1. Find the area of the region enclosed by the graph of the function $y = x^3$ and the line $y = 9x$.

   A.) 27 B.) $\frac{27}{2}$ C.) $\frac{27}{4}$ D.) 81 E.) $\frac{81}{2}$ F.) $\frac{81}{4}$

   The correct answer is E.)

   The graphs of $y = x^3$ and $y = 9x$ intersect at $x = -3, 0, +3$, and the area enclosed by these graphs consists of two equal parts, one from $x = -3$ to $x = 0$, the other from $x = 0$ to $x = 3$. For one of these parts we find the area by means of integration,

$$\int_0^3 (9x - x^3)dx = \left[\frac{9}{2}x^2 - \frac{1}{4}x^4\right]_0^3 = \frac{81}{4}.$$

   The area is 2 times this integral, so $\frac{81}{2}$.

2. Find the volume of the solid obtained by rotating the region bounded by the curve $y = x^2$ and the lines $y = 0$ and $x = 2$ and $x = 4$ about the y-axis as the axis of rotation.

   A.) 105π B.) 110π C.) 115π D.) 120π E.) 125π F.) 130π

   The correct answer is D.)

   The method of cylindrical shells is preferable here. The volume of a single shell is $2\pi rh\,dx$. We see from a sketch of the graph that the radius of the shell is $r = x$, while the height equals $h = x^2$. Therefore,

$$V = 2\pi \int_2^4 x\,x^2\,dx = 2\pi \int_2^4 x^3 dx = 2\pi \left[\frac{1}{4}x^4\right]_2^4 = 120\pi.$$

3. Find the volume of the solid obtained by rotating the region bounded by the curve $y = \sqrt{x}$ and the lines $y = 0$ and $x = 1$, about the axis of rotation $y = -1$.

A.) $\frac{11}{5}\pi$ B.) $\frac{11}{6}\pi$ C.) $\frac{17}{8}\pi$ D.) $\frac{25}{12}\pi$ E.) $\frac{32}{15}\pi$ F.) $\frac{29}{16}$

The correct answer is B.)

You can use washers or shells for this problem. Using shells with volume $2\pi rh\,dy$ (a y-integral!), we have radius $r = 1 + y$, height $h = 1 - y^2$ (because $y = \sqrt{x}$ gives $x = y^2$). And so,

$$V = 2\pi \int_0^1 (1+y)(1-y^2)dy = 2\pi \int_0^1 (1 + y - y^2 - y^3)dy$$
$$= 2\pi \left[y + \frac{1}{2}y^2 - frac13y^3 - \frac{1}{4}y^4 \right]_0^1 = \frac{11}{6}\pi.$$

If you want to use washers with volume $\pi(R^2 - r^2)\,dx$, you have to set up an x-integral. Inner radius $r = 1$, outer radius $R = 1 + x^{1/2}$, and for a single washer you find,

$$\pi[(1 + x^{1/2}]^2 - 1^2 = \pi(2x^{1/2} + x)dx.$$

Integration gives the total volume,

$$V = \pi \int_0^1 (2x^{1/2} + x)dx = \pi \left[\frac{4}{3}x^{3/2} + \frac{1}{2}x^2 \right]_0^1 = \frac{11}{6}\pi.$$

4. Find the volume of the solid obtained by rotating the region bounded by the curves $y^4 = x$ and $y = x^2$ about the y-axis.

A.) $\frac{7}{18}\pi$ B.) $\frac{7}{15}\pi$ C.) $\frac{1}{2}\pi$ D.) $\frac{2}{3}\pi$ E.) $\frac{7}{12}\pi$ F.) $\frac{13}{24}\pi$

The correct answer is A.)

As before, for this problem can be solved using either washers or shells. A single shell has volume $2\pi rh\,dx$, and we must set up an x-integral. We find $r = x$, and $h = x^{1/4} - x^2$, and

$$V = 2\pi \int_0^1 (x^{5/4} - x^3)dx = 2\pi \left[\frac{4}{9}x^{9/4} - \frac{1}{4}x^4 \right] = \frac{7}{18}\pi.$$

Using washers, you must evaluate the y-integral $V = \pi \int_0^1 (R^2 - r^2)dy$, where the outer radius is $R = y^{1/2}$, and the inner radius $r = y^4$. So,

$$V = \pi \int_0^1 (y - y^8)dy = \pi \left[\frac{1}{2}y^2 - \frac{1}{9}y^9 \right]_0^1 = \frac{7}{18}\pi.$$

5. Find the average value of the function

$$f(x) = x\sin(x^2)$$

on the interval $[0, \sqrt{\pi}]$.

A.) $\frac{1}{\sqrt{2\pi}}$ B.) $\frac{1}{2\pi}$ C.) $\frac{1}{2\sqrt{\pi}}$ D.) $\frac{2}{\pi}$ E.) $\frac{1}{\sqrt{\pi}}$ F.) $\frac{1}{\pi}$

The correct answer is E.)

