APPLIED ALGEBRAIC TOPOLOGY & SENSOR NETWORKS

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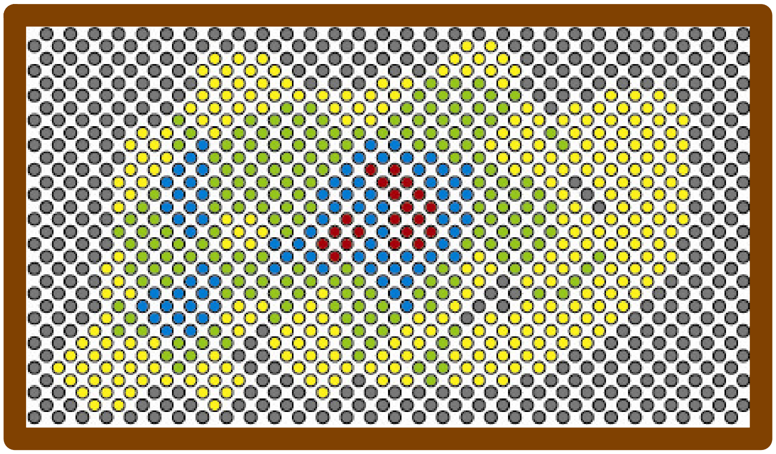
NOTES FOR AMS SHORT COURSE

These notes consist of lectures by R Ghrist based on mini-courses given at the IMA, Cleveland State, Malaga, & Colorado College

The author wishes to thank the many patient audiences/students who have sat through this introductory material.

R.G. is supported by DARPA, ONR, AFOSR, & NSF & is grateful for such...

These notes contain results with Yuliy Baryshnikov, Vin de Silva, and Michael Robinson:



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INTRODUCTION

The problems of sensor networks are inherently local-to-global: how does one integrate local data about an environment into global understanding? These problems become all-the-more difficult in the context of ad hoc wireless networks & GPS-denied settings (underwater, in buildings, etc.). In these settings, one wishes for a Mathematics that is coordinate-free, integrative, and qualitative. Oh, and computable.

Topology -- Algebraic Topology -- is that Mathematics.

These notes give a brief introduction to Applied Algebraic Topology for sensor networks. The reader who wishes to learn algebraic topology properly will want to consult a real text & begin rigorous training therein [e.g., in HATCHER'S text].

PART 1 : MOTIVATIONS FROM SYSTEMS

Topological methods are by no means confined to problems arising from data & statistics. Systems engineering is full of important challenges that demand global understanding/coordination/planning.

SYSTEMS	CHALLENGES	
ROBOTICS	MOTION PLANNING	(how do I move safely?)
	LOCALIZATION	(what is my location?)
	MAPPING	(where am I? "Hello, Cleveland?")
	PURSUIT/EVASION	(how do I catch/escape?)
	⋮	
SENSOR NETWORKS	COVERAGE	(are there holes in network?)
	AGGREGATION	(how to integrate redundant sensor data?)
	⋮	
COMMUNICATION NETWORKS	ROUTING	(how do I send signals to a specific location?)
	CODING	(how to encode/decode signals for broadcast?)
⋮	CONSENSUS	(can't we all just agree on something?)
	⋮	

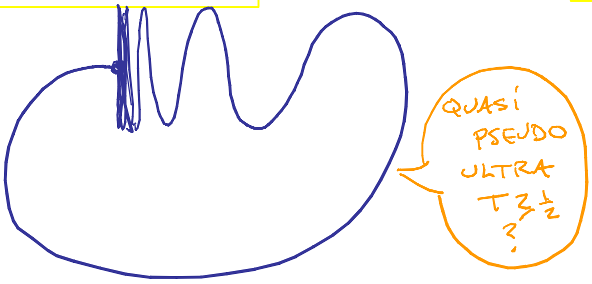
These & many^3 more problems in systems/science are not merely amenable to global methods -- in some cases current problems seem to demand a topological solution.

