1. Evaluate the limit

$$\lim_{x\to 0} \frac{x^2+x}{\sin^{-1} 3x}.$$

A.) $\frac{1}{9}$ B.) $\frac{1}{3}$ C.) 0 D.) 3 E.) 9 F.) ∞

Using l'Hospital's rule once, we find

$$\lim_{x\to 0} \frac{x^2+x}{\sin^{-1} 3x} = \lim_{x\to 0} \frac{2x+1}{\frac{3}{\sqrt{1-9x^2}}}.$$

The new limit is directly evaluated by plugging in $x=0$. We get,

$$\frac{0+1}{\frac{3}{\sqrt{1-0}}} = \frac{1}{3}.$$

The correct answer is B.)

2. Evaluate the integral

$$\int_0^{\ln 2} e^x \ln\left(e^x+1\right) dx.$$

A.) $-2+\ln\frac{9}{2}$ B.) $-1+\ln\frac{27}{2}$ C.) $\ln 8$ D.) $-2+\ln\frac{9}{4}$ E.) $-1+\ln\frac{27}{4}$ F.) ∞

The substitution $u=e^x+1$, $du=e^x dx$ leads to

$$\int_0^{\ln 2} e^x \ln\left(e^x+1\right) dx = \int_2^3 \ln u \, du.$$

Here we have replaced the boundaries of integration ($x=0$ gives $u=2$, while $x=\ln 2$ gives $u=3$). The new integral is solved by integration by parts

$$\int 1\cdot \ln u \, du = u\ln u - \int u\frac{1}{u}\, du = u\ln u - u.$$

The definite integral is

$$[u\ln u - u]_2^3 = [3\ln 3 - 3] - [2\ln 2 - 2] = \ln 3^3 - 3 - \ln 2^2 + 2 = -1 + \ln\frac{27}{4}.$$

The correct answer is E.)

3. Evaluate the integral

$$\int_2^4 \frac{x^2+x+1}{x^2+x-2}\,dx.$$

A.) $3+\ln\frac{2}{3}$ B.) $\frac{5}{2}+\ln\frac{3}{2}$ C.) $2+\ln 2$ D.) $\frac{3}{2}+\ln\frac{9}{4}$ E.) $1+\ln 4$ F.) ∞

The first step is a long division, followed by factorization of the denominator,

$$\frac{x^2+x+1}{x^2+x-2} = 1+\frac{3}{x^2+x-2} = 1+\frac{3}{(x+2)(x-1)}.$$

The method of partial fraction gives

$$\frac{3}{(x+2)(x-1)} = \frac{A}{x+2}+\frac{B}{x-1}.$$

Equating denominators, we get $A(x-1)+B(x+2)=3$, which is solved by $B=1$ and $A=-1$. The result is the integral

$$\begin{aligned}\int_2^4 \left(1-\frac{1}{x+2}+\frac{1}{x-1}\right)dx &= [x-\ln(x+2)+\ln(x-1)]_2^4 \\ &= [4-\ln 6+\ln 3]-[2-\ln 4+\ln 1] = 2+\ln\frac{3\cdot 4}{6\cdot 1} \\ &= 2+\ln 2.\end{aligned}$$

The correct answer is C.)

4. Evaluate the integral

$$\int_1^e \frac{1}{x\ln x}\,dx.$$

A.) $\ln\frac{1}{3}$ B.) $\ln\frac{2}{3}$ C.) $\ln\frac{3}{2}$ D.) $\ln\frac{1}{4}$ E.) $\ln\frac{3}{4}$ F.) ∞

Substitute $u=\ln x$, $du=\frac{1}{x}\,dx$. You get,

$$\int_1^e \frac{1}{x\ln x}\,dx = \int_0^1 \frac{1}{u}\,du = [\ln u]_0^1.$$

This is an improper integral, and since

$$\lim_{t\downarrow 0} \ln u = -\infty,$$

the integral diverges. The correct answer is F.)

