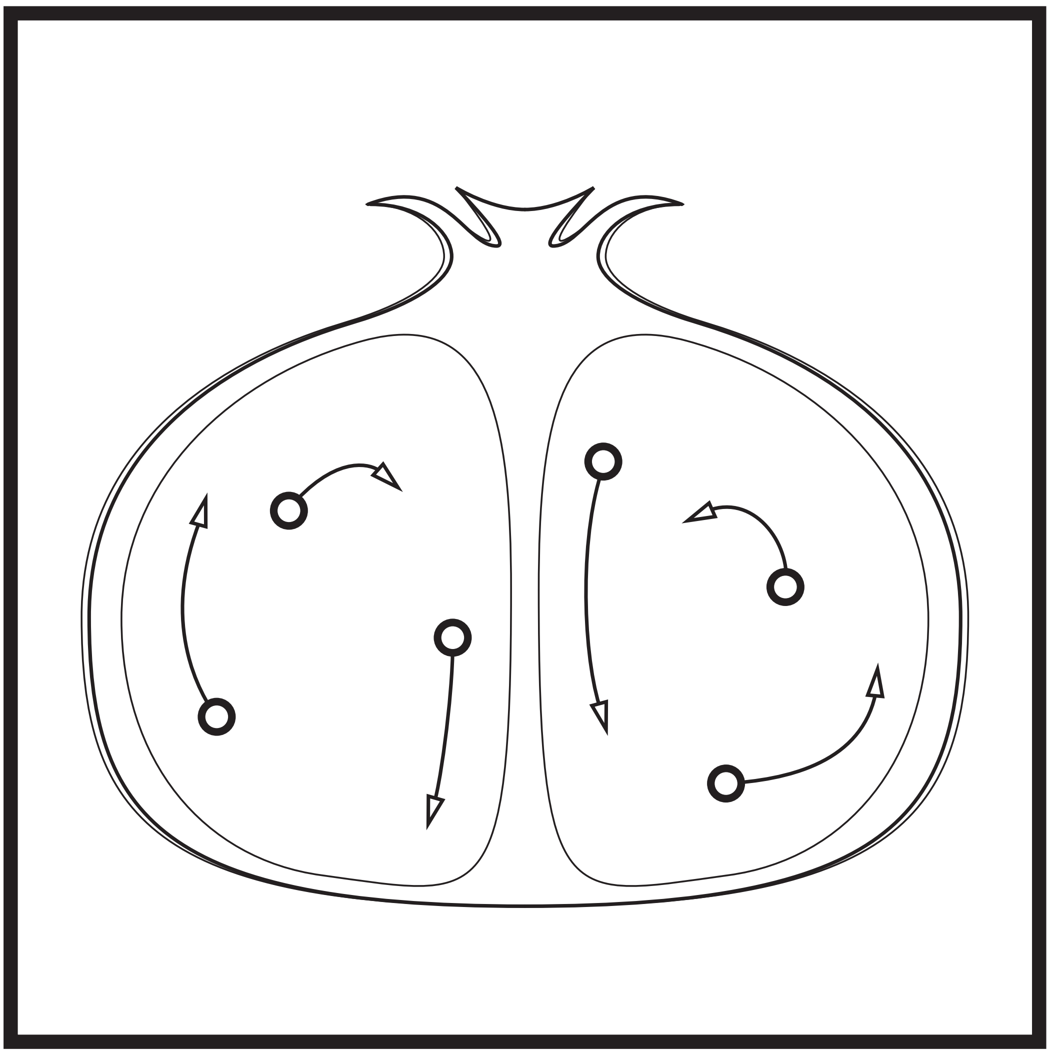
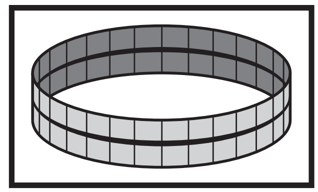
Preface

