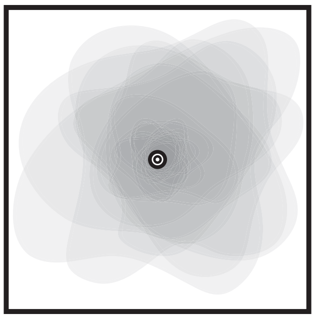
Appendix A

Background

A.1 On point-set topology

It is an unfortunate fact that most students' exposure to topology begins and ends with the set-theoretic foundations. To minimize such would be akin to minimizing grammar or arithmetic in primary school (which produces bad writers and no mathematicians). This text cannot cover all the foundations; however, a few definitions will perhaps provide appropriate pointers, with, *e.g.*, [238], as a source for more thorough coverage.



A **topology** on a set X is a collection $\mathcal{T}_{\mathsf{op}}$ of subsets of X declared to be the open sets. The topology must satisfy three properties:

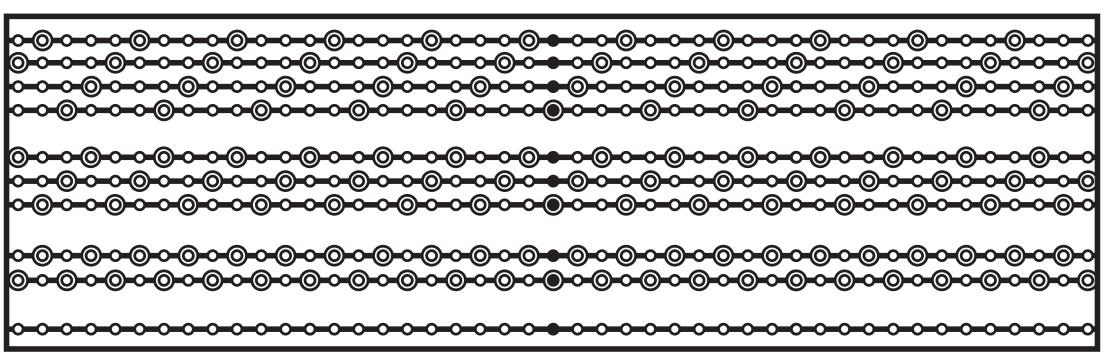
1. $\mathcal{T}_{\mathsf{op}}$ is closed under arbitrary unions;
2. $\mathcal{T}_{\mathsf{op}}$ is closed under finite intersections; and
3. $\mathcal{T}_{\mathsf{op}}$ contains both X and the empty set $\varnothing$.

From this spartan frame is the subject supported. A set $A \subset X$ is **open** if $A \in \mathcal{T}_{\mathsf{op}}$ and **closed** if its complement $X - A \in \mathcal{T}_{\mathsf{op}}$ is open. The **standard topology** on $\mathbb{R}^n$ is the topology whose elements consist of all possible unions of all open metric balls of all sufficiently small radii at all points. There are well-defined ways to generate topologies from such a **basis** for a topology.

The concept of a topology allows one to encode proximity without specifying a metric. This feature is increasingly relevant to applications in data, networks, and biology, where natural metrics may be obscured or nonexistent. The definition of a topology, though primal, is powerful and subtle.

Example A.1 (Prime numbers) ⊙

The following brilliant proof is due to Furstenberg; it proves the infinitude of prime numbers using only the definition of primeness and basic properties of a topology. Consider the topology on $\mathbb{Z}$ generated by affine subsets $\{a\mathbb{Z} + b : a \neq 0\}$, meaning that nonempty open sets are generated by unions and finite intersections of these basis sets. Let the reader verify that: (1) this indeed forms a topology on $\mathbb{Z}$; (2) each basis affine set $\{a\mathbb{Z} + b : a \neq 0\}$ is both open and closed in this topology; and (3) no



nonempty finite subset of $\mathbb{Z}$ is open. Then, consider the set:

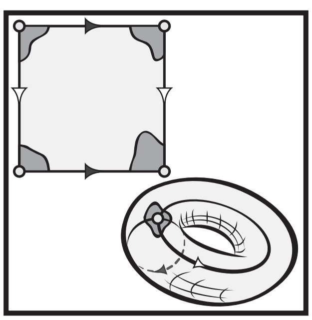
$$S = \bigcup_p p\mathbb{Z} \quad p \ \text{ prime} .$$

