# Characterizing obstacle-avoiding paths using cohomology theory

Paweł Dłotko[1], Walter G. Kropatsch[2] and Hubert Wagner[1]

[1] Institute of Computer Science, Jagiellonian University, Poland
hubert.wagner@ii.uj.edu.pl and dlotko@ii.uj.edu.pl,
[2] Pattern Recognition and Image Processing Group, Vienna University of Technology, Austria
krw@prip.tuwien.ac.at

**Abstract.** In this paper, we investigate the problem of analyzing the shape of obstacle-avoiding paths in a space. Given a $d$-dimensional space with holes, representing obstacles, we ask if certain paths are equivalent, informally if one path can be continuously deformed into another, within this space. Algebraic topology is used to distinguish between topologically different paths. A compact yet complete *signature* of a path is constructed, based on cohomology theory. Possible applications include assisted living, residential, security and environmental monitoring. Numerical results will be presented in the final version of this paper.

**Keywords:** obstacle-avoidance, cohomology generators, trajectory planning problem

## 1 Introduction.

In the recent years, there has been growing interest in topics such as assisted living, residential, security and environmental monitoring [1, 2]. This is closely related to the area of remote sensing, which aims at delivering a description of the chosen aspects of the sensed environment by aggregating information from an array of sensors.

The information gathered by individual sensors ranges from visual data (Visual Sensor Networks [3]) to the presence of smoke in the air. Visual Sensor Networks are the most closely related to the computer vision field. In this paper we treat the sensors in an abstract way, therefore the method should be applicable in a number of settings.

One important question that arises is how to arrange such sensors. In [1], which largely inspired us to write this paper, straight laser beams are used as sensors. Prompted by some of the questions posed in the summary of that paper, we consider the following questions. How does this scenario generalize to sensors of different shapes? Can we generalize these concepts to higher dimensions (the original considerations were done in 2D)?

Recently a field of *minimal sensing* has been actively developed. It assumes that sensors are very limited in their capabilities. For example, no GPS module is available, which prevents the sensors from knowing their own position. On the one hand, it makes the sensors much cheaper, enabling deploying sensor-networks of much greater capacities. On the other hand, many new challenges appear. For example, determining if the network covers (without holes) a given area is much more difficult [2].

In the setting of minimal sensing, sensors are typically not able to capture the actual geometry of the space. The challenging task of inferring the properties of a sensed scene based only on its topological features was successfully tackled in a number of cases (see [2] and references therein). Algebraic topology is a powerful tool that is used increasingly often in such a setting.

Our aim is also to expose the *cohomology theory* to the CAIP community. We believe that the mathematical robustness and intuitibity makes it an interesting tool.

The paper is structured as follows: In Section 2 a rigorous formulation of the considered problem is presented. In Section 3 the complexes used in this paper are discussed. In Section 4 an intuitive introduction to homology and cohomology theory is given. In Section 5 the main result of this paper is stated. In Section 6 an algorithm to compute signature of a given path is presented. Finally in Section 7 the conclusions are drawn.

## 2 Problem formulation.

The problem of analyzing paths of moving agents in a 2-dimensional space, in the presence of obstacles and linear (beam) sensors was introduced in [1]. We present a variation for a $d$-dimensional, orientable space, where "sensors" are represented by certain $(d-1)$-dimensional hypersurfaces (possibly with self-intersections). For the $2-$dimensional case the difference is that our sensors can have arbitrary shape, possibly with self-intersections. Different sensors are also allowed to intersect. While the idea of a sensor of arbitrary shape might seem contrived, imagine that such a sensor is actually composed of a number of small sensing units covering a given curve (or hypersurface in general). These sensors, as will be shown, need to be placed in the support of cohomology generators.

