1. At which points (x, y) is the tangent to the graph of the parametrized curve

$$x = t^3 - 3t$$
$$y = 3t^2$$

a vertical line?

A.) $(0, 0)$

B.) $(2, 3)$

C.) $(2, 3)$ and $(-2, 3)$

D.) $(2, 12)$

E.) $(2, 12)$ and $(-2, 12)$

F.) $(2, 3)$ and $(2, 12)$

The slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t}{3t^2 - 3}.$$

A vertical tangent corresponds to an infinite slope, i.e., we want the denominator $3t^2 - 3 = 0$, and the numerator $6t$ not equal to zero. This means that $t = \pm 1$. If $t = +1$, you get $(x, y) = (-2, 3)$, and if $t = -1$ you get $(2, 3)$.

The correct answer is C.)

2. Find the arc length of the parametrized curve

$$x = t^3 - 3t$$
$$y = 3t^2$$

for the segment of the curve between the points $(x, y) = (0, 0)$ and $(2, 12)$.

A.) 13 B.) 14 C.) 15 D.) 16 E.) 17 F.) 18

The point $(0, 0)$ corresponds to $t = 0$, while $(2, 12)$ corresponds to $t = 2$. You find these t values by solving $y = 3t^2 = ...$, and then verifying if $x = t^3 - 3t$ has the correct value.

The arc length is now calculated by an integral,

$$\begin{aligned} L &= \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^2 \sqrt{(3t^2 - 3)^2 + (6t)^2}\, dt = \int_0^2 \sqrt{9t^4 + 18t^2 + 9}\, dt \\ &= \int_0^2 \sqrt{(3t^2 + 3)^2}\, dt = \int_0^2 (3t^2 + 3)\, dt = [t^3 + 3t]_0^2 \\ &= 14. \end{aligned}$$

The correct answer is B.)

3. Find the area of a single leaf (contained inside a single loop) of the 'calculus flower'

$$r = 2\cos 3\theta.$$

A.) $\frac{\pi}{4}$ B.) $\frac{\pi}{3}$ C.) $\frac{\pi}{2}$ D.) π E.) $\frac{5\pi}{4}$ F.) $\frac{3\pi}{2}$

Solving $r = 0$ gives $\cos 3\theta = 0$, or $3\theta = \frac{\pi}{2} + n\pi$, and so

$$\theta = \frac{\pi}{6} + n\frac{\pi}{3}, \quad n = 0, 1, 2, \ldots.$$

For a single loop we can take θ between $\pi/6$ and $\pi/6 + \pi/3 = \pi/2$. The area is calculated by means of the following integral,

$$\begin{aligned} A &= \int_{\pi/6}^{\pi/2} \frac{1}{2} r^2 \, d\theta \\ &= \int_{\pi/6}^{\pi/2} \frac{1}{2} \cdot 4\cos^2 3\theta \, d\theta = \int_{\pi/6}^{\pi/2} 2\cos^2 3\theta \, d\theta \\ &= \int_{\pi/6}^{\pi/2} (1 + \cos 6\theta) \, d\theta = [\theta + \frac{1}{6}\sin 6\theta]_{\pi/6}^{\pi/2} \\ &= \frac{\pi}{3}. \end{aligned}$$

The correct answer is B.)

4. Evaluate the limit

$$\lim_{n\to\infty} \frac{2n+1}{\sqrt{1+2n^2}}.$$

A.) 0 B.) $\frac{1}{2}$ C.) 1 D.) $\sqrt{2}$ E.) 2 F.) ∞

Concentrate on the dominant terms in the numerator and denominator,

$$\lim_{n\to\infty} \frac{2n+1}{\sqrt{1+2n^2}} = \lim_{n\to\infty} \frac{2n}{\sqrt{2n^2}} = \lim_{n\to\infty} \frac{2n}{n\sqrt{2}} = \lim_{n\to\infty} \frac{2}{\sqrt{2}} = \sqrt{2}.$$

The correct answer is D.)

