

Lebesgue Measure

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1 Overview

Measure theory aims to quantify the size of sets of points. It generalizes the notion of length, area, and volume of classical geometric shapes to more general types of sets that appear in the analysis of functions. Besides squares and spheres and cylinders, we want to be able to measure Cantor sets and fractals and other sets that do not have smooth boundaries.

The most important measure for the purposes of analysis is *Lebesgue measure*, introduced by Henri Lebesgue in his doctoral thesis of 1902, as part of the foundation for his new technique of integration. In these notes we give a careful construction of the Lebesgue measure in Euclidean space \mathbb{R}^n . Most constructions of Lebesgue measure take as their point of departure the familiar notion of the volume of a rectangular box. From there, one gradually extends to more and more complicated sets.

In section 2, we study how to measure sets that can be divided into rectangular cells. This is one way to analyze the area inside a circle, for example: by dividing it up as an infinite collection of smaller and smaller squares. It turns out that all open sets can be measured in this way, but that not all closed sets are amenable to this approach.

Examples like the Cantor set show that closed sets may have no interior, and therefore no rectangle at all will fit inside. To deal with such sets, in section 3 we define outer and inner measures, which are defined for arbitrary sets. Now open and closed sets end up playing complementary roles.

We are now ready to define Lebesgue measure, which is done in section 4. Essentially, to measure a set we squeeze it between a compact set (from inside) and an open set (from outside). If the difference in measure between the compact and open set can be made arbitrarily small, the inner and outer measures of the set are the same, and the set is Lebesgue measurable. The key property of Lebesgue measure, proven in section 4, is that it is additive for countable disjoint unions of measurable sets.

Finally, in section 5 we give a characterization of measurable sets, and will see that virtually all sets that play a role in analysis are Lebesgue measurable.

2 Content of Cellular Sets

As usual $[a, b)$ denotes the half-open interval of numbers $x \in \mathbb{R}$ with $a \leq x < b$. We allow $a \leq b$. The *length* of a half-open interval $[a, b)$ is simply $b - a$. Because there are no numbers x with $a \leq x < a$, we have $[a, a) = \emptyset$. We see that the length of $\emptyset = [a, a)$ is zero.

A *cell* in \mathbb{R}^n is a product of half-open intervals

$$S = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n),$$

where $a_i \leq b_i$. The intervals $[a_i, b_i)$ are called the *sides* of S . The *content* of S is the product of the length of its sides,

$$\nu(S) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

Again, $\emptyset = [a_1, a_1) \times \cdots [a_n, a_n)$ is a cell with content zero. With this convention the intersection of any two cells is again a cell (possibly empty).

A *partition* of an interval $[a, b) \subset \mathbb{R}$ is a finite collection of disjoint intervals $[x_i, x_{i+1})$ whose union is $[a, b)$. Such a partition corresponds to a finite strictly increasing sequence,

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

A partition of a cell S in \mathbb{R}^n is obtained by partitioning each of the sides of S ,

$$a_i = x_1^i < x_2^i < \cdots < x_{N_i}^i = b_i,$$

resulting in a collection P of $N = N_1 N_2 \cdots N_n$ disjoint cells of the form

$$[x_{j_1}^1, x_{j_1+1}^1) \times [x_{j_2}^2, x_{j_2+1}^2) \times \cdots \times [x_{j_n}^n, x_{j_n+1}^n).$$

If we refer to “a partition P of S ”, then P denotes the set of subcells of S in the partition, and we will often assume that those subcells are numbered, at random,

$$P = \{S_1, S_2, \dots, S_N\}.$$

The construction of Lebesgue measure in Euclidean space \mathbb{R}^n has as its starting point the idea that the measure of a cell S must be equal to its content $\nu(S)$. We will first prove that content is well-behaved for disjoint unions of cells.

Lemma 1 *Let the cell $S = S_1 \cup S_2 \cup \dots \cup S_N$ be the disjoint union of finitely many subcells (not necessarily a partition). Then*

$$\nu(S) = \nu(S_1) + \nu(S_2) + \dots + \nu(S_N).$$

Proof. If the union of cells S_i is a *partition* of S , then this follows immediately from the distributive property $a(b + c) = ab + ac$.