First evaluate the integral

$$\int_0^{\sqrt{\pi}} x\sin x^2 dx.$$

To simplify the sine expression, we use the substition $u = x^2$, $du = 2x\,dx$. To fix the boundaries of the new integral, note that if $x = 0$ then $u = 0$, and if $x = \sqrt{\pi}$ then $u = \pi$. The new integral becomes

$$\int_0^{\pi} \frac{1}{2}\sin u\,du = \frac{1}{2}[\cos u]_0^{\pi} = 1.$$

To get the average, we must divide by the length of the x-interval, which is $\sqrt{\pi}$. The result is $1/\sqrt{\pi}$.

6. The function $f(x) = 2x^3 - 3x^2 + 6x$ is one-to-one. Find $(f^{-1})'(5)$.

A.) $\frac{1}{6}$ B.) $\frac{1}{3}$ C.) $\frac{1}{2}$ D.) $\frac{2}{3}$ E.) $\frac{5}{6}$ F.) $\frac{3}{2}$

The correct answer is A.)

We use the formula

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))}.$$

We know that $f'(x) = 6x^2 - 6x + 6$, and we must find the value of $f^{-1}(5)$. But $f^{-1}(5)$ is just the solution to $f(x) = 5$, or $2x^3 - 3x^2 + 6x = 5$. Guess-and-check gives the solution $x = 1$. So $f(1) = 5$ and therefore $f^{-1}(5) = 1$. Then

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(1)} = \frac{1}{6}.$$

7. Find the formula for the inverse function $f^{-1}(x)$, if

$$f(x) = \frac{e^x}{e^x + 2}.$$

A.) $\ln \frac{2x}{1-x}$ B.) $\ln \frac{1-x}{2x}$ C.) $\ln \frac{3x}{x+2}$ D.) $\ln \frac{x+2}{3x}$ E.) $\ln \frac{3x}{x-2}$ F.) $\ln \frac{x-2}{3x}$

The correct answer is A.)

Swapping x and y we have

$$x = \frac{e^y}{e^y + 2}.$$

Solving for y will give us the inverse function step-by-step,

$$\begin{aligned}
&x(e^y + 2) = e^y,\ xe^y + 2x = e^y,\\
&xe^y - e^y = -2x,\ (x-1)e^y = -2x,\\
&e^y = \frac{-2x}{x-1} = \frac{2x}{1-x},\\
&y = \ln \frac{2x}{1-x}.
\end{aligned}$$

8. Let

$$f(x) = \ln x\sqrt{x^2+1}.$$

Find $f'(1)$.

A.) $\frac{1}{6}$ B.) $\frac{1}{3}$ C.) $\frac{1}{2}$ D.) $\frac{2}{3}$ E.) $\frac{5}{6}$ F.) $\frac{3}{2}$

The correct answer is F.)

The easiest method is to first rewrite the logarithm as,

$$\ln x\sqrt{x^2+1} = \ln x + \ln \sqrt{x^2+1} = \ln x + \frac{1}{2}\ln(x^2+1).$$

Now take derivatives, using the chain rule for the second term,

$$\frac{1}{x} + \frac{1}{2}\cdot 2x \cdot \frac{1}{x^2+1} = \frac{1}{x} + \frac{x}{x^2+1}.$$

Plug in $x = 1$ and you get $1 + \frac{1}{2} = \frac{3}{2}$.

9. At what x-value does the function

$$f(x) = x^3 e^{-x}$$

attain it's maximum? *(Recall that at a maximum, the slope of the graph equals zero.)*

A.) $x = \sqrt{3}$ B.) $x = 2$ C.) $x = \frac{5}{2}$ D.) $x = e$ E.) $x = 3$ F.) $x = \frac{7}{2}$

The correct answer is E.)

To find $f'(x)$ we use the product rule,

$$f'(x) = 3x^2 e^{-x} + x^3(-e^{-x}) = (3x^2 - x^3)e^{-x}.$$

Here we have used the fact that the derivative of e^{-x} is $-1 \cdot e^{-x}$ by application of the chain rule. Because e^{-x} is always positive, $f'(x) = 0$ means that $3x^2 - x^3 = 0$. This factors as $x^2(3 - x) = 0$, and the solutions are $x = 0$ and $x = 3$. At $x = 0$ we have $f(0) = 0$, while at $x = 3$ the function-value is positive. So at $x = 3$ the function attains it's maximum value.

(Note: strictly speaking you would have to verify that at $x = 3$ the second derivative $f''(3)$ is negative, so that you really have a maximum, and not a minimum. But on a multiple choice test that is not required.)

10. Evaluate the integral

$$\int_0^1 \frac{x^2}{x^3 + 1} dx.$$

A.) $\frac{1}{4}\ln 2$ B.) $\frac{1}{2}\ln 4$ C.) $\frac{1}{4}\ln 3$ D.) $\frac{1}{3}\ln 4$ E.) $\frac{1}{2}\ln 3$ F.) $\frac{1}{3}\ln 2$

The correct answer is F.)

Substitute $u = x^3 + 1$. Then $du = 3x^2\,dx$, and we get

$$\int_0^1 \frac{x^2}{x^3 + 1} dx = \frac{1}{3}\int_1^2 \frac{1}{u} du = \frac{1}{3}\left[\ln x\right]_1^2 = \frac{1}{3}[\ln 2 - \ln 1] = \frac{1}{3}\ln 2.$$

The boundaries for this integral are obtained by calculating $u = x^3 + 1$ for $x = 0$ and $x = 1$.