5. Find the value of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2}^n}.$$

A.) $\frac{1}{2-\sqrt{2}}$ B.) $\frac{\sqrt{2}}{2-\sqrt{2}}$ C.) $\frac{1}{\sqrt{2}-1}$ D.) $\frac{\sqrt{2}}{\sqrt{2}-1}$ E.) $\frac{1}{n!\sqrt{2}}$ F.) ∞

This is a geometric series,

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n.$$

The first term in the series is $a = 1/\sqrt{2}$ (here that is the $n = 1$ term), and ratio $r = 1/\sqrt{2}$. By the formula for geometric series, the sum is

$$\frac{a}{1-r} = \frac{1/\sqrt{2}}{1-1/\sqrt{2}} = \frac{1}{\sqrt{2}-1}.$$

The correct answer is C.)

6. Determine how many of the following five series are convergent,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}, \quad \sum_{n=1}^{\infty} \frac{n}{\sqrt{n+1}}, \quad \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n.$$

A.) None B.) One C.) Two D.) Three E.) Four F.) Five

The comparison theorem applies to the first three cases. They are comparable to the following series, respectively,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2}}, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}.$$

All of these are p-series, with $p = \frac{1}{2}, 1, \frac{3}{2}$, respectively. So the first two diverge ($p \leq 1$), and the third one converges ($p > 1$).

Number four is divergent by the divergence test, because

$$\lim_{n\to\infty} \frac{n}{\sqrt{n+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{n} = \infty.$$

Number five is a geometric series, with ratio $r = \pi/4$, which is less than 1, and therefore the geometric series is convergent.

In summary, only numbers three and five are convergent. The correct answer is C.)

7. For the three series

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n} \qquad (I)$$
$$\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n} \qquad (II)$$
$$\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n^3} \qquad (III)$$

which of the following statements is true?

A.) (I), (II) and (III) converge conditionally

B.) (I), (II) and (III) converge absolutely

C.) (I) converges conditionally, (II) and (III) converge absolutely

D.) (I) converges conditionally, (II) diverges, (III) converges condionally

E.) (I) and (II) diverge, (III) converges absolutely

F.) (I) and (II) converge conditionally, (III) converges absolutely

All three series are alternating series, and since the terms converge to zero, the series is convergent. We need to see which of the three converge absolutely, i.e., we need to inspect the series

$$\sum_{n=1}^{\infty}\frac{1}{n},\ \sum_{n=1}^{\infty}\frac{\ln n}{n},\ \sum_{n=1}^{\infty}\frac{\ln n}{n^3}.$$

The first one diverges (p-series with $p = 1$). The second one diverges, because

$$\frac{\ln n}{n} > \frac{1}{n}$$

(at least if $n \geq 3$), which allows us to compare the second series to the first series. The third series converges, again by comparison. Because $\ln n < n$, we have

$$\frac{\ln n}{n^3} < \frac{1}{n^2}.$$

And so we can compare with a p-series with $p = 2$, which converges.

In summary, (I) and (II) converge, but not absolutely, and only (III) converges absolutely. The correct answer is F.)

8. Determine the interval of convergence for the following power series,

$$\sum_{n=1}^{\infty}(2x-4)^n.$$

A.) $(0,4)$ B.) $[0,4)$ C.) $(0,4]$ D.) $(1\frac{1}{2},2\frac{1}{2})$ E.) $[1\frac{1}{2},2\frac{1}{2})$ F.) $(1\frac{1}{2},2\frac{1}{2}]$

First we need to bring the series in the standard form

$$\sum c_n(x-a)^n.$$

To do this, we write

$$2x-4=2(x-2),$$

and then

$$\sum(2x-4)^n=\sum[2(x-2)]^n=\sum 2^n(x-2)^n.$$

We see that $a=2$, and $c_n=2^n$. The center of the interval of convergence is therefore at $a=2$. The radius of convergence follows from $R=1/L$, where L is the limit obtained by the ratio test (for power series),

$$L=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=\lim_{n\to\infty}\frac{2^{n+1}}{2^n}=2,$$

and so $R=1/2$. It follows that the series converges at least on the interval $(a-R,a+R)=(1\frac{1}{2},2\frac{1}{2})$. Finally, we need to decide whether to include the boundary points in the interval. We try each one in turn.