In the general case, for $i = 1, 2, \dots, N$, let

$$S_i = [c_1^{(i)}, d_1^{(i)}] \times \dots \times [c_n^{(i)}, d_n^{(i)}].$$

By putting the various boundaries of sides with the same coordinate

$$\{c_j^{(1)}, d_j^{(1)}, \dots, c_j^{(N)}, d_j^{(N)}\}$$

in ascending order (removing duplicates), we obtain a partition of each x_j -coordinate in \mathbb{R}^n , and a corresponding partition P of S .

The crucial property of this partition is that each of the cells in P is entirely contained in one of the sets S_i . This proves the lemma, because the content $\nu(S)$ is equal to the content of all the cells in P , while the content $\nu(S_i)$ is equal to the sum of the contents of the cells in the partition P that are contained in S_i . □

From this simple fact, a much more interesting result is derived.

Definition 2 *A subset $E \subseteq \mathbb{R}^n$ is called cellular if it can be divided as a disjoint countable union of cells.*

Proposition 3 *Let E be a cellular set, and let R_1, R_2, R_3, \dots be a sequence of cells (not necessarily disjoint) that covers E , i.e.,*

$$E \subseteq R_1 \cup R_2 \cup R_3 \cup \dots .$$

If $E = S_1 \cup S_2 \cup S_3 \cup \dots$ is any disjoint union of cells equal to E , then

$$\sum_{i=1}^{\infty} \nu(S_i) \leq \sum_{j=1}^{\infty} \nu(R_j).$$

Remark. We allow for the possibility that the sum $\sum^{\infty} \nu(S_i)$ diverges. In that case the inequality in the proposition states that $\sum^{\infty} \nu(R_j)$ also diverges.

Proof. We first prove that the statement is true if $E = S$ is just a single set.

Let A be a cell which is slightly smaller than $E = S$, with $\nu(A) = (1 - \varepsilon)\nu(S)$, and such that the compact closure \overline{A} is contained in S . And let B_j be a cell which is slightly larger than R_j , with $\nu(B_j) = (1 + \varepsilon)\nu(R_j)$, and such that the interior B_j^o contains R_j .

Then the open cover $\{B_j^o\}$ of the compact set \overline{A} has a finite subcover, and we find that

$$B_1 \cup \cdots \cup B_M \supset B_1^o \cup \cdots \cup B_M^o \supset \overline{A} \supset A.$$

Because we only have a finite number of cells now, a partition argument shows that

$$\nu(B_1) + \cdots + \nu(B_M) \geq \nu(A),$$

and so for every $\varepsilon > 0$, we have

$$(1 + \varepsilon) \sum_{j=1}^{\infty} \nu(R_j) \geq (1 - \varepsilon)\nu(S),$$

which gives the desired inequality.

For the general case $E = S_1 \cup S_2 \cup \cdots$, we argue by contradiction. If $\sum^{\infty} \nu(R_j) = \infty$, there is nothing to prove. So assume that this sum converges, and that

$$\sum_{i=1}^{\infty} \nu(S_i) > \sum_{j=1}^{\infty} \nu(R_j).$$

Then, by definition of the infinite sum

$$\sum_{i=1}^{\infty} \nu(S_i) = \lim_{N \rightarrow \infty} \sum_{i=1}^N \nu(S_i),$$

and because $\sum^{\infty} \nu(R_j)$ is finite, there is an integer N so that the partial sum satisfies

$$\sum_{i=1}^N \nu(S_i) > \sum_{j=1}^{\infty} \nu(R_j).$$

Define new cells (possibly empty)

$$C_{ij} = S_i \cap R_j.$$

Because the cells S_i are disjoint, the cells

$$C_{1j}, C_{2j}, \dots, C_{Nj}$$

are also disjoint. All of the cells C_{ij} with a fixed value for j are contained in the cell R_j . Therefore the previous lemma implies that

$$\nu(R_j) \geq \sum_{i=1}^N \nu(C_{ij}).$$

Therefore

$$\sum_{i=1}^N \nu(S_i) > \sum_{j=1}^{\infty} \nu(R_j) \geq \sum_{j=1}^{\infty} \left(\sum_{i=1}^N \nu(C_{ij}) \right) = \sum_{i=1}^N \left(\sum_{j=1}^{\infty} \nu(C_{ij}) \right).$$

This implies that for at least one value of $i = 1, \dots, N$ we must have

$$\nu(S_i) > \sum_{j=1}^{\infty} \nu(C_{ij}).$$

This inequality contradicts what we proved before if we can see that the cells C_{ij} with fixed i cover the cell S_i . This is true, because S_i is covered by the cells R_j . Every point $x \in S_i$ is contained in at least one R_j , but then also $x \in C_{ij} = S_i \cap R_j$. Therefore

$$S_i = \bigcup_{j=1}^{\infty} C_{ij}.$$

□

This result has important implications.

