Take-home Midterm Exam *Math 361, Fall 2007*

1. Prove that a simple function is Riemann integrable if and only if the set $s^{-1}(y)$ has volume, for every $y \neq 0$ in the range.

2. If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are measurable sets with finite measure, then $A \times B \subset \mathbb{R}^{n+m}$ is measurable with

$$\mu(A \times B) = \mu(A)\mu(B).$$

For a measurable function $f\colon \mathbb{R}^n \to \mathbb{R}$, let $C_f \subset \mathbb{R}^{n+1}$ be the set of points under the graph, defined as

$$C_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid 0 < y < f(x)\}.$$

If $s \geq 0$ is a non-negative simple function, show that $\int s = \mu(C_s)$.

If $f \geq 0$ is a non-negative integrable function, prove that there exists an increasing sequence $0 \leq s_1 \leq s_2 \leq s_3 \leq \cdots$ of simple functions, that converge to f, and such that

$$\lim_{i\to\infty} \int s_i = f.$$

Hint: first create a sequence of simple functions $0 \leq \tilde{s}_i \leq f$ that satisfy the limit. Then let $s_i(x) = \max\{\tilde{s}_1(x), \ldots, \tilde{s}_i(x)\}$.

Prove that

$$\int f = \mu(C_f).$$

3. The rational numbers are a dense subset of the real numbers. Nevertheless, not all irrational numbers are equally far removed from being rational. This statement may, at first, not seem to make much sense, because the ordinary "distance" from any irrational number x to the set $\mathbb{Q}$ of rational numbers is, of course, zero. But there are more subtle ways to measure how far removed a number is from being rational.

Consider the following collection of functions. For any real number x and positive integer n define $f_n(x)$ as the *absolute value* of the *smallest* possible remainder r in an expression of the type

$$x = \frac{m + r}{n},$$

where m, like n, is an integer. Clearly, $-\frac{1}{2} \leq r \leq \frac{1}{2}$, and so $0 \leq f_n(x) \leq \frac{1}{2}$. For a fixed denominator n, the fraction m/n is simply the best possible approximation of x, and the value $f_n(x) = |r| = |nx - m|$ measures the difference from the multiple nx to the nearest integer m—which is the best possible numerator in m/n. The value $f_n(x)$ is an indication of how good the numerator n "works" for approximating x. (I recommend that you sketch the graph of f_n for some n, to get a feeling for these functions).

Take, for example, $x = \pi$. From $8\pi = 25.132741\cdots$ we see that the nearest integer is $m = 25$. The best approximation of π with denominator 8 is $\frac{25}{8}$, and the numerator is off by

$$f_8(\pi) = 0.132741\cdots.$$

With $n = 7$ we have $7\pi = 22.008851\cdots$ so we get the approximation $\pi \approx \frac{22}{7}$, and

$$f_7(\pi) = 0.008851\cdots.$$

Apparently 7 is a much better denominator for approximation of π than 8. Archimedes's approximation $\pi \approx \frac{355}{113}$ gives an even better result. From

$$113\pi = 354.999969\cdots$$

we derive

$$f_{113}(\pi) = 0.000030\cdots$$

It may seem that, by trying ever larger denominators, the infimum of the remainders $f_n(x)$ will eventually go to zero. But while this may or may not be true for π, it is certainly not true for all irrational numbers.

For irrational x we will never have $f_n(x) = 0$, but perhaps if we choose n sufficiently large, we can make $f_n(x)$ as small as we like?

To study this phenomenon, we define

$$f(x) = \liminf_{n\to\infty} f_n(x).$$

The value of $f(x)$ is a measure for how well the number x can be approximated by rational numbers.

Why is $f(x) = 0$ *for all* rational x*?*

Why is f *a measurable function?*

Prove that the average value of $f(x)$ *on the interval* $0 \le x \le 1$ *is exactly* $\frac{1}{4}$.

Prove that this implies that the set of irrational numbers $0 < x < 1$ *for which* $f(x) > 0$ *has at least measure* $\frac{1}{2}$.

Prove that f *is* not *Riemann integrable.*

Hint: Calculate the Lebesgue integral $\int_0^1 f$ using Fatou's Lemma.