As the union of open sets, S is open. If the number of primes is finite, then S is also closed, by the fact that a finite union of closed sets (being the complement of a finite intersection of open sets) is closed. However, the complement $\mathbb{Z}-S = \{-1, 1\}$ is finite: contradiction. Conclusion: there are infinitely many primes. Note that the only point at which arithmetic was used was in the definition of S and the verification of the definition of a topology. ⊙

Other definitions common in point-set topology are:

1. The **interior** of a subset $A \subset X$ is the largest open subset in A, or, equivalently, the union of all open subsets of A.
2. The **closure** of $A \subset X$ is the smallest closed superset of A, or, equivalently, the intersection of all closed supersets of A.
3. The **boundary** of $A \subset X$, ∂A, is the complement of the interior in the closure.
4. A space X is **compact** if every open cover of X restricts to a finite subcover.
5. A function is **continuous** if the inverse image of open sets is open.
6. A **homeomorphism** is a continuous bijection with continuous inverse.

A topology on X can induce topologies on spaces built from X. Given a pair of spaces X and Y, the (cartesian) **product** $X \times Y$ is the set of ordered pairs (x, y); the **product topology** is that with basis $U \times V$ for $U \in \mathcal{T}_{\mathrm{op}_X}$ and $V \in \mathcal{T}_{\mathrm{op}_Y}$. For A any subset of a space X, the **subspace topology** on A consists of the collection $\{U \cap A\}$ for all $U \in \mathcal{T}_{\mathrm{op}}$. This is the smallest or *weakest* topology on A making the inclusion map $A \hookrightarrow X$ continuous. The subspace topology is assumed whenever one discusses a subset of a space as a space in its own right.



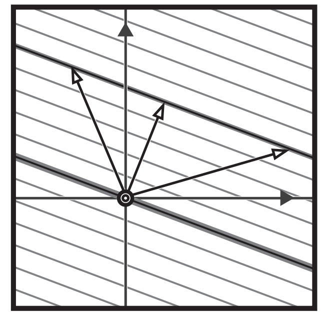
When collapsing a space X to a quotient by identifying certain subsets, one imposes a topology on the image. Given a surjective function $q\colon X \to Y$, the **quotient** topology on Y via q is the collection of sets $V \subset Y$ such that $q^{-1}(V)$ is open in X. This is the largest or *strongest* topology on Y making the quotient map continuous.

Most simple spaces have an obvious quotient or subspace topology, and the arcana of the subject emphasized in early texts is, for most practical purposes, inessential. There are, however, subtleties associated with topologies on infinite-dimensional spaces, particularly function spaces. The beginner should focus on the **compact-open** topology on the space $C(X, Y)$ of maps $f\colon X \to Y$. This topology is the smallest generated by sets of the form $C(K, U)$, where $K \subset X$ is compact and $U \subset Y$ is open.

A.2 On linear and abstract algebra

Deep knowledge of abstract algebra is not a prerequisite for reading this book, nor for learning many basic aspects of algebraic topology. The reader who *does* know algebra well will find many of the important tools not in this book (Hom, Ext, Tor, $\otimes$, *etc.*) natural and implicit. A few basic concepts suffice for those coming from a minimal background. At the very least, the reader needs proper training in linear algebra. From this subject, little need be said, with two crucial exceptions.

First: quotient spaces. Given a vector space V and subspace W, the quotient space V/W consists of equivalence classes $[v]$ of vectors in V modulo vectors in W. Specifically, $[v] = [v']$ if and only if $v' - v \in W$. These **cosets** are often illustrated in terms of the orthogonal complement $W^\perp$ of W in V. This is erroneous, as the orthogonal complement requires the existence of a well-defined inner product on V and *such is not required* to define V/W. Another common error is, via conflation of V/W and $W^\perp$, to assume that V/W is a subspace of V. *It is not.*



Second: transformations. More important that vector spaces themselves are linear transformations between them. One characterizes $A\colon V \to W$ in terms of auxiliary vector spaces and simple transformations. The **image** of A is a subspace $\text{im}\, A$ of the codomain W; the **kernel** of A is $A^{-1}(0)$, a subspace $\ker A$ of the domain V. Less familiar to students is the equally-important **cokernel**, $\text{coker}\, A$, the quotient space $W/\text{im}\, A$ of the codomain by the image.

While many of the constructs in this text can be accomplished using only the language of linear algebra, it quickly becomes important to grasp the more general algebraic structures available. A **group** is a set $\mathbf{G}$ together with a binary operation $\bullet\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ (often called *multiplication*) that satisfies the following:

1. The operation is associative.
2. There is an identity element $e \in \mathbf{G}$, with $e \bullet g = g \bullet e = g$ for all $g \in \mathbf{G}$.
3. There is an inverse operation $g \mapsto \overline{g}$ so that $\overline{g} \bullet g = e = g \bullet \overline{g}$ for all $g \in \mathbf{G}$.