Corollary 4 *The total content of a cellular set $E = S_1 \cup S_2 \cup \dots$, defined as*

$$\nu(E) = \nu(S_1) + \nu(S_2) + \dots,$$

is independent of the way E is divided into disjoint cells.

Proof. If $E = R_1 \cup R_2 \cup \dots$ is another disjoint union of cells, then $\bigcup S_i$ is covered by $\bigcup R_j$, and so $\sum \nu(S_i) \leq \sum \nu(R_j)$. But now $\bigcup R_j$ is also covered by $\bigcup S_i$, which gives $\sum \nu(R_j) \leq \sum \nu(S_i)$.

□

Thus, we have a well-defined notion of the content of cellular sets.

Corollary 5 *If E is a cellular set covered by a countable union E_1, E_2, \dots of cellular sets, then*

$$\nu(E) \leq \sum_{i=1}^{\infty} \nu(E_i).$$

If the cellular set E is a countable disjoint union of cellular sets E_1, E_2, \dots , then

$$\nu(E) = \sum_{i=1}^{\infty} \nu(E_i).$$

Proof. Just write the sets E and E_i as disjoint unions of cells, and apply the proposition. □

A difficulty in developing content into a measure for arbitrary sets arises from the fact that the set-difference of two cellular sets is not necessarily cellular. In general, open sets are cellular, but compact sets are not.

Proposition 6 *Every open set in \mathbb{R}^n is cellular.*

Proof Let \mathbb{P} be the countable collection of cells of the form

$$S = \left[\frac{m_1}{2^k}, \frac{m_1 + 1}{2^k} \right) \times \dots \times \left[\frac{m_n}{2^k}, \frac{m_n + 1}{2^k} \right),$$

where k is a positive integer, and $m_i \in \mathbb{Z}, i = 1, \dots, n$. This collection \mathbb{P} has the property that for two cells S_1, S_2 in \mathbb{P} , either $S_1 \subseteq S_2$, or $S_2 \subseteq S_1$, or else $S_1 \cap S_2 = \emptyset$.

Let U be an open set. Every $x \in U$ is contained in an open ball $B_x \subseteq U$. For every $x \in U$ there exists a cell $S \in \mathbb{P}$ with $x \in S$ such that S is small enough to fit inside B_x . Then $S \subseteq U$. Let S_x be the largest cell from the collection \mathbb{P} that contains x , and that is contained in U .

Then for two points $x, y \in U$, either $S_x = S_y$, or else $S_x \cap S_y = \emptyset$. Thus, the cells S_x form a disjoint collection. It is a countable collection, because \mathbb{P} is countable. Since every $S_x \subseteq U$, and every $x \in U$ is contained in some S_x , the union of all S_x is precisely U . □

Proposition 7 *No compact set in \mathbb{R}^n is cellular.*

Proof. We argue by contradiction, assuming that the compact set K is a disjoint union $K = \bigcup S_i$ of cells.

Consider the x_1 coordinate as a continuous function

$$f(x) = f(x_1, x_2, \dots, x_n) = x_1.$$

If K is a compact set, there is a point $p \in K$ where $f(p) = p_1$ is maximal. By assumption, there exists a cell in K

$$S_i = [a_1, b_1) \times \cdots \times [a_n, b_n)$$

that contains p . From $p \in S_i$ we get $p_1 \in [a_1, b_1)$, and so in particular $p_1 < b_1$. But that means there are points in S_i , and hence in K , with larger x_1 -coordinate than p , contradicting the assumption. □

Exercise 1. Prove that the interior of a cell has the same content as the cell itself.

