Homework 3 *Math 361, Fall 2007*

1. Prove that an open interval (a, b) is a cellular set in $\mathbb{R}$.

2. Prove that the content of the open interval $E = (a, b)$ is $\nu(E) = b - a$ (assuming $a < b$).

3. Why is a disjoint union of cellular sets again a cellular set?

4. Prove that the complement $[0, 1] \setminus C$ of the Cantor set C is cellular.

5. Let A be a "fat" Cantor set, constructed as was shown in class. Show that the content of the complement $[0, 1] \setminus A$ is strictly less than 1.

   Reminder. Let $A_0 = [0, 1]$, and let U_1 be an open interval of length 4^{-1} contained in A_0, and $A_1 = A_0 \setminus U_1$. Then A_1 has two connected components. Let U_2^1, U_2^2 be two open intervals of length 4^{-2} each, one contained in each connected component of A_1. Write $U_2 = U_2^1 \cup U_2^2$, and $A_2 = A_1 \setminus U_2$. Then A_2 has four connected components. Now keep going. Let $U_3^1, \ldots, U_3^4$ be four open intervals of length 4^{-3} each, one contained in each connected component of A_2. Write $U_3 = U_3^1 \cup \cdots \cup U_3^4$, and $A_3 = A_2 \setminus U_3$. Etcetera.

   The intersection

   $$A = \bigcap_{i=1}^{\infty} A_i,$$

   is a "fat" Cantor set.

6. Why is it impossible to contain any cell $[a, b)$, however small, in a "fat" Cantor set?

7. Prove the following proposition.

   Proposition. *Any countable union (not necessarily disjoint) of cellular sets is cellular.*

   Step I. First show that every countable union of cells $R_1 \cup R_2 \cup \cdots$ is equal to a *disjoint* countable union of cells $S_1 \cup S_2 \cdots$. Construct the cells S_i by adding one cell R_j at a time.

   Step II. Show how the result of Step I implies the proposition.

8. Prove the following proposition.

   Proposition *Every finite intersection of cellular cells is cellular.*

   Hint: start with the union of two cellular sets.

9. Let $q_1, q_2, q_3, \ldots$ be an arbitrary sequence of numbers in $[0, 1]$. Consider the union of cells

   $$E = \bigcup_{i=1}^{\infty} [q_i, q_i + \frac{1}{3^i}).$$

   Prove that there exists a number $x \in [0, 1]$ that is not contained in the set E.

   Step I. By the previous exercise, E is a cellular set. Use Proposition 3 in the Lecture Notes to prove that

   $$\nu(E) < 1.$$

   Step II. Now use the same Proposition again to show that this implies that E cannot contain all points in $[0, 1]$.