xi'| - \xi_{n+1}) \text{Id} \end{pmatrix} - \alpha\beta \begin{pmatrix} \text{Id} & \mathbf{0} \\ \mathbf{0} & \text{Id} \end{pmatrix} \right] \quad (90)$$

Various identity and zero matrices appear in these symbolic computations. Precisely which matrix is needed depends on the dimension, the parity, etc. We do not encumber our notation with these distinctions.

In order to compute  ${}^H\sigma(\mathcal{T}_+^{\text{eo}})$ , we represent the Heisenberg symbols as model operators and use operator composition. To that end we need to quantize  $\mathfrak{d}(\eta')$  as well as the terms coming from the diagonals in (89)–(90). For the pseudoconvex side, we need to consider the symbols on positive Heisenberg face, where the function  $|\xi'| + \xi_{n+1}$  vanishes.

We express the various terms in the symbol  $\mathfrak{p}_+^{\text{co}}$ , near the positive contact line as sums of Heisenberg homogeneous terms

$$\begin{aligned} |\xi'| &= \frac{\eta_0}{2\alpha}(1 + \mathfrak{D}_{-2}^H) \\ |\xi'| - \xi_{n+1} &= \frac{\eta_0}{\alpha}(1 + \mathfrak{D}_{-2}^H), \quad |\xi'| + \xi_{n+1} = \frac{\alpha|\eta'|^2}{\eta_0}(1 + \mathfrak{D}_{-2}^H) \\ \mathfrak{d}(\xi'') &= \sum_{j=1}^{n-1} [(i\eta_j + \eta_{n+j-1})e_j - (i\eta_j - \eta_{n+j-1})\epsilon_j]. \end{aligned} \quad (91)$$

Recall that the notation  $\mathfrak{D}_j^H$  denotes a term of Heisenberg order at most  $j$ . To find the model operators, we use the quantization rule, equation (20) in [10] (with the + sign), obtaining

$$\begin{aligned} \eta_j - i\eta_{n+j-1} &\leftrightarrow C_j \stackrel{d}{=} (w_j - \partial_{w_j}) \\ \eta_j + i\eta_{n+j-1} &\leftrightarrow C_j^* \stackrel{d}{=} (w_j + \partial_{w_j}) \\ |\eta'|^2 &\leftrightarrow \mathfrak{H} \stackrel{d}{=} \sum_{j=1}^{n-1} w_j^2 - \partial_{w_j}^2. \end{aligned} \quad (92)$$

The following standard identities are useful

$$\sum_{j=1}^{n-1} C_j^* C_j - (n-1) = \mathfrak{H} = \sum_{j=1}^{n-1} C_j C_j^* + (n-1) \quad (93)$$

We let  $\mathfrak{D}_+$  denote the model operator defined, using the + quantization, by  $\mathfrak{d}(\xi'')$ , it is given by

$$\mathfrak{D}_+ = i \sum_{j=1}^{n-1} [C_j e_j - C_j^* \epsilon_j]. \quad (94)$$

This operator can be split into even and odd parts,  $\mathfrak{D}_+^{\text{eo}}$  and these chiral forms of the operator are what appear in the model operators below.

With these preliminaries, we can compute the model operators for  $\mathcal{P}_+^e$  and  $\text{Id} - \mathcal{P}_+^e$  in the positive contact direction. They are:

$${}^{eH}\sigma(\mathcal{P}_+^e)(+) = \begin{pmatrix} \text{Id} & \frac{\alpha\mathfrak{D}_+^{\text{eo}}}{\eta_0} \\ \frac{\alpha\mathfrak{D}_+^e}{\eta_0} & \frac{\alpha^2\mathfrak{H} - \alpha^2\beta\eta_0}{\eta_0^2} \end{pmatrix} \quad {}^{eH}\sigma(\text{Id} - \mathcal{P}_+^e)(+) = \begin{pmatrix} \frac{\alpha^2\mathfrak{H} + \alpha^2\beta\eta_0}{\eta_0^2} & -\frac{\alpha\mathfrak{D}_+^{\text{eo}}}{\eta_0} \\ -\frac{\alpha\mathfrak{D}_+^e}{\eta_0} & \text{Id} \end{pmatrix}. \quad (95)$$



The denominators involving  $\eta_0$  are meant to remind the reader of the Heisenberg orders of the various blocks:  $\eta_0^{-1}$  indicates a term of Heisenberg order  $-1$  and  $\eta_0^{-2}$  a term of order  $-2$ . Similar computations give the model operators in the odd case:

$${}^{eH}\sigma(\mathcal{P}_+^o)(+) = \begin{pmatrix} \frac{\alpha^2\mathfrak{H}-\alpha^2\beta\eta_0}{\eta_0^2} & \frac{\alpha\mathcal{D}_+^o}{\eta_0} \\ \frac{\alpha\mathcal{D}_+^e}{\eta_0} & \text{Id} \end{pmatrix} \quad {}^{eH}\sigma(\text{Id}-\mathcal{P}_+^o)(+) = \begin{pmatrix} \text{Id} & -\frac{\alpha\mathcal{D}_+^o}{\eta_0} \\ -\frac{\alpha\mathcal{D}_+^e}{\eta_0} & \frac{\alpha^2\mathfrak{H}+\alpha^2\beta\eta_0}{\eta_0^2} \end{pmatrix}. \quad (96)$$

Let  $\pi'_0 = {}^{eH}\sigma(+)(\mathcal{S})$ ; it is a self adjoint rank one projection defined by a compatible almost complex structure on  $H$ . We use the  $'$  to distinguish this rank one projection, from the rank one projection  $\pi_0$  defined by the CR-structure on the fiber of cotangent bundle at  $p$ . The model operators for  $\mathcal{R}_+^{\text{eo}}$  in the positive contact direction are:

$${}^{eH}\sigma(\mathcal{R}_+^e)(+) = \begin{pmatrix} \pi'_0 & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Id} & \mathbf{0} \end{pmatrix}, \quad {}^{eH}\sigma(\mathcal{R}_+^o)(+) = \begin{pmatrix} 1-\pi'_0 & 0 & \mathbf{0} \\ 0 & \text{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \quad (97)$$

We can now compute the model operators for  $\mathcal{T}_+^{\text{eo}}$  on the upper Heisenberg face.

**Proposition 6.** *If  $(X, J, g, \rho)$  is normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold, then, at  $p \in bX$ , the model operators for  $\mathcal{T}_+^{\text{eo}}$ , in the positive contact direction, are given by*

$${}^{eH}\sigma(\mathcal{T}_+^e)(+) = \begin{pmatrix} \pi'_0 & 0 & -\left[ \begin{array}{cc} 1-2\pi'_0 & 0 \\ 0 & \text{Id} \end{array} \right] \frac{\alpha\mathcal{D}_+^o}{\eta_0} \\ 0 & \mathbf{0} & \frac{\alpha^2\mathfrak{H}-\alpha^2\beta\eta_0}{\eta_0^2} \\ \frac{\alpha\mathcal{D}_+^e}{\eta_0} & & \end{pmatrix} \quad (98)$$

$${}^{eH}\sigma(\mathcal{T}_+^o)(+) = \begin{pmatrix} \pi'_0 & 0 & \left[ \begin{array}{cc} 1-2\pi'_0 & 0 \\ 0 & \text{Id} \end{array} \right] \frac{\alpha\mathcal{D}_+^o}{\eta_0} \\ 0 & \mathbf{0} & \frac{\alpha^2\mathfrak{H}+\alpha^2\beta\eta_0}{\eta_0^2} \\ -\frac{\alpha\mathcal{D}_+^e}{\eta_0} & & \end{pmatrix}. \quad (99)$$

*Proof.* Observe that the Heisenberg orders of the blocks in (98) and (99) are

$$\begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}. \quad (100)$$

Proposition 6 in [10] shows that all other terms in the symbol of the Calderon projector lead to diagonal terms of Heisenberg order at most  $-4$ , and off diagonal terms of order at most  $-2$ . This, along with the computations above, completes the proof of the proposition.  $\square$

This brings us to the generalization, in the non-Kähler case, of Theorem 1 in [10]:

**Theorem 1.** *Let  $(X, J, g, \rho)$  be a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold, and  $\mathcal{S}$  a generalized Szegő projector, defined by a compatible deformation of the almost complex structure on  $H$  induced by the embedding of  $bX$  as the boundary of  $X$ . The comparison operators,  $\mathcal{T}_+^{\text{eo}}$ , are elliptic elements of the extended Heisenberg calculus, with parametrices having Heisenberg orders*

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \quad (101)$$

*Proof.* The proof is identical to the proof of Theorem 1 in [10]: we need to show that the principal symbols of  $\mathcal{T}_+^{\text{eo}}$  are invertible, which is done in the next section.  $\square$

## 5 Invertibility of the model operators

In this section we produce inverses for the model operators  $\sigma^H(\mathcal{T}_+^{\text{eo}})(+)$ . We begin by writing down inverses for the model operators using the projector compatible with the CR-structure induced at  $p$  by  $J$ . We denote this projector by  $\pi_0$  to distinguish it from  $\pi'_0 = \sigma_0^H(p)(\mathcal{S})$ . In this section, we let  ${}^{eH}\sigma(\mathcal{T}_+^{\text{eo}})(+)$ , denote the model operators with this projector to distinguish it from  ${}^{eH}\sigma(\mathcal{T}'_+{}^{\text{eo}})(+)$ , the model operators with  $\pi'_0$ . As before, the inverse in the general case is a finite rank perturbation of this case. For the computations in this section we recall that  $\alpha$  is a positive number.

The operators  $\{C_j\}$  are called the creation operators and the operators  $\{C_j^*\}$  the annihilation operators. They satisfy the commutation relations

$$[C_j, C_k] = [C_j^*, C_k^*] = 0, \quad [C_j, C_k^*] = -2\delta_{jk} \quad (102)$$

The operators  $\mathcal{D}_{\pm}$  act on sums of the form

$$\omega = \sum_{k=0}^{n-1} \sum_{I \in \mathcal{J}'_k} f_I \bar{\omega}^I, \quad (103)$$

here  $\mathcal{J}'_k$  are increasing multi-indices of length  $k$ . We refer to the terms with  $|I| = k$  as the terms of degree  $k$ . For an increasing  $k$ -multi-index  $I = 1 \leq i_1 < i_2 < \dots < i_k \leq n-1$ ,  $\bar{\omega}^I$  is defined by

$$\bar{\omega}^I = \frac{1}{2^{\frac{k}{2}}} d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_k}. \quad (104)$$

The projector  $\pi_0$  and the operator  $\mathcal{D}_+$  satisfy the following relation:

**Lemma 6.** *Let  $\pi_0$  be the symbol of the generalized Szegő projector compatible with the CR-structure defined on the fiber of  $T_p bX$  by the almost complex structure, then*

$$\begin{bmatrix} \pi_0 & 0 \\ 0 & \mathbf{0} \end{bmatrix} \mathfrak{D}_+^o = 0 \quad (105)$$

*Proof.* See Lemma 7 in [10] □

This lemma simplifies the analysis of the model operators for  $\mathcal{T}_+^{\text{eo}}$ . The following lemma is useful in finding their inverses.

**Lemma 7.** *Let  $\Pi_q$  denote projection onto the terms of degree  $q$ ,*

$$\Pi_q \omega = \sum_{I \in \mathcal{J}'_q} f_I \bar{\omega}^I. \quad (106)$$

*The operators  $\mathfrak{D}_+$  satisfies the identity*

$$\mathfrak{D}_+^2 = \sum_{j=1}^{n-1} C_j C_j^* \otimes \text{Id} + \sum_{q=0}^{n-1} 2q \Pi_q \quad (107)$$

*Proof.* See Lemma 9 in [10]. □

As before  ${}^e H \sigma(\mathcal{T}_+^{\text{eo}})(+)$  are Fredholm elements (in the graded sense), in the isotropic algebra. Notice that this is a purely symbolic statement in the isotropic algebra. The blocks have isotropic orders

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}. \quad (108)$$

The leading order part in the isotropic algebra is independent of the choice of generalized Szegő projector. In the former case we can think of the operator as defining a map from  $H^1(\mathbb{R}^{n-1}; E_1) \oplus H^2(\mathbb{R}^{n-1}; E_2)$  to  $H^1(\mathbb{R}^{n-1}; F_1) \oplus H^0(\mathbb{R}^{n-1}; F_2)$  for appropriate vector bundles  $E_1, E_2, F_1, F_2$ . It is as maps between these spaces that the model operators are Fredholm.

**Proposition 7.** *The model operators,  ${}^e H \sigma(\mathcal{T}_+^{\text{eo}})(+)$ , are graded Fredholm elements in the isotropic algebra.*

*Proof.* See Proposition 7 in [10]. □

The operators  $\mathcal{D}_+^e$  and  $\mathcal{D}_+^o$  are adjoint to one another. From (107) and the well known properties of the harmonic oscillator, it is clear that  $\mathcal{D}_+^e \mathcal{D}_+^o$  is invertible. As  $\mathcal{D}_+^e$  has a one dimensional null space this easily implies that  $\mathcal{D}_+^o$  is injective with image orthogonal to the range of  $\pi_0$ , while  $\mathcal{D}_+^e$  is surjective. With these observations we easily invert the model operators.

Let  $[\mathcal{D}_+^e]^{-1}u$  denote the unique solution to the equation

$$\mathcal{D}_+^e v = u,$$

orthogonal to the null space of  $\mathcal{D}_+^e$ . We let

$$\overset{\square}{u} = \begin{pmatrix} 1 - \pi_0 & 0 \\ 0 & \text{Id} \end{pmatrix} u; \quad (109)$$

this is the projection onto the range of  $\mathcal{D}_+^o$  and

$$u_0 = \begin{pmatrix} \pi_0 & 0 \\ 0 & \mathbf{0} \end{pmatrix} u, \quad (110)$$

denotes the projection onto the nullspace of  $\mathcal{D}_+^e$ . We let  $[\mathcal{D}_+^o]^{-1}$  denote the unique solution to

$$\mathcal{D}_+^o v = \overset{\square}{u}.$$

Proposition 7 shows that these partial inverses are isotropic operators of order  $-1$ .

With this notation we find the inverse of  ${}^{eH}\sigma(\mathcal{T}_+^e)(+)$ . The vector  $[u, v]$  satisfies

$${}^{eH}\sigma(\mathcal{T}_+^e)(+) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad (111)$$

if and only if

$$\begin{aligned} u &= a_0 + [\alpha \mathcal{D}_+^e]^{-1}(\alpha^2 \mathfrak{H} - \alpha^2 \beta)[\alpha \mathcal{D}_+^o]^{-1} \overset{\square}{a} + [\alpha \mathcal{D}_+^e]^{-1} b \\ v &= -[\alpha \mathcal{D}_+^o]^{-1} \overset{\square}{a}. \end{aligned} \quad (112)$$

Writing out the inverse as a block matrix of operators, with appropriate factors of  $\eta_0$  included, gives:

$$[{}^{eH}\sigma(\mathcal{T}_+^e)(+)]^{-1} = \begin{bmatrix} \begin{pmatrix} \pi_0 & 0 \\ 0 & \mathbf{0} \end{pmatrix} + [\mathcal{D}_+^e]^{-1}(\mathfrak{H} - \beta)[\mathcal{D}_+^o]^{-1} \begin{pmatrix} 1 - \pi_0 & 0 \\ 0 & \text{Id} \end{pmatrix} & \eta_0 [\alpha \mathcal{D}_+^e]^{-1} \\ -\eta_0 [\alpha \mathcal{D}_+^o]^{-1} \begin{pmatrix} 1 - \pi_0 & 0 \\ 0 & \text{Id} \end{pmatrix} & \mathbf{0} \end{bmatrix} \quad (113)$$

The isotropic operators  $[\alpha \mathcal{D}_+^{\text{eo}}]^{-1}$  are of order  $-1$ , whereas the operator,

$$[\mathcal{D}_+^{\text{e}}]^{-1}(\mathfrak{H} - \beta)[\mathcal{D}_+^{\text{o}}]^{-1},$$

is of order zero. The Schwartz kernel of  $\pi_0$  is rapidly decreasing. From this we conclude that the Heisenberg orders, as a block matrix, of the parametrix for  $[{}^e H \sigma(\mathcal{T}_+^{\text{e}})(+)]$  are

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \quad (114)$$

We get a 1 in the lower right corner because the principal symbol of this entry, a priori of order 2, vanishes. As a result, the inverses of the model operators have Heisenberg order at most 1, which in turn allows us to use this representation of the parametrix to deduce the standard subelliptic  $\frac{1}{2}$ -estimates for these boundary value problems.

The solution for the odd case is given by

$$\begin{aligned} u &= a_0 + [\mathcal{D}_+^{\text{e}}]^{-1}(\mathfrak{H} + \beta)[\mathcal{D}_+^{\text{o}}]^{-1} a - [\alpha \mathcal{D}_+^{\text{e}}]^{-1} b \\ v &= [\alpha \mathcal{D}_+^{\text{o}}]^{-1} a. \end{aligned} \quad (115)$$

Once again the  $(2, 2)$  block of  $[{}^e H \sigma(\mathcal{T}_+^{\text{o}})(+)]^{-1}$  vanishes, and the principal symbol has the Heisenberg orders indicated in (114).

For the case that  $\pi'_0 = \pi_0$ , Lemma 6 implies that the model operators satisfy

$$[{}^e H \sigma(\mathcal{T}_+^{\text{eo}})(+)]^* = {}^e H \sigma(\mathcal{T}_+^{\text{oe}})(+). \quad (116)$$

From Proposition 7, we know that these are Fredholm operators. Since we have shown that all the operators  ${}^e H \sigma(\mathcal{T}_+^{\text{eo}})(+)$  are surjective, i.e., have a left inverse, it follows that all are in fact injective and therefore invertible. In all cases this completes the proof of Theorem 1 in the special case that the principal symbol of  $\mathcal{S}$  equals  $\pi_0$ .

We now show that the parametrices for  ${}^e H \sigma(\mathcal{T}_+^{\text{eo}})(+)$  differ from those with classical Szegő projectors by operators of finite rank. The Schwartz kernels of the correction terms are in the Hermite ideal, and so do not affect the Heisenberg orders of the blocks in the parametrix. As before the principal symbol in the  $(2, 2)$  block vanishes.

With these preliminaries and the results from the beginning of Section 7 in [10], we can now complete the proof of Theorem 1. As noted above,  ${}^e H \sigma(\mathcal{T}_+^{\text{eo}})(+)$  denotes the model operators with the projector  $\pi_0$ , and  ${}^e H \sigma(\mathcal{T}_+^{\text{oe}})(+)$  the model operators with projector  $\pi'_0$ .

**Proposition 8.** *If  $\pi'_0$  is the principal symbol of a generalized Szegő projection, which is a deformation of  $\pi_0$ , then  ${}^{eH}\sigma(\mathcal{T}'_{+}{}^{eo})(+)$  are invertible elements of the isotropic algebra. The inverses satisfy*

$$[{}^{eH}\sigma(\mathcal{T}'_{+}{}^{eo})(+)]^{-1} = [{}^{eH}\sigma(\mathcal{T}_{+}{}^{eo})(+)]^{-1} + \begin{pmatrix} c_1 & c_2 \\ c_3 & 0 \end{pmatrix}. \quad (117)$$

Here  $c_1, c_2, c_3$  are finite rank operators in the Hermite ideal.

*Proof.* In the formulæ below we let  $z_0$  denote the unit vector spanning the range of  $\pi_0$  and  $z'_0$ , the unit vector spanning the range of  $\pi'_0$ .

Proposition 7 implies that  ${}^{eH}\sigma(\mathcal{T}'_{+}{}^{eo})(+)$  are Fredholm operators. Since, as isotropic operators, the differences

$${}^{eH}\sigma(\mathcal{T}'_{+}{}^{eo})(+) - {}^{eH}\sigma(\mathcal{T}_{+}{}^{eo})(+)$$

are finite rank operators, it follows that  ${}^{eH}\sigma(\mathcal{T}'_{+}{}^{eo})(+)$  have index zero. It therefore suffices to construct a left inverse.