Exercise 2. Let $C \subset [0, 1]$ be a Cantor set, constructed as follows. Let $C_0 = [0, 1]$, and let U_0 be an open interval of length ε contained in C_0 , and $C_1 = C_0 \setminus U_0$. Then C_1 has two connected components. Let U_1^1, U_1^2 be two open intervals of length ε^2 each, one contained in each component of C_1 . Write $U_1 = U_1^1 \cup U_1^2$, and $C_2 = C_1 \setminus U_1$. Then C_2 has four connected components. In general, let $U_k^1, \dots, U_k^{2^k}$ be intervals of length ε^{k+1} each, one contained in every connected component of C_k . Write $U_k = U_k^1 \cup \cdots \cup U_k^{2^k}$, and $C_{k+1} = C_k \setminus U_k$.

The intersection

$$C = \bigcap_{i=1}^{\infty} C_i,$$

is a “fat” Cantor set. Show that the complement $[0, 1] \setminus C$ is cellular, with content strictly less than 1.

Exercise 3. Prove the following proposition.

Proposition 8 *Any countable union (not necessarily disjoint) of cellular sets is cellular.*

Exercise 4. Prove the following proposition.

Proposition 9 *Every finite intersection of cellular cells is cellular.*

Exercise 5. Let q_1, q_2, q_3, \dots be an arbitrary sequence of numbers in $[0, 1]$. Consider the union

$$E = \bigcup_{i=1}^{\infty} \left[q_i, q_i + \frac{1}{3^i} \right).$$

Prove that there exists a number $x \in [0, 1]$ that is not contained in the set E .

Hint: Use Proposition 8 and Proposition 3 to show that E cannot contain all points in $[0, 1]$.

3 Outer and Inner Measure

A convenient way to quantify the size of arbitrary sets is by comparison with cellular sets.

Definition 10 *Let $E \subseteq \mathbb{R}^n$. The outer measure $\mu^*(E)$ is the infimum*

$$\mu^*(E) = \inf \{ \nu(U) \mid E \subseteq U, U \text{ open} \}.$$

Proposition 11 *If E is cellular, then*

$$\mu^*(E) = \nu(E).$$

Proof. Since all open sets are cellular, $E \subseteq U$ implies $\nu(E) \leq \nu(U)$, and therefore $\nu(E) \leq \mu^*(E)$. We must prove the reverse inequality.

Let $E = S_1 \cup S_2 \cup \dots$ be a disjoint union of cells. Then

$$\nu(E) = \sum \nu(S_i).$$

For each cell S_i choose a slightly larger cell R_i , such that $\nu(R_i) = (1 + \varepsilon)\nu(S_i)$ and S_i is contained in the interior R_i^o of R_i .

Let $U_\varepsilon = \bigcup R_i^o$. Then $E \subseteq U$, and

$$\nu(U_\varepsilon) \leq \sum \nu(R_i) \leq (1 + \varepsilon) \sum \nu(S_i) \leq (1 + \varepsilon)\nu(E).$$

This can be done for any $\varepsilon > 0$, so

$$\mu^*(E) \leq \nu(E).$$

□

Definition 12 Let $E \subseteq \mathbb{R}^n$. The inner measure $\mu_*(E)$ is the supremum

$$\mu_*(E) = \sup \{ \mu^*(K) \mid K \subseteq E, K \text{ compact} \}.$$

If $K \subseteq E \subseteq U$, with K compact and U open, then $\mu^*(K) \leq \mu^*(U) = \nu(U)$. This implies the inequality,

$$\mu_*(E) \leq \mu^*(E).$$

Another obvious fact is that for compact sets,

$$\mu_*(K) = \mu^*(K).$$

We now show that for cells, all defined measures agree.

Lemma 13 Let $S \in \mathbb{R}^n$ be a cell, with interior S^o and closure \bar{S} .

Then the inner and outer measures of S , S^o , \bar{S} are all equal to the content of S .

