Homework 5: Solutions

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Question 8.2.20: Evaluate the integral

$$\int cos^2(x)\ sin(2x)\ dx$$

Solution: The most straightforward approach is to rewrite the integrand as a polynomial in the functions $\sin(x)$ and $\cos(x)$; the change of variables $u(x) := \cos(x)$ then reduces the problem to that of integrating a polynomial in u:

$$\begin{aligned}\int \cos^2(x)\sin(2x)\ dx &= \int \cos^2(x)\ (2\sin(x)\cos(x))\ dx \\ &= \int\ 2\cos^3(x)\ (-dcos(x)) \\ &= -\frac{1}{2}\cos^4(x) + C\end{aligned}$$

Alternatively, one might write the integrand as a polynomial in the functions $\sin(2x)$ and $\cos(2x)$:

$$\begin{aligned}\int \cos^2(x)\sin(2x)\ dx &= \int \frac{1+\cos(2x)}{2}\sin(2x)dx \\ &= \frac{1}{2}\int sin(2x)\ dx - \frac{1}{2}\int \cos(2x)\ \frac{1}{2}dcos(2x) \\ &= -\frac{1}{4}\cos(2x) - \frac{1}{8}\cos^2(2x) + C\end{aligned}$$

Question 8.2.42: Evaluate

$$\int \sin(3x)\cos(x)dx.$$

Solution: Using the identity $\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B)+\sin(A-B))$, we have

$$\begin{aligned}\int \sin(3x)\cos(x)dx &= \frac{1}{2}\int(\sin(3x+x)+\sin(3x-x))dx \\ &= \frac{1}{2}\int(\sin(4x)+\sin(2x))dx \\ &= \frac{1}{2}(-\frac{\cos(4x)}{4} - \frac{\cos(2x)}{2}) + C \\ &= -\frac{\cos(4x)}{8} - \frac{\cos(2x)}{4} + C\end{aligned}$$

Question 8.3.2: Evaluate

$$\int x^3\sqrt{9-x^2}\ dx$$

using the change of variables $x := 3\sin\theta$

Solution: The change of variables $x = 3\sin\theta$ together with the identities $1-\sin^2(\theta) = \cos^2(\theta)$ and $d(3\sin(\theta)) = 3\cos(\theta)d\theta$ give:

$$\begin{aligned}
\int x^3\sqrt{9-x^2}\ dx &= \int (3\sin(\theta))^3\sqrt{3^2(1-\sin^2(\theta))}\ (3\cos(\theta)d\theta) \\
&= 3^5\int \sin^3(\theta)\cos^2(\theta)\ d\theta \\
&= -3^5\int \cos^2(\theta)(1-\cos^2(\theta))\ d\cos(\theta) \\
&= 3^5\int \cos^4(\theta)-\cos^2(\theta)\ d\theta \\
&= \frac{3}{4}(3\cos(\theta))^4 - 3(3\cos(\theta))^3 \\
&= \frac{3}{4}(3^2-(3\sin(\theta))^2)^2 - 3(3^2-3\sin(\theta)^2)^{\frac{3}{2}} \\
&= \frac{3}{4}(9-x^2)^2 - 3(9-x^2)^{\frac{3}{2}}
\end{aligned}$$

Question 8.3.4: Evaluate $\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\ dx$

Solution: Since the expression in the denominator is of the form $\sqrt{a^2-x^2}$ we might try making the change of variables $x = a\sin(t)$ (or $x = a\cos(t)$, which is essentially the same thing). Let $x(t) := 4\sin(t)$. Then $t(x) = sin^{-1}(\frac{x}{4})$. So $t(0) = sin^{-1}(\frac{0}{4}) = 0$ and $t(2\sqrt{3}) = sin^{-1}(\frac{2\sqrt{3}}{4}) = sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$. So our integral becomes

$$\begin{aligned}
\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\ dx &= \int_0^{\frac{\pi}{6}} \frac{(4\sin(t))^3}{\sqrt{4^2-(4\sin^2(t))}} d(4\sin(t)) \\
&= \int_0^{\frac{\pi}{6}} \frac{4^4\sin^3(t)\cos(t)}{4\cos(t)} dt \\
&= 4^3\int_0^{\frac{\pi}{6}} \sin^3(t)dt \\
&= 4^3\int_0^{\frac{\pi}{6}} (1-\cos^2(t))\sin(t)dt \\
&= 4^3\left(\int_0^{\frac{\pi}{6}} \sin(t)dt + \int_0^{\frac{\pi}{6}} \cos^2(t)d\cos(t)\right) \\
&= 4^3\left[-\cos(t)+\frac{\cos^3(t)}{3}\right]_0^{\frac{\pi}{6}} \\
&= \frac{1}{2}
\end{aligned}$$

