Math 425 **Dr. DeTurck**
Practice Midterm 2 **April 11, 2008**

1. (a) Say that a function is *oddly odd* if it satisfies both the conditions

$$f(-x) = -f(x), \qquad f(L+x) = f(L-x)$$

Show that such a function is periodic with period $4L$.

(b) Draw the graph of a non-zero oddly odd function. Is it symmetric around the line $x = L$?

(c) Show that the Fourier series of an oddly odd function is of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)\pi x}{2L}.$$

Give a formula for the coefficients b_n.

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2. Let

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \end{cases}$$

Solve the heat equation $u_t = u_{xx}$ for $x \in [0, 2]$ and $t \in [0, \infty)$ with initial condition $u(x, 0) = f(x)$ and boundary conditions $u(0, t) = u(2, t) = 0$. Draw a sketch of the graph of $u(x, \epsilon)$ for a fixed, very small value of ϵ and $0 \le x \le 2$.

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3. (a) Find the Fourier (cosine) series of the function $f(x) = x^2$, $-\pi < x < \pi$.

(b) Draw the graph of the function to which your series converges. Explain how you know the series converges pointwise to this function. Does it converge uniformly?

(c) Use the series to show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$$

and

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots + \frac{(-1)^{n+1}}{n^2} + \cdots = \frac{\pi^2}{12}$$

(d) Use the results in part (c) to deduce

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots = \frac{\pi^2}{8}$$

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4. Solve the initial-boundary value problem for the wave equation:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0$$

where $u(x, 0) = \sin \pi x$, $u_t(x, 0) = 0$, $u(0, t) = 0$, $u(1, t) = 1$.

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5. (a) Find the eigenvalues and eigenfunctions of the boundary-value problem:

$$u'' + \lambda u = 0, \qquad u(0) = 0, \qquad u'(3) + u(3) = 0$$

for $u(x)$ defined on the interval [0,3].

(b) If we number the eigenvalues in increasing order, so that $\lambda_1 < \lambda_2 < \lambda_3 < ...$, find A and B so that

$$\lim_{n\to\infty} \left(\lambda_n - (An + B)^2\right) = 0.$$

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6. Solve $u_{tt} = u_{xx} - 2xt$, $u(x, 0) = 0$, $u_t(x, 0) = 0$ for $t > 0$, $-\infty < x < \infty$.

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7. Find $u(x, t)$ if $u_t = u_{xx}$ for $0 < x < \infty$ and $t > 0$, $u(0, t) = 0$ for all t, and

$$u(x, 0) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

(Use the fundamental solution of the heat equation and a suitable reflection.)

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8. For the exam, be sure you know the integrals:

$$\int \cos ax \, \cos bx \, dx, \qquad \int \sin ax \, \cos bx \, dx$$

and

$$\int x^2 \sin ax \, dx, \qquad \int x \sin ax \, dx.$$