We begin with the  $+$  even case by rewriting the equation

$${}^{eH}\sigma(\mathcal{T}'_{+}{}^e)(+) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad (118)$$

as

$$\begin{aligned} \begin{bmatrix} \pi'_0 & 0 \\ 0 & \mathbf{0} \end{bmatrix} [u + \alpha \mathcal{D}'_{+} v] &= \begin{bmatrix} \pi'_0 & 0 \\ 0 & \mathbf{0} \end{bmatrix} a \\ \begin{bmatrix} 1 - \pi'_0 & 0 \\ 0 & \text{Id} \end{bmatrix} \alpha \mathcal{D}'_{+} v &= - \begin{bmatrix} 1 - \pi'_0 & 0 \\ 0 & \text{Id} \end{bmatrix} a \\ \alpha \mathcal{D}'_{+} u + (\alpha^2 \mathfrak{H} - \alpha^2 \beta) v &= b. \end{aligned} \quad (119)$$

We solve the middle equation in (119) first. Let

$$A_1 = \left( \frac{z'_0 \otimes z'_0}{\langle z'_0, z_0 \rangle} - \pi_0 \right) \Pi_0 a, \quad (120)$$

and note that  $\pi_0 A_1 = 0$ . Corollary 2 in [10] shows that the model operator in (120) provides a globally defined symbol. The section  $v$  is determined as the unique solution to

$$\alpha \mathcal{D}'_{+} v = -(\overset{\square}{a} - A_1). \quad (121)$$

By construction  $(1 - \pi'_0)(a_0 + A_1) = 0$  and therefore the second equation is solved.

The section  $\overset{\square}{u}$  is now uniquely determined by the last equation in (119):

$$\overset{\square}{u} = [\alpha \mathcal{D}'_{+}]^{-1} (b + (\alpha^2 \mathfrak{H} - \alpha^2 \beta) [\alpha \mathcal{D}'_{+}]^{-1} (\overset{\square}{a} - A_1)). \quad (122)$$

This leaves only the first equation, which we rewrite as

$$\begin{bmatrix} \pi'_0 & 0 \\ 0 & \mathbf{0} \end{bmatrix} u_0 = \begin{bmatrix} \pi'_0 & 0 \\ 0 & \mathbf{0} \end{bmatrix} (a - \alpha \mathcal{D}_+^0 v - \overset{\square}{u}). \quad (123)$$

It is immediate that

$$u_0 = \frac{z_0 \otimes z'_0}{\langle z_0, z'_0 \rangle} \Pi_0 (a - \alpha \mathcal{D}_+^0 v - \overset{\square}{u}). \quad (124)$$

By comparing these equations to those in (112) we see that  $[{}^e H \sigma(\mathcal{T}'_+{}^e)(+)]^{-1}$  has the required form. The finite rank operators are finite sums of terms involving  $\pi_0$ ,  $z_0 \otimes z'_0$  and  $z'_0 \otimes z_0$ , and are therefore in the Hermite ideal.

The solution in the + odd case is given by

$$\begin{aligned} v &= [\alpha \mathcal{D}_+^0]^{-1} (\overset{\square}{a} - A_1) \\ \overset{\square}{u} &= [\alpha \mathcal{D}_+^e]^{-1} [(\alpha^2 \mathfrak{H} + \alpha^2 \beta) v - b] \\ u_0 &= \frac{z_0 \otimes z'_0}{\langle z_0, z'_0 \rangle} \Pi_0 (a + \alpha \mathcal{D}_+^0 v - \overset{\square}{u}) \end{aligned} \quad (125)$$

As before  $A_1$  is given by (120). Again the inverse of  ${}^e H \sigma(\mathcal{T}'_+{}^o)(+)$  has the desired form.  $\square$

As noted above, the operators  ${}^e H \sigma(\mathcal{T}'_+{}^{eo})(+)$  are Fredholm operators of index zero. Hence, solvability of the equations

$${}^e H \sigma(\mathcal{T}'_+{}^{eo})(+) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad (126)$$

for all  $[a, b]$  implies the uniqueness and therefore the invertibility of the model operators. This completes the proof of Theorem 1. We now turn to applications of these results.

*Remark 5.* For the remainder of the paper  $\mathcal{T}'_+{}^{eo}$  is used to denote the comparison operator defined by  $\mathcal{R}'_+{}^{eo}$ , where the rank one projections are given by the principal symbol of  $\mathcal{S}$ .

## 6 Consequences of Ellipticity

As in the Kähler case, the ellipticity of the operators  $\mathcal{T}'_+{}^{eo}$  implies that the graph closures of  $(\bar{\partial}'_+{}^{eo}, \mathcal{R}'_+{}^{eo})$  are Fredholm and moreover,

$$(\bar{\partial}'_+{}^{eo}, \mathcal{R}'_+{}^{eo})^* = \overline{(\bar{\partial}'_+{}^{oe}, \mathcal{R}'_+{}^{oe})} \quad (127)$$

Given the ellipticity of  $\mathcal{T}_+^{\text{eo}}$ , the proofs of these statements are identical to the proofs in the Kähler case. For later usage, we introduce some notation and state these results.

Let  $\mathcal{U}_+^{\text{eo}}$  denote a 2-sided parametrix defined so that

$$\begin{aligned}\mathcal{T}_+^{\text{eo}}\mathcal{U}_+^{\text{eo}} &= \text{Id} - K_1^{\text{eo}} \\ \mathcal{U}_+^{\text{eo}}\mathcal{T}_+^{\text{eo}} &= \text{Id} - K_2^{\text{eo}},\end{aligned}\tag{128}$$

with  $K_1^{\text{eo}}, K_2^{\text{eo}}$  finite rank smoothing operators. The principal symbol computations show that  $\mathcal{U}_+^{\text{eo}}$  has classical order 0 and Heisenberg order at most 1. Such an operator defines a bounded map from  $H^{\frac{1}{2}}(bX)$  to  $L^2(bX)$ .

**Proposition 9.** *The operators  $\mathcal{U}_+^{\text{eo}}$  define bounded maps*

$$\mathcal{U}_+^{\text{eo}} : H^s(bX; F) \rightarrow H^{s-\frac{1}{2}}(bX; F)$$

for  $s \in \mathbb{R}$ . Here  $F$  is an appropriate vector bundle over  $bX$ .

The mapping properties of the boundary parametrices allow us to show that the graph closures of the operators  $(\bar{\partial}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}})$  are Fredholm.

**Theorem 2.** *Let  $(X, J, g, \rho)$  define a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold. The graph closures of  $(\bar{\partial}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}})$ , are Fredholm operators.*

*Proof.* The proof is exactly the same as the proof of Theorem 2 in [10].  $\square$

We also obtain the standard subelliptic Sobolev space estimates for the operators  $(\bar{\partial}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}})$ .

**Theorem 3.** *Let  $(X, J, g, \rho)$  define a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold. For each  $s \geq 0$ , there is a positive constant  $C_s$  such that if  $u$  is an  $L^2$ -solution to*

$$\bar{\partial}_+^{\text{eo}}u = f \in H^s(X) \text{ and } \mathcal{R}_+^{\text{eo}}[u]_{bX} = 0$$

in the sense of distributions, then

$$\|u\|_{H^{s+\frac{1}{2}}} \leq C_s[\|\bar{\partial}_+^{\text{eo}}u\|_{H^s} + \|u\|_{L^2}].\tag{129}$$

*Proof.* Exactly as in the Kähler case.  $\square$

**Remark 6.** In the case  $s = 0$ , there is a slightly better result: the Poisson kernel maps  $L^2(bX)$  into  $H_{(1, -\frac{1}{2})}(X)$  and therefore the argument shows that there is a constant  $C_0$  such that if  $u \in L^2$ ,  $\bar{\partial}_+^{\text{eo}}u \in L^2$  and  $\mathcal{R}_+^{\text{eo}}[u]_{bX} = 0$ , then

$$\|u\|_{(1, -\frac{1}{2})} \leq C_0[\|f\|_{L^2} + \|u\|_{L^2}]\tag{130}$$

This is just the standard  $\frac{1}{2}$ -estimate for the operators  $(\bar{\partial}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}})$



It is also possible to prove localized versions of these results. The higher norm estimates have the same consequences as for the  $\bar{\partial}$ -Neumann problem. Indeed, under certain hypotheses these estimates imply higher norm estimates for the second order operators considered in [9]. We identify the adjoints:

**Theorem 4.** *Let  $(X, J, g, \rho)$  define a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold, then we have the following relations:*

$$(\bar{\partial}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}})^* = \overline{(\bar{\partial}_+^{\text{oe}}, \mathcal{R}_+^{\text{oe}})}. \quad (131)$$

As a corollary of Theorem 4, we get estimates for the second order operators  $\bar{\partial}_+^{\text{oe}}\bar{\partial}_+^{\text{eo}}$ , with subelliptic boundary conditions.

**Corollary 2.** *Let  $(X, J, g, \rho)$  define a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold. For  $s \geq 0$  there exist constants  $C_s$  such that if  $u \in L^2$ ,  $\bar{\partial}_+^{\text{eo}}u \in L^2$ ,  $\bar{\partial}_+^{\text{oe}}\bar{\partial}_+^{\text{eo}}u \in H^s$  and  $\mathcal{R}_+^{\text{eo}}[u]_{bX} = 0$ ,  $\mathcal{R}_+^{\text{oe}}[\bar{\partial}_+^{\text{eo}}u] = 0$  in the sense of distributions, then*

$$\|u\|_{H^{s+1}} \leq C_s [\|\bar{\partial}_+^{\text{oe}}\bar{\partial}_+^{\text{eo}}u\|_{H^s} + \|u\|_{L^2}]. \quad (132)$$

We close this section by considering  $(\mathcal{P}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}})$  as a tame Fredholm pair, as defined in the appendix. To apply the functional analytic framework set up in the appendix, we use as the family of separable Hilbert spaces the  $L^2$ -Sobolev spaces  $H^s(bX; F)$ , where  $F$  are appropriate vector bundles. The norms on these spaces can be selected to satisfy the conditions, (263) and (264). In this setting the algebra of tame operators certainly includes the extended Heisenberg calculus. In this setting the smoothing operators are operators in  ${}^eH\Psi^{-\infty, -\infty, -\infty}(bX; F, G)$ , i.e., operators from sections of  $F$  to sections of  $G$  (two vector bundles) with a Schwartz kernel in  $\mathcal{C}^\infty(bX \times bX)$ .

An immediate corollary of Theorem 1 is:

**Corollary 3.** *Let  $(X, J, g, \rho)$  define a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold, and let  $\mathcal{P}_+^{\text{eo}}$  be the Calderon projectors for  $\bar{\partial}_+^{\text{eo}}$ . If  $\mathcal{R}_+^{\text{eo}}$  are projectors defining modified  $\bar{\partial}$ -Neumann boundary conditions, then  $(\mathcal{P}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}})$  are tame Fredholm pairs.*

If  $\mathcal{U}_+^{\text{eo}}$  are parametrices for  $\mathcal{T}_+^{\text{eo}}$ , and  $K_1^{\text{eo}}, K_2^{\text{eo}}$  are smoothing operators that satisfy

$$\mathcal{T}_+^{\text{eo}}\mathcal{U}_+^{\text{eo}} = \text{Id} - K_1^{\text{eo}}, \quad \mathcal{U}_+^{\text{eo}}\mathcal{T}_+^{\text{eo}} = \text{Id} - K_2^{\text{eo}}, \quad (133)$$

then Theorem 15 immediately implies:

**Theorem 5.** *Let  $(X, J, g, \rho)$  define a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold, and let  $\mathcal{P}_+^{\text{eo}}$  be Calderon projectors for  $\bar{\partial}_+^{\text{eo}}$ . If  $\mathcal{R}_+^{\text{eo}}$  are projectors defining a modified  $\bar{\partial}$ -Neumann boundary conditions, then*

$$\text{R-Ind}(\mathcal{P}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}}) = \text{tr}_{L^2}(\mathcal{P}_+^{\text{eo}}K_2^{\text{eo}}\mathcal{P}_+^{\text{eo}}) - \text{tr}_{L^2}(\mathcal{R}_+^{\text{eo}}K_1^{\text{eo}}\mathcal{R}_+^{\text{eo}}) \quad (134)$$

If  $\mathcal{P}_+^{\text{eo}} K_2^{\text{eo}} \mathcal{P}_+^{\text{eo}}$  have Schwartz kernels  $\kappa_2^{\text{eo}}(x, y)$  and  $\mathcal{R}_+^{\text{eo}} K_1^{\text{eo}} \mathcal{R}_+^{\text{eo}}$  have Schwartz kernels  $\kappa_1^{\text{eo}}(x, y)$ , then Lidskii's theorem, see [22], implies that

$$\text{R-Ind}(\mathcal{P}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}}) = \int_{bX} \kappa_2^{\text{eo}}(x, x) dS(x) - \int_{bX} \kappa_1^{\text{eo}}(x, x) dS(x). \quad (135)$$

This formula, coupled with (19), is very useful for showing the constancy of the index under smooth isotopies of the structures involved in its definition.

## 7 Invertible doubles and the Calderon Projector

In order to better understand the functorial properties of sub-elliptic boundary value problems and prove the Atiyah-Weinstein conjecture, it is important to be able to deform the  $\text{Spin}_{\mathbb{C}}$ -structure and projectors without changing the indices of the operators. We now consider the dependence of the various operators on the geometric structures. Of particular interest is the dependence of the Calderon projector on  $(J, g, \rho)$ . To examine this we need to consider the invertible double construction from [3] in greater detail. We also want to express the indices of  $(\tilde{\partial}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}})$  as the relative indices of the tame Fredholm pairs  $(\mathcal{P}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}})$ .

We now recount the invertible double construction from [3]. We begin with a compact manifold  $X$  with boundary, with a metric  $g$ , complex spinor bundles  $\mathcal{S}^{\text{eo}} \rightarrow X$ , and  $h$  a Hermitian metric on  $\mathcal{S}^{\text{eo}}$ . Let  $Y = bX$  and suppose that an identification of a neighborhood  $U$  of  $bX$  with  $Y \times [-1, 0]_t$  is fixed. We assume that  $dt$  is a outward pointing unit co-vector. With respect to this collar neighborhood, we say that  $X$  has a *cylindrical end* if  $\mathcal{S}^{\text{eo}}, h$  and  $g$  are independent of the ‘‘normal variable,’’  $t$ . In this case, the invertible double of  $(X, g, h, \mathcal{S}^{\text{eo}})$  is defined to be  $\tilde{X} = X \amalg_Y \bar{X}$ , here  $\bar{X}$  is  $X$  with the opposite orientation. We denote the components of  $\tilde{X} \setminus Y \times \{0\}$  by  $X_+(t < 0)$ ,  $X_-(t > 0)$ . The smooth structure on  $\tilde{X}$  is obtained by gluing  $Y \times [-1, 0] \subset X_+$  to  $Y \times [0, 1] \subset X_-$ , along  $Y \times \{0\}$ . As  $\mathcal{S}^{\text{eo}}, h$  and  $g$  are independent of  $t$  it is clear that they extend smoothly to  $\tilde{X}$ .

Because the orientation of  $X_-$  is reversed, to get a smooth bundle of complex spinors we glue  $\mathcal{S}^{\text{eo}}|_Y$  to  $\mathcal{S}^{\text{oe}}|_Y$  using

$$c(-dt) \cdot \sigma_+|_{Y \times 0^-} \sim \sigma_-|_{Y \times 0^+}. \quad (136)$$

In [3] it is shown that this defines a smooth Clifford module over  $\tilde{X}$  and hence a  $\text{Spin}_{\mathbb{C}}$ -structure. We let  $\tilde{\partial}_X^{\text{eo}}$  denote the Dirac operator, and use the notation

$$\tilde{\partial}_{X_{\pm}}^{\text{eo}} \stackrel{d}{=} \tilde{\partial}_X^{\text{eo}}|_{\mathcal{C}^\infty(X_{\pm}; \mathcal{S}^{\text{eo}})}. \quad (137)$$

From the construction and results in [3], the following identities are obvious

$$\ker \bar{\partial}_{X\pm}^{\text{eo}} = \ker \bar{\partial}_{X\mp}^{\text{oc}}. \quad (138)$$

In [3] it is shown that  $\bar{\partial}_X^{\text{eo}}$  are invertible operators, we denote the inverses by  $Q^{\text{eo}}$ .

If  $(X, J, g)$  is an almost complex manifold with boundary, then it can be included into a larger manifold  $X'$  that has a cylindrical end. It is clear that this can be done with smooth dependence on  $(J, g)$ . We fix an identification of a neighborhood  $U$  of  $bX$  with  $[-3, -2] \times Y$ . Using this identification we smoothly glue  $Y \times [-2, 0]$  to  $X$ . Denote this manifold by  $X'$ . Using Lemma 8 below we easily show that the almost complex structure can be extended to  $X'$  so that by the time we reach  $t = -1$  it is independent of  $t$ . Hence we can also extend  $\mathcal{F}^{\text{eo}}$  to  $X'$ . Using the Seeley extension theorem we can extend  $(g, h)$  to  $Y \times [-2, -1]$  in such a way that the extended metric tensors depend continuously, in the  $\mathcal{C}^\infty$ -topology, on  $(g, h) \upharpoonright_{Y \times [-3, -2]}$ , and  $(g, h)$  also have a product structure by the time we reach  $Y \times \{-1\}$ . Everything can be further extended to  $Y \times [-1, 0]$  so that it is independent of  $t$ , and hence  $X'$ , with this hermitian spin-structure, has a cylindrical end. Compatible connections can be chosen on  $\mathcal{F}^{\text{eo}}$  so that both the metric and spin geometries of  $\tilde{X}' = X' \amalg_{Y \times \{0\}} \overline{X'}$  depends smoothly on the geometry of  $(X, J, g, h)$ . In particular the symbols of  $\bar{\partial}_{\tilde{X}'}^{\text{eo}}$  depend smoothly on the symbols of  $\bar{\partial}_X^{\text{eo}}$ .

Fix a collar neighborhood,  $U$ , of  $bX$  so that  $TX \upharpoonright_U$  is independent of  $t$ . We can also normalize so that the 1-jet of  $t$  along  $t = 0$  equals that of  $\rho$ . We first homotope the Hermitian metric through a family  $\{g_s : s \in [0, \frac{1}{2}]\}$  so that the  $g_0 = g$ , and  $g_{\frac{1}{2}}$  has a product structure in  $U$ . Moreover we can fix the metric on  $bX$  throughout this homotopy. For each  $s$  there is a unique positive definite endomorphism  $A_s$  of  $TX$  so that, for all vector fields  $V, W$ , we have

$$g(V, W) = g_s(A_s V, A_s W). \quad (139)$$

If we set  $J_s = A_s J A_s^{-1}$ , then this is a smooth family of almost complex structures  $\{J_s\}$  compatible with  $g_s$ , and  $J_s \upharpoonright_{t=0}$  remaining fixed. Finally, with the metric in  $U$  fixed to equal  $g_{\frac{1}{2}}$ , we can deform  $J_{\frac{1}{2}}$  through a family  $\{J_s : s \in [\frac{1}{2}, 1]\}$  so that:

1.  $J_s = J_{\frac{1}{2}}$  outside of a small neighborhood of  $bX$ .
2.  $J_1$  has a product structure within a smaller neighborhood of  $bX$ .
3.  $J_s$  is compatible with  $g_{\frac{1}{2}}$  for  $s \in [\frac{1}{2}, 1]$ .
4.  $J_s \upharpoonright_{t=0} = J \upharpoonright_{t=0}$ .

That this is possible follows from the fact that the space of almost complex structures compatible with  $g_{\frac{1}{2}}$  can be represented as sections of a smooth fiber bundle  $\mathcal{F}$

with fiber equal to  $SO(2n)/U(n)$ . This representation is obtained by using  $J \upharpoonright_{t=0}$ , pulled back to the collar neighborhood, to define a reference structure. By compactness, there is an  $\epsilon < 0$  so that, the section of  $\mathcal{F} \upharpoonright_{\epsilon \leq t \leq 0}$  defined by  $J_{\frac{1}{2}}$  lies in a neighborhood of  $\mathcal{F}$  retractable onto the “zero section,” defined by the reference structure. Hence we can perform the homotopy described. We let  $g_s = g_{\frac{1}{2}}$  for  $s \in [\frac{1}{2}, 1]$  and  $\rho_s = t$  for all  $s$ . For later application we summarize the results of this discussion as a lemma. We refer to this process as *flattening the end*.