5. Evaluate the integral

$$\int_0^{\pi/12} \frac{\tan^2 3x}{\cos^2 3x}\, dx.$$

A.) $\frac{1}{3}$ B.) $\frac{1}{4}$ C.) $\frac{1}{6}$ D.) $\frac{1}{9}$ E.) $\frac{1}{12}$ F.) ∞

The trick here is to use

$$\frac{\tan^2 3x}{\cos^2 3x} = \tan^2 3x \sec^2 3x.$$

Then substitute $u = \tan 3x$, $du = 3\sec^2 3x\, dx$, as follows,

$$\int_0^{\pi/12} \tan^2 3x \sec^2 3x\, dx = \int_0^1 \frac{1}{3}u^2 du = \left[\frac{1}{9}u^3\right]_0^1 = \frac{1}{9}.$$

We used the fact that if $x = \frac{\pi}{12}$, then $u = \tan\frac{\pi}{4} = 1$, while $x = 0$ gives $u = \tan 0 = 0$.

The correct answer is D.)

6. Evaluate the integral

$$\int_0^1 \frac{1}{\sqrt{x^2+4x}} dx.$$

A.) $\ln\frac{3+\sqrt{5}}{2}$ B.) $\ln(2+\sqrt{6})$ C.) $\ln\frac{1+\sqrt{7}}{4}$ D.) $1-\ln 2$ E.) $\ln\frac{-1+\sqrt{10}}{3}$ F.) ∞

Complete the square to,

$$x^2 + 4x = (x+2)^2 - 4.$$

Then substitute $u = x + 2$ to get

$$\int_0^1 \frac{1}{\sqrt{(x+2)^2-4}} dx = \int_2^3 \frac{1}{\sqrt{u^2-4}} du.$$

This integral calls for a trigonometric substitution

$$u = 2\sec t,\ du = 2\sec t\tan t,\ \sqrt{u^2-4} = 2\tan t.$$

Ignoring the boundaries of the integral, we find

$$\int \frac{1}{\sqrt{u^2-4}} du = \int \frac{2\sec t\tan t}{2\tan t} dt = \int \sec t\, dt = \ln|\sec t + \tan t| = \ln\left|\frac{1}{2}u + \frac{1}{2}\sqrt{u^2-4}\right|.$$

Then evaluate

$$\left[\ln\left|\frac{1}{2}u + \frac{1}{2}\sqrt{u^2-4}\right|\right]_2^3 = \ln\left|\frac{3+\sqrt{5}}{2}\right| - \ln\left|\frac{2+0}{2}\right| = \ln\frac{3+\sqrt{5}}{2}.$$

The correct answer is A.)

7. Find an *approximate* value for the integral

$$\int_1^5 \frac{1}{x}\,dx$$

by using *Simpson's Rule*, dividing the interval of integration into 4 equal parts. (In other words, calculate S_4.)

Remark. The *exact* value of the integral is $\ln 5 = 1.609\cdots$.

A.) $\frac{8}{5} = 1.600\cdots$ B.) $\frac{19}{12} = 1.583\cdots$ C.) $\frac{29}{18} = 1.611\cdots$
D.) $\frac{51}{32} = 1.593\cdots$ E.) $\frac{73}{45} = 1.622\cdots$ F.) $\frac{97}{60} = 1.616\cdots$

We have $\Delta x = 1$, so

$$\begin{aligned} S_4 &= \frac{1}{3}\left(f(1) + 4f(2) + 2f(3) + 4f(4) + f(1)\right) \\ &= \frac{1}{3}\left(1 + 4\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + 4\cdot\frac{1}{4} + \frac{1}{5}\right) \\ &= \frac{1}{3}\cdot\frac{73}{15} = \frac{73}{45}. \end{aligned}$$

The correct answer is E.)