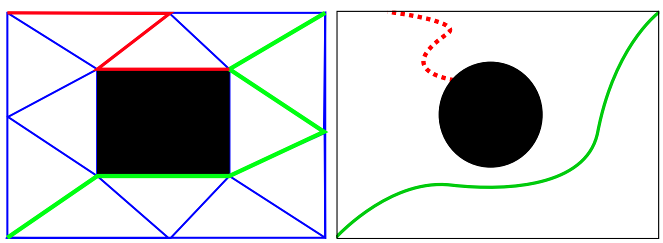
We are primarily interested in compactly describing a path of an *agent* moving through a $d-$dimensional space. Using *cohomology theory* we produce a signature of a path which describes the motion up to its homology class. It means that two paths connecting the same end-points and bounding a surface without holes, are considered the same.

## 3 Representing spaces with holes.

In this section we present some theory related to computational topology, used later in the paper. For simplicity the concept of *simplicial complex* is used to represent the space, but any kind of so-called *regular CW-complex* can be used (see [4]). The definition of simplicial complex can be found in [5]. Imagine that a simplicial complex is a decomposition of the space into a set of simplices, that is vertices, edges, triangles etc. In general, $n-$simplex is a convex hull of $n+1$ points lying in general position. The number $n$ is the *dimension* of a simplex $S$ and is denoted by $dim(S)$. We assume that vertices of a simplicial complex are uniquely enumerated with integers, allowing to index each simplex with the set of its vertices. Each simplex in the simplicial complex has an orientation (this is discussed in details in [6]). In the implementation presented in [7], enumeration of vertices of complex $\mathcal{K}$ is used in orienting the edges and higher dimensional simplices. For instance every edge $E$ is oriented from its higher vertex to lower vertex. From now on the orientation of all simplices in the complex is assumed to be fixed. A subset of simplices is chosen to represent the obstacles. During the computation of cohomology, the interior of obstacles is removed from the complex. Later by $\mathcal{K}$ we will denote the complex after this removal.

There are two vertices chosen in our complex, marked as $S$ and $T$ from $S$ource and $T$arget. An oriented path is the formal sum of edges joining those points with $+1, -1$ coefficients, which induce orientation.

The goal is to provide an efficient algorithm to describe and distinguish paths from $S$ to $T$, which avoid all the obstacles[3]. An example of a $2-$dimensional simplicial complex can be found in Figure 1(a).



(a) (b)

**Fig. 1.** a) Simple example of a complex. Obstacles are marked with black, paths with green (solid). b) Graphical representation of complexes that we will use for clarity of images. Imagine that the complex is very finely subdivided, but paths and generators are still composed of edges of the complex, which is not displayed. Cohomology generator is depicted as the red (dotted) curve. In both cases point $S$ is placed in the lower left, and point $T$ in the upper right corner of the picture.

## 4 Cohomology theory.

In this section an intuitive exposition of homology and cohomology theory is given. For a full introduction consult [6]. Both homology and cohomology groups give a compact description of topology of a simplicial complex.

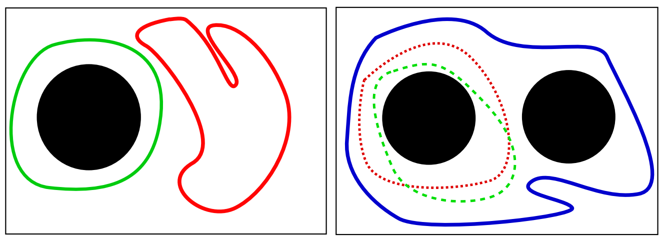
[3] Note that the number of homologically different paths is unbounded for non-trivial cases.

In homology theory one uses a concept of *chain*, being a formal sum of simplices with integer coefficients. A group of chains of dimension $n$ is denoted by $C_n(\mathcal{K}) := \{\sum_{S \in \mathcal{K}, dim(S)=n} \alpha_S S\}$. A boundary operator $\partial : C_n \to C_{n-1}$ is then introduced for a simplex $S = [v_0, \ldots, v_n]$:

$$\partial S = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n] \quad (1)$$

and extended linearly to $C_n(\mathcal{K})$. As an example, let us calculate the boundary of a full triangle: $\partial[0,1,2] = [1,2] - [0,2] + [0,1]$.