**Lemma 8.** *Let  $(X, J, g, \rho)$  be a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold. There exists a smooth family  $\{(X, J_s, g_s, \rho_s) : s \in [0, 1]\}$  of normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifolds with*

1. *The structure at  $s = 0$  is equal to the given structure.*
2. *Throughout the homotopy, the data remains fixed along  $bX$ .*
3. *The space  $(X, J_1, g_1, \rho_1)$  has a cylindrical end.*

We let  $Q_{X'}^{\text{co}}$  be the inverses of  $\tilde{\partial}_{X'}^{\text{co}}$ . These are classical pseudodifferential operators of order  $-1$ , whose symbols depend smoothly on the symbols of  $\tilde{\partial}_{X'}^{\text{co}}$  and therefore, in turn on the geometric data on  $X$ . Throughout the discussion below we use the fact that the operator norms of a pseudodifferential operator depend continuously on finite semi-norms of the “full” symbol of the operator, see [18].

We state a general result:

**Proposition 10.** *Let  $M$  be a compact manifold and  $E, F$  complex vector bundles over  $M$ . Let  $\{A_\tau \in \Psi^1(M; E, F) : \tau \in \mathfrak{T}\}$  be a compact smooth family of invertible elliptic pseudodifferential operators. For any  $s \in \mathbb{R}$  the family of inverses  $A_\tau^{-1}$  is a norm continuous family of operators from  $H^s(M; F)$  to  $H^{s+1}(M; E)$ .*

*Proof.* As  $\{A_\tau\}$  is a smooth family of pseudodifferential operators of order 1, for each  $s \in \mathbb{R}$ ,  $\tau \mapsto [A_\tau : H^s \rightarrow H^{s-1}]$  is norm continuous. First we show that

$$C_0 = \inf_{\tau \in T} \inf_{v \neq 0 \in H^1(M; E)} \frac{\|A_\tau v\|_{L^2}}{\|v\|_{L^2}}, \quad (140)$$

is positive. Let  $\{B_\tau\}$  denote parametrices for  $\{A_\tau\}$ , with smoothing errors:

$$A_\tau B_\tau = \text{Id} - K_{1\tau}, \quad B_\tau A_\tau = \text{Id} - K_{2\tau}. \quad (141)$$

Because the construction of  $B_\tau$  is symbolic, it follows from the general principle above that, for any  $s \in \mathbb{R}$ , the maps  $\tau \mapsto [B_\tau : H^s \rightarrow H^{s+1}]$  are continuous in the norm topology. Moreover for any  $s, t \in \mathbb{R}$ ,  $j = 1, 2$ ,  $\tau \mapsto [K_{j\tau} : H^s \rightarrow H^t]$

are norm continuous. The relations in (141) and these results imply that there is a constant  $C_1$  so that, for any  $\tau \in \mathfrak{T}$  and  $v \in H^1$  we have:

$$\|v\|_{H^1} \leq C_1 [\|A_\tau v\|_{L^2} + \|v\|_{L^2}]. \quad (142)$$

Suppose that  $C_0 = 0$ , then there exists a convergent sequence  $\langle \tau_n \rangle$  and a sequence  $\langle v_n \rangle$  with  $\|v_n\|_{L^2} = 1$  so that  $\langle A_{\tau_n} v_n \rangle$  converges to zero in  $L^2$ . From (142) it follows that  $\|v_n\|_{H^1}$  is uniformly bounded and so, by extracting a subsequence, we can assume that  $\langle v_n \rangle$  converges to  $v_0 \neq 0$  in  $L^2$ . Let  $\tau_0$  denote the limit of  $\langle \tau_n \rangle$ . The second relation in (141) shows that

$$v_n = B_{\tau_n} A_{\tau_n} v_n + K_{2\tau_n} v_n, \quad (143)$$

which, along with the Rellich compactness lemma, implies that  $\langle v_n \rangle$  converges to  $v_0$  in  $H^1$ . This shows that  $A_{\tau_0} v_0 = 0$ , which contradicts the invertibility of  $A_{\tau_0}$ . Hence  $C_0 > 0$ .

The relations in (141) imply that

$$A_\tau^{-1} = B_\tau + K_{2\tau} B_\tau + K_{2\tau} A_\tau^{-1} K_{1\tau}. \quad (144)$$

The first two terms on the right hand side of (144) satisfy the conclusion of the proposition. As  $K_{1\tau}$ ,  $K_{2\tau}$  are norm continuous families of smoothing operators, to complete the proof, we only need to show that  $\tau \mapsto [A_\tau^{-1} : L^2 \rightarrow L^2]$  is strongly continuous. That is, if  $\tau_n \rightarrow \tau_0$ , and  $v \in L^2$  then  $A_{\tau_n}^{-1} v \rightarrow A_{\tau_0}^{-1} v$  in  $L^2$ . Let  $w_n = A_{\tau_n}^{-1} v$ ; the fact that  $C_0 > 0$  and (142) imply that

$$\|w_n\|_{H^1} \leq C_1 [\|v\|_{L^2} + \|A_{\tau_n}^{-1} v\|_{L^2}] \leq C_1 [\|v\|_{L^2} + C_0^{-1} \|v\|_{L^2}]. \quad (145)$$

This estimate shows that  $\|w_n\|_{H^1}$  is uniformly bounded and therefore  $\langle w_n \rangle$  has a  $H^1$ -weakly convergent subsequence  $\langle w_{n_j} \rangle$ . Once again using (141) we see that

$$w_n = B_{\tau_n} v + K_{2\tau_n} w_n, \quad (146)$$

which implies that  $\langle w_{n_j} \rangle$  converges strongly in  $H^1$  to  $w_0$ . As  $A_{\tau_{n_j}} w_{n_j} = v$ , we see that  $A_{\tau_0} w_0 = v$ , i.e.  $w_0 = A_{\tau_0}^{-1} v$ . This shows that every convergent subsequence of  $\langle w_n \rangle$  converges to  $A_{\tau_0}^{-1} v$ . The precompactness of the sequence  $\langle w_n \rangle$ , which follows from (146), shows that  $\langle w_n \rangle$  itself converges to this limit. Thus completing the proof of the proposition.  $\square$

Once the collar neighborhood is fixed, we can define the Calderon projectors for the hypersurface  $t = 0$ : The Calderon projectors  $\mathcal{P}_\pm^{\text{co}}$  are defined by

$$\mathcal{P}_\pm^{\text{co}} g = \lim_{\epsilon \rightarrow 0^+} \gamma_{\mp\epsilon} Q^{\text{co}} \gamma_0^* \mathbf{c}(\mp dt) g. \quad (147)$$

Here  $\gamma_\epsilon$  is the operation of restriction to the submanifold  $t = \epsilon$ . In fact, we can define a pair of Calderon projectors for any hypersurface  $t = t_0$  lying in the collared part of the manifold. In [3] it is shown that,

$$\mathcal{P}_+^{\text{eo}} + \mathcal{P}_-^{\text{eo}} = \text{Id}. \quad (148)$$

We can extend  $t$  smoothly to all of  $\tilde{X}'$  so that it is negative on the original manifold  $X$ , and positive on the interior of  $\tilde{X}' \setminus X'$ . The following result is very useful in our analysis.

**Proposition 11.** *Let  $X$  be a  $\text{Spin}_\mathbb{C}$ -manifold with boundary and let  $\tilde{X}'$  be an invertible double for  $X$ . If  $\mathcal{P}_\pm^{\text{eo}}$  are the Calderon projectors defined by the above prescription, then the adjoints satisfy:*

$$\mathcal{P}_\pm^{\text{eo}*} = \mathbf{c}(\pm dt)\mathcal{P}_\mp^{\text{oe}}\mathbf{c}(\pm dt)^{-1}. \quad (149)$$

*Proof.* We give the proof for the  $+$  even case; the odd and  $-$  cases are identical. Let  $f$  and  $g$  be smooth sections of  $\mathcal{F}^\circ|_{bX}$ , and let  $G$  denote the extension of  $g$  to the collar neighborhood of  $bX$  that is constant in  $t$ . As  $[Q^{\text{eo}}]^* = Q^{\text{oe}}$ , the definition implies that

$$\begin{aligned} \langle f, \mathcal{P}_+^{\text{e}*}g \rangle &= \langle \mathcal{P}_+^{\text{e}}f, g \rangle = \lim_{\epsilon \rightarrow 0^-} \langle \gamma_\epsilon Q^\circ \gamma_0^* \mathbf{c}(-dt)f, G \rangle \\ &= \lim_{\epsilon \rightarrow 0^-} \langle \mathbf{c}(-dt)f, \gamma_0 Q^\circ \gamma_\epsilon^* G \rangle \end{aligned} \quad (150)$$

The proof of the proposition follows from the observations that  $\epsilon < 0$  in (150),  $\mathbf{c}(-dt)^* = \mathbf{c}(dt)$ , and

$$\lim_{\epsilon \rightarrow 0^-} \gamma_0 Q^\circ \gamma_\epsilon^* G = \mathcal{P}_-^{\text{o}} \mathbf{c}(dt)^{-1} g. \quad (151)$$

This follows because  $u_\epsilon = Q^\circ \gamma_\epsilon^* G$  solves  $\tilde{\partial}^\circ u_\epsilon = 0$  in the subset

$$\tilde{X}'_\epsilon = \{x : t(x) > \epsilon\}.$$

It is easy to see that, as  $\epsilon \uparrow 0$ , this is a uniformly bounded family of solutions, which converges uniformly on  $\tilde{X}'_-$  to  $u_0 = Q^\circ \gamma_0^* G$ . Clearly the restrictions to  $t = 0$  converge to  $\mathcal{P}_-^{\text{o}} \mathbf{c}(dt)^{-1} g$ .  $\square$

This proposition has an interesting and useful corollary

**Corollary 4.** *Suppose that  $X$  is  $\text{Spin}_\mathbb{C}$ -manifold, with a cylindrical end and  $\mathcal{P}_\pm^{\text{eo}}$  are defined using the inverse of  $\tilde{\partial}_X^{\text{eo}}$  on the invertible double  $\tilde{X}$ . Then these are self adjoint projection operators.*

*Proof.* We do the even-+ case, the others are identical. Proposition 11 shows that  $\mathcal{P}_+^{e*} = \mathbf{c}(dt)\mathcal{P}_-^o\mathbf{c}(dt)^{-1}$ . On the other hand, because  $\tilde{X}$  is obtained by doubling across  $bX$ , we have (138), implying that

$$\text{range } \mathcal{P}_+^e = \text{range } \mathcal{P}_+^{e*}. \quad (152)$$

Generally we have that

$$[\text{range } \mathcal{P}_+^e]^\perp = \ker \mathcal{P}_+^{e*} = \text{range}(\text{Id} - \mathcal{P}_+^{e*}) = \text{range}(\text{Id} - \mathcal{P}_+^e), \quad (153)$$

and therefore  $\langle \mathcal{P}_+^e f, (\text{Id} - \mathcal{P}_+^e)g \rangle = 0$ , for all pairs,  $f, g$ . These relations imply that

$$\mathcal{P}_+^{e*}\mathcal{P}_+^e = \mathcal{P}_+^e \text{ and } \mathcal{P}_+^{e*}(\text{Id} - \mathcal{P}_+^e) = 0, \quad (154)$$

from which the conclusion is immediate.  $\square$

The symbols of  $\mathcal{P}_\pm^{eo}$  are smooth functions of the symbols of  $Q_{\tilde{X}'}^{eo}$ . Using the norm continuity of  $Q_{\tilde{X}'}^{eo}$ , we conclude that the Calderon projectors also depend continuously, in the uniform norm topology, on the geometric data on  $X$ .

**Proposition 12.** *Suppose that  $\{(X, J_\tau, g_\tau, \rho_\tau) : \tau \in \mathfrak{T}\}$  is a compact smooth family of normalized strictly pseudoconvex  $\text{Spin}_\mathbb{C}$ -manifolds. The Calderon projectors,  $\mathcal{P}_{\pm\tau}^{eo}$ , defined by the invertible double construction are smooth families of pseudodifferential operators of order zero, and*

$$\tau \mapsto [\mathcal{P}_{\pm\tau}^{eo} : L^2(bX; \mathfrak{S}^{eo}) \rightarrow L^2(bX; \mathfrak{S}^{eo})] \quad (155)$$

are continuous in the uniform norm topology.

*Proof.* First we show that  $\mathcal{P}_{\pm\tau}^{eo}$  is a norm continuous family of operators on  $L^2$ . Let  $Q_\tau^{eo}$  denote the inverse of  $\mathfrak{D}_\tau^{eo}$ , the  $\text{Spin}_\mathbb{C}$ -Dirac operator on the invertible double defined by the data  $(X, J_\tau, g_\tau, \rho_\tau)$ , and  $\tilde{Q}_\tau^{eo}$  a parametrix with

$$\mathfrak{D}_\tau^{eo}\tilde{Q}_\tau^{eo} = \text{Id} - K_{1\tau}, \quad \tilde{Q}_\tau^{eo}\mathfrak{D}_\tau^{eo} = \text{Id} - K_{2\tau}. \quad (156)$$

Proposition 10 shows that  $Q_\tau^{eo} : L^2(X; \mathfrak{S}^{eo}) \rightarrow H^1(X; \mathfrak{S}^{eo})$  are norm continuous families. The inverse and the parametrix are related by

$$Q_\tau^{eo} = \tilde{Q}_\tau^{eo} + Q_\tau^{eo}K_{1\tau}. \quad (157)$$

Recall that the restriction maps from  $H_{(1, -\frac{1}{2})}(X_\pm; \mathfrak{S}^{eo}) \rightarrow L^2(bX_\pm; \mathfrak{S}^{eo})$  are continuous. The second statement of the proposition is an easy consequence of this fact, the relation (157), and the observation that

$$\tilde{Q}_\tau^{eo} : L^2(bX; \mathfrak{S}^{eo}) \rightarrow H_{(1, -\frac{1}{2})}(X_\pm; \mathfrak{S}^{eo}) \text{ and } K_{1\tau} : L^2(bX; \mathfrak{S}^{eo}) \rightarrow H^1(X; \mathfrak{S}^{eo}) \quad (158)$$

are norm continuous families. The proof that the maps in (158) define norm continuous families is a simple adaptation of the argument showing that the norm of a pseudodifferential operator is bounded by a finite semi-norm of its full symbol, which we leave to the interested reader.

To see that  $\mathcal{P}_{\pm\tau}^{\text{eo}}$  is a smooth family of pseudodifferential operators we use the relation, (144) to conclude that

$$\mathcal{P}_{\pm\tau}^{\text{eo}} = \tilde{\mathcal{P}}_{\pm\tau}^{\text{eo}} + k_{2\tau}\mathcal{P}_{\pm\tau}^{\text{eo}}k_{1\tau}, \quad (159)$$

where  $\tilde{\mathcal{P}}_{\pm\tau}^{\text{eo}}$  is the smooth family of pseudodifferential operators defined by using the parametrices,  $\tilde{Q}_{\tau}^{\text{eo}}$ , in the definition of the Calderon projector, (147), and  $k_{1\tau}, k_{2\tau}$  are smooth families of smoothing operators. This relation, combined with the  $L^2$ -norm continuity of  $\tau \mapsto \mathcal{P}_{\pm\tau}^{\text{eo}}$ , show that  $\mathcal{P}_{\pm\tau}^{\text{eo}}$  is a smooth family of pseudodifferential operators.  $\square$

We can suppose that the contact structure induced on the boundary of the family  $(X, J_{\tau}, g_{\tau}, \rho_{\tau})$  is fixed and we let  $\mathcal{R}_{+\tau}^{\text{eo}}$  denote a smooth family of modified  $\bar{\partial}$ -Neumann conditions. The non-trivial part of such a family is a smooth family of generalized Szegő projectors  $\tau \mapsto \mathcal{S}_{\tau}$ . In [12] it is shown that such a family is norm continuous as a family of maps  $\tau \mapsto [\mathcal{S}_{\tau} : L^2(bX) \rightarrow L^2(bX)]$ .

**Theorem 6.** *If  $\{(X, J_{\tau}, g_{\tau}, \rho_{\tau}) : \tau \in \mathfrak{T}\}$  is a compact smooth family of normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifolds and  $\mathcal{R}_{+\tau}^{\text{eo}}$  is a smooth family of modified  $\bar{\partial}$ -Neumann conditions, then  $\text{R-Ind}(\mathcal{P}_{+\tau}^{\text{eo}}, \mathcal{R}_{+\tau}^{\text{eo}})$  is constant.*

*Proof.* Proposition 12 shows that the operators  $\mathcal{T}_{+\tau}^{\text{eo}}$  are a smooth family of extended Heisenberg operators and therefore so are the parametrices  $\mathcal{U}_{+\tau}^{\text{eo}}$ . Hence the residual terms

$$K_{1\tau}^{\text{eo}} = \text{Id} - \mathcal{T}_{+\tau}^{\text{eo}} \mathcal{U}_{+\tau}^{\text{eo}}, \quad K_{2\tau}^{\text{eo}} = \text{Id} - \mathcal{U}_{+\tau}^{\text{eo}} \mathcal{T}_{+\tau}^{\text{eo}} \quad (160)$$

are smooth families of smoothing operators. As  $\tau \mapsto \mathcal{P}_{+\tau}^{\text{eo}}$  and  $\tau \mapsto \mathcal{R}_{+\tau}^{\text{eo}}$  are norm continuous as maps from  $L^2$  to itself, the operators  $\mathcal{P}_{+\tau}^{\text{eo}} K_{2\tau}^{\text{eo}} \mathcal{P}_{+\tau}^{\text{eo}}$  and  $\mathcal{R}_{+\tau}^{\text{eo}} K_{1\tau}^{\text{eo}} \mathcal{R}_{+\tau}^{\text{eo}}$  are continuous in the trace norm. It follows from (134) that  $\text{R-Ind}(\mathcal{P}_{+\tau}^{\text{eo}}, \mathcal{R}_{+\tau}^{\text{eo}})$  depends continuously on  $\tau$ . As it is integer valued, it is constant.  $\square$

## 8 The relative index formula

In this section we prove the formula in (19), expressing  $\text{Ind}(\bar{\partial}_{+}^{\text{eo}}, \mathcal{R}_{+}^{\text{eo}})$  as the relative index of the tame Fredholm pair  $(\mathcal{P}_{+}^{\text{eo}}, \mathcal{R}_{+}^{\text{eo}})$ , and derive several consequences of this formula.