Question 8.3.6: Evaluate $\int_0^2 x^3\sqrt{x^2+3}dx$

Solution:

$$\int_0^2 x^3\sqrt{x^2+4}dx \quad = \quad \int_0^2 2x^3\sqrt{\left(\frac{x}{2}\right)^2+1}\ dx$$

Since $\tan^2(t)+1=\sec^2(t)$, we try the change of variables $\frac{x}{2}=tan(t)$. Then

$$dx = 2\sec^2(t)dt$$

and since $t=\tan^{-1}\left(\frac{x}{2}\right)$, we have

$$t(0) = tan^{-1}(0) = 0$$
$$t(2) = tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$$

So in the t coordinate, our integral above is

$$\int_0^{\frac{\pi}{4}} (2\tan(t))^3\sqrt{2^2\tan^2(t)+2^2}\ d(2tan(t)) = 2^5\int_0^{\frac{\pi}{4}} \tan^3(t)\sec(t)\sec^2(t)dt$$
$$= 2^5\int_0^{\frac{\pi}{4}} \tan^2(t)\sec^2(t)\sec(t)\tan(t)dt$$
$$= 2^5\int_0^{\frac{\pi}{4}} (\sec^2(t)-1)\sec^2(t)\ d(\sec(t))$$
$$2^5\left[\frac{\sec^5(t)}{5}-\frac{sec^3(t)}{3}\right]_0^{\frac{\pi}{4}}$$
$$= 2^5\left[\frac{2^{\frac{5}{2}}}{5}-\frac{2^{\frac{3}{2}}}{3}\right]$$
$$= \frac{7}{15}2^6\sqrt{2}$$

Question 8.3.12 Evaluate $\int_0^1 x\sqrt{x^2+2}\;dx$

Solution: This problem is very similar to 8.3.6. We make exactly the same change of variables, and we have

$$\int x\sqrt{x^2+2}\;dx \;=\; \int (2\tan(t))(2\sec(t))(2\sec^2(t)dt)$$

$$= 2^3\int \sec^2(t)d(\sec(t)) \;=\; 2^3\frac{\sec^3(t)}{3}$$

$$= \frac{2^3}{3}(1+tan^2(t))^{\frac{3}{2}} \;=\; \frac{2^3}{3}\left(1+\left(\frac{x}{2}\right)^2\right)^{\frac{3}{2}}$$

So we have

$$\int_0^1 x\sqrt{x^2+2}\;dx = \left[\frac{2^3}{3}\left(1+\left(\frac{x}{2}\right)^2\right)\right]_0^1 = \frac{1}{3}$$

Question 8.3.16: Evaluate the integral $\int \frac{1}{x^2\sqrt{16x^2-9}}dx$

Solution: We rewrite the integrand so that the expression under the square root sign is of the form u^2-1:

$$\int \frac{1}{x^2\sqrt{16x^2-9}}\;dx \;=\int \frac{1}{x^2\sqrt{(4x)^2-3^2}}\;dx \;=\frac{1}{3}\int \frac{1}{x^2\sqrt{(\frac{4x}{3})^2-1}}\;dx$$