For $x=1\frac{1}{2}$, the power series becomes an ordinary series,

$$\sum_{n=1}^{\infty}(2\cdot(1\frac{1}{2})-4)^n=\sum_{n=1}^{\infty}(-1)^n.$$

This is an alternating series, but it diverges (by the divergence test). For $x=1$ we get

$$\sum_{n=1}^{\infty}(2\cdot(2\frac{1}{2})-4)^n=\sum_{n=1}^{\infty}1^n=\infty,$$

which is also divergent. We conclude that neither of the boundary points is included in the interval. The correct answer is D.)

9. Find the MacLaurin Series for the function $F(x)$ defined by the integral

$$F(x) = \int_0^x \frac{1}{1+t^3}\, dt.$$

A.) $x + x^3 + x^5 + x^7 + \cdots$

B.) $1 - x^3 + x^6 - x^9 + \cdots$

C.) $x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \cdots$

D.) $1 + \frac{1}{3}x^3 + \frac{1}{5}x^6 + \frac{1}{7}x^9 + \cdots$

E.) $x - \frac{1}{7!}x^7 + \frac{1}{13!}x^{13} - \frac{1}{19!}x^{19} + \cdots$

F.) $1 - x^6 + x^{12} - x^{18} + \cdots$

We proceed step by step, starting with the standard series,

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots .$$

Substituting $-t^3$ for t, we get

$$\begin{aligned}\frac{1}{1+t^3} &= 1 + (-t^3) + (-t^3)^2 + (-t^3)^3 + \cdots \\ &= 1 - t^3 + t^6 - t^9 + \cdots\end{aligned}$$

Now we can integrate term by term, as if the series is an ordinary polynomial, and get

$$\int \frac{1}{1+t^3} == t - \frac{1}{4}t^4 + \frac{1}{7}t^7 - \frac{1}{10}t^{10} + \cdots$$

The definite integral then is simply

$$\begin{aligned}\int_0^x \frac{1}{1+t^3} &= \left[t - \frac{1}{4}t^4 + \frac{1}{7}t^7 - \frac{1}{10}t^{10} + \cdots\right]_0^x \\ &= x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \cdots\end{aligned}$$

The correct answer is C.)

10. Find the MacLaurin Series for the function

$$f(x) = \sin(x^2) + \cos(x^2).$$

A.) $1 + x^2 - \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots$

B.) $1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$

C.) $x + \frac{1}{3}x^3 - \frac{1}{10}x^5 - \frac{1}{14}x^7 + \cdots$

D.) $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$

E.) $1 + \frac{1}{2}x^4 + \frac{1}{4}x^8 + \frac{1}{6}x^{12} + \frac{1}{8}x^{16} + \cdots$

F.) $1 + \frac{1}{2!}x^4 - \frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \frac{1}{8!}x^{16} + \cdots$

Starting from the standard series,

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$
$$\cos x = 1 + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

we get new series by substituting x^2 for x,

$$\sin x^2 = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \frac{1}{7!}x^{14} + \cdots$$
$$\cos x^2 = 1 - \frac{1}{2!}x^4 + \frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \frac{1}{8!}x^{16} + \cdots$$

We can add these series. By rearranging the terms, so they appear in order of the exponent of x, the sum becomes,

$$\sin x^2 + \cos x^2 = 1 + x^2 - \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \frac{1}{5!}x^{10} - \frac{1}{6!}x^{12} - \frac{1}{7!}x^{14} + \frac{1}{8!}x^{16} + \cdots$$

If you calculate the values of the factorials, $2! = 2$, $3! = 6$, $4! = 24$, and ignore the terms starting at x^{10}, you get the desired answer. The correct answer is A.)