A group of $n$ dimensional *cycles* $Z_n(\mathcal{K}) := \{c \in C_n(\mathcal{K}) \mid \partial c = 0\}$. In short, a cycle is a chain whose boundary vanishes. A group of $n-$ dimensional *boundaries* $B_n(\mathcal{K}) := \{c \in C_n(\mathcal{K}) | \exists d \in C_{n+1}(\mathcal{K}) | \partial d = c\}$. The idea behind cycles and boundaries is presented in Figure 2(a).



(a) (b)

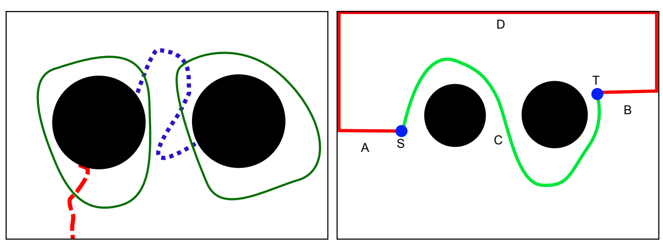
**Fig. 2.** a) Right chain is a cycle and a boundary. Left cycle surrounds a hole, so it is not a boundary b) Red (dotted) and green (dashed) cycles are homologous. Blue (bold) is not homologous with any of them. Red (or green) and blue cycles constitute a homology basis.

It is straightforward to verify from Formula 1 that $\partial\partial = 0$. Therefore we have $B_n(\mathcal{K}) \subset Z_n(\mathcal{K})$ and we can define the *homology group* $H_n(\mathcal{K})$ as a classes of cycles which are not boundaries, namely $H_n(\mathcal{K}) := Z_n(\mathcal{K})/B_n(\mathcal{K})$. Two $n$-cycles $c_1$ and $c_2$ such that $c_1 - c_2 \in B_n(\mathcal{K})$ are said to be *homologous*. By homology generators we mean any representants of classes of cycles that generate $H_n(\mathcal{K})$. In absence of *torsions* the rank of homology group can be interpreted as number of holes in the considered space. Idea of homology groups is given in Figure 2(b).

In this paper we restrict ourselves to connected simplicial complexes $\mathcal{K}$ which are torsion-free in dimension one (i.e. after the obstacles are removed from the complex, the resulting complex is connected and torsion free). Torsions in homology mean that elements of a homology group have finite order (they generate a subgroup $\mathbb{Z}_p$ of homology group for $p \in \mathbb{Z}$ being the order of an element). It might appear that for torsion-free spaces all (co)homology computations could be performed with $\mathbb{Z}_p$ coefficients for $p \in \mathbb{Z}$, $p \geq 2$. This is not the case. Without going into details: we must use $\mathbb{Z}$ coefficients to handle the case of paths crossing certain cohomology generators $np$-times for $n \in \mathbb{Z}$.

For a formal introduction to the cohomology theory please consult [6], for an intuitive introduction consult [7]. Further in the paper we need a concept of *n-cochain* $c^*$ being a map assigning any chain $c \in Z_n(\mathcal{K})$ a number[4] $\langle c^*, c\rangle \in \mathbb{Z}$. A group of $n-$cochains is denoted as $C^n(\mathcal{K})$. Dually to homology, a so-called *coboundary operator* $\delta : C^n(\mathcal{K}) \to C^{n+1}(\mathcal{K})$ is introduced. It is defined as $\langle \delta c^*, c\rangle = \langle c^*, \partial c\rangle$ for every $c^* \in C^{n-1}(\mathcal{K})$ and $c \in C_n(\mathcal{K})$. Again, cochain $c^*$ is a *cocycle* if $\delta c^* = 0$. Cochain $c^*$ is a *coboundary* if there exists a cochain $d^* \in C^{n-1}(\mathcal{K})$ such that $\delta d^* = c^*$. Cocycles are denoted as $Z^n(\mathcal{K})$, and coboundaries as $B^n(\mathcal{K})$. Finally, *cohomology group* is defined as the quotient $H^n(\mathcal{K}) := Z^n(\mathcal{K})/B^n(\mathcal{K})$.