Proof. Because $S^o \subset S \subset \bar{S}$, we have inequalities

$$\mu^*(S^o) \leq \mu^*(S) \leq \mu^*(\bar{S}).$$

If we can show that $\mu^*(\bar{S}) \leq \mu^*(S^o)$, then the three outer measures are equal. Take a cell R which is slightly larger than S , say $\nu(R) = \nu(S) + \varepsilon$, such that $\bar{S} \subset R^o$. We get $\mu^*(\bar{S}) \leq \mu^*(R^o)$. The interior R^o is an open set, and therefore a cellular set. Because R^o is cellular we have $\mu^*(R^o) = \nu(R^o)$, and by Exercise 1 we know that $\nu(R^o) = \nu(R)$. Therefore

$$\mu^*(\bar{S}) \leq \mu^*(R^o) = \nu(R^o) = \nu(R) = \nu(S) + \varepsilon = \nu(S^o) + \varepsilon = \mu^*(S^o) + \varepsilon.$$

This establishes

$$\nu(S) = \mu^*(S^o) = \mu^*(S) = \mu^*(\bar{S}).$$

Next we must consider the inner measures. Again, we can start with

$$\mu_*(S^o) \leq \mu_*(S) \leq \mu_*(\bar{S}).$$

This time take a cell R slightly smaller than S , with $\nu(R) = \nu(S) - \varepsilon$, such that $\bar{R} \subset S^o$. Then, by definition, $\mu^*(\bar{R}) \leq \mu_*(S^o)$. Because of what we proved about outer measures, we get

$$\mu^*(\bar{S}) = \nu(S) = \nu(R) + \varepsilon \leq \mu_*(S^o) + \varepsilon,$$

which proves $\mu^*(\bar{S}) \leq \mu_*(S^o)$. Finally, because the closure \bar{S} is a compact set, we have $\mu_*(\bar{S}) = \mu^*(\bar{S})$. So we have $\mu_*(\bar{S}) \leq \mu_*(S^o)$, which implies

$$\mu_*(S^o) = \mu_*(S) = \mu_*(\bar{S}) = \mu^*(\bar{S}).$$

□

4 Finite Measurable Sets

The content of open sets was defined by means of a cellular decomposition. An outer measure for arbitrary sets was defined by means of the content of opens sets. Finally, an inner measure for arbitrary sets was defined by means of the outer measure of compact sets. It turns out that for virtually all sets, inner and outer measure agree.

Definition 14 A set $E \subseteq \mathbb{R}^n$ with finite outer measure $\mu^*(E)$ is called a finite measurable set, if its inner measure is equal to its outer measure. In this case we define

$$\mu(E) = \mu_*(E) = \mu^*(E),$$

and call $\mu(E)$ the Lebesgue measure of E .

All we know so far is that compact sets and single cells are measurable. To get results about other sets, our aim in this section is to show how Lebesgue measure behaves with respect to countable unions and set differences.

We first prove *subadditivity* of outer measure. (Outer measure is not countably additive for general sets.)

Proposition 15 Let $E = E_1 \cup E_2 \cup E_3 \cup \dots$, be a union (not necessarily disjoint) of subsets of \mathbb{R}^n . Then

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

Proof. Let U_i be an open set such that $E_i \subseteq U_i$, and

$$\mu^*(E_i) > \mu(U_i) - \varepsilon 2^{-i}.$$

Then

$$\sum_{i=1}^{\infty} \mu^*(E_i) > \sum_{i=1}^{\infty} \mu(U_i) - \varepsilon.$$

Let $U = \bigcup_{i=1}^{\infty} U_i$, then U is an open set that covers E . From the theory of cellular sets, we know that

$$\sum_{i=1}^{\infty} \mu(U_i) \geq \mu(U),$$

and, of course, $\mu(U) \geq \mu^*(E)$. In summary, for every $\varepsilon > 0$

$$\sum_{i=1}^{\infty} \mu^*(E_i) > \mu^*(E) - \varepsilon.$$

□

As a corollary of this Proposition, we see that all *cellular sets*—and therefore, in particular, all open sets—are measurable.