8. What is the arc length of the curve

$$y = \ln(\cos x),$$

on the interval

$$\frac{\pi}{4} \le x \le \frac{\pi}{3}.$$

A.) $\ln(4+\sqrt{3})$ B.) $\ln\frac{3}{4+\sqrt{2}}$ C.) $\ln\frac{\sqrt{3}}{\sqrt{2}}$ D.) $\ln(2+\sqrt{2})$ E.) $\ln\frac{2+\sqrt{3}}{1+\sqrt{2}}$ F.) $\ln\frac{4}{1+\sqrt{3}}$

Using the chain rule we find the derivative

$$\frac{dy}{dx} = \sin x \cdot \frac{1}{\cos x} = \tan x.$$

Then the arc length formula gives

$$\begin{aligned} L &= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_{\pi/4}^{\pi/3} \sqrt{1+\tan^2 x}\,dx = \int_{\pi/4}^{\pi/3} \sqrt{\sec^2 x}\,dx = \int_{\pi/4}^{\pi/3} \sec x\,dx \\ &= \left[\ln|\sec x + \tan x|\right]_{\pi/4}^{\pi/3} = \ln(2+\sqrt{3}) - \ln(\sqrt{2}+1). \end{aligned}$$

The correct answer is E.)

9. What is the surface area of the solid of revolution obtained by rotating the graph of $x = y^3$ about the y-axis? Consider the piece of the graph between the points $(x, y) = (0, 0)$ and $(1, 1)$.

A.) $\frac{26}{27}\pi$ B.) $\frac{-1+10\sqrt{10}}{27}\pi$ C.) π D.) $\frac{16\sqrt{3}}{9}\pi$ E.) $\frac{10\sqrt{3}}{9}\pi$ F.) 2π

Using the "zipped" formula $S = \int 2\pi x \, ds$, and one of the formulas for the line element

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)} \, dy,$$

we get the "unzipped" surface area formula

$$\begin{aligned} S &= \int 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \\ &= \int_0^1 2\pi y^3 \sqrt{1 + (3y^2)^2} \, dy = 2\pi \int_0^1 y^3 \sqrt{1 + 9y^4} \, dy. \end{aligned}$$

Substitution $u = 1 + 9y^4$, $du = 36y^3$, gives

$$S = \frac{2\pi}{36} \int_1^{10} u^{1/2} du = \left[\frac{\pi}{18} \cdot \frac{2}{3} u^{3/2}\right]_1^{10} = \frac{\pi}{27}(10\sqrt{10} - 1).$$

The correct answer is B.)

10. The possible values of a continuous stochastic variable x range between $x = 0$ and $x = \pi/2$. The probability density function is given by

$$f(x) = \sin x, \quad 0 \leq x \leq \frac{\pi}{2}.$$

Which of the following statements is correct? (Select only one answer.)

A.) The *mean* of x is $\mu = 1$.

B.) The *mean* of x is $\mu = \pi/6$.

C.) The *mean* of x is $\mu = 2/\pi$.

D.) The *median* of x is $m = 1$.

E.) The *median* of x is $m = \pi/6$.

F.) The *median* of x is $m = \pi/4$.

We find the median m by solving

$$\int_0^m \sin x \, dx = \frac{1}{2}.$$

From

$$\int_0^m \sin x \, dx = [-\cos x]_0^m = -\cos m + \cos 0 = -\cos m + 1.$$

we get the equation $-\cos m + 1 = \frac{1}{2}$, with solution $m = \pi/3$. This is not one of the answer choices.

The mean μ simply follows from the formula

$$\mu = \int x f(x) \, dx = \int_0^{\pi/2} x \sin x \, dx.$$

Integration by parts solves the integral

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x.$$

Thus,

$$\mu = [-x \cos x + \sin x]_0^{\pi/2} = [-\frac{\pi}{2} \cdot 0 + 1] - [-0 \cdot 1 + 0] = 1.$$

The correct answer is A.).