Now since we know that $sec^2(t)-1=tan^2(t)$, we make the change of variables $\frac{4x}{3}=sec(t)$. In terms of the new variable our integral is

$$\frac{1}{3}\int \frac{d\left(\frac{3}{4}sec(t)\right)}{(\frac{3}{4}sec(t))^2\sqrt{sec^2(t)-1}} \;=\; \frac{4}{3^2}\int \frac{sec(t)\;tan(t)\;dt}{sec^2(t)\;tan(t)}$$

$$= \frac{4}{3^2}\int \frac{1}{sec(t)}\;dt \;=\frac{4}{3^2}\int cos(t)\;dt \;=\frac{4}{3^2}\sin(t)+C$$

$$= \frac{4}{3^2}\sqrt{1-cos^2(t)}+C \;=\; \frac{4}{3^2}\sqrt{1-\frac{1}{sec^2(t)}}+C$$

$$= \frac{4}{3^2}\sqrt{1-(\frac{3}{4x})^2} \;=\; \frac{1}{9x}\sqrt{16x^2-9}+C$$

Question 8.3.20: Evaluate the integral $\int \frac{t}{\sqrt{25-t^2}} dt$

Solution: Since $d(25-t^2) = -2tdt$, and tdt appears in the numerator of the integrand, we change variables to $u(t) := 25 - t^2$:

$$\int \frac{t}{\sqrt{25-t^2}} dt = -\frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

$$= -\sqrt{u} + C = -\sqrt{25-t^2} + C$$

Question 8.3.30: Evaluate $\int_0^{\frac{\pi}{2}} \frac{cos(t)}{\sqrt{1+sin^2(t)}} dt$

Solution: Since $d(sin(t)) = cos(t)dt$ we make the change of variables $u(t) := sin(t)$. $u(0) = sin(0) = 0$ and $u(\frac{\pi}{2}) = sin(\frac{\pi}{2}) = 1$, so in the new variable the integral reads $\int_0^1 \frac{du}{\sqrt{1+u^2}}$. To evaluate this integral, we must make another change of variable. Since $1 + tan^2(\theta) = sec^2(\theta)$, the appropriate subsititution is $u(\theta) = tan(\theta)$. We have

$$\int \frac{du}{\sqrt{1+u^2}} = \int \frac{d(tan(\theta))}{\sqrt{1+tan^2(\theta)}}$$

$$\int \frac{sec^2(\theta)d\theta}{sec(\theta)} = \int \sec(\theta)d\theta$$

A priori, there is no reason to expect that the antiderivative(s) of $sec(\theta)$ admits a closed form expression in terms of elementary functions that we know. It turns out that it does admit such an expression; one sees this if one accidentally makes the following observation:

$$d(sec(\theta) + tan(\theta)) = (sec^2(\theta) + sec(\theta)tan(\theta))\ d\theta = (sec(\theta) + tan(\theta))(sec(\theta)d\theta)$$

Or equivalently,

$$sec(\theta)d\theta = \frac{d(sec(\theta) + tan(\theta))}{sec(\theta) + tan(\theta)}$$

So we have

$$\int sec(\theta)d\theta = \int \frac{d(sec(\theta) + tan(\theta))}{sec(\theta) + tan(\theta)}$$

$$= \ln|sec(\theta) + tan(\theta)| + C$$

We would like our answer in terms of u. Since $u := tan(\theta)$, we have $sec(\theta) = \sqrt{1 + tan^2(\theta)} = \sqrt{1+u^2}$. Plugging this into our solution above, we have

$$\int_0^1 \frac{du}{\sqrt{1+u^2}} = \left[ln \left| \sqrt{1+u^2} \ + \ u \right| \right]_0^1 = ln \left| \sqrt{2} + 1 \right|$$

Multiple Choice Questions

Question 1: Evaluate the following integral:

$\int_0^{\frac{\pi}{4}} \frac{\sec^2(x)}{\sqrt{1-\tan^2(x)}}dx$

A.) 1 B.) 2 C.) 4 D.) $\frac{\pi}{4}$ E.) $\frac{\pi}{2}$ F.) π

The correct answer is **E**

Solution: Since $d(\tan(x)) = \sec^2(x)dx$, we make the change of variables $u(x) := tan(x)$. Then $u(0) = 0$ and $u(\frac{\pi}{4}) = 1$, so we have:

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2(x)}{\sqrt{1-\tan^2(x)}}dx = \int_0^1 \frac{du}{\sqrt{1-u^2}}$$

$$= \left[\sin^{-1}(u)\right]_0^1$$

$$= \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \frac{\pi}{2} - 0$$

To see that $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C$, we make the change of variables $u = sin(t)$, which gives

$$\int \frac{du}{\sqrt{1-u^2}} = \int \frac{d(sin(t))}{\sqrt{1-sin^2(t)}} = \int \frac{cos(t)dt}{cos(t)} = \int dt = t + C = \sin^{-1}(u) + C$$

Question 2 Find the volume of a solid of revolution obtained by rotating the region enclosed by the graph of $y = \sin(x)$ and the x-axis between $x = 0$ and $x = \pi$ about the y-axis as the axis of revolution.