**Theorem 7.** *Let  $(X, J, g, \rho)$  be a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold, and  $\mathcal{R}_+^{\text{eo}}$  projections defining a modified  $\bar{\partial}$ -Neumann problems, then*

$$\text{Ind}(\bar{\partial}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}}) = \text{R-Ind}(\mathcal{P}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}}). \quad (161)$$

*Proof.* We give the proof for the even case, the odd case is identical. The kernel of  $(\bar{\partial}_+^{\text{e}}, \mathcal{R}_+^{\text{e}})$  consists of smooth forms  $\sigma$  that satisfy

$$\bar{\partial}_+^{\text{e}} \sigma = 0 \text{ and } \mathcal{R}_+^{\text{e}}[\sigma]_{bX} = 0. \quad (162)$$

The first condition implies that  $\mathcal{P}_+^{\text{e}}[\sigma]_{bX} = [\sigma]_{bX}$ , and therefore  $\sigma \in \ker \mathcal{R}_+^{\text{e}} \mathcal{P}_+^{\text{e}}$ . Conversely, by the unique continuation theorem for  $\ker \bar{\partial}_+^{\text{e}}$ , any form in the range of  $\mathcal{P}_+^{\text{e}}$  that lies in the kernel of  $\mathcal{R}_+^{\text{e}}$  defines a unique form in  $\ker(\bar{\partial}_+^{\text{e}}, \mathcal{R}_+^{\text{e}})$ . Thus

$$\ker(\bar{\partial}_+^{\text{e}}, \mathcal{R}_+^{\text{e}}) \simeq \ker \mathcal{R}_+^{\text{e}} \mathcal{P}_+^{\text{e}} \upharpoonright_{\text{range } \mathcal{P}_+^{\text{e}}}.$$

The cokernel of  $\mathcal{R}_+^{\text{e}} \mathcal{P}_+^{\text{e}}$  is isomorphic to the null space of

$$\mathcal{P}_+^{\text{e}*} : \text{range } \mathcal{R}_+^{\text{e}} \rightarrow \text{range } \mathcal{P}_+^{\text{e}*}. \quad (163)$$

Proposition 11 and equation (148) show that

$$\mathcal{P}_+^{\text{e}*} = \mathbf{c}(dt)(\text{Id} - \mathcal{P}_+^{\text{o}})\mathbf{c}(dt)^{-1} \quad (164)$$

This identity, along with (97) show that the cokernel of  $\mathcal{R}_+^{\text{e}} \mathcal{P}_+^{\text{e}}$  is isomorphic to the null space of  $\mathcal{R}_+^{\text{o}}$  acting on  $\text{range } \mathcal{P}_+^{\text{o}}$ , which, by the first part of this argument, is isomorphic to  $\ker(\bar{\partial}_+^{\text{o}}, \mathcal{R}_+^{\text{o}})$ . Applying Theorem 4, we complete the proof of the theorem.  $\square$

As a corollary we have

**Corollary 5.** *Let  $(X, J, g, \rho)$  be a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold, and  $\mathcal{R}_+^{\text{eo}}$  projection operators defining modified  $\bar{\partial}$ -Neumann conditions, then*

$$\text{R-Ind}(\mathcal{P}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}}) = -\text{R-Ind}(\text{Id} - \mathcal{P}_+^{\text{eo}}, \text{Id} - \mathcal{R}_+^{\text{eo}}) \quad (165)$$

*Proof.* We give the proof for the even case, the odd case is identical. First suppose that  $(X, J, g, \rho)$  has a cylindrical end. In this case  $\mathcal{P}_+^{\text{e}} = \mathcal{P}_+^{\text{e}*}$  and therefore (148) and Proposition 11 imply that

$$\text{Id} - \mathcal{P}_+^{\text{e}} = \mathbf{c}(dt)\mathcal{P}_+^{\text{o}}\mathbf{c}(dt)^{-1} \quad (166)$$

As  $\text{Id} - \mathcal{R}_+^{\text{e}} = \mathbf{c}(dt)\mathcal{R}_+^{\text{o}}\mathbf{c}(dt)^{-1}$ , the relation (161) implies that

$$\text{R-Ind}(\text{Id} - \mathcal{P}_+^{\text{e}}, \text{Id} - \mathcal{R}_+^{\text{e}}) = \text{R-Ind}(\mathcal{P}_+^{\text{o}}, \mathcal{R}_+^{\text{o}}) = \text{Ind}(\bar{\partial}_+^{\text{o}}, \mathcal{R}_+^{\text{o}}) = -\text{Ind}(\bar{\partial}_+^{\text{e}}, \mathcal{R}_+^{\text{e}}). \quad (167)$$

The corollary, in this case, follows with one further application of (161).

We treat the general case by deformation. Lemma 8 gives a smooth family  $\{(X, J_s, g_s, \rho_s) : s \in [0, 1]\}$  such that  $(X, J_1, g_1, \rho_1)$  has a cylindrical end and the data along  $bX$  is fixed. As the metric, almost complex structure and defining function remain constant along  $bX$ , the family  $\{(X, J_s, g_s, \rho_s) : s \in [0, 1]\}$  satisfies the hypotheses of Theorem 6 and therefore  $\mathbf{R}\text{-Ind}(\text{Id} - \mathcal{P}_{+s}^{\text{eo}}, \text{Id} - \mathcal{R}_{+s}^{\text{eo}})$  and  $\mathbf{R}\text{-Ind}(\mathcal{P}_{+s}^{\text{eo}}, \mathcal{R}_{+s}^{\text{eo}})$  are independent of  $s$ . The argument above shows that

$$\mathbf{R}\text{-Ind}(\text{Id} - \mathcal{P}_{+1}^{\text{eo}}, \text{Id} - \mathcal{R}_{+1}^{\text{eo}}) = -\mathbf{R}\text{-Ind}(\mathcal{P}_{+1}^{\text{eo}}, \mathcal{R}_{+1}^{\text{eo}}), \quad (168)$$

completing the proof of the theorem.  $\square$

As a corollary of the corollary we have the following result.

**Corollary 6.** *The operators  $\mathcal{T}_+^{\text{eo}}$  are tame Fredholm operators of index zero.*

*Proof.* The first statement follows from the ellipticity of  $\mathcal{T}_+^{\text{eo}}$  in the extended Heisenberg calculus. The indices of  $\mathcal{T}_+^{\text{eo}}$  are computed using the trace formula:

$$\text{Ind}(\mathcal{T}_+^{\text{eo}}) = \text{tr } K_2 - \text{tr } K_1. \quad (169)$$

It is an elementary computation to show that the right hand side of (169) equals  $\mathbf{R}\text{-Ind}(\mathcal{P}_+^{\text{eo}}, \mathcal{R}_+^{\text{eo}}) + \mathbf{R}\text{-Ind}(\text{Id} - \mathcal{P}_+^{\text{eo}}, \text{Id} - \mathcal{R}_+^{\text{eo}})$  plus a sum of terms of the form  $\text{tr}[K, A]$  where  $K$  is a smoothing operator and  $A$  is bounded. As such terms vanish, this corollary follows from Corollary 5.  $\square$

## 9 The Agranovich-Dynin formula

In [9] we proved a generalization of the Agranovich-Dynin formula for subelliptic boundary conditions, assuming that the  $\text{Spin}_{\mathbb{C}}$ -structure arises from an integrable almost complex structure. In this section we show that the integrability is not necessary.

**Theorem 8.** *Let  $(X, J, h, \rho)$  be a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold. Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be two generalized Szegő projections and  $\mathcal{R}_{+1}^e, \mathcal{R}_{+2}^e$  the modified  $\bar{\partial}$ -Neumann conditions they define, then*

$$\mathbf{R}\text{-Ind}(\mathcal{S}_1, \mathcal{S}_2) = \text{Ind}(\bar{\partial}^e, \mathcal{R}_{+2}^e) - \text{Ind}(\bar{\partial}^e, \mathcal{R}_{+1}^e). \quad (170)$$

*Proof.* We apply Theorem 6 to conclude that

$$\text{Ind}(\bar{\partial}_+^e, \mathcal{R}_{+2}^e) - \text{Ind}(\bar{\partial}_+^e, \mathcal{R}_{+1}^e) = \mathbf{R}\text{-Ind}(\mathcal{P}_+^e, \mathcal{R}_{+2}^e) - \mathbf{R}\text{-Ind}(\mathcal{P}_+^e, \mathcal{R}_{+1}^e). \quad (171)$$

By relabeling we may assume that  $\text{R-Ind}(\mathcal{S}_1, \mathcal{S}_2) \geq 0$ . In [12] it is shown that  $\mathcal{S}_2$  can be deformed, through a smooth family of generalized Szegő projections to a projection  $\mathcal{S}_3$  which is a sub-projection of  $\mathcal{S}_1$ . That is  $\mathcal{S}_1 = \mathcal{S}_3 + P$ , where  $P$  is a finite rank orthogonal projection, with a smooth kernel and

$$\mathcal{S}_3 P = P \mathcal{S}_3 = 0. \quad (172)$$

If  $\{\mathcal{R}_{+s}^e : s \in [2, 3]\}$  is the associated family of modified  $\bar{\partial}$ -Neumann conditions, then the proof of Theorem 6 applies equally well to show that  $\text{R-Ind}(\mathcal{P}_+^e, \mathcal{R}_{+s}^e)$  is constant. So we are reduced to showing that

$$\text{R-Ind}(\mathcal{S}_1, \mathcal{S}_2) = \text{R-Ind}(\mathcal{S}_1, \mathcal{S}_3) = \text{R-Ind}(\mathcal{P}_+^e, \mathcal{R}_{+3}^e) - \text{R-Ind}(\mathcal{P}_+^e, \mathcal{R}_{+1}^e), \quad (173)$$

with  $\mathcal{S}_3$  a finite corank sub-projection of  $\mathcal{S}_1$ . For convenience we apply Lemma 8 to deform to a structure with a cylindrical end, so that we can assume that  $\mathcal{P}_+^e$  is self adjoint. Theorem 16 in the appendix applies to show that  $\text{R-Ind}(\mathcal{P}_+^e, \mathcal{R}_{+3}^e) - \text{R-Ind}(\mathcal{P}_+^e, \mathcal{R}_{+1}^e)$  is the index of the operator

$$\mathcal{R}_{+3}^e \mathcal{P}_+^e \mathcal{R}_{+1}^e : L^2 \cap \text{range } \mathcal{R}_{+1}^e \longrightarrow \mathcal{H} \cap \text{range } \mathcal{R}_{+3}^e, \quad (174)$$

where  $\mathcal{H}$  is an appropriately defined Hilbert space. Let  $\mathcal{U}_{+j}^e$ ,  $j = 1, 3$  denote parametrices for  $\mathcal{T}_{+j}^e$ . The space  $\mathcal{H}$  is defined as the closure of  $\mathcal{C}^\infty$  with respect to the inner product defined by

$$\|u\|_{\mathcal{H}}^2 = \|u\|_{L^2}^2 + \|\mathcal{U}_{+1}^{e*} \mathcal{U}_{+3}^e u\|_{L^2}^2. \quad (175)$$

We can rewrite

$$\mathcal{R}_{+3}^e \mathcal{P}_+^e \mathcal{R}_{+1}^e = \mathcal{R}_{+3}^e \mathcal{T}_{+3}^e \mathcal{T}_{+1}^{e*} \mathcal{R}_{+1}^e \quad (176)$$

We recall that the difference  $\mathcal{R}_{+1}^e - \mathcal{R}_{+3}^e$  is a smoothing operator and therefore so is  $\mathcal{T}_{+1}^e - \mathcal{T}_{+3}^e$ . Hence the operator on the right hand side of (176) has the same index as

$$\mathcal{R}_{+3}^e \mathcal{T}_{+3}^e \mathcal{T}_{+3}^{e*} \mathcal{R}_{+1}^e \quad (177)$$

It is a simple matter to show that  $[\mathcal{R}_{+3}^e, \mathcal{T}_{+3}^e \mathcal{T}_{+3}^{e*}] = 0$  and therefore, as  $[\mathcal{R}_{+3}^e]^2 = \mathcal{R}_{+3}^e$ , we are reduced to computing the index of

$$\mathcal{R}_{+3}^e \mathcal{T}_{+3}^e \mathcal{T}_{+3}^{e*} \mathcal{R}_{+3}^e \mathcal{R}_{+1}^e. \quad (178)$$

We think of this as a composition

$$\mathcal{R}_{+3}^e \mathcal{R}_{+1}^e : L^2 \cap \text{range } \mathcal{R}_{+1}^e \rightarrow L^2 \cap \text{range } \mathcal{R}_{+3}^e \quad (179)$$

with

$$\mathcal{W} = \mathcal{R}_{+3}^e \mathcal{T}_{+3}^e \mathcal{T}_{+3}^{e*} \mathcal{R}_{+3}^e : L^2 \cap \text{range } \mathcal{R}_{+3}^e \rightarrow \text{range } \mathcal{R}_{+3}^e \cap \mathcal{H}. \quad (180)$$

The index of the operator in (179) is  $\text{R-Ind}(\mathcal{S}_1, \mathcal{S}_3) = \text{R-Ind}(\mathcal{S}_1, \mathcal{S}_2)$ ; hence we are left to show that  $\mathcal{W}$  has index zero.

As  $\mathcal{T}_{+1}^e - \mathcal{T}_{+3}^e$  is a smoothing operator, the space defined by the inner product in (175) is unchanged if  $\mathcal{U}_{+1}^{e*}$  is replaced by  $\mathcal{U}_{+3}^{e*}$ ; thus  $\mathcal{W}$  is a bounded operator. The operator

$$\mathcal{V} = \mathcal{R}_{+3}^e \mathcal{U}_{+3}^{e*} \mathcal{U}_{+3}^e \mathcal{R}_{+3}^e : \text{range } \mathcal{R}_{+3}^e \cap \mathcal{H} \rightarrow L^2 \cap \text{range } \mathcal{R}_{+3}^e \quad (181)$$

is a parametrix for  $\mathcal{R}_{+3}^e \mathcal{T}_{+3}^e \mathcal{T}_{+3}^{e*} \mathcal{R}_{+3}^e$ . If

$$\mathcal{V}\mathcal{W} = \mathcal{R}_{+3}^e - \mathcal{R}_{+3}^e K \mathcal{R}_{+3}^e, \quad (182)$$

where  $K$  is smoothing, then

$$\mathcal{W}\mathcal{V} = \mathcal{R}_{+3}^e - \mathcal{R}_{+3}^e K^* \mathcal{R}_{+3}^e. \quad (183)$$

Hence the  $\text{Ind}(\mathcal{W})$  is given by

$$\text{Ind}(\mathcal{W}) = \text{tr}(\mathcal{R}_{+3}^e K^* \mathcal{R}_{+3}^e) - \text{tr}(\mathcal{R}_{+3}^e K \mathcal{R}_{+3}^e). \quad (184)$$

But this means it must be zero, because the right hand side of (184) is a purely imaginary number. This completes the proof of Theorem 8.  $\square$

*Remark 7.* The proof of the Atiyah-Weinstein conjecture is a small modification of the proof of the Agranovich-Dynin formula.

## 10 The Atiyah-Weinstein conjecture

In [37] Weinstein considers the following situation: let  $X_0, X_1$  be strictly pseudoconvex Stein manifolds with boundary. The CR-structures on  $bX_0$ , and  $bX_1$  define Szegő projectors  $\mathcal{S}_0, \mathcal{S}_1$  as projectors onto the nullspaces of  $\bar{\partial}_b$ -operators acting on functions. Suppose that there is a contact diffeomorphism  $\phi : bX_1 \rightarrow bX_0$ . Weinstein describes a construction, using stable almost complex structures, for gluing  $X_0$  to  $X_1$  via  $\phi$ , to obtain a compact manifold  $X$  with a well defined  $\text{Spin}_{\mathbb{C}}$ -structure. Weinstein conjectures that

$$\text{R-Ind}(\mathcal{S}_0, [\phi^{-1}]^* \mathcal{S}_1 \phi^*) = \text{Ind}(\bar{\partial}_X^e). \quad (185)$$

He also gives a conjecture for a formula when  $X_0$ , or  $X_1$  is a Stein space and not a Stein manifold. As described in the introduction, these conjectures evolved from conjectures, made jointly with Michael Atiyah in the 1970s, for the indices

of elliptic Fourier integral operators defined by contact transformations between co-sphere bundles of compact manifolds.

In this section we prove a more general formula, covering all these cases. As noted earlier, we do not use the stable almost complex structure construction to build a  $\text{Spin}_{\mathbb{C}}$ -structure on  $X$ , but rather a simple extension of the invertible double construction. It seems clear that these two constructions lead to the same  $\text{Spin}_{\mathbb{C}}$ -structure on the glued space.

Our set-up is the following: we have two normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifolds,  $(X_k, J_k, g_k, \rho_k)$ ,  $k = 0, 1$ , and a co-orientation preserving contact diffeomorphism of their boundaries,  $\phi : bX_1 \rightarrow bX_0$ . On each of these boundaries we choose a generalized Szegő projector,  $\mathcal{S}_0, \mathcal{S}_1$ . Using the contact diffeomorphism we obtain a second Szegő projector on  $bX_0$ , where

$$\mathcal{S}'_1 = \phi^{-1*} \mathcal{S}_1 \phi^*. \quad (186)$$

We can extend the contact diffeomorphism to a diffeomorphism of collar neighborhoods of the boundaries. Let us heretofore suppose that such an identification is fixed. The two compatible almost complex structures can now be regarded as being defined on a neighborhood of one and the same contact manifold. We show below that there is a homotopy,  $\{J_s : s \in [0, 1]\}$ , through compatible almost complex structures joining  $J_0$  to  $J_1$ . Using this and our collar neighborhood construction we can add collars to  $X_0, X_1$  obtaining  $\text{Spin}_{\mathbb{C}}$ -manifolds  $\widehat{X}_0, \widehat{X}_1$  with cylindrical ends and identical structures on a collar neighborhood of their boundaries. We describe this construction more precisely below.

Using the obvious extension of the invertible double construction, we can now build a compact space

$$\widetilde{X}_{01} = \widehat{X}_0 \sqcup_{b\widehat{X}_j} \overline{\widehat{X}_1}, \quad (187)$$

with a well defined isotopy class of  $\text{Spin}_{\mathbb{C}}$ -structures and Dirac operator  $\bar{\partial}_{\widetilde{X}_{01}}^c$ . The operators  $\bar{\partial}_{\widetilde{X}_{01}}^{\text{eo}}$  need not be invertible, nor have index zero. In the sequel we refer to this as the *extended* invertible double construction.

**Theorem 9.** *Let  $(X_k, j_k, g_k, \rho_k)$ ,  $k = 0, 1$  be normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifolds, and suppose that  $\phi : bX_1 \rightarrow bX_0$  is a co-orientation preserving contact diffeomorphism. Suppose that  $\mathcal{S}_0, \mathcal{S}_1$  are generalized Szegő projections on  $bX_0, bX_1$ , respectively, which define modified  $\bar{\partial}$ -Neumann conditions  $\mathcal{R}_{+j}^c$  on  $X_j$ . With  $\mathcal{S}'_1$  as defined in (186) and  $\widetilde{X}_{01}$  as defined in (187) we have*

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}'_1) = \text{Ind}(\bar{\partial}_{\widetilde{X}_{01}}^c) - \text{Ind}(\bar{\partial}_{X_0}^c, \mathcal{R}_{+0}^c) + \text{Ind}(\bar{\partial}_{X_1}^c, \mathcal{R}_{+1}^c). \quad (188)$$

In the sequel, we refer to the indices of the boundary value problems in (188) as the *boundary terms*.

*Remark 8.* This includes Weinstein's conjectured formula: we take the CR-structures  $J_0, J_1$  to be integrable, and  $X_0, X_1$  to be compact complex manifolds with the given boundary. The classical Szegő projectors are used for  $\mathcal{S}_0, \mathcal{S}_1$ , respectively. In this case, it is shown in Section 7 of [9] that the indices of the boundary value problems in (188) are renormalized holomorphic Euler characteristics:

$$\text{Ind}(\bar{\partial}_{X_j}^c, \mathcal{R}_{\nu_j}^c) = \chi'_0(X_j) = \sum_{q=1}^n \dim H^{0,q}(X_j)(-1)^q. \quad (189)$$

Hence (188) gives

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = \text{Ind}(\bar{\partial}_{\tilde{X}_{01}}^c) - \chi'_0(X_0) + \chi'_0(X_1). \quad (190)$$

If  $X_0$  and  $X_1$  are Stein manifolds then  $\chi'_0(X_j) = 0$ ,  $j = 0, 1$ .

Applying the Atiyah-Singer theorem for a Dirac operator we obtain a (partially) cohomological formula for the index.

**Corollary 7.** *With the hypotheses of Theorem 9, we have*

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}'_1) = \langle e^{\frac{1}{2}c_1} \hat{\mathbf{A}}(\tilde{X}_{01}), \tilde{X}_{01} \rangle - \text{Ind}(\bar{\partial}_{X_0}^c, \mathcal{R}_{+0}^c) + \text{Ind}(\bar{\partial}_{X_1}^c, \mathcal{R}_{+1}^c), \quad (191)$$

with  $c_1 = c_1(\mathcal{S}^c)$  the canonical class of the  $\text{Spin}_{\mathbb{C}}$ -structure on  $\tilde{X}_{01}$ .

