Homework 7 *Math 361, Fall 2007*

1. Prove that $\|f\|_1$ and $\|f\|_\infty$ satisfy the axioms of a norm on the vector space of equivalence classes of measurable functions (where $f \sim g$ if $f = g$ a.e.)

   Recall that $\|f\|_1 = \int |f|$, and that $\|f\|_\infty$ is the *essential supremum* of f. The essential supremum of a function f is defined as the least essential upper bound of the absolute value $|f(x)|$. An essential upper bound is a number $c \in \mathbb{R}$ such that the set $\{x \mid f(x) > c\}$ has measure zero. The point of taking the essential supremum is that if $f = g$ a.e., then $\|f\|_\infty = \|g\|_\infty$.

2. Prove the inclusions
   $$L^\infty([0,1]) \subseteq L^2([0,1]) \subseteq L^1([0,1]).$$

3. Prove the inclusions
   $$l^1 \subseteq l^2 \subseteq l^\infty.$$
   Recall that l^p is the Banach space of sequences $v = (v_1, v_2, \cdots)$ with finite norm $\|v\|_p$. The space l^∞ consists of sequences that are bounded.

4. Prove that neither of the inclusions $L^1(\mathbb{R}) \subseteq L^2(\mathbb{R})$ or $L^2(\mathbb{R}) \subseteq L^1(\mathbb{R})$ is correct. Hint: find an explicit function that is in L^1 but not in L^2, and one that is in L^2 but not in L^1.

5. Find an example of a sequence of continuous functions $f_1, f_2, f_3, \cdots$ on the interval $[0,1]$ that converges to 0 (i.e., the function $f(x) = 0$) pointwise, but that does *not* converge in the Banach space $L^1([0,1])$.

6. Find an example of a sequence of continuous functions $f_1, f_2, f_3, \cdots$ that converges to 0 in the Banach space $L^1([0,1])$, but that does *not* converge pointwise.

7. Find a sub-sequence of your example in exercise 6 that *does* converge pointwise (at least almost everywhere).

8. Find an example of a sequence of continuous functions $f_1, f_2, f_3, \cdots$ that converges to 0 in the Banach space $L^1([0,1])$, but not in the Hilbert space $L^2([0,1])$.

   Hint for exercises 5–8: It is easiest to work with functions that are piecewise linear, i.e., that are of the form $y = a_i x + b_i$ on intervals $[x_i, x_{i+1}]$.

9. Let $S = \{e_1, e_2, \cdots, e_k\}$ be a finite orthonormal set of vectors in a Hilbert space H. Let M be the span of S, i.e., the subspace consisting of all linear combinations $c_1e_2 + \cdots + c_ke_k$ of vectors in S. The subspace M is, of course, finite dimensional. Prove that M is *closed* in H.

10. In Euclidean space $\mathbb{R}^n$, every closed and bounded set is compact, and vice versa. But this is not true in infinite dimensional Hilbert space. Prove that if $S = \{e_1, e_2, e_3, \ldots\}$ is a countable orthonormal set, then (a) S is closed, (b) S is bounded, but (c) S is not compact.

    Remarks: (b) A set is bounded if it is contained in a ball of finite radius. (c) To prove that a set is *not* compact, you must find a cover by open sets for which there clearly is no finite subcover.