Examples include number systems ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, but not $\mathbb{N}$) under addition (but not multiplication); vector spaces under vector addition; square matrix groups (SL_n, SO_n, U_n) under matrix multiplication; and polynomials $\mathbb{Z}[x]$ under addition.

Groups are broader than vector spaces, yet have some structural similarities. A **subgroup** $\mathbf{H} < \mathbf{G}$ is a subset of a group which is itself a group under the group operation, meaning, in particular, that it is closed under the group operation. A **homomorphism** is a function $\phi\colon \mathbf{G} \to \mathbf{K}$ between groups which is linear in the sense that $\phi(g \bullet g') = \phi(g) \circ \phi(g')$, where $\circ$ denotes the group operation in $\mathbf{K}$. The **kernel** of a homomorphism is the subgroup $\ker \phi = \phi^{-1}(e)$, where e denotes the identity element of $\mathbf{K}$.

Groups are often specified using a **presentation**, in which a collection of **generators** $\{g_\alpha\}$ and their inverses form finite words with the usual associativity, identity, and inverse rules applying. To these words are applied a set of **relations**, thought of as replacement rules of the form $r_\beta = e$, for some collection of finite words r_β. For example, the group $\mathbb{Z}^2$ under addition has the presentation

$$\mathbb{Z}^2 \cong \langle x, y \ :\ x \bullet y \bullet x^{-1} \bullet y^{-1} = e \rangle$$

Rewriting the (sole) relation yields the usual commutativity rule for multiplication of x and y. The group presented with N generators and *no* relations is the **free**

group F_N; in general, the free product $G * H$ of two finitely presented groups has presentation given by combining the generators and relations of each factor, with no additional relations. Presentations can be very complicated, implicating infinitely many generators and/or relations. Determining when two presentations yield isomorphic groups is uncomputable in general.

For purposes of doing homology and cohomology, it is convenient to work with **abelian** groups – groups for which the operation is commutative. An abelian group operation is almost always written in additive notation: $+\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, the identity is written as "0", and the inverse of $g \in \mathbf{G}$ is written "$-g$." The **quotient** of an abelian group $\mathbf{G}$ by a subgroup $\mathbf{H} < \mathbf{G}$ consists of equivalence classes $[g]$ for $g \in \mathbf{G}$, where $[g] = [g']$ if and only if $g' - g \in \mathbf{H}$.

Abelian groups generalize to a hierarchy of algebraic structures ascending to vector spaces. A **ring** is an abelian group $\mathbf{R}$ whose group operation $+$ is paired with a multiplication $\bullet\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ that is associative and distributive with a multiplicative identity. The canonical examples are $\mathbb{Z}$ and $\mathbb{Z}[x]$ with 1 as multiplicative inverse. When multiplication is commutative and a multiplicative inverse exists (on the complement of the additive identity 0), a ring ascends to a **field**. The familiar $\mathbb{R}$-vector space generalizes to have scalars in an arbitrary field $\mathbb{F}$. One can relax the definition of a vector space yet further, allowing scalars to reside not in a field but rather a ring $\mathbf{R}$. The resulting structure $\mathbf{M}$ is called an **R-module**. Other relaxations of algebraic objects can be beneficial, even those which yield structures more primitive than groups: for example, dropping from the definition of a group the existence of inverses leads to a **monoid**; further dropping the existence of an identity yields a **semigroup**.

Other structures, though simpler, can capture aspects of algebraic operations that are crucial in applications. For example, a **poset**, or **partially-ordered set**, is a set P together with a binary relation $\trianglelefteq$ that is reflexive, antisymmetric, and transitive. Such a structure encodes, *e.g.*, inclusion in a topology, or the face relation of cells.

As this text demonstrates at several points, the rewards of even moderately increased algebraic generality are substantial. Coefficients in finite fields can provide computational accuracy as compared to real coefficients. Homology and cohomology of chain complexes as $\mathbb{Z}$-modules are critical for defining winding numbers, degrees, and more. Ring structures enable cup products in cohomology. Monoids are the critical structures for encoding constraints in network flow problems. Deeper truths about homology and cohomology – in particular, the understanding and management of torsional elements – require yet deeper tools from homological algebra (Ext, Tor) that linear algebra does not immediately presage.