*Remark 9.* The terms  $\text{Ind}(\bar{\partial}_{X_0}^c, \mathcal{R}_{+0}^c)$  and  $\text{Ind}(\bar{\partial}_{X_1}^c, \mathcal{R}_{+1}^c)$  are essential parts of this formula as they capture the non-symbolic nature of the relative index. In the case that  $X_0, X_1$  are the co-ball bundles of compact manifolds, an equivalent formula appears in [23]. A related formula for a contact self map is given in [12].

If  $\phi'$  is a different choice of contact diffeomorphism, then  $\psi = \phi' \circ \phi^{-1}$  is a contact automorphism of  $bX_0$ . The projector

$$\mathcal{S}'_1 = [\phi']^{-1*} \mathcal{S}_1 \phi'^* = \psi^{-1*} \mathcal{S}'_1 \psi^*. \quad (192)$$

The cocycle formula, proved in [12] shows that

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}''_1) = \text{R-Ind}(\mathcal{S}_0, \mathcal{S}'_1) + \text{c-deg}(\psi), \quad (193)$$

where  $\text{c-deg}(\psi)$  is the contact degree. This is shown to be a topological invariant of the isotopy class of  $\psi$  in the contact mapping class group. A formula for  $\text{c-deg}(\psi)$  as the index of a Dirac operator on the mapping torus of  $\psi$ ,  $\bar{\partial}_{Z_\psi}$  is also provided, i.e.

$$\text{c-deg}(\psi) = \text{Ind}(\bar{\partial}_{Z_\psi}). \quad (194)$$

Using the cohomological expression for  $\text{Ind}(\bar{\partial}_{Z_\psi})$ , one can easily show that the contact degree always vanishes if  $\dim Y = 3$ . Hence, if  $\dim_{\mathbb{R}} X_j = 4$ , then the  $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}'_1)$  does not depend on the choice of contact diffeomorphism.

*Proof of Theorem 9.* We now turn to the details of the proof of Theorem 9. We suppose that the extension of  $\phi$  has been applied to identify a neighborhood of  $bX_0$  with a neighborhood of  $bX_1$ . We then apply Lemma 8 to reduce to the situation that  $(X_k, J_k, g_k, \rho_k)$ ,  $k = 0, 1$  have cylindrical ends. This deformation does not change the index of the operators  $(\bar{\partial}_{X_k}^c, \mathcal{R}_k^c)$ .

We use  $j_0, j_1$  to denote the compatible almost CR-structures defined on  $H$  by  $J_0, J_1$ , respectively. The set of almost CR-structures compatible with a given contact form is contractible. Let  $\{j_s : s \in [0, 1]\}$  denote a deformation of the compatible almost CR-structure  $j_0$  to the compatible almost CR-structure  $j_1$  through compatible almost CR-structures. The contact form,  $\theta$  is fixed throughout this deformation and hence, so is the Reeb vector field  $T$ . Let  $\pi_H$  denote the projection of  $TY \times [0, 1]$  to  $H$  (pulled back to  $Y \times [0, 1]$ ) along  $\text{span}\{T, \partial_t\}$ . We extend the almost CR-structure to an almost complex structure,  $J$  on  $TY \times [0, 1]$  by setting  $JT = \partial_t$ . With this definition, the function  $t$  satisfies

$$\theta \upharpoonright_{Y \times \{t\}} = \frac{i}{2} \bar{\partial} t \upharpoonright_{Y \times \{t\}} \quad (195)$$

We extend  $\theta$  to  $\Theta$  defined on  $TY \times [0, 1]$  by setting

$$\Theta(\partial_t) = 0. \quad (196)$$

The metric on the collar is given by

$$ds_{Y \times \{s\}}^2 = dt^2 + \Theta \cdot \Theta + d\theta(\pi_H \cdot, j_s \pi_H \cdot). \quad (197)$$

Observe that we can reparametrize the family  $\{j_s\}$  so that both ends of  $Y \times [0, 1]$  are cylindrical. To do this we choose a smooth non-decreasing function  $\varphi : [0, 1] \rightarrow [0, 1]$  with  $\varphi(s) = 0$  for  $s \in [0, \frac{1}{4}]$  and  $\varphi(s) = 1$  for  $s \in [\frac{3}{4}, 1]$ . If we replace  $\{j_s\}$  with  $\{j_{\varphi(s)}\}$ , then the forgoing construction defines an almost complex structure on  $Y \times [0, 1]$  with both ends cylindrical and agreeing with the given structures. We summarize this construction in a lemma.

**Lemma 9.** *Suppose that  $(Y, H)$  is a contact manifold with contact form  $\theta$  and two compatible almost CR-structures  $j_0, j_1$ . Then there is an almost complex structure on  $Y \times [0, 1]$  with both ends cylindrical. The structure induced on  $Y \times \{1\}$  is strictly pseudoconvex and agrees with  $j_1$ , while that on  $Y \times \{0\}$  is strictly pseudoconcave and agrees with  $j_0$ , with its co-orientation reversed. For all members of the family the relation (195) holds.*

We use Lemma 9 to define an almost complex structure on  $Y \times [0, 1]$ . For each  $0 \leq \tau \leq 1$  we set  $\widehat{X}_0^\tau = X_0 \sqcup_{Y \times \{\tau\}} (Y \times [0, \tau])$ . The relative index formula and Theorem 6 imply that  $\text{Ind}(\bar{\partial}_{\widehat{X}_0^\tau}^c, \mathcal{R}_{+0}^c)$  is independent of  $\tau$ . The boundary of  $\widehat{X}_0^1$  is

cylindrical and isomorphic to a neighborhood of the (flattened) boundary of  $X_1$ , hence we can glue  $\widehat{X}_0^1$  to  $\overline{X_1}$  to obtain

$$\widetilde{X}_{01} = \widehat{X}_0^1 \amalg_{Y \times \{1\}} \overline{X_1}. \quad (198)$$

This is a manifold with a  $\text{Spin}_{\mathbb{C}}$ -structure and Dirac operator  $\widetilde{\partial}_{\widetilde{X}_{01}}^e$ . Let  $\mathcal{P}_{+0}^e$  denote Calderon projector on  $b\widehat{X}_0^1$  defined by the invertible double construction and  $\mathcal{P}_{+1}^e$  that defined on  $bX_1$  via the invertible double construction. Since these two manifolds agree in neighborhoods of their respective boundaries, it is clear that  $\mathcal{P}_{+0}^e - \mathcal{P}_{+1}^e$  is a smoothing operator.

We can use the boundary projector  $\mathcal{R}_{+1}^{e'}$  to define a boundary condition on  $\widehat{X}_0^1$ . Because  $\text{Ind}(\widetilde{\partial}_{\widehat{X}_0^1}^e, \mathcal{R}_{+0}^e) = \text{Ind}(\partial_{X_0}^e, \mathcal{R}_{+0}^e)$ , the Agranovich–Dynin formula implies that

$$\begin{aligned} \text{R-Ind}(\mathcal{S}_0, \mathcal{S}'_1) &= -\text{Ind}(\widetilde{\partial}_{\widehat{X}_0^1}^e, \mathcal{R}_{+0}^e) + \text{Ind}(\widetilde{\partial}_{\widehat{X}_0^1}^e, \mathcal{R}_{+1}^{e'}) \\ &= -\text{Ind}(\partial_{X_0}^e, \mathcal{R}_{+0}^e) + \text{Ind}(\widetilde{\partial}_{\widehat{X}_0^1}^e, \mathcal{R}_{+1}^{e'}), \end{aligned} \quad (199)$$

and therefore

$$\begin{aligned} \text{R-Ind}(\mathcal{S}_0, \mathcal{S}'_1) + \text{Ind}(\partial_{X_0}^e, \mathcal{R}_{+0}^e) - \text{Ind}(\partial_{X_1}^e, \mathcal{R}_{+1}^{e'}) &= \\ \text{Ind}(\widetilde{\partial}_{\widehat{X}_0^1}^e, \mathcal{R}_{+1}^{e'}) - \text{Ind}(\partial_{X_1}^e, \mathcal{R}_{+1}^{e'}). \end{aligned} \quad (200)$$

The relative index formula implies that

$$\begin{aligned} \text{Ind}(\widetilde{\partial}_{\widehat{X}_0^1}^e, \mathcal{R}_{+1}^{e'}) &= \text{R-Ind}(\mathcal{P}_{+0}^e, \mathcal{R}_{+1}^{e'}) \\ \text{Ind}(\partial_{X_1}^e, \mathcal{R}_{+1}^{e'}) &= \text{R-Ind}(\mathcal{P}_{+1}^e, \mathcal{R}_{+1}^{e'}). \end{aligned} \quad (201)$$

To complete the proof we need to show that

$$\begin{aligned} \text{R-Ind}(\mathcal{P}_{+0}^e, \mathcal{R}_{+1}^{e'}) - \text{R-Ind}(\mathcal{P}_{+1}^e, \mathcal{R}_{+1}^{e'}) &= \\ \text{R-Ind}(\mathcal{P}_{+0}^e, \mathcal{P}_{+1}^e) &= \text{R-Ind}(\mathcal{P}_{+0}^e, \text{Id} - \mathcal{P}_{-1}^e). \end{aligned} \quad (202)$$

The last equality holds as  $\mathcal{P}_{+1}^e = \text{Id} - \mathcal{P}_{-1}^e$ . Applying Bojarski's theorem, we could therefore conclude that:

$$\text{R-Ind}(\mathcal{P}_{+0}^e, \text{Id} - \mathcal{P}_{-1}^e) = \text{Ind}(\widetilde{\partial}_{\widetilde{X}_{01}}^e). \quad (203)$$

The proof of (202) is essentially the same as the proof of the Agranovich–Dynin formula.



**Proposition 13.** *Let  $X_0, X_1$  be normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifolds with cylindrical ends. Suppose that a collar neighborhood of  $bX_0$  is isomorphic to a collar neighborhood of  $bX_1$ . Let  $\mathcal{R}_+^e$  be a modified  $\bar{\partial}$ -Neumann condition defined on  $bX_0 \simeq bX_1$  and  $\mathcal{P}_{+j}^e, j = 0, 1$  are the Calderon projectors defined via the invertible double construction on  $X_j, j = 0, 1$ . We have that*

$$\text{R-Ind}(\mathcal{P}_{0+}^e, \mathcal{R}_+^e) - \text{R-Ind}(\mathcal{P}_{+1}^e, \mathcal{R}_+^e) = \text{R-Ind}(\mathcal{P}_{+0}^e, \mathcal{P}_{+1}^e). \quad (204)$$

*Proof.* We give an outline for the proof. As in the proof of Theorem 8, we consider the map

$$A = \mathcal{P}_{+1}^e \mathcal{R}_+^e \mathcal{P}_{+0}^e = \mathcal{P}_{+1}^e \mathcal{T}_{1+}^{e*} \mathcal{T}_{+0}^e \mathcal{P}_{+0}^e. \quad (205)$$

Step 1 We show that  $A$  is a Fredholm map from  $L^2 \cap \text{range } \mathcal{P}_{+0}^e$  to  $\mathcal{H} \cap \text{range } \mathcal{P}_{+1}^e$ , with index

$$\text{R-Ind}(\mathcal{P}_{0+}^e, \mathcal{R}_+^e) - \text{R-Ind}(\mathcal{P}_{+1}^e, \mathcal{R}_+^e).$$

Here  $\mathcal{H}$  is the closure of  $\mathcal{C}^\infty$  with respect to the inner product defined by

$$\|u\|_{\mathcal{H}}^2 = \|u\|_{L^2}^2 + \|\mathcal{U}_{+0}^e \mathcal{U}_{+1}^{e*} u\|_{L^2}^2. \quad (206)$$

Step 2 We then observe that, as  $\mathcal{P}_{+0}^e - \mathcal{P}_{+1}^e$  is a smoothing operator, so is  $\mathcal{T}_{+0}^e - \mathcal{T}_{+1}^e$ . Hence the index of  $A$  equal to that of

$$B = \mathcal{P}_{+1}^e \mathcal{T}_{1+}^{e*} \mathcal{T}_{+1}^e \mathcal{P}_{+0}^e : L^2 \cap \text{range } \mathcal{P}_{+0}^e \rightarrow \mathcal{H} \cap \text{range } \mathcal{P}_{+1}^e. \quad (207)$$

Step 3 As before, the commutator  $[\mathcal{P}_{+1}^e, \mathcal{T}_{1+}^{e*} \mathcal{T}_{+1}^e] = 0$ , hence we can replace  $B$  with

$$B = \mathcal{P}_{+1}^e \mathcal{T}_{1+}^{e*} \mathcal{T}_{+1}^e \mathcal{P}_{+1}^e \mathcal{P}_{+0}^e, \quad (208)$$

which we think of as a composition of Fredholm maps

$$\begin{aligned} \mathcal{P}_{+1}^e \mathcal{P}_{+0}^e &: L^2 \cap \text{range } \mathcal{P}_{+0}^e \rightarrow L^2 \cap \text{range } \mathcal{P}_{+1}^e \text{ and} \\ \mathcal{W} = \mathcal{P}_{+1}^e \mathcal{T}_{1+}^{e*} \mathcal{T}_{+1}^e \mathcal{P}_{+1}^e &: L^2 \cap \text{range } \mathcal{P}_{+1}^e \rightarrow \mathcal{H} \cap \text{range } \mathcal{P}_{+1}^e. \end{aligned} \quad (209)$$

The index of the first term is  $\text{R-Ind}(\mathcal{P}_{+0}^e, \mathcal{P}_{+1}^e)$ .

Step 4 To complete the proof we need to show that  $\text{Ind}(\mathcal{W}) = 0$ . Again, this is a formally self adjoint operator, so the vanishing of the index follows from the trace formula.

Step 1 essentially follows from Theorem 16 in the Appendix. For clarity we outline the argument. To show that  $A$  is a Fredholm map, as indicated, follows easily from the commutation relations

$$\begin{aligned} (\text{Id} + K)\mathcal{P}_{+0}^c \mathcal{U}_{+0}^c &= \mathcal{U}_{+0}^c \mathcal{R}_+^c (\text{Id} + K) \\ (\text{Id} + K)\mathcal{R}_+^c \mathcal{U}_{+1}^{c*} &= \mathcal{U}_{+1}^{c*} \mathcal{P}_{+1}^c (\text{Id} + K), \end{aligned} \quad (210)$$

where we use  $K$  to denote a variety of smoothing operators. To compute its index we factor it as the composition of

$$\begin{aligned} \mathcal{R}_+^c \mathcal{P}_{+0}^c : L^2 \cap \text{range } \mathcal{P}_{+0}^c &\rightarrow \mathcal{H} \cap \text{range } \mathcal{R}_+^c \text{ and} \\ \mathcal{P}_{+1}^c \mathcal{R}_+^c : \mathcal{H} \cap \text{range } \mathcal{R}_+^c &\rightarrow \mathcal{H} \cap \text{range } \mathcal{P}_{+1}^c. \end{aligned} \quad (211)$$

Here  $\mathcal{H}$  is the closure of  $\mathcal{C}^\infty$  with respect to the inner product defined by

$$\|u\|_{\mathcal{H}}^2 = \|u\|_{L^2}^2 + \|\mathcal{U}_{+0}^c u\|_{L^2}^2, \quad (212)$$

and  $\mathcal{H}$  is the closure of  $\mathcal{C}^\infty$  with respect to the inner product defined by

$$\|u\|_{\mathcal{H}}^2 = \|u\|_{L^2}^2 + \|\mathcal{U}_{+0}^c \mathcal{U}_{+1}^{c*} u\|_{L^2}^2. \quad (213)$$

That  $\mathcal{R}_+^c \mathcal{P}_{+0}^c$  is a Fredholm map with respect to these spaces and has index

$$\text{R-Ind}(\mathcal{P}_{+0}^c, \mathcal{R}_+^c)$$

follows immediately from the results in the appendix.

We now show that the second map in (211) is Fredholm and has index

$$-\text{R-Ind}(\mathcal{P}_{+1}^c, \mathcal{R}_+^c).$$

Using the commutation relations, (210), it follows easily that this map is bounded. The commutation relations imply that the map  $\mathcal{R}_+^c \mathcal{U}_{+1}^{c*} \mathcal{P}_{+1}^c : \mathcal{H} \rightarrow \mathcal{H}$  is bounded and satisfies

$$\begin{aligned} \mathcal{P}_{+1}^c \mathcal{R}_+^c [\mathcal{R}_+^c \mathcal{U}_{+1}^{c*} \mathcal{P}_{+1}^c] &= \mathcal{P}_{+1}^c (\text{Id} - K_1) \mathcal{P}_{+1}^c \text{ and} \\ [\mathcal{R}_+^c \mathcal{U}_{+1}^{c*} \mathcal{P}_{+1}^c] \mathcal{P}_{+1}^c \mathcal{R}_+^c &= \mathcal{R}_+^c (\text{Id} - K_2) \mathcal{R}_+^c \end{aligned} \quad (214)$$

for  $K_1, K_2$  smoothing operators. This shows that  $\mathcal{P}_{+1}^c \mathcal{R}_+^c$  is Fredholm. As follows from the results in the appendix, the index of this operator is given by

$$\text{Ind}(\mathcal{P}_{+1}^c \mathcal{R}_+^c) = \text{tr } \mathcal{P}_{+1}^c K_1 \mathcal{P}_{+1}^c - \text{tr } \mathcal{R}_+^c K_2 \mathcal{R}_+^c, \quad (215)$$

where as usual, we can use the  $L^2$ -topology to compute the traces. Comparing this to (298) we see that

$$\text{Ind}(\mathcal{P}_{+1}^c \mathcal{R}_+^c) = -\text{R-Ind}(\mathcal{P}_{+1}^c, \mathcal{R}_+^c). \quad (216)$$

As  $\mathcal{P}_{+1}^e \mathcal{R}_+^e \mathcal{P}_{+0}^e$  is the composition of the Fredholm maps in (211), this completes step 1.

Step 2 is obvious as

$$A - B = \mathcal{P}_{+1}^e K \mathcal{P}_{+0}^e, \quad (217)$$

for  $K$  a smoothing operator. We now turn to step 3. Using the commutation relations in (210), we easily show that  $\mathcal{V}W$  is a Fredholm map with parametrix  $\mathcal{V} = \mathcal{P}_{+1}^e \mathcal{Q}_{+1}^e \mathcal{U}_{+1}^{e*} \mathcal{P}_{+1}^e$ . As before, if

$$\mathcal{V}W = \mathcal{P}_{+1}^e - \mathcal{P}_{+1}^e K \mathcal{P}_{+1}^e, \quad (218)$$

for  $K$  a smoothing operator, then

$$\mathcal{W}\mathcal{V} = \mathcal{P}_{+1}^e - \mathcal{P}_{+1}^e K^* \mathcal{P}_{+1}^e. \quad (219)$$

Step 4 is completed, as before, by using the trace formula

$$\text{Ind}(\mathcal{W}) = \text{tr } \mathcal{P}_{+1}^e K \mathcal{P}_{+1}^e - \text{tr } \mathcal{P}_{+1}^e K^* \mathcal{P}_{+1}^e. \quad (220)$$

Because the right hand side is a purely imaginary number, this relation implies that the index is zero. This completes the proof of the proposition.  $\square$

As noted above, the Atiyah–Weinstein conjecture is an immediate consequence of the proposition.  $\square$

## 11 Vector Bundle Coefficients

The foregoing analysis applies equally well if we consider the  $\text{Spin}_{\mathbb{C}}$ -Dirac operators acting on sections of  $E \otimes \mathcal{S}$ , where  $E \rightarrow X$  is a complex vector bundle. The results in this paper rest entirely on the properties of the principal symbols of the comparison operators,  $\mathcal{T}_+^{\text{eo}}$ . The computations of these principal symbols follow from equations (44)–(48). These, in turn, are consequences of the geometric statements in equations (41)–(43), as well as (33). These geometric normalizations continue to be possible if we twist the spin bundle with a Hermitian vector bundle.