Lemma 16 *If K_1, K_2 are two disjoint compact sets in \mathbb{R}^n , then*

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2).$$

Proof. By definition of the outer measure μ^* , there exists an open set U such that $K_1 \cup K_2 \subset U$, and

$$\mu(K_1 \cup K_2) > \mu(U) - \varepsilon.$$

A standard separation property says that for two disjoint compact sets K_1, K_2 there exists disjoint open sets V_1, V_2 such that $K_i \subset V_i$. Then let $U_i = U \cap V_i$, so that U_1, U_2 are disjoint open sets with $U_1 \cup U_2 \subseteq U$ and $K_i \subset U_i$. Then

$$\mu(K_1 \cup K_2) > \mu(U) - \varepsilon > \mu(U_1 \cup U_2) - \varepsilon.$$

But the measure of open sets is additive (because they are cellular sets), and we obtain

$$\mu(K_1 \cup K_2) > \mu(U_1) + \mu(U_2) - \varepsilon > \mu(K_1) + \mu(K_2) - \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we get the desired result.

□

This was just a special case that we need to prove the following more general statement.

Proposition 17 *If E_1, E_2, E_3, \dots are disjoint finite measurable sets, such that the sum $\sum_{i=1}^{\infty} \mu(E_i)$ is finite, then the countable union $E = E_1 \cup E_2 \cup E_3 \dots$ is measurable with finite measure*

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i).$$

Proof. By definition of the inner measure $\mu_*(E_i)$, there exists a compact set $K_i \subseteq E_i$ such that

$$\mu(K_i) > \mu(E_i) - 2^{-i}\varepsilon.$$

The previous lemma implies that

$$\mu(K_1 \cup \dots \cup K_N) = \sum_{i=1}^N \mu(K_i).$$

Then, by definition of inner measure,

$$\mu_*(E) \geq \sup_N \mu(K_1 \cup \dots \cup K_N) = \sum_{i=1}^{\infty} \mu(K_i) > \sum_{i=1}^{\infty} \mu(E_i) - \varepsilon.$$

Therefore

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i) = \sum_{i=1}^{\infty} \mu(E_i) \leq \mu_*(E).$$

□

We see how the proof of countable additivity of Lebesgue measure relies on an interesting back-and-forth between inner and outer measures, and compact and open sets. Another example of this phenomenon is the proof that Lebesgue measure is subtractive.

Lemma 18 *Let $E \subseteq \mathbb{R}^n$ be a finite measurable set. For every $\varepsilon > 0$ there exist compact K and open U such that,*

$$K \subseteq E \subseteq U$$

and

$$\mu(U \setminus K) < \varepsilon.$$

Proof. By definition of measurability, and of inner and outer measure, we can choose K, U such that $K \subseteq E \subseteq U$ while $\mu_*(E) - \mu(K) < \varepsilon$, and $\mu(U) - \mu^*(E) < \varepsilon$. If $\mu^*(E) = \mu_*(E)$, then

$$\mu(U) - \mu(K) < 2\varepsilon.$$

But the set $U \setminus K$ is open, hence measurable, and we have $\mu(U \setminus K) + \mu(K) = \mu(U)$, or

$$\mu(U \setminus K) = \mu(U) - \mu(K) < 2\varepsilon.$$

□

Proposition 19 *If E_1, E_2 are finite measurable sets, then so is the set difference $E_1 \setminus E_2$, and*

$$\mu(E_1 \setminus E_2) = \mu(E_1) - \mu(E_2).$$

Proof. Choose $K_i \subseteq E_i \subseteq U_i$ with $\mu(U_i \setminus K_i) < \varepsilon$. Because of the disjoint union

$$U_1 \setminus K_2 \subseteq (U_1 \setminus K_1) \cup (K_1 \setminus U_2) \cup (U_2 \setminus K_2),$$

we have

$$\mu^*(E_1 \setminus E_2) \leq \mu^*(U_1 \setminus K_2) \leq \mu^*(K_1 \setminus U_2) + 2\varepsilon.$$

Now $K_1 \setminus U_2$ is a compact subset of $E_1 \setminus E_2$, and therefore

$$\mu^*(E_1 \setminus E_2) \leq \mu_*(E_1 \setminus E_2) + 2\varepsilon.$$

Since this holds for all $\varepsilon > 0$, it follows that $E_1 \cup E_2$ is measurable.

The formula for the measure is a consequence of the disjoint union

$$E_1 = E_2 \cup (E_1 \setminus E_2).$$

□

5 Sets of Infinite Measure

To complete the construction of Lebesgue measure, we discuss the distinction between sets with *infinite measure* and sets that are *not measurable*. This is analogous to the difference between a series like $\sum_{i=1}^{\infty} i = \infty$, which can meaningfully be said to converge to infinity, and the divergent series $\sum_{i=1}^{\infty} (-1)^i$, which cannot.