For our purposes it is sufficient to consider cohomology group basis in dimension one. For torsion-free spaces, there is a straightforward correspondence between homology and cohmology group generators (see Theorem 4.8, [8]). Theorem 4.8 states that for any set of cycles representing homology generators $h_1, \ldots, h_n$ there exist dual cohomology generators $h^1, \ldots, h^n$ such that $\langle h^i, h_j\rangle = \delta_{ij}$. This theorem allows us to use the so-called "cutting analogy" to describe a cohomology basis. In fact, in the considered case the generator $h^i$, for $i \in \{1, \ldots, n\}$, can be seen as a fence that blocks any cycles in the class of $h_i$. This idea is illustrated in Figure 3(a). The concept of the presented "cut analogy" was developed in the so-called *Discrete Geometrical Approach* to Maxwell's equations [7].




(a) (b)

**Fig. 3.** a) The "cut analogy". When one cuts a complex along the red (dashed) cohomology generator, the left homology class vanishes. Cutting along blue (dotted) generator makes the right homology class vanish. b) Completion of chain $c$.

With the algorithm described in [7], we obtain cohomology generators (represented as a set of pairs (edge, integer)) of any simplicial complex. Note that cohomology generators are allowed to intersect. See the Borromean Rings phe-

[4] Operation $\langle c^*, c\rangle$ is called evaluation of a cocycle $c^*$ on a cycle $c$. In order to compute $\langle c^*, c\rangle$, note that the set of maps $\{S^* | \langle S^*, K\rangle = \delta_{SK}$ for any $K \in \mathcal{K}\}_{S\in\mathcal{K}}$ constitutes a basis of $C^n(\mathcal{K})$. Therefore every cochain $c^*$ is equal to $\sum_{S\in\mathcal{K}} \alpha_S S^*$ for $dim\ S = n$. Then for a chain $c = \sum_{S\in\mathcal{K}} \beta_S S$ we have $\langle c^*, c\rangle = \langle \sum_{S\in\mathcal{K}} \alpha_S S^*, \sum_{S\in\mathcal{K}} \beta_S S\rangle = \sum_{S\in\mathcal{K}} \alpha_S \beta_S$.

nomenon in [9] for an example of a $3-\textit{dimensional}$ space, where it is impossible to find a non-intersecting cohomology basis.

## 5 Path characterization using signatures.

In this section a formal proof of the main result of the paper is provided. Suppose a simplicial complex $\mathcal{K}$ is given. As previously, we assume that $H_1(\mathcal{K})$ is torsion-free and $\mathcal{K}$ itself is connected. Let $h^1, \ldots, h^n$ be cocycles representing first cohomology group generators of $\mathcal{K}$. Moreover, let $h_1, \ldots, h_n$ be the homology generators dual to $h^1, \ldots, h^n$ according to Theorem 4.8 in [8] (they are only needed for the proof). We fix $h^1, \ldots, h^n$ and their dual $h_1, \ldots, h_n$ for the rest of this section. Let $c \in C_1(\mathcal{K})$ be a path from $S$ to $T$.

**Definition 1.** *For a path $c$ the vector $S_c = [a_1, \ldots, a_n]$ such that $a_i = \langle h^i, c\rangle$, for $i \in \{1, \ldots, n\}$, is called a* signature *of $c$.*

In this section we show that paths having the same signature are homologous and that paths having different signature are non-homologous. It is necessary to assume that all paths lead from $S$ to $T$. A signature of a path provides an efficient way of distinguishing non-homologous paths and identifying homologous ones. Let us start with a lemma, the proof of which can be found in [8].