Let  $\nabla^{\mathcal{S}}$  denote the compatible connection on the  $\text{Spin}_{\mathbb{C}}$ -bundle and  $\nabla^E$ , an Hermitian connection on  $E$ . The connection

$$\nabla^{\mathcal{S} \otimes E} = \nabla^{\mathcal{S}} \otimes \text{Id}_E + \text{Id}_{\mathcal{S}} \otimes \nabla^E, \quad (221)$$

is a compatible connection on  $\mathcal{S} \otimes E$ . We fix a point  $p \in bX$ , and let  $(x_1, \dots, x_{2n})$  be normal coordinates at  $p$ . Let  $\{\sigma_J\}$  be a local frame field for  $\mathcal{S}$ , satisfying (43), and  $\{e_l\}$  a local framing for  $E$  with

$$\nabla^E e_l = O(|x|). \quad (222)$$

In this case  $\{\sigma_J \otimes e_l\}$  is a local framing for  $\mathcal{F} \otimes E$  that satisfies

$$\nabla^{\mathcal{F} \otimes E} \sigma_J \otimes e_l = O(|x|). \quad (223)$$

Let  $\tilde{\partial}_E$  be the  $\text{Spin}_\mathbb{C}$ -Dirac operator acting on sections of  $\mathcal{F} \otimes E$ . Because the coordinates are normal, (223) implies that

$$\begin{aligned} \tilde{\partial}_E \sum_{J,l} a_{J,l} \sigma_J \otimes e_l &= \frac{1}{2} \sum_{J,l} \sum_{j=1}^{2n} \mathbf{c}(dx_j) \cdot [(\partial_{x_j} a_{J,l}) \sigma_J \otimes e_l + a_{J,l} \nabla^{\mathcal{F} \otimes E} \sigma_J \otimes e_l] \\ &= \sum_{J,l} \sum_{j=1}^{2n} [\bar{\partial} + \bar{\partial}^*]_{\mathbb{C}^n} [a_{J,l} d\bar{z}^J \otimes e_l] + \mathfrak{D}_1(|x|) + \mathfrak{D}_0(|x|). \end{aligned} \quad (224)$$

It is a general result about Dirac operators that  $[\tilde{\partial}^E]^2 = \Delta + R$ , where  $\Delta$  is the Laplace operator and  $R$  is an operator of order zero. We compute the action of the Laplace operator in the normal coordinates at  $p$  :

$$\begin{aligned} \Delta \sum_{J,l} a_{J,l} \sigma_J \otimes e_l &= \\ & \sum_{j,k=1}^{2n} (\delta_{jk} + O(|x|^2)) [\nabla_{\partial_{x_j}}^{\mathcal{F} \otimes E} \nabla_{\partial_{x_k}}^{\mathcal{F} \otimes E} a_{J,l} \sigma_J \otimes e_l - \nabla_{\nabla_{\partial_{x_j}}^{\mathcal{F} \otimes E} \partial_{x_k}} a_{J,l} \sigma_J \otimes e_l] \\ &= \sum_{J,l} \sum_{j=1}^{2n} \partial_{x_j}^2 a_{J,l} \sigma_J \otimes e_l + \mathfrak{D}_2(|x|^2) + 2\partial_{x_j} a_{J,l} \nabla_{\partial_{x_j}}^{\mathcal{F} \otimes E} \sigma_J \otimes e_l + \\ & \quad \mathfrak{D}_1(|x|) + \mathfrak{D}_0(1). \end{aligned} \quad (225)$$

Using once again that  $\nabla_{\partial_{x_j}}^{\mathcal{F} \otimes E} \sigma_J \otimes e_l = O(|x|)$  we see that

$$\Delta \sum_{J,l} a_{J,l} \sigma_J \otimes e_l = \sum_{J,l} \sum_{j=1}^{2n} \partial_{x_j}^2 a_{J,l} \sigma_J \otimes e_l + \mathfrak{D}_2(|x|^2) + \mathfrak{D}_1(|x|) + \mathfrak{D}_0(1). \quad (226)$$

These formulæ demonstrate that the necessary symbolic conditions are satisfied by  $\tilde{\partial}_E$  and  $\tilde{\partial}_E^2$ . As described in [10], complex vector bundle coefficients are easily incorporated into the generalized Szegő projector formalism, and therefore the results proved in the previous sections apply equally well when vector bundle coefficients are included. We do not wish to exhaustively enumerate these generalizations, but simply list a few of these results.

We let  $\mathcal{P}_{\pm E}^{\text{co}}$  denote the Calderon projectors with bundle coefficients and  $\mathcal{R}_{+E}^{\text{co}}$  the modified  $\bar{\partial}$ -Neumann conditions defined by a generalized Szegő projector  $\mathcal{S}_E$ . As before we set

$$\mathcal{T}_{+E}^{\text{co}} = \mathcal{R}_{+E}^{\text{co}} \mathcal{P}_{+E}^{\text{co}} + (\text{Id} - \mathcal{R}_{+E}^{\text{co}})(\text{Id} - \mathcal{P}_{+E}^{\text{co}}). \quad (227)$$

The basic analytic result is:

**Theorem 10.** *Let  $(X, J, g, \rho)$  be a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold,  $(E, h) \rightarrow X$  a Hermitian vector bundle and  $\mathcal{S}_E$  a generalized Szegő projector acting on sections of  $E$ . The principal symbol of  $\mathcal{S}_E$  is defined by a compatible deformation of the almost complex structure on  $H$  induced by the embedding of  $bX$  as the boundary of  $X$ . Then the operators  $\mathcal{T}_{+E}^{\text{co}}$  are elliptic, in the extended Heisenberg calculus, with parametrices having Heisenberg orders*

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \quad (228)$$

As before, this result shows that the graph closures of  $(\bar{\partial}_{+E}^{\text{co}}, \mathcal{R}_{+E}^{\text{co}})$  are Fredholm and

$$(\bar{\partial}_{+E}^{\text{co}}, \mathcal{R}_{+E}^{\text{co}})^* = \overline{(\bar{\partial}_{+E}^{\text{oc}}, \mathcal{R}_{+E}^{\text{oc}})}. \quad (229)$$

Moreover, the  $(\mathcal{P}_{+E}^{\text{co}}, \mathcal{R}_{+E}^{\text{co}})$  are tame Fredholm pairs with

$$\text{Ind}(\bar{\partial}_{+E}^{\text{co}}, \mathcal{R}_{+E}^{\text{co}}) = \mathbf{R}\text{-Ind}(\mathcal{P}_{+E}^{\text{co}}, \mathcal{R}_{+E}^{\text{co}}). \quad (230)$$

We have the Agranovich-Dynin Formula:

**Theorem 11.** *Let  $(X, J, h, \rho)$  be a normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifold, and  $E \rightarrow X$  a Hermitian vector bundle. Let  $\mathcal{S}_{E_1}$  and  $\mathcal{S}_{E_2}$  be two generalized Szegő projections and  $\mathcal{R}_{+E_1}^{\text{e}}, \mathcal{R}_{+E_2}^{\text{e}}$  the modified  $\bar{\partial}$ -Neumann conditions they define, then*

$$\mathbf{R}\text{-Ind}(\mathcal{S}_{E_1}, \mathcal{S}_{E_2}) = \text{Ind}(\bar{\partial}_{+E}^{\text{e}}, \mathcal{R}_{+E_2}^{\text{e}}) - \text{Ind}(\bar{\partial}_{+E}^{\text{e}}, \mathcal{R}_{+E_1}^{\text{e}}). \quad (231)$$

Finally, we have the Atiyah-Weinstein conjecture for this case. Over  $X_0, X_1$  we have bundles  $E_0 \rightarrow X_0, E_1 \rightarrow X_1$ . If  $\Phi$  denotes the extension of the contact diffeomorphism to a neighborhood of  $bX_j$ , then we need to assume that, in the collar neighborhood,  $\Phi^*E_1$  is isomorphic to  $E_0$ , via an bundle map  $\Psi$ . Altogether we get a vector bundle  $E \rightarrow \tilde{X}_{01}$ , which may depend on the choice of  $\Psi$ .

**Theorem 12.** *Let  $(X_k, j_k, g_k, \rho_k), k = 0, 1$  be normalized strictly pseudoconvex  $\text{Spin}_{\mathbb{C}}$ -manifolds, with  $E_j \rightarrow X_j, j = 0, 1$  Hermitian vector bundles. Suppose that  $\phi : bX_1 \rightarrow bX_0$  is a co-orientation preserving contact diffeomorphism, and*

$\Psi : E_0 \rightarrow \Phi^* E_1$ , is a bundle equivalence, covering  $\phi$ . Suppose that  $\mathcal{S}_{E_0}, \mathcal{S}_{E_1}$  are generalized Szegő projections on  $bX_0, bX_1$ , respectively, which define modified  $\bar{\partial}$ -Neumann conditions  $\mathcal{R}_{+E_j}^e$  on  $X_j$ . If we let  $\mathcal{S}'_{E_1} = \Psi^{-1} \mathcal{S}_{E_1} \Psi$ ,  $\tilde{X}_{01}$  as defined in (187), and  $E$  the bundle over  $\tilde{X}_{01}$  defined by gluing  $E_0$  to  $\Phi^* E_1$  via  $\Psi$ , then we have

$$\mathbf{R}\text{-Ind}(\mathcal{S}_{E_0}, \mathcal{S}'_{E_1}) = \text{Ind}(\bar{\partial}_{\tilde{X}_{01}E}^e) - \text{Ind}(\bar{\partial}_{X_0E_0}^e, \mathcal{R}_{+E_0}^e) + \text{Ind}(\bar{\partial}_{X_1E_1}^e, \mathcal{R}_{+E_1}^e). \quad (232)$$

## 12 The Relative Index Conjecture

In [7] we introduced the relative index for pairs of embeddable CR-structures on 3-dimensional manifolds with the same underlying contact structure. In those papers we used the opposite convention to that employed in the current series of papers and therefore the relative index defined there is minus that defined here. With the present convention, Proposition 8.1 in [7] implies that if  $\mathcal{S}_0$  is the Szegő projector defined by a “reference” embeddable CR-structure  ${}^0T^{0,1}Y$  on  $(Y, H)$ , and  $\mathcal{S}_1$  is the Szegő projector defined by a *sufficiently small*, embeddable deformation,  ${}^1T^{0,1}Y$ , of this CR-structure, then

$$\mathbf{R}\text{-Ind}(\mathcal{S}_0, \mathcal{S}_1) \geq 0. \quad (233)$$

In [7] we showed that, for  $n \in \mathbb{N}$ , the set of embeddable deformations of  ${}^0T^{0,1}Y$  that satisfy  $\mathbf{R}\text{-Ind}(\mathcal{S}_0, \mathcal{S}_1) \leq n$  is closed in the  $\mathcal{C}^\infty$ -topology. This motivated our relative index conjecture, which asserts (with our current sign convention) that  $\mathcal{S}_1 \rightarrow \mathbf{R}\text{-Ind}(\mathcal{S}_0, \mathcal{S}_1)$  is bounded from above, among sufficiently small, embeddable deformations,  ${}^1T^{0,1}Y$ , of  ${}^0T^{0,1}Y$ . In [8] we establish this conjecture for CR-structures that bound pseudoconcave manifolds  $X_-$ , satisfying either

$$\begin{aligned} H_c^2(X_-; \Theta \otimes [-Z]) &= 0 \text{ or,} \\ H_c^2(X_-; \Theta) &= 0 \text{ and } H^1(Z; N_Z) = 0, \end{aligned} \quad (234)$$

here  $Z \subset\subset X_-$  is a smooth, compact holomorphic curve with positive normal bundle, and  $\Theta$  is the tangent sheaf of  $X_-$ .

Suppose that  $j_0, j_1$  define CR-structures on a 3-dimensional contact manifold  $(Y, H)$ , which bound strictly pseudoconvex complex manifolds  $(X_0, J_0), (X_1, J_1)$ . If  $\mathcal{S}_0, \mathcal{S}_1$  are the classical Szegő projectors defined by  $j_0, j_1$ , respectively, then formula (190) gives:

$$\mathbf{R}\text{-Ind}(\mathcal{S}_0, \mathcal{S}_1) = \text{Ind}(\bar{\partial}_{\tilde{X}_{01}}^e) + \dim H^{0,1}(X_0, J_0) - \dim H^{0,1}(X_1, J_1). \quad (235)$$

The Atiyah-Singer index theorem provides a cohomological formula for  $\text{Ind}(\bar{\partial}_{\tilde{X}_{01}}^e)$  :

$$\text{Ind}(\bar{\partial}_{\tilde{X}_{01}}^e) = \frac{c_1^2(\mathcal{F}^e)[\tilde{X}_{01}] - \text{sig}[\tilde{X}_{01}]}{8}. \quad (236)$$

Here  $\mathcal{F}^e$  is the  $+$ -spinor bundle defined on  $\tilde{X}_{01}$  by the extended invertible double construction, and  $\text{sig}[M]$  is the signature of the oriented 4-manifold  $M$ .

In [31] a general formula is given relating the characteristic numbers on a compact  $\text{Spin}_C$  4-manifold,  $M$ :

$$c_2(\mathcal{F}^e)[M] = \frac{c_1^2(\mathcal{F}^e)[M] - 3 \text{sig}[M] - 2\chi[M]}{4}, \quad (237)$$

Here  $[M]$  denotes the fundamental class of the oriented manifold  $M$ . This formula is stated as an exercise, whose solution we briefly explain: One first shows that, if  $L$  is a Hermitian line bundle, then the Chern-Weil representative of  $c_2(\mathcal{F}^e \otimes L) - c_1^2(\mathcal{F}^e \otimes L)/4$  does not depend on  $L$ . Hence we can *locally* represent  $\mathcal{F}^e \otimes L$  as  $\mathcal{F}_0^e$ , where  $\mathcal{F}_0^e$  is the  $+$ -spinor bundle coming from a *locally defined* Spin-bundle. Using the expression for the curvature of  $\mathcal{F}^e \otimes L$  that arises from such a local representation, we show that the Chern-Weil representative of  $c_2(\mathcal{F}^e) - c_1^2(\mathcal{F}^e)/4$  agrees with that of  $-(p_1(M) + 2e(M))$ , where  $p_1$  is the first Pontryagin class and  $e$  is the Euler class.

Putting (237) into (236) gives:

$$\text{Ind}(\tilde{\partial}_{\tilde{X}_{01}}^e) = \frac{2c_2(\mathcal{F}^e)[\tilde{X}_{01}] + \text{sig}[\tilde{X}_{01}] + \chi[\tilde{X}_{01}]}{4}. \quad (238)$$

Note that on  $X_0$ ,  $\mathcal{F}^e \simeq \Lambda^{0,0}X_0 \oplus \Lambda^{0,2}X_0$ , and on  $X_1$ ,  $\mathcal{F}^e \simeq \Lambda^{0,1}X_1$ . Since  $\mathcal{F}^e$  has a global section over  $X_0$  and over the part of  $\tilde{X}_{01}$  coming from the neck joining  $X_0$  to  $X_1$ , and a neighborhood of the boundary of  $X_1$ , we can choose a metric for  $\mathcal{F}^e$  so that the Chern-Weil representative of  $c_2(\mathcal{F}^e)$  is supported in the interior of  $X_1$ . Over  $X_1$

$$\mathcal{F}^e = \Lambda^{0,1}X_1 = [T^{0,1}X_1]^* \simeq T^{1,0}X_1,$$

and therefore it follows that  $c_2(\mathcal{F}^e)|_{X_1} = e(X_1)$ . Recalling that the orientation of  $X_1$  is reversed in  $\tilde{X}_{01}$ , and using the additivity of the signature and Euler characteristic we obtain

$$\begin{aligned} \text{Ind}(\tilde{\partial}_{\tilde{X}_{01}}^e) &= \frac{2e(X_1)[-X_1] + \text{sig}[X_0] - \text{sig}[X_1] + \chi[X_0] + \chi[X_1]}{4} \\ &= \frac{\text{sig}[X_0] - \text{sig}[X_1] + \chi[X_0] - \chi[X_1]}{4}. \end{aligned} \quad (239)$$

Using this formula in (235) completes the proof of the following theorem.

**Theorem 13.** *Let  $(Y, H)$  be a compact 3-dimensional co-oriented, contact manifold, and let  $j_0, j_1$  be CR-structures with underlying plane field  $H$ . Suppose that  $(X_0, J_0), (X_1, J_1)$  are strictly pseudoconvex complex manifolds with boundary*

$(Y, H, j_0), (Y, H, j_1)$ , respectively. If  $\mathcal{P}_0, \mathcal{P}_1$  are the classical Szegő projectors defined by these CR-structures then

$$\mathbf{R}\text{-Ind}(\mathcal{P}_0, \mathcal{P}_1) = \dim H^{0,1}(X_0, J_0) - \dim H^{0,1}(X_1, J_1) + \frac{\text{sig}[X_0] - \text{sig}[X_1] + \chi[X_0] - \chi[X_1]}{4}. \quad (240)$$

*Remark 10.* If  $(M, \mathcal{F}^c, \mathcal{F}^o)$  is a compact  $\text{Spin}_{\mathbb{C}}$  4-manifold, then

$$d_{\text{SW}}(M) = \frac{c_1^2(\mathcal{F}^c)[M] - 3 \text{sig}[M] - 2\chi[M]}{4} \quad (241)$$

is the formal dimension of the moduli space of solutions to the Seiberg-Witten equations, see [26]. Our calculations show that for manifolds  $\tilde{X}_{01} \simeq X_0 \amalg \overline{X_1}$ , with  $\text{Spin}_{\mathbb{C}}$ -structure defined by the invertible double construction,

$$d_{\text{SW}}(\tilde{X}_{01}) = -\chi[X_1]. \quad (242)$$

Reversing the orientation of  $\tilde{X}_{01}$  gives  $\tilde{X}_{10}$  and interchanges  $\mathcal{F}^c$  with  $\mathcal{F}^o$ , so that

$$d_{\text{SW}}(\tilde{X}_{10}) = -\chi[X_0]. \quad (243)$$

Note finally that, under the hypotheses of Theorem 13, equation (240) implies that

$$\frac{\text{sig}[X_0] - \text{sig}[X_1] + \chi[X_0] - \chi[X_1]}{4} \in \mathbb{Z}. \quad (244)$$

Formula (240) has a direct bearing on the relative index conjecture.

**Corollary 8.** *Let  $(Y, H)$  be a compact 3-dimensional, co-oriented contact manifold. Suppose that among Stein manifolds  $(X, J)$  with pseudoconvex boundary  $(Y, H)$  the signature,  $\text{sig}(X)$  and Euler characteristic  $\chi(X)$  assume only finite many values, then, among embeddable deformations,  $\mathcal{P}_1$  of a given embeddable reference CR-structure,  $\mathcal{P}_0$ , the relative index  $\mathbf{R}\text{-Ind}(\mathcal{P}_0, \mathcal{P}_1)$  is bounded from above.*

*Proof.* Suppose that  $X$  is a strictly pseudoconvex complex manifold with boundary  $(Y, H)$ . We can assume that  $X$  is minimal, i.e. all inessential compact varieties are blown down. Bogomolov and DeOliveira proved that one can deform the complex structure on  $X$  to obtain a Stein manifold, see [1]. Such a deformation does not change the topological invariants  $\text{sig}[X]$ ,  $\chi[X]$ . Thus among minimal strictly pseudoconvex complex manifolds  $X$ , with boundary  $(Y, H)$ , the numbers  $\text{sig}[X]$ ,  $\chi[X]$  assume only finitely many values, and the corollary follows from (240).  $\square$



If  $X_0$  is diffeomorphic to  $X_1$ , then (240) implies that

$$\mathbf{R}\text{-Ind}(\mathcal{S}_0, \mathcal{S}_1) = \dim H^{0,1}(X_0, J_0) - \dim H^{0,1}(X_1, J_1). \quad (245)$$

For such deformations of the CR-structure we see that

$$\mathbf{R}\text{-Ind}(\mathcal{S}_0, \mathcal{S}_1) \leq \dim H^{0,1}(X_0, J_0). \quad (246)$$

This becomes an equality if  $(X, J_1)$  is a Stein manifold. It says that a singular Stein surface, with  $H^{0,1}(X_0, J_0) \neq 0$ , has a larger algebra of holomorphic functions than its smooth (Stein) deformations. If both  $(X_0, J_0)$  and  $(X_1, J_1)$  are Stein manifolds, then the relative index is zero.