Definition 20 *A measurable set is any countable union of finite measurable sets.*

The following proposition shows that there can be no confusion of terminology when we speak of ‘finite measurable set’ in the sense of Definition 14, or a ‘measurable set’ in the sense of Definition 20 that has finite (inner or outer) measure.

Proposition 21 *If $E \subseteq \mathbb{R}^n$ is a measurable set, then the following are equivalent: (1) E is a finite measurable set, in the sense of Definition 14, (2) $\mu^*(E)$ is finite, (3) $\mu_*(E)$ is finite.*

Proof. That (1) implies (2) is true by definition, and that (2) implies (3) is also obvious.

To see that (3) implies (1), we argue by contradiction. Let $E = E_1 \cup E_2 \cup \dots$ be a disjoint union, with E_i finite measurable, and $\sum \mu(E_i) = \infty$. Choose disjoint compact sets $K_i \subseteq E_i$ with $\mu(K_i) = (1 - \varepsilon)\mu(E_i)$. Then $K_1 \cup \dots \cup K_N$ is a compact set with measure $(1 - \varepsilon)\sum_{i=1}^N \mu(E_i)$, which converges to ∞ as $N \rightarrow \infty$. □

Observe that it is possible that $\mu_*(E) = \mu^*(E)$, while E is *not* measurable.

Definition 22 *If E is measurable, and $\mu_*(E) = \mu^*(E) = \infty$, then we say that E has infinite measure, and write $\mu(E) = \infty$.*

We can extend Proposition 17, about countable additivity of Lebesgue measure, to include infinite measurable sets.

Proposition 23 *Any countable union $E = E_1 \cup E_2 \cup \dots$ of measurable sets is measurable. Moreover, if any one $\mu(E_i) = \infty$, or if $\sum \mu(E_i) = \infty$, then $\mu(E) = \infty$.*

Proof. This follows directly from Definition 20 and Proposition 21. □

The following property of measurable sets is the main reason that we distinguish measurable sets with infinite measure from non-measurable sets.

Proposition 24 *The complement of a measurable set is measurable.*

Proof. Let $\mathbb{R}^n = S_1 \cup S_2 \cup \dots$ be some disjoint union of cells that fills the entire space \mathbb{R}^n . Suppose E is a measurable set, with $E = E_1 \cup E_2 \cup \dots$, where $\mu(E_i) < \infty$. Let

$$G_j = S_j \setminus \bigcup_{i=1}^{\infty} (E_i \cap S_j).$$

Because E_i and S_i are finite measurable, and $G_j \subset S_j$, the set G_j is also finite measurable (Proposition 17). The complement E^c of E is the countable union $E^c = G_1 \cup G_2 \cup \dots$. So E^c is measurable. □

An immediate corollary is the following.

Corollary 25 *Countable unions and countable intersections of measurable sets are measurable.*

Proof. This is so because $\cap E_i = (\cap E_i^c)^c$. □

We can now extend Lemma 18 to sets with infinite measure.

Proposition 26 *Let $E \subseteq \mathbb{R}^n$ be a measurable set. For every $\varepsilon > 0$ there exists a closed set F and an open set G such that,*

$$F \subseteq E \subseteq G$$

and

$$\mu(G \setminus F) < \varepsilon.$$

Proof. As in the proof of the previous proposition, let $\mathbb{R}^n = S_1 \cup S_2 \cup \dots$ with $\mu(S_i)$ finite. The sets $E_i = E \cap S_i$ are disjoint, and $E = \bigcup E_i$. Each E_i is measurable (Proposition 26), and of finite measure (Proposition 21, using $\mu^*(E_i) \leq \mu^*(S_i)$).

By Lemma 18 there exists an open set U_i with $E_i \subseteq U_i$ and $\mu(U_i \setminus E_i) < \varepsilon 2^{-i}$, for arbitrary choice of $\varepsilon > 0$. The open set $U = \bigcup U_i$ contains E , while $U \setminus E \subseteq \bigcup (U_i \setminus E_i)$ implies

$$\mu(U \setminus E) < \varepsilon.$$

We can take $G = U$. Applying the same idea to the measurable set E^c , we find an open set V with $E^c \subseteq V$ and $\mu(V \setminus E^c) < \varepsilon$. Take $F = V^c$. □

6 Borel Sets

The essential properties of the collection of measurable sets can be summed up in a definition.