A.) π B.) 2π C.) 4π D.)π^2 E.) $2\pi^2$ F.) $4\pi^2$

The correct answer is **E**

Solution: Let us use the shell method. The volume V is given by the formula

$$V = \int_0^\pi 2\pi r h dx$$

where $r(x)$ and $h(x)$ are, respectively, the radius and height of the cylindrical shell "located at x". In our example, $r(x) = x$ and $h(x) = \sin(x)$. So we have $V = \int_0^\pi 2\pi x \sin(x)dx$. In order to evaluate this integral, we integrate by parts with $u = 2\pi x$ and $dv = \sin(x)dx$ (this gives $du = 2\pi dx$ and $v = -\cos(x)$):

$$\begin{aligned} \mathrm{V} &= \int_0^\pi 2\pi x \sin(x)dx \\ &= [-2\pi x \cos(x)]_0^\pi - \int_0^\pi 2\pi(-\cos(x))dx \\ &= 2\pi \left[-x\cos(x) + \sin(x)\right]_0^\pi \\ &= 2\pi^2 \end{aligned}$$

In general, if the integrand is a product $p(x)f(x)$ where $p(x)$ is a polynomial of degree d and $f(x)$ is a function for which we can compute explicity at least $d+1$ iterated antiderivatives, then we can compute $\int p(x)f(x)$ by iterating the process above. In the first step one takes $u = p(x)$ and $dv = f(x)dx$. Then $du = p'(x)dx$, and $p'(x)$ is a polynomial of degree $d-1$. So after d steps, one ends up with a polynomial of degree 0, namely a constant. Thus, one sees that by integrating by parts iteratively, one can "get rid of the polynomial part" of the integrand.

Question 3: Solve the following indefinite integral

$\int \frac{\sqrt{4x^2-9}}{x} dx$

A.) $\frac{x}{2} + sec^{-1}\sqrt{4x^2-9} + C$

B.) $ln\left|\sec\sqrt{4x^2-9} + \tan\sqrt{4x^2-9}\right| + C$

C.) $ln\sqrt{4x^2-9} - 3tan^{-1}\frac{2x}{3} + C$

D.) $\sqrt{4x^2-9} - 3sec^{-1}\frac{2x}{3} + C$

E.) $\frac{1}{2}\tan^{-1}\sqrt{4x^2-9} + C$

F.) $\frac{1}{4x^2-9} + \frac{1}{2}\tan^{-1}\frac{2x}{3} + C$

The correct answer is **D**

Solution:

$$\int \frac{\sqrt{4x^2-9}}{x} dx = \int 3\frac{\sqrt{(\frac{2x}{3})^2-1}}{x} dx$$

Since the numerator is of the form $\sqrt{u^2-1}$, and we know that $\sec^2(t) - 1 = \tan^2(t)$, we try the change of variables $\frac{2x}{3} = \sec(t)$:

$$\int 3\frac{\sqrt{(\frac{2x}{3})^2-1}}{x} dx = \int 3\frac{\sqrt{\sec^2(t)-1}}{\frac{3}{2}\sec(t)} d(\tfrac{3}{2}\sec(t))$$

$$= \int \frac{3\tan(t)}{\sec(t)}\sec(t)\tan(t)dt$$

$$= 3\int \tan^2(t)dt$$

$$= 3\int(\sec^2(t) - 1)dt$$

$$= 3\tan(t) - 3t + C$$

$$= \sqrt{3^2\sec^2(t) - 3^2} - 3t + C$$

$$= \sqrt{3(\tfrac{2x}{3})^2 - 3^2} - 3\sec^{-1}(\tfrac{2x}{3}) + C$$

$$= \sqrt{4x^2-9} - 3\sec^{-1}(\tfrac{2x}{3}) + C$$