For some time it was believed that a given compact, contact 3-manifold should have, at most, finitely many diffeomorphism classes of Stein fillings. Indeed this expectation has been established for some classes of contact 3-manifolds. Eliashberg proved that the 3-sphere, with its standard contact structure, has a unique Stein filling. McDuff extended this to the lens spaces  $L(p, 1)$  for  $p \neq 4$ , with contact structure induced from  $S^3$ . Stipsicz showed the uniqueness of the Stein filling for the only fillable contact structure on the 3-torus, see [34]. Lisca showed that other lens spaces have finitely many diffeomorphism classes of fillings, see [25]. Ohta and Ono proved the finiteness statement for simple and simple elliptic isolated singularities, see [28, 29]. For most of these cases, the local form of the relative index conjecture was proved by other means in [8].

Unfortunately, I. Smith and Ozbagci and Stipsicz have produced examples of compact, contact 3-manifolds that arise as the boundaries of infinitely many diffeomorphically distinct Stein manifolds, see [30]. In the examples of Ozbagci and Stipsicz, the signatures and Euler characteristics of the 4-manifolds are all equal. In a later paper Stipsicz conjectured that, given a 3-dimensional compact, contact manifold,  $(Y, H)$ , the signatures and Euler characteristics for Stein manifolds with boundary  $(Y, H)$  should assume only finitely many distinct values, see [35]. Using Corollary 8, this conjecture would clearly imply the strengthened form of the relative index conjecture.

Stipsicz has established his conjecture in a variety of cases, among them, the boundaries of circle bundles in line bundles of degree  $n$  over surfaces of genus  $g$ , provided

$$|n| > 2g - 2. \quad (247)$$

This allows us to extend the result proved in [8], where the relative index conjecture was established for circle bundles under the assumption that  $|n| \geq 4g - 3$ . Stipsicz also proved his conjecture for the Seifert fibered 3-manifold  $\Sigma(2, 3, 11)$ , and so the relative index conjecture holds in this case as well.

Even if the Stipsicz conjecture is false, it would not necessarily invalidate our conjecture. The relative index conjecture is a conjecture for *sufficiently small* embeddable deformations of the CR-structure. It could well be that as one moves through the infinitely many diffeomorphism types, arising among the fillings, the deformations of the CR-structure on the boundary become large. Conversely, when such bounds do exist, a global upper bound for the relative index follows from (240) and Corollary 8.

## 13 Further Interesting Special Cases

We consider some other special cases of the index formulæ proved above.

### 13.1 Co-ball Bundles

First we consider the original Atiyah-Weinstein conjecture which concerns pairs of co-ball bundles,  $X_0 = B^*M_0$ ,  $X_1 = B^*M_1$  and a contact diffeomorphism of their boundaries,  $\phi : S^*M_1 \rightarrow S^*M_0$ . In this case there is a complex structure on each of the manifolds, well defined up to isotopy. In these structures,  $X_0, X_1$  are Stein manifolds and therefore

$$\chi'_0(X_0) = \chi'_0(X_1) = 0. \quad (248)$$

If  $\mathcal{S}_0, \mathcal{S}_1$  are the classical Szegő projectors defined by the complex structures on  $X_0, X_1$ , respectively, then equations (190) and (248) imply that, with  $\mathcal{S}'_1 = [\phi^{-1}]^*\mathcal{S}_1\phi^*$ ,

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}'_1) = \text{Ind}(\bar{\partial}_{X_\phi}^e), \quad (249)$$

where

$$X_\phi = B^*M_0 \sqcup_\phi \overline{B^*M_1}. \quad (250)$$

As noted in the introduction, the mapping  $\phi$  and the complex structures on the ball bundles define a class of elliptic Fourier integral operators. Let  $F^\phi : \mathcal{C}^\infty(M_0) \rightarrow \mathcal{C}^\infty(M_1)$  be an element of this class. From the definition of  $F^\phi$  it is clear that its index equals  $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}'_1)$ , and therefore

$$\text{Ind}(F^\phi) = \text{Ind}(\bar{\partial}_{X_\phi}^e). \quad (251)$$

Let  $\Phi$  denote the homogeneous extension of  $\phi$  to  $B^*M_1 \setminus \{0\}$ . If  $\Phi$  extends smoothly across the zero section, i.e.,  $\phi$  is defined as the differential of a diffeomorphism  $f : M_0 \rightarrow M_1$ , then the glued space  $X_\phi$  is essentially an invertible double. Hence  $\text{Ind}(\bar{\partial}_{X_\phi}^e) = 0$ , and we have:

**Proposition 14.** *If  $\Phi$  extends smoothly across the zero section of  $B^*M_1$ , then  $\text{Ind } F^\phi = 0$ .*

The case of a pair of co-ball bundles is treated in [23], where equivalent results are proved.

In the 3-dimensional case, we can use formula (239) to obtain:

$$\text{Ind}(F^\phi) = \frac{\text{sig}[X_0] - \text{sig}[X_1] + \chi[X_0] - \chi[X_1]}{4}, \quad (252)$$

where  $X_0 = B^*M_0$  and  $X_1 = B^*M_1$ . Note that  $X_j$  retracts onto  $M_j$ . Hence  $H_2(X_j, \mathbb{Z})$  is one dimensional, and generated by the zero section,  $[M_j]$ . The self intersection of the zero section in a vector bundle equals the Euler characteristic of the bundle. In this case it is clear that, for  $j = 0, 1$ ,

$$\chi[X_j] = \chi[M_j] \text{ and } \text{sig}[X_j] = \begin{cases} -1 & \text{if } \chi[M_j] < 0 \\ 0 & \text{if } \chi[M_j] = 0 \\ 1 & \text{if } \chi[M_j] > 0. \end{cases} \quad (253)$$

Using (252) and (253) we easy prove that, in the 3-dimensional case,  $\text{Ind}(F^\phi)$  vanishes.

**Theorem 14.** *Let  $M_0, M_1$  be compact 2-manifolds and suppose that  $\phi$  is a co-orientation preserving contact diffeomorphism,  $\phi : S^*M_1 \rightarrow S^*M_0$ , then*

$$\text{Ind}(F^\phi) = 0.$$

*Proof.* The theorem follows from the formulæ above and the following lemma:

**Lemma 10.** *If  $Y$  is the co-sphere bundle of an oriented compact surface,  $M$ , then*

$$H_1(M; \mathbb{Q}) \simeq H_1(Y; \mathbb{Q}). \quad (254)$$

*Proof of lemma.* The Leray spectral sequence implies that there is a short exact sequence:

$$0 \longrightarrow \mathbb{Z}/\chi[M]\mathbb{Z} \longrightarrow H_1(Y; \mathbb{Z}) \longrightarrow H_1(M; \mathbb{Z}) \longrightarrow 0. \quad (255)$$

Taking tensor product with  $\mathbb{Q}$  leaves the sequence exact and proves the lemma.  $\square$

The theorem follows from the lemma as it implies that  $\chi[M_0] = \chi[M_1]$ .  $\square$

*Remark 11.* A similar vanishing theorem in dimension 3 is proved in [12] for  $\phi : Y \rightarrow Y$  a contact self map.

### 13.2 The Atiyah-Singer Index Theorem

Let  $M$  be a compact manifold without boundary, and let  $E, F$  be complex vector bundles over  $M$ . Let  $P$  be an elliptic pseudodifferential operator of order zero,  $P : \mathcal{C}^\infty(M; E) \rightarrow \mathcal{C}^\infty(M; F)$ . We can use our theorem to give an analytic proof of the K-theoretic step in the original proof of the Atiyah-Singer, which states that the index of  $P$  equals that of a Dirac operator on the glued space:

$$\mathbb{T}M = B^*M \amalg_{S^*M} \overline{B^*M}. \quad (256)$$

Because  $B_\epsilon^*M$  is a Stein manifold, Oka's principle implies that the lifted bundles  $\pi^*E, \pi^*F$  have well defined complex structures. Let  $G_{b\epsilon}^E, G_{b\epsilon}^F$  be the maps from  $\mathcal{O}_b(B_\epsilon^*M; E), \mathcal{O}_b(B_\epsilon^*M; F)$  to  $\mathcal{C}^\infty(M; E), \mathcal{C}^\infty(M; F)$ , respectively. defined by pushforward. As before these maps are isomorphisms for small enough  $\epsilon$ .

Let  $\mathcal{S}_\epsilon^E, \mathcal{S}_\epsilon^F$  be classical Szegő projectors onto the boundary values of holomorphic sections of  $\pi^*E, \pi^*F$ , respectively. If

$$\sigma_P \in \mathcal{C}^\infty(S^*M; \text{Hom}(\pi^*E, \pi^*F))$$

is the principal symbol of  $P$ , then, for sufficiently small  $\epsilon > 0$ , the composition,

$$F_P S = G_{b\epsilon}^F \mathcal{S}_\epsilon^F \sigma_P \mathcal{S}_\epsilon^E [G_{b\epsilon}^E]^{-1} S \quad (257)$$

is a pseudodifferential operator with principal symbol  $\sigma_P$ , see [4]. The operators  $G_{b\epsilon}^F, G_{b\epsilon}^E$  are invertible and therefore

$$\text{Ind}(P) = \text{Ind}(F_P) = \text{Ind}(\mathcal{S}_\epsilon^F \sigma_P \mathcal{S}_\epsilon^E), \quad (258)$$

where the last term in (258) is the index of a Toeplitz operator.

The Toeplitz index in (258) is easily seen to equal the relative index

$$\text{Ind}(\mathcal{S}_\epsilon^F \sigma_P \mathcal{S}_\epsilon^E) = \text{R-Ind}(\mathcal{S}_\epsilon^E, [\sigma_P]^{-1} \mathcal{S}_\epsilon^F \sigma_P).$$

This relative index is computed in Theorem 12. Equation (137) in [9] and the fact that  $B_\epsilon^*M$  is a Stein manifold imply that the boundary terms in (232) vanish, and therefore

$$\text{Ind}(P) = \text{Ind}(\tilde{\mathcal{D}}_{\mathbb{T}M, V_P}^c). \quad (259)$$

Here  $V_P$  is the bundle obtained by gluing  $\pi^*E$  to  $\pi^*F$  along  $S^*M$  via the symbol of  $P$ . This assertion is an important step, proved using K-theory, in the original proof of the Atiyah-Singer theorem. The relative index formalism in this paper, along with results from [9] provide a completely analytic proof of this statement.

### 13.3 Higher Dimensional Complex Manifolds

Suppose that  $(X, J_0)$  is a strictly pseudoconvex complex  $n$ -dimensional manifold, and  $J_1$  is a deformation of the complex structure on  $X$  that is again strictly pseudoconvex. For our purposes it is sufficient if the deformation takes place through a smooth family,  $\{J_s : s \in [0, 1]\}$ , of strictly pseudoconvex almost complex structures. Let  $\tilde{X}_{01}$  denote the  $\text{Spin}_{\mathbb{C}}$ -manifold obtained by gluing  $(X, J_0)$  to  $(X, J_1)$ , via the extended invertible double construction. Because  $J_0$  is homotopic to  $J_1$ , it follows that the space  $\tilde{X}_{01}$  is isotopic, as a  $\text{Spin}_{\mathbb{C}}$ -manifold, to the invertible double,  $\tilde{X}_0$ , of  $(X, J_0)$ . Using the results of Chapter 9 in [3] we conclude that

$$\text{Ind}(\bar{\partial}_{\tilde{X}_{01}}^c) = \text{Ind}(\bar{\partial}_{\tilde{X}_0}^c) = 0. \quad (260)$$

In this case, (188) implies that

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = \chi'_0(X, J_1) - \chi'_0(X, J_0). \quad (261)$$

Thus for CR-structures that can be obtained by deformation of the complex structure through almost complex structures on a fixed manifold, the relative index is simply the change in the renormalized holomorphic Euler characteristic. The relative index is again non-negative for small integrable deformations. Hence, if the deformation arises from a deformation of the almost complex structure on  $X$ , then we get the semi-continuity result for the renormalized Euler characteristic:

$$\chi'_0(X, J_1) \geq \chi'_0(X, J_0). \quad (262)$$

## 14 Appendix A: Tame Fredholm Pairs

In this appendix we present a generalization of the theory of Fredholm pairs and the index theory for such pairs. We give this discussion in a fairly general functional analytic setting. We suppose that we have a nested family of separable Hilbert spaces  $(H_s, \|\cdot\|_s)$ , labeled by  $s \in \mathbb{R}$ . If  $s < t$ , then  $H_t \subset H_s$  and

$$\|x\|_s \leq \|x\|_t \text{ for all } x \in H_t. \quad (263)$$

The intersection

$$H_\infty = \bigcap_{s=-\infty}^{\infty} H_s$$

is assumed to be dense in  $H_s$  for all  $s \in \mathbb{R}$ . We also define

$$H_{-\infty} = \bigcup_{s=-\infty}^{\infty} H_s.$$

The inner product on  $H_0$  is assumed to satisfy the generalized Hölder inequality, for all  $x, y \in H_\infty$  we have:

$$|\langle x, y \rangle| \leq \|x\|_s \|y\|_{-s}. \quad (264)$$

The fact that  $H_\infty$  is dense in each  $H_s$  and this estimate implies that  $\langle \cdot, \cdot \rangle$  extends to define a non-degenerate pairing  $H_s \times H_{-s} \rightarrow \mathbb{C}$ , which identifies  $H'_s \simeq H_{-s}$ : The density shows that  $H_{-s} \subset H'_s$  for every  $s \in \mathbb{R}$ . Applying this to  $-s$  and taking duals shows that  $H'_s \subset H_{-s}$  as well, hence they are equal.

We consider several classes of operators generalizing notions from the theory of pseudodifferential operators.

**Definition 2.** A tame operator  $T$  is an operator defined on  $H_{-\infty}$  for which there is a fixed  $m \in \mathbb{R}$  such that, for all  $s \in \mathbb{R}$ ,

$$T H_s \subset H_{s-m}, \quad (265)$$

and the map  $T : H_s \rightarrow H_{s-m}$  is bounded. In this case we say that  $T$  has order  $m$ .

For  $x, y$  in  $H_\infty$  we define the *formal adjoint*,  $T^*$  of the tame operator  $T$  by duality:

$$\langle Tx, y \rangle = \langle x, T^*y \rangle. \quad (266)$$

In fact, this extends by continuity to  $x \in H_\infty$  and  $y \in H_{-\infty}$  or  $x \in H_{-\infty}$  and  $y \in H_\infty$ . The definition of tameness implies that  $Tx \in H_0$ , for  $x \in H_\infty$ ; so this notion of adjoint is consistent with the  $L^2$ -adjoint. In this appendix the notation  $T^*$  always refers to the formal adjoint.

**Lemma 11.** *If  $T$  is a tame operator of order  $m$ , then its formal adjoint  $T^*$  is as well.*

*Proof.* We use the fact that  $H'_s \simeq H_{-s}$ . Let  $x, y \in H_\infty$  and fix a value of  $s$ . For  $x, y \in H_\infty$  we have  $\langle Tx, y \rangle = \langle x, T^*y \rangle$ . The generalized Cauchy-Schwarz inequality implies that

$$|\langle x, T^*y \rangle| \leq \|Tx\|_{-s} \|y\|_s \leq C \|x\|_{-s+m} \|y\|_s. \quad (267)$$

This inequality implies that for  $y \in H_\infty$  we have the estimate

$$\|T^*y\|_{s-m} \leq C \|y\|_s, \quad (268)$$

as  $H_\infty$  is dense in  $H_s$  this proves the lemma.  $\square$

It is clear that, under composition, the set of tame operators defines a star algebra.

**Definition 3.** A tame operator  $K$  that maps  $H_{-\infty}$  to  $H_{\infty}$  is called a smoothing operator. For any  $s \in \mathbb{R}$ , we suppose that, when acting on  $H_s$ , a smoothing operator is a trace class operator.

A tame operator  $K$  is smoothing if and only if it is a tame operator of order  $m$  for every  $m \in (0, -\infty)$ . It is clear that the class of smoothing operators is closed under adjoints and defines a two sided ideal in the algebra of tame operators.

**Definition 4.** An tame operator  $T$  is tamely elliptic if there is a tame operator  $U$  so that

$$TU = \text{Id} - K_1, \quad UT = \text{Id} - K_2, \quad (269)$$

where  $K_1, K_2$  are smoothing operators. The operator  $U$  is called a parametrix for  $T$ .

As the smoothing operators are closed under taking adjoints, if  $T$  is tamely elliptic, then so is  $T^*$ . If  $T$  is invertible in any reasonable sense and tamely elliptic, then it follows easily that  $U - T^{-1}$  is a smoothing operator.

If  $T$  is a tame operator of non-positive order, then the operator  $\text{Id} + T^*T$  is self adjoint and invertible on  $H_0$ . We make the following additional assumption: if  $T$  has non-positive order then  $\text{Id} + T^*T$  is tamely elliptic. This is true of any operator calculus with a good symbol map. Tame smoothing operators are compact operators and therefore, acting on  $H_{-\infty}$ , an elliptic operator has a finite dimensional kernel contained in  $H_{\infty}$ . If  $T$  is elliptic then so is  $T^*$  and it also has a finite dimensional kernel, which is contained in  $H_{-\infty}$ . We define the tame index of a tamely elliptic operator to be

$$\text{t-Ind}(T) = \dim \ker T - \dim \ker T^*. \quad (270)$$

We define the  $H_0$ -domain of tame operator  $T$  to be the graph closure, in the  $H_0$ -norm, of  $T$  acting on  $H_{\infty}$ .

$$\text{Dom}_0(T) = \{x \in H_0 : \exists \langle x_n \rangle \subset H_{\infty} \text{ with } x_n \rightarrow x \text{ and } Tx_n \rightarrow Tx \text{ in } H_0\}. \quad (271)$$

As  $H_{\infty} \subset \text{Dom}_0(T)$  it is clear that  $\text{Dom}_0(T)$  is dense in  $H_0$ .

**Lemma 12.** *The operator  $(T, \text{Dom}_0(T))$  is a densely defined, closed operator.*

*Proof.* If the order of  $T$  is non-positive, then  $T$  is bounded on  $H_0$  and therefore  $\text{Dom}_0(T) = H_0$ . Now assume that the order of  $T$  is  $m > 0$ . Let  $\langle x_n \rangle \subset \text{Dom}_0(T)$ . Suppose that  $\langle x_n \rangle$  converges to  $x$  in  $H_0$  and  $\langle Tx_n \rangle$  converges to  $y$  in  $H_0$ . For each  $n$  we can choose a sequence  $\langle x_{nj} \rangle \subset H_{\infty}$  so that  $x_{nj} \rightarrow x_n$  and  $Tx_{nj} \rightarrow Tx_n$  in  $H_0$ . We can easily extract a subsequence  $\langle x_{nkjk} \rangle$  that converges

to  $x$  and so that  $\langle Tx_{n_k j_k} \rangle$  converges to  $y$ . As  $T$  has order  $m \geq 0$ ,  $\langle Tx_{n_k j_k} \rangle$  converges to  $Tx$  in  $H_{-m}$ . By duality and the density of  $H_\infty$ , it is immediate that  $y = Tx$ , and therefore  $x \in \text{Dom}_0(T)$ .  $\square$

If  $U$  is an operator of positive order,  $\text{Dom}_0(U)$  is a Hilbert space with the inner product

$$\langle x, y \rangle_U = \langle x, y \rangle + \langle Ux, Uy \rangle. \quad (272)$$

For  $x \in H_\infty$  this can be rewritten,

$$\langle x, (\text{Id} + U^*U)y \rangle, \quad (273)$$

where  $(\text{Id} + U^*U)y$  is an element of  $H_{-\infty}$ . This identifies the Hilbert space dual to  $\text{Dom}_0(U)$  with  $(\text{Id} + U^*U)\text{Dom}_0(U) \subset H_{-\infty}$ .