Definition 27 *A σ -algebra Σ in a set X is a collection of subsets of X that has the following properties,*

- (1) *If E is in Σ , then the complement $E^c = X \setminus E$ is in Σ ,*
- (2) *if E_1, E_2, \dots are in Σ then the union $E_1 \cup E_2 \cup \dots$ is in Σ .*

Observe that the axioms of a σ -algebra imply that finite or countable intersections $E_1 \cap E_2 \cap \dots$ are also in Σ , because of the formula

$$\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{\infty} E_i^c \right)^c.$$

From what we have seen, the collection of measurable sets is a σ -algebra in \mathbb{R}^n . With the inclusion of sets of infinite measure, all open and closed sets are measurable. Countable intersections of open sets are called G_δ sets, while countable unions of closed sets are called F_σ sets. In turn, countable unions of G_δ sets are called $G_{\delta\sigma}$ sets, and countable intersections of F_σ sets are called $F_{\sigma\delta}$ sets. All sets that can be constructed from open and closed sets by repeated application of these basic set operations are called *Borel sets*. To make precise what we mean by ‘repeated application’ would involve the theory of transfinite ordinals (a fundamental aspect of the theory of infinite sets). But a very straightforward definition of Borel sets is as follows.

Definition 28 *The collection of Borel sets is the smallest σ -algebra in \mathbb{R}^n that contains all open sets.*

The following proposition describes the relationship between measurable sets and Borel sets.

Proposition 29 *A set $E \subseteq \mathbb{R}^n$ is measurable if and only if there exists an F_σ set A , and a G_δ set B with $A \subseteq E \subseteq B$ and such that $\mu(B \setminus A) = 0$.*

Less specifically, a set is measurable if and only if it differs from a Borel set by a set of measure zero.

Proof. Suppose E is measurable. For $j = 1, 2, \dots$ choose closed sets F_j and open sets G_j with $F_j \subseteq E \subseteq G_j$ and $\mu(G_j) - \mu(F_j) < j^{-1}$. Then $A = F_1 \cup F_2 \cup \dots$, and $B = G_1 \cap G_2 \cap \dots$ satisfy the conditions.

Conversely, suppose $A \subseteq E \subseteq B$, as stated. Then, since every F_σ and G_δ set is measurable, A and B are measurable. It follows that $\mu^*(E \setminus A) \leq \mu^*(B \setminus A) = 0$, and therefore $E \setminus A$ is measurable with measure zero. Then also $E = A \cup (E \setminus A)$ is measurable. □

The crucial properties of Lebesgue measure are abstracted in the following definition of a general measure.

Definition 30 *A measure μ on a σ -algebra Σ in a set X is a function*

$$\mu: \Sigma \rightarrow [0, \infty]$$

that assigns a non-negative real number $\mu(E)$ (possibly ∞) to every set $E \in \Sigma$. A measure must be countably additive, which means that if E_1, E_2, E_3, \dots is a sequence of disjoint sets in Σ , then

$$\mu(E_1 \cup E_2 \cup \dots) = \sum_{i=1}^{\infty} \mu(E_i).$$

A measure space is a set X , together with a σ -algebra Σ of subsets in X , and a measure μ defined for sets in Σ .

We have seen that Lebesgue measure satisfies all the necessary properties. Its σ -algebra is the collection of all measurable sets, which includes all Borel sets, and all sets of measure zero. While it is true that there exist subsets of \mathbb{R}^n that are not Lebesgue measurable, it is not possible to construct an example of such a set by means of explicit operations. Non-measurable sets can only be defined indirectly, by means of a notorious (if not controversial) axiom of set theory called the *axiom of choice*. Therefore, in practice, every set you will ever encounter will be measurable.

In short, Lebesgue measure is a very powerful extension of the classical notion of volume (or length, or area). It applies to virtually every kind of set that plays a role in analysis, while at the same time retaining the important property of (countable) additivity.