Take-home Final Exam (Second Mid-Term) *Math 361, Fall 2007*

1. Consider the *Heat Equation*,

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

for a temperature distribution $u(x,t)$ in a one-dimensional segment $x \in [0, 2\pi]$, that evolves in time $t \geq 0$. We impose the boundary conditions

$$\begin{aligned} u(0,t) &= u(2\pi, t) = 0 \\ u(x,0) &= f(x). \end{aligned}$$

Here $f(x)$ is a function that represents the initial temperature dustribution at time $t = 0$. The temperature is held at zero at the end points.

We assume that f is a function that is *piecewise continuous*, i.e., the interval $[0, 2\pi]$ is divided into a finite number of sub-intervals, and f is continuous on each of those intervals. Moreoever, we assume that f is *bounded* (there are no infinite temperatures).

Look up the explicit formula for the solution to this boundary value problem (math 241). (If you want to derive it from scratch: The solution method is separation of variables.)

By analyzing the solution formula for $u(x,t)$, prove that, no matter what the initial temperature distribution $f(x)$ is at the starting point $t = 0$ (given the restrictions mentioned at the beginning), the temperature distribution $u(x,t)$ is a smooth function (i.e., all derivatives exist) in the variable x, at every point in time $t > 0$ past the starting point. (In physical terms, the heat equation immediately *diffuses* all discontinuities).

2. Let $u(x,y)$ be a complex valued function in two spatial variables $(x,y) \in \mathbb{R}^2$. We assume that u is doubly periodic, i.e.,

$$u(x + 2\pi, y) = u(x, y + 2\pi) = u(x,y),$$

and that $u(x,y)$ is smooth. (If you think of u as a function on the square $[0, 2\pi] \times [0, 2\pi]$, then the smoothness of u as a doubly periodic function involves certain boundary conditions.)

Consider the elliptic differential equation

$$3 \frac{\partial^2 u}{\partial x^2} + 5 \frac{\partial^2 u}{\partial y^2} + 32u = 0.$$

Solve this equation using Fourier series in 2 variables.

3. With the same set-up as in exercise 2, consider the *eigenvalue problem*

$$3 \frac{\partial^2 u}{\partial x^2} + 5 \frac{\partial^2 u}{\partial y^2} = \lambda u.$$

Prove that all eigenvalues λ for which this equation has a solution are *non-positive*.

Find the five largest eigenvalues $\lambda_0 > \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ for which the equation has a solution (including $\lambda_0 = 0$).

What are the *dimensions* of the eigenspaces corresponding to each of the eigenvalues that you found?

Are there any eigenvalues at all for which the corresponding eigenspace is infinite dimensional? Why, or why not?

How many eigenvalues does the equation have in total?

4. Let $f(x, y)$ be a second function of the same type as $u(x, y)$ (see exercise 2). Now consider the PDE,

$$3\,\frac{\partial^2 u}{\partial x^2} + 5\,\frac{\partial^2 u}{\partial y^2} + 32u = f.$$

Describe the conditions that f must satisfy in order for this equation to have a solution. Express these conditions by means of orthogonality relations (i.e., f must be orthogonal to $\ldots$).