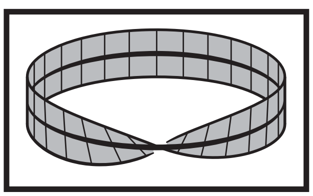


What topology is

Spaces and maps between spaces generate that branch of Mathematics known as *Topology* – the natural evolution of the notions of proximity and continuity. Nested systems of open neighborhoods communicate nearness without necessitating a metric distance. The collection of all open sets in a space is (confusingly) called its **topology**. This thin notion of closeness suffices to define continuity, convergence, and connectivity familiar to students of calculus. A **map** between two spaces carries the implication of continuity.



Topology (the *subject*) explores deformations of maps and spaces, and the basic equivalence relations of topology emphasize qualitative features. Certainly, a **homeomorphism** (a bijective map with continuous inverse) counts as a topological isomorphism. A more general equivalence emerges at the level of maps. A **homotopy** of maps $f_0 \simeq f_1$ from X to Y is a continuous 1-parameter family of maps $f_t\colon X \to Y$ interpolating f_0 and f_1. **Homotopy equivalent** (or *homotopic*) spaces are those $X \simeq Y$ having maps $g\colon X \to Y$ and $h\colon Y \to X$ with $h \circ g \simeq \mathsf{Id}_X$ and $g \circ h \simeq \mathsf{Id}_Y$, where Id denotes an identity map. The simplest (nonempty) space is one which is **contractible**, *i.e.*, homotopic to a single point.



One example of homotopy equivalence is a **deformation retraction** of X to a subspace $A \subset X$. This is defined by a map $r\colon X \to A$ such that the inclusion map $\iota\colon A \hookrightarrow X$ satisfies $r \circ \iota = \mathsf{Id}_A$ (that is, r is a **retraction**), and $\iota \circ r \simeq \mathsf{Id}_X$ (the deformation). The homotopic simplification of X to A in a deformation retraction illustrates nicely the difference between homotopy and homeomorphism. The reader to whom these concepts are new should take some time to play with a few examples. Chapter 0 of the excellent text of Hatcher [176] is recommended.

Topology does its work by means of distinguishing spaces and maps to various degrees of resolution. A **topological invariant** of spaces is an assignment of some (usually algebraic) datum to spaces which respects the equivalence relation of homotopy: homotopic spaces are sent to the same invariant. Counting the number of connected components of a space is a simple topological invariant.

What topology can do

Topology was built to distinguish qualitative features of spaces and mappings. It is good for, *inter alia*:

1. **Characterization:** Topological properties encapsulate qualitative signatures. For example, the genus of a surface, or the number of connected components of an object, give global characteristics important to classification.
2. **Continuation:** Topological features are robust. The number of components or holes is not something that should change with a small error in measurement. This is vital to applications in scientific disciplines, where data is never not noisy.

3. **Integration:** Topology is the premiere tool for converting local data into global properties. As such, it is rife with principles and tools (Mayer-Vietoris, Excision, spectral sequences, sheaves) for integrating from local to global.
4. **Obstruction:** Topology often provides tools for answering feasibility of certain problems, even when the answers to the problems themselves are hard to compute. These characteristics, classes, degrees, indices, or obstructions take the form of algebraic-topological entities.

What topology cannot do

Topology is fickle. There is no recourse to tweaking epsilons should desiderata fail to be found. If the reader is a scientist or applied mathematician hoping that topological tools are a quick fix, take this text with caution. The reward of reading this book with care may be limited to the realization of new questions as opposed to new answers. It is not uncommon that a new mathematical tool contributes to applications not by answering a pressing question-of-the-day but by revealing a different (and perhaps more significant) underlying principle.

What this text is

This text is a quick tour of applied topology, with just enough detail to motivate further study elsewhere. The intent is breadth in ideas, tools, perspectives, and applications; this precludes depth, both in the mathematics and in its applications. The subject of *applied topology* is in its infancy, and it seems certain that a more detailed treatment of the examples given here would appear quaint in less than a decade. The best approach, perhaps, is to make the text intentionally shallow, in the hopes that it will lure the unsuspecting reader to greater depths and prepare for the field as it will be. The author would have called this text *"Cartoons in Applied Topology"* were it not for the resulting confusion.

The chapters are organized according to mathematical topic, rather than according to application domain. This raises an interesting philosophical question about the nature of applied mathematics: is it how different branches of mathematics embed in the physical world, or is it how different applications implicate and are aided by mathematics? The organization of the text reflects a firmly-held belief: *applied mathematics concerns the incarnation of mathematical objects and structures*.

The text begins with an informal introduction to spaces, emphasizing examples and avoiding the set-theoretic technicalities which, though not unnecessary, may overly discourage the interested scientist. The goal is to get to applications as quickly as possible. This reflects the author's learning of the subject of topology: despite the best efforts of brilliant topologists (including Profs. Dranishnikov, Hatcher, Kahn, Krstić, and Vogtmann), the author never learned much of anything in the subject without first finding some physical manifestation of the principle, no matter how cartoonish. This book represents a partial collection of such cartoons.

This is not a mathematics text in the classical sense: some theorems are in a stripped-down version, and proofs are usually skipped, for the sake of making the exposition quick and painless. The reader should not conclude that the subject is quickly or painlessly learned. This text, properly used, is the impetus for future work: hard, slow, and fruitful.

This is not a text in computational topology: the reader may look to several excellent sources [104, 186] for the problem of algorithmic complexity of the topological objects explored in this text. The questions of *"What is it good for?"* and *"How do I compute that?"* are neither independent nor inseparable.

Experts may be exasperated with this text, for many reasons. The text is meant not for experts but for beginners, to point them in the direction of better things to come.

How to use this text

With the advent of *Wikipedia*, *MathOverflow*, and other searchable resources, the need for comprehensive reference texts has perhaps diminished and may continue to do so. To some extent, the demand for the classical definition-theorem-proof text is also somewhat lessened, since one can look up a standard proof on-demand. What has not been eliminated is the need for a story with drama and characters. Overarching narratives are not easily modularized; connections and applications between areas require a global view.

This text attempts to tell a story. Even excised from applications, this story is unorthodox, in content and in tone. Manifolds are quickly marched past a chorus of cell complexes. The Euler characteristic is given stage time far out of measure with its more discerning invariant cousins, homology and, subsequently, cohomology. Classical Morse theory is glossed, interrupted by the ghost of stratified Morse theory, colliding with the Conley index. The shock of introducing the fundamental group after homology and cohomology is surpassed by the scandal of a stripped sheaf theory, preempting the categorical language that would have made for a simpler-seeming entrance.

Perhaps this text will be best used if simply read, for pleasure. It may also serve as a basis text for a graduate-level course in applied topology for mathematical scientists, in which case the lack of a formal theorem-proof delivery seems no impediment. If used as the text for a course in applied topology for mathematicians, this book should provide structure and lots of examples: the instructor for such a course can add details and proofs of the classical material to taste. It is hoped that this book will also make a good accompaniment to a principal text in a traditional algebraic topology course. Those students who struggle in this subject may find some motivation to persevere here, and even those students not interested in applications may find the story entertaining.

A good text should have numerous exercises. This text does not (save for some cryptic figures), for good reasons. First, exercises should be tested and refined via years of teaching from the text. Applied topology is too new a subject, and this author's teaching vocation is, at present, calculus. Second, the diversity of the audience for this text prompts a partitioning of the exercises into those meant for applied mathematical scientists and those meant for mathematicians in a more classical topology course.

Such an array of exercises will require experimentation with level of rigor demanded: it is best not to conduct that experiment in print. The author *will* do this experiment on-line: for the present, this will take the form of an evolving list of exercises linked to the web site for this text. The reader is encouraged to use and comment on these exercises.

Acknowledgements

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