**Lemma 1.** *Let $c^* \in Z^1(\mathcal{K})$ be a cocycle and let $b \in B_1(\mathcal{K})$ be a boundary. Then $\langle c^*, b\rangle = 0$.*

Let us now define the *completion* of a chain. Let us take any chain $A$ joining point $S$ with the boundary of the complex $\mathcal{K}$, $B$ joining point $T$ with the boundary of a complex and $D$ lying entirely on the boundary of $\mathcal{K}$ joining endpoints of chains $A$ and $B$. With any path $c \in C_1(\mathcal{K})$ from $S$ to $T$ we can assign a cycle $c \cup A \cup B \cup D$. This cycle is called a *completion* of chain $c$ (see Figure 3(b)).

Now we are ready to give the two main theorems of this paper.

**Theorem 1.** *Two homologous paths $c_1$ and $c_2$ have the same signature, $S_{c_1} = S_{c_2}$.*

*Proof.* Since $c_1$ and $c_2$ are homologous, there exists $b \in C_2(\mathcal{K})$ such that $\partial b = c_1 - c_2$. Therefore $c_1 = c_2 + \partial b$. From Lemma 1 we have, that $\langle h^i, c_1\rangle = \langle h^i, c_2 + \partial b\rangle = \langle h^i, c_2\rangle + \langle h^i, \partial b\rangle = \langle h^i, c_2\rangle + 0 = \langle h^i, c_2\rangle$ for every $i \in \{1, \ldots, n\}$. Therefore $S_{c_1} = S_{c_2}$. □

**Theorem 2.** *Two non-homologous paths $c_1$ and $c_2$ have different signatures, $S_{c_1} \neq S_{c_2}$.*

*Proof.* Suppose by contrary that $c_1$ and $c_2$ are non-homologous and $S_{c_1} = S_{c_2}$. Therefore $d_1 = c_1 \cup A \cup B \cup D$ and $d_2 = c_2 \cup A \cup B \cup D$ are also non-homologous. But $h_1, \ldots, h_n$ is a homology basis dual to cohomology basis $h^1, \ldots, h^n$. Then we have $d_1 = \sum_{i=1}^{n} \alpha_i h_i + \partial e$ and $d_2 = \sum_{i=1}^{n} \beta_i h_i + \partial f$ for some $e, f \in C_2(\mathcal{K})$ and $\alpha_i, \beta_i \in \mathbb{Z}$ for $i \in \{1, \ldots, n\}$. Since $d_1$ and $d_2$ are not homologous there exists an index $i \in \{1, \ldots, n\}$ such that $\alpha_i \neq \beta_i$. But from the hypothesis we have $S_{c_1} = S_{c_2}$. It implies, that $S_{d_1} = S_{d_2}$. We have $\langle h^i, d_1 \rangle = \langle h^i, \sum_{i=1}^{n} \alpha_i h_i \rangle = \alpha_i$ and $\langle h^i, d_2 \rangle = \langle h^i, \sum_{i=1}^{n} \beta_i h_i \rangle = \beta_i$. Therefore from the hypothesis we have $\alpha_i = \beta_i$ for every $i \in \{1, \ldots, n\}$, which gives a contradiction. □

## 6 Computing the Signature of a path.

In this section we present an algorithm which, for fixed cocycles $h^1, \ldots, h^n$, constituting a cohomology basis and a path $c$ from $A$ to $B$ outputs $S_c$, the signature of $c$. We assume that simplicial complex is represented as a pointer-based data-structure as in [7]. Moreover, let each edge $E$ of simplicial complex $\mathcal{K}$ be equipped with a vector $v$ of $n$ integers such that $v_E[i] = \langle h^i, E \rangle$ for every $i \in \{1, \ldots, n\}$. Let a path $c$ be given as a vector of pointers to edges in $\mathcal{K}$.

It remains to resolve the subtlety of orientation of simplices versus an orientation of a path $c$. The path is oriented from point $S$ to $T$. Let us define $o(c, E)$ in the following way: $o(c, E) := 1$ if orientation of $c$ is the same as orientation of $E$ and $-1$ otherwise. Now we list the algorithm. Also, see Figure 4 for a visual example.

**Algorithm 1** Computing signature of a path

**Input:** path $c$, simplicial complex $\mathcal{K}$ with cohomology generators $h^1, \ldots, h^n$
**Output:** $s$ - signature of path $c$
1: Let $v$ be the vector encoding the intersections of $c$ with cohomology generators
2: Let $s$ be an $n$-tuple
3: **for** $i \in \{1, \ldots, n\}$ **do**
4: $\quad s[i] \leftarrow \sum_{E \in c} o(c, E) v_E[i]$
5: **return** $s$

## 7 Conclusions

The ideas presented in this paper generalize the approach using "laser beams" presented in [1]. We use topological tools to distinguish between different obstacle-avoiding paths, based only on their intersections with selected *sensors*. The usage of algebraic topology enables us to use sensors of arbitrary shape and abstract away from the actual geometry of the space. Topological information (cohomology generators and their intersections with paths) sufficiently represents the space. Additionally, the usage of algebraic topology makes our method dimension-independent, which extends the area of applications.

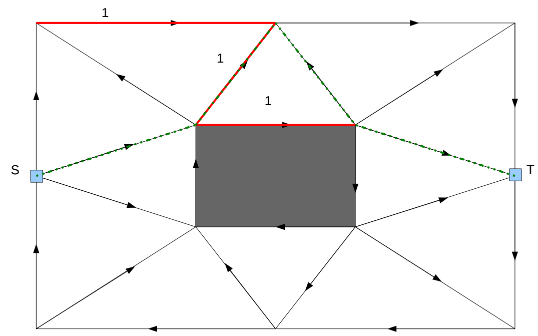



**Fig. 4.** We use the presented procedure to compute $s[1]$ for the green (dotted) path. $v_E[1]$ is nonzero only for the edges in the support of the cohomology generator. Therefore $s[1] = 1$, as the orientation of this path is the same as the orientation of cohomology generator (marked with red).

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## References

1. B. Tovar, F. Cohen and S. M. LaValle, "Sensor Beams, Obstacles, and Possible Paths", Proc. Workshop on the Algorithmic Foundations of Robotics (WAFR), 2008
2. V. de Silva and R. Ghrist, "Coverage in sensor networks via persistent homology", Alg. & Geom. Topology, 7, 339-358, 2007.
3. S. Soro and W. Heinzelman, "A Survey of Visual Sensor Networks", Advances in Multimedia vol. 2009, 2009
4. P. Dłotko, T. Kaczynski, M. Mrozek, Th. Wanner, Coreduction homology algorithm for regular CW-complexes, Discrete & Computational Geometry, to appear.
5. H. Edelsbrunner, Geometry and Topology for Mesh Generation, Cambridge University Press, 2001, ISBN 9780521793094.
6. A. Hatcher, Algebraic Topology, available online, Cambridge University Press 2002.
7. P. Dłotko, R. Specogna, "Efficient cohomology computation for electromagnetic modeling", CMES: Computer Modeling in Engineering & Sciences, Vol. 60, No. 3, 2010, pp. 247-278.
8. P. Dłotko, R. Specogna, "Critical analysis of the spanning tree techniques", SIAM Journal of Numerical Analysis (SINUM), Vol. 48, No. 4, 2010, pp. 1601-1624.
9. P. R. Cromwell, E. Beltrami and M. Rampichini, "The Borromean Rings", Mathematical Intelligencer 20 no 1 (1998) 5362.