The following observation is useful:

**Lemma 13.** *Let  $U$  be a tame operator of non-negative order and  $K$  a smoothing operator. The map  $K : \text{Dom}_0(U) \rightarrow H_0$  is compact.*

*Proof.* This follows because the unit ball in  $\text{Dom}_0(U)$  is contained in the unit ball in  $H_0$ . Thus its image under  $K$  is a precompact subset of  $H_0$ .  $\square$

**Proposition 15.** *Let  $T$  be a tame elliptic operator of non-positive order. Let  $U$  denote a parametrix for  $T$ , then  $T : H_0 \rightarrow \text{Dom}_0(U)$  is a Fredholm operator. Moreover we have*

$$\text{Ind}(T) = \text{t-Ind}(T). \quad (274)$$

*Proof.* We first observe that  $TH_0 \subset \text{Dom}_0(U)$ . Let  $x \in H_0$  and let  $\langle x_n \rangle \subset H_\infty$  converge to  $x$  in  $H_0$ . Note that  $\langle Tx_n \rangle \subset H_\infty$ . As  $T$  is of non-positive order  $Tx_n \rightarrow Tx$  in  $H_0$  as well. As  $UTx_n = x_n - K_2x_n$  and  $UTx = x - K_2x$ , it is clear that  $UTx_n$  converges to  $UTx$  in  $H_0$ ; thus verifying that  $TH_0 \subset \text{Dom}_0(U)$ . Hence, the operator  $U$  maps  $\text{Dom}_0(U)$  boundedly onto  $H_0$ . It follows from Lemma 13 that  $T : H_0 \rightarrow \text{Dom}_0(U)$  has a left and right inverse, up to a compact error, and is therefore a Fredholm operator. The null space of  $T$  acting on  $H_{-\infty}$  is contained in  $H_\infty$ ; hence  $\ker T$  does not depend on the topology. To complete the proof we need to show that the coker  $T$  is isomorphic to the kernel of the formal adjoint  $T^*$ .

As  $T^*T(\text{Id} + U^*U) = I + T^*T + K_1$ , and  $(\text{Id} + U^*U)T^*T = I + T^*T + K_2$ , for smoothing operators  $K_1, K_2$  it follows from the assumption that  $\text{Id} + T^*T$  is tamely elliptic that  $\text{Id} + U^*U$  is tamely elliptic as well. Hence the null-space of  $(\text{Id} + U^*U)$  is contained in  $H_\infty$  and is therefore trivial. Indeed, as an operator on  $H_0$  we can identify  $(\text{Id} + U^*U)$  as the self adjoint operator defined by Friedrichs' extension



from the symmetric quadratic form  $\langle \cdot, \cdot \rangle_U$ . This operator is self adjoint and therefore invertible. The cokernel of  $T$  is isomorphic to the set of  $y$  in  $\text{Dom}_0(U)$  with

$$\langle Tx, y \rangle_U = 0 \quad (275)$$

for all  $x \in H_\infty$ . Using the extension of the pairing  $\langle \cdot, \cdot \rangle$  to  $H_\infty \times H_{-\infty}$ , this implies that  $T^*(\text{Id} + U^*U)y = 0$ . As  $T^*(\text{Id} + U^*U)$  is tamely elliptic we see that  $y \in H_\infty$ . Thus the coker  $T$  is isomorphic to  $(\text{Id} + U^*U)^{-1} \ker T^*$ , completing the proof of the proposition.  $\square$

A bounded operator  $P$  is a projection if  $P^2 = P$ . The following elementary fact about bounded projections is very useful:

**Lemma 14.** *Suppose that  $H$  is a Hilbert space and  $P : H \rightarrow H$  is a bounded projection operator, then range  $P$  is a closed subspace of  $H$ .*

*Proof.* Let  $\langle x_n \rangle \subset \text{range } P$  be a convergent sequence with limit  $x$ . Since  $P$  is a projection  $x_n = Px_n$ . As  $P$  is continuous

$$x = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} Px_n = P \lim_{n \rightarrow \infty} x_n = Px. \quad (276)$$

This proves the lemma.  $\square$

**Definition 5.** A tame projection  $P$  is a projection that is also a tame operator of order 0. A pair of tame projections  $(P, R)$  defines a tame Fredholm pair if the operator

$$T = RP + (I - R)(I - P) \quad (277)$$

is tamely elliptic. We let  $U$  denote a parametrix for  $T$ .

**Proposition 16.** *If  $(P, R)$  are a tame Fredholm pair, then*

$$RP : H_0 \cap \text{range } P \longrightarrow \text{Dom}_0(U) \cap \text{range } R \quad (278)$$

*is a Fredholm operator*

*Proof.* We first need to show that  $RP$  is bounded from  $H_0$  to  $\text{Dom}_0(U)$ . Let  $x \in H_\infty$ , as  $URP = UTP$  we have

$$\begin{aligned} \|RPx\|_U^2 &= \|RPx\|_0^2 + \|UTPx\|_0^2 \\ &= \|RPx\|_0^2 + \|(\text{Id} - K_2)Px\|_0^2 \\ &\leq C\|x\|_0^2. \end{aligned} \quad (279)$$

This establishes the boundedness. Moreover  $R : \text{Dom}_0(U) \rightarrow \text{Dom}_0(U)$  is a bounded map. This follows from the identity  $RT = TP$ , which implies that

$$UR(\text{Id} - K_1) = (\text{Id} - K_2)PU \quad (280)$$

As  $URK_1$  is a smoothing operator and  $P$  is order 0, for  $x \in \text{Dom}_0(U)$  we have

$$\|URx\| = \|(PU + URK_1 - K_2PU)x\| \leq C[\|Ux\| + \|x\|]. \quad (281)$$

Hence Lemma 14 implies that  $H_0 \cap \text{range } P$  and  $\text{Dom}_0(U) \cap \text{range } R$  are closed subspaces of their respective Hilbert spaces and are therefore Hilbert spaces in their own rights. Similarly we can show that  $PUR : \text{Dom}_0(U) \rightarrow H_0$  is bounded. That the map is Fredholm follows from the identities:

$$\begin{aligned} (PUR)(RP) &= PUTP = P(\text{Id} - K_2)P \\ (RP)(PUR) &= RTUR = R(\text{Id} - K_1)R, \end{aligned} \quad (282)$$

which imply that the map in (278) is a bounded map between Hilbert spaces, invertible up to a compact error.  $\square$

**Definition 6.** For a  $(P, R)$  a tame Fredholm pair we let  $\text{R-Ind}(P, R)$  denote the Fredholm index of the operator in (278). The number  $\text{R-Ind}(P, R)$  is also called the *relative index* of  $P$  and  $R$ .

The relative index can be identified with difference of the dimensions of null spaces. This is a relative index analogue of Proposition 15.

**Proposition 17.** *Let  $(P, R)$  be a tame Fredholm pair, then*

$$\text{R-Ind}(P, R) = \dim[\ker RP \upharpoonright_{PH_{-\infty}}] - \dim[\ker P^*R^* \upharpoonright_{R^*H_{-\infty}}] \quad (283)$$

*Proof.* By definition

$$\text{R-Ind}(P, R) = \dim[\ker RP \upharpoonright_{PH_0}] - \dim[\text{coker } PR \upharpoonright_{PH_0}]. \quad (284)$$

Ellipticity of  $T$  implies that

$$\ker RP \upharpoonright_{PH_{-\infty}} \subset PH_{\infty}, \quad (285)$$

so the first terms in (283) and (284) agree. Using the representation of  $\text{Dom}_0(U)'$  as  $(\text{Id} + U^*U) \text{Dom}_0(U)$  we see that

$$\text{coker } PR \upharpoonright_{PH_0} \simeq \{v \in \text{Dom}_0(U)' \cap \text{range } R^* : \langle RPu, v \rangle = 0 \text{ for all } u \in H_0\}. \quad (286)$$

The right hand side of (286) is certainly contained in  $\ker P^*R^* \upharpoonright_{R^*H_{-\infty}}$ . Again ellipticity implies that this subspace is contained in  $H_{\infty}$  and therefore is also contained in  $\text{Dom}_0(U)' \cap \text{range } R^*$ . This proves (283)  $\square$

*Remark 12.* If  $K : H \rightarrow H$  is a trace class operator, then we use  $\text{tr}_H(K)$  to denote its trace as an operator on  $H$ . A priori the definition of the trace class and the value of the trace appear to depend on the inner product on  $H$ . In many cases one can show that the value of the trace is independent of the inner product.

The operators  $RK_1R$  and  $PK_2P$  are smoothing operators and therefore trace class operators on  $H_0$ . In order to use the standard trace formula for the index we need to show that  $RK_1R : \text{Dom}_0(U) \rightarrow \text{Dom}_0(U)$  is trace class and that

$$\text{tr}_{\text{Dom}_0(U)}(RK_1R) = \text{tr}_{H_0}(RK_1R). \quad (287)$$

Theorem VI.2.23 in [20] implies that  $\sqrt{\text{Id} + U^*U}$  is a (possibly unbounded) self adjoint operator with domain  $\text{Dom}_0(U)$ . Moreover  $\sqrt{\text{Id} + U^*U}$  has a bounded inverse,  $(\text{Id} + U^*U)^{-\frac{1}{2}}$  and

$$\langle x, y \rangle_U = \langle \sqrt{\text{Id} + U^*U}x, \sqrt{\text{Id} + U^*U}y \rangle, \quad \forall x, y \in \text{Dom}_0(U). \quad (288)$$

To show that  $RK_1R$  is a trace class operator on  $\text{Dom}_0(U)$  we need to verify that there is an  $M$  so that, for every pair of orthonormal bases  $\{f_j^1\}, \{f_j^2\}$  of  $\text{Dom}_0(U)$ , we have

$$\sum_{j=1}^{\infty} |\langle RK_1Rf_j^1, f_j^2 \rangle_U| \leq M. \quad (289)$$

As  $RK_1R$  is a smoothing operator,  $RK_1Rf_j^1 \in H_\infty \subset \text{Dom}_0(\text{Id} + U^*U)$ , hence the Friedrichs extension theorem and (288) imply that

$$\sum_{j=1}^{\infty} |\langle RK_1Rf_j^1, f_j^2 \rangle_U| = \sum_{j=1}^{\infty} |\langle (\text{Id} + U^*U)RK_1Rf_j^1, f_j^2 \rangle|. \quad (290)$$

For  $i = 1$  or  $2$  we let

$$e_j^i = (\text{Id} + U^*U)^{\frac{1}{2}} f_j^i, \quad (291)$$

where  $\{e_j^1\}, \{e_j^2\}$  are orthonormal bases for  $H_0$ . As  $(\text{Id} + U^*U)^{-\frac{1}{2}}$  is a bounded self adjoint operator, we obtain:

$$\begin{aligned} \sum_{j=1}^{\infty} |\langle RK_1Rf_j^1, f_j^2 \rangle_U| &= \\ \sum_{j=1}^{\infty} |\langle (\text{Id} + U^*U)^{-\frac{1}{2}}(\text{Id} + U^*U)RK_1R(\text{Id} + U^*U)^{-\frac{1}{2}}e_j^1, e_j^2 \rangle|. \end{aligned} \quad (292)$$

The operator  $(\text{Id} + U^*U)RK_1R$  is a smoothing operator and therefore a trace class operator on  $H_0$ . The operator  $(\text{Id} + U^*U)^{-\frac{1}{2}}$  is bounded on  $H_0$ , which shows that

$(\text{Id} + U^*U)^{-\frac{1}{2}}(\text{Id} + U^*U)RK_1R(\text{Id} + U^*U)^{-\frac{1}{2}}$  is also a trace class operator on  $H_0$ . Hence there exists an upper bound  $M$  for the sum on the right hand side of (292) valid for any pair of orthonormal bases  $\{e_j^1\}, \{e_j^2\}$ . This completes the proof that

$$RK_1R : \text{Dom}_0(U) \rightarrow \text{Dom}_0(U)$$

is a trace class operator.

To compute the trace we select an orthonormal basis  $\{f_j\}$  of  $\text{Dom}_0(U)$  and let  $\{e_j\}$  be the corresponding orthonormal basis of  $H_0$ . Arguing as above we conclude that

$$\begin{aligned} \text{tr}_{\text{Dom}_0(U)}(RK_1R) &= \sum_{j=1}^{\infty} \langle (\text{Id} + U^*U)^{-\frac{1}{2}}(\text{Id} + U^*U)RK_1R(\text{Id} + U^*U)^{-\frac{1}{2}}e_j, e_j \rangle \\ &= \text{tr}_{H_0} \left( (\text{Id} + U^*U)^{-\frac{1}{2}}(\text{Id} + U^*U)RK_1R(\text{Id} + U^*U)^{-\frac{1}{2}} \right). \end{aligned} \quad (293)$$

The operator  $(\text{Id} + U^*U)^{-\frac{1}{2}}(\text{Id} + U^*U)RK_1R$  is  $H_0$ -trace class and the operator  $(\text{Id} + U^*U)^{-\frac{1}{2}}$  is  $H_0$  bounded. Therefore

$$\begin{aligned} \text{tr}_{H_0} \left( (\text{Id} + U^*U)^{-\frac{1}{2}}(\text{Id} + U^*U)RK_1R(\text{Id} + U^*U)^{-\frac{1}{2}} \right) &= \\ \text{tr}_{H_0} [(\text{Id} + U^*U)^{-\frac{1}{2}}]^2 (\text{Id} + U^*U)RK_1R &= \text{tr}_{H_0} RK_1R. \end{aligned} \quad (294)$$

This completes the proof of the following result:

**Proposition 18.** *If  $(P, R)$  is a tame Fredholm pair then  $RK_1R : \text{Dom}_0(U) \rightarrow \text{Dom}_0(U)$  is a trace class operator and*

$$\text{tr}_{\text{Dom}_0(U)}(RK_1R) = \text{tr}_{H_0}(RK_1R). \quad (295)$$

To obtain a trace formula for  $\text{R-Ind}(P, R)$  we need to compute the traces of  $RK_1R$  and  $PK_2P$  restricted to the ranges of  $R$  and  $P$  respectively. Let  $\{e_j^1\}$  be an orthonormal basis for the range of  $P$  and  $\{e_j^2\}$  be a orthonormal basis for the orthocomplement of the range. Note that  $P^*e_j^2 = 0$  for all  $j$  and therefore

$$\langle PK_2Pe_j^2, e_j^2 \rangle = \langle PK_2Pe_j^2, P^*e_j^2 \rangle = 0. \quad (296)$$

This shows that

$$\text{tr}_{H_0}(PK_2P) = \sum_{j=1}^{\infty} \langle PK_2Pe_j^1, e_j^1 \rangle. \quad (297)$$

Similar considerations apply to  $\text{tr}_{\text{Dom}_0(U)}(RK_1R)$  and  $\text{tr}_{H_0}(RK_1R)$ . From these observations we obtain:

**Theorem 15.** *If  $(P, R)$  is a tame Fredholm pair, then*

$$\text{R-Ind}(P, R) = \text{tr}_{H_0}(PK_2P) - \text{tr}_{H_0}(RK_1R). \quad (298)$$

*Proof.* This follows from (282), Proposition 18 and Theorem 15 in Chapter 30 of [22].  $\square$

In applications,  $H_0$  is often  $L^2(M; E)$  where  $M$  is a compact manifold and  $E \rightarrow M$  is a vector bundle. Theorem 15 and Lidskii's Theorem imply that  $\text{R-Ind}(P, R)$  can be computed by integrating the Schwartz kernels of  $PK_2P$  and  $RK_1R$  along the diagonal. This is a very useful fact.

Finally we have a logarithmic property for relative indices.

**Theorem 16.** *Let  $(P, Q)$  and  $(Q, R)$  be tame Fredholm pairs. For an appropriately defined Hilbert space,  $H_{UV}$ ,  $RQP$  is a tame Fredholm operator from  $\text{range } P \cap H_0$  to  $\text{range } Q \cap H_{UV}$  and*

$$\text{Ind}(RQP) = \text{R-Ind}(P, Q) + \text{R-Ind}(Q, R). \quad (299)$$

*Proof.* For  $S = QP + (\text{Id} - Q)(\text{Id} - P)$  and  $T = RQ + (\text{Id} - R)(\text{Id} - Q)$ , let  $U, V$  be parametrices, with

$$\begin{aligned} SU &= \text{Id} - K_1 & US &= \text{Id} - K_2 \\ TV &= \text{Id} - K_3 & VT &= \text{Id} - K_4. \end{aligned} \quad (300)$$

We define the Hilbert spaces  $H_U, H_V$  and  $H_{UV}$  as the closures of  $H_\infty$  with respect to the inner products

$$\|x\|_U^2 = \|x\|^2 + \|Ux\|^2, \quad \|x\|_V^2 = \|x\|^2 + \|Vx\|^2, \quad \|x\|_{UV}^2 = \|x\|^2 + \|UVx\|^2. \quad (301)$$

By definition  $QP : \text{range } P \cap H_0 \rightarrow \text{range } Q \cap H_U$  and  $RQ : \text{range } Q \cap H_U \rightarrow \text{range } R \cap H_V$  are Fredholm with indices  $\text{R-Ind}(P, Q), \text{R-Ind}(Q, R)$  respectively. To prove the theorem we need to show that

$$RQ : \text{range } Q \cap H_U \longrightarrow \text{range } R \cap H_{UV} \quad (302)$$

is Fredholm and has index  $\text{R-Ind}(Q, R)$ . The proofs of these statements make use of the commutation relations  $QS = SP, RT = TQ$ , which, along with (300) imply that

$$UQ(\text{Id} - K_1) = (\text{Id} - K_2)PU \text{ and } VR(\text{Id} - K_3) = (\text{Id} - K_4)QV. \quad (303)$$

First we show that  $Q$  acts boundedly on  $H_U$  and  $R$  acts boundedly on  $H_{UV}$ , so that we can apply Lemma 14. Using (303) and the fact that the smoothing operators are a two sided ideal we see that for  $x \in H_\infty$ , we have

$$\|UQx\| = \|(UQK_1 + PU - K_2PU)x\| \leq C_1[\|Ux\| + \|x\|]. \quad (304)$$

In the following computation  $K$  denotes a variety of smoothing operators:

$$\begin{aligned} \|UVRx\| &= \|(UQV + K)x\| \\ &= \|(PUV + K)x\| \leq C_2[\|UVx\| + \|x\|]. \end{aligned} \quad (305)$$

The constants  $C_1, C_2$  are independent of  $x$ . This shows that  $\text{range } Q \cap H_U$  and  $\text{range } R \cap H_{UV}$  are closed subspaces.

The operator  $QVR$  is a parametrix for the operator in (302). First we see that  $QVR : \text{range } R \cap H_{UV} \rightarrow \text{range } Q \cap H_U$  is bounded:

$$\|UQVRx\| = \|(UQK_1 + PU - K_2PU)VR\| \leq C\|x\|_{UV}. \quad (306)$$

That it is a parametrix follows from

$$(QVR)(RQ) = Q(\text{Id} - K_4)Q \text{ and } (RQ)(QVR) = R(\text{Id} - K_3)R. \quad (307)$$

The null-space of  $RQ \upharpoonright_{QH_U}$  agrees with the null-space of  $RQ$  acting on  $QH_\infty$ . We use the operator  $(\text{Id} + V^*U^*UV)$  to identify the dual space  $H'_{UV}$  of  $H_{UV}$  as a subspace of  $H_\infty$  via the pairing  $\langle \cdot, \cdot \rangle$ . With this identification, the cokernel of the operator in (302), is isomorphic to the null-space of  $Q^*R^*$  acting on  $R^*H'_{UV}$ . As in the proof of Proposition 17, we see that the cokernel is therefore isomorphic to the null-space of  $Q^*R^*$  acting on  $R^*H_\infty$ . This, along with Proposition 17, shows that the index of the operator in (302) equals  $\text{R-Ind}(Q, R)$ . The theorem now follows from the standard logarithmic property for the Fredholm indices of operators acting on Hilbert spaces. □

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