Many of the aforementioned problems involve the passage from something local [sensing / actuation / communication / ...] to a corresponding global feature. In many respects, topology is the mathematical response to the local ↔ global transition.

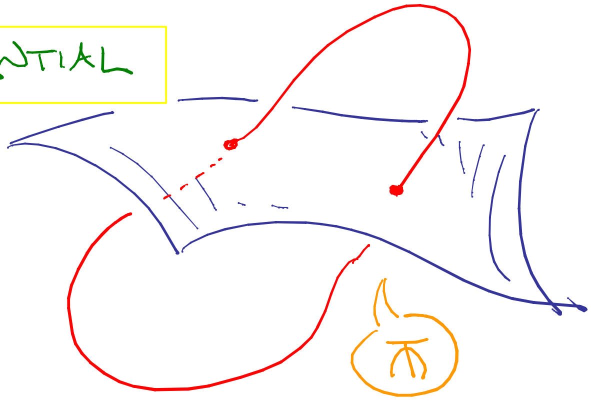
TOPOLOGY...

...has a rich history, with many emanations, including

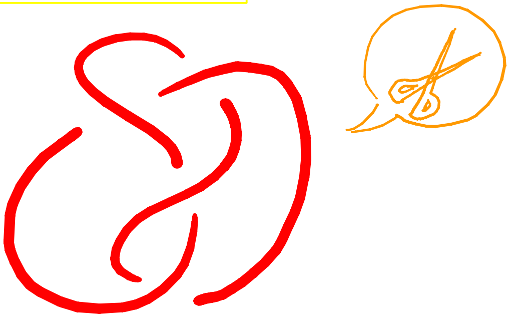
POINT-SET



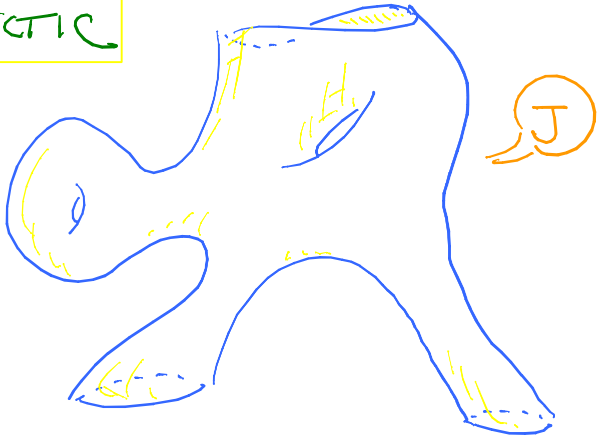
DIFFERENTIAL



GEOMETRIC



SYMPLECTIC



ALGEBRAIC

$$\begin{tikzcd}
{} \arrow[r] & H_n(X) \arrow[r] \arrow[d] & H_n(X,A) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & {}
\end{tikzcd}$$

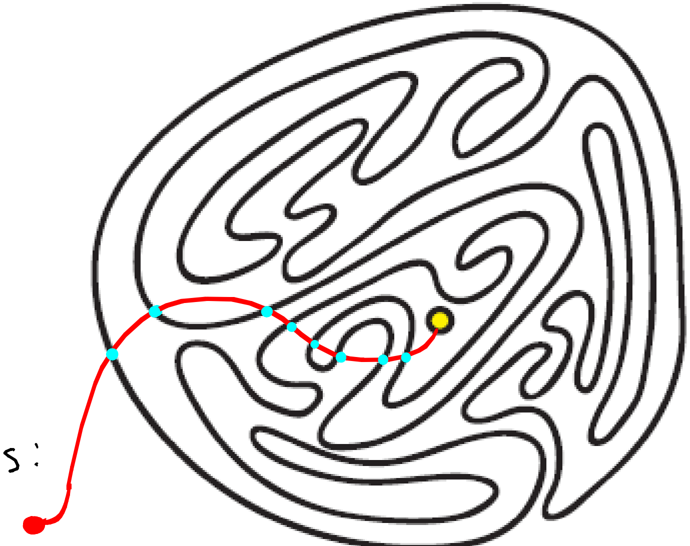
We will focus on applications of algebraic topology, a branch whose strengths are perhaps best adapted to the challenges of algorithmically managing large, high-dimensional systems.

The motivated reader is encouraged to consult the wonderful text of A. HATCHER: "Algebraic Topology" for more depth than is given here.

HOMOLOGICAL THINKING...

The core idea from algebraic topology to be detailed in these notes is Homology. As an introduction to the subject, we consider a few examples of simple situations where homological quantities are easily seen.

1. WINDING NUMBER

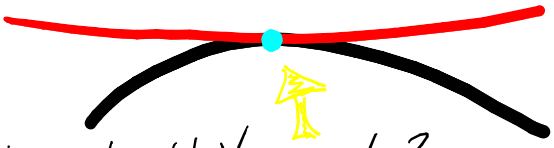


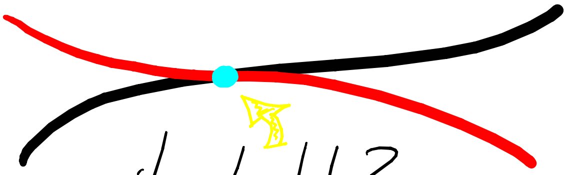
Consider a simple closed curve in the plane and a point off the curve. Is it inside or outside?

One simple method is as follows: draw a curve from the point to a point "far" from the curve. ("Infinity" if you like...)

Count the # of intersections of the loop & the path: EVEN means outside, ODD means inside.

OK. But... I've lied a bit -- what about a tangency?



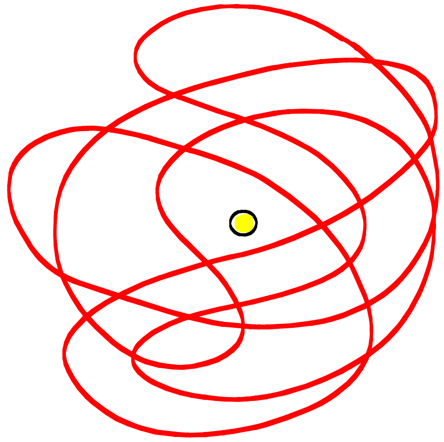


Maybe this shouldn't count? Or perhaps it should?

A better perspective is to assign an INDEX to intersections between the loop & curve, in such a manner that the sum of the indices is INVARIANT under deformations. One way to do that in this context is to assign the index as an element of $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} =$ the field $\{0,1\}$ with addition mod 2. This accounts for even/odd.

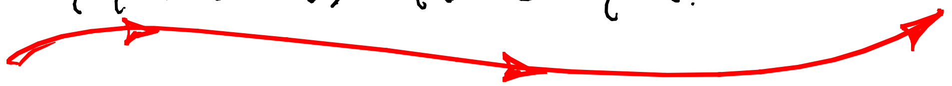
NOTA BENE: this mod-2 index is a LOCAL COMPUTATION (you don't need to know exactly what the curves are doing outside of near the intersection points); and it is COORDINATE-FREE (you don't need equations for the curves -- qualitative data suffice.

Mod-2 arithmetic is efficacious for problems of inside/outside. If, instead, one has a non-simple closed curve, then the appropriate question is not "inside or out?" but rather "how many times does the curve wind about?



This is solvable by the same technique with two extra ingredients:

1) One enriches the index from $\mathbb{F}_2$ to $\mathbb{Z}$.
2) Since $+1 \neq -1$ in $\mathbb{Z}$, one must account for this geometrically by ORIENTING the curve(s) with a direction.

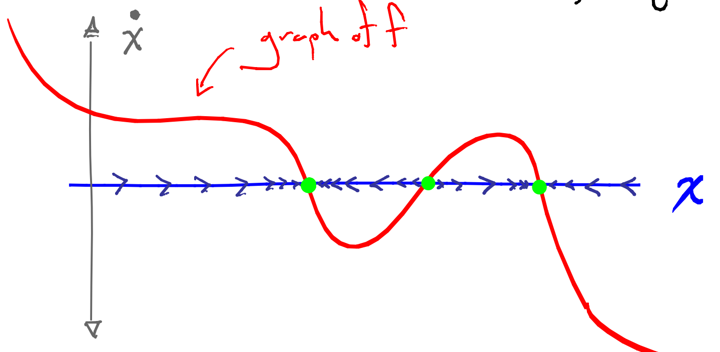


2. EULER-POINCARÉ INDEX

Consider a vector field & its flow on the real line; i.e. a differential equation of the form $\dot{x} = f(x)$

For simplicity, let's say that f is "dissipative" -- $\lim_{x \to \pm\infty} f(x) = \mp\infty$.

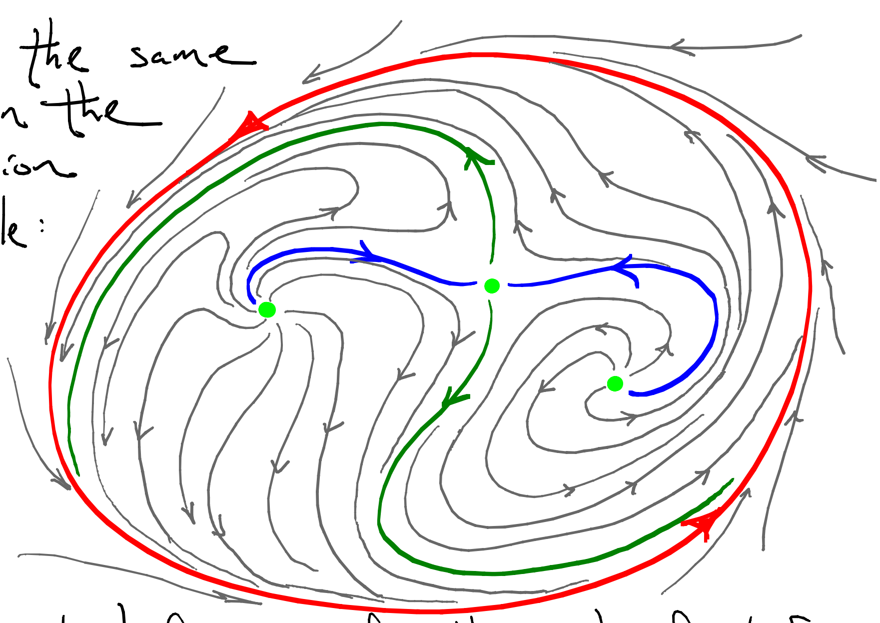
Then, the fixed points ("zeros", or "equilibria") have a natural index.

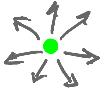


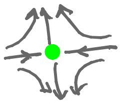
Assuming non-degenerate fixed points, there are, clearly, an odd number of equilibria: the sum of the fixed points mod 2 is 1.

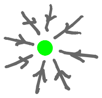
Enriching the algebra to $\mathbb{Z}$, we assign an index of $+1$ to a SOURCE ← ← ● → → and -1 to a SINK → → → ● ← ← ← ; 0 is assigned to a degenerate equilibrium: → → → → ● → → →

Of course, this index is the same "intersection" index as used in the winding number. The extension to vector fields on $\mathbb{R}^2$ is simple:



SOURCE:  $+1$

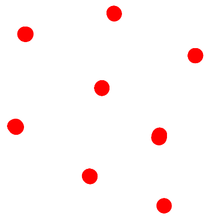
SADDLE:  -1

SINK:  $+1$

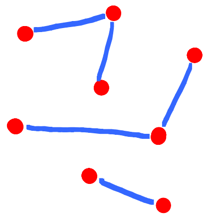
This index is local, needs no explicit formulae for the vector field [i.e. coordinates], and the sum of these numbers provides a "global" invariant. E.g., for a dissipative vector field on $\mathbb{R}^n$ ("infinity" repels) the sum of the indices of the fixed points equals $+1$.

3. EULER CHARACTERISTIC

Continuing this pattern, we examine spaces built from simple pieces -- "simplices" -- and find a way to count intelligently. This count, the EULER CHARACTERISTIC, is denoted χ.

$\chi \Bigg\{$  $\Bigg\} = 8$;

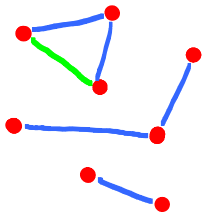
the number of objects, or "vertices" or "0-simplices"

$\chi \Bigg\{$  $\Bigg\} = 3$;

we've added 5 "ties" or "edges" or "1-simplices", leaving 3 connected pieces left to count

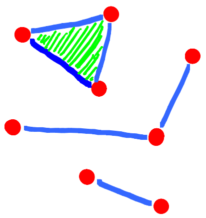
χ gives index $+1$ to vertices; -1 to edges; add it up!

However, the addition of one more edge creates difficulties...

$\chi \Big\{$  $\Big\} = 3 - 1 = 2$

if we follow our weighting scheme, we no longer have a count of the number of components

Note the presence of the "hole" in the graph. It appears that χ takes this into consideration. Let us fill in the hole with a "face" or a "2-simplex" and adjust χ accordingly...

$\chi \Big\{$  $\Big\} = 2 + 1 = 3$

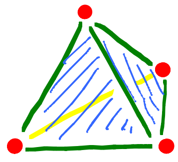
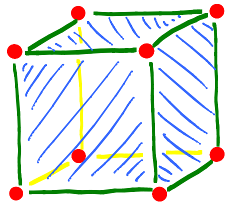
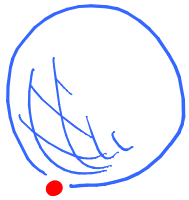
Assigning a weight of +1 to a 2-simplex accounts for the hole disappearing...

Proceeding inductively, one defines χ for any "complex" X built from "cells" as:

$$\boxed{\chi(X) = \sum_{\substack{\sigma \text{ cell} \\ \text{in } X}} (-1)^{\dim(\sigma)}}$$

$\left(\begin{array}{ll} \# & \text{vertices} \\ -\# & \text{edges} \\ +\# & \text{faces } \ldots \end{array}\right)$

THEOREM: For a compact "triangulable" space X, χ is independent of the [finite] simplicial structure imposed, as well as the topological type of X.

$\chi \Bigg\{$  $\begin{array}{c} +4 \\ -6 \\ +4 \\ \hline 2 \end{array}$  $\begin{array}{c} +8 \\ -12 \\ +6 \\ \hline 2 \end{array}$  $\begin{array}{c} +1 \\ +1 \\ \hline 2 \end{array} \Bigg\} = 2$

This theorem, and the other implicit theorems on invariance of an index, all rely on HOMOLOGY THEORY. We will cover homology in detail. For the present, suffice to say that all homology has the following ingredients:

① COUNTING

One counts objects of some relevance to the space-at-hand.

② GRADING

The objects counted possess some intrinsic "size" or "dimension" that is used in...

③ CANCELLING

By the correct application of algebra, one specifies how to cancel pairs of objects.

Done properly, these ingredients combine to form an algebraic device which is a global invariant...

PART 2: A SEQUENCE OF HOMOLOGIES

Homology has three essential ingredients:

1) CHAINS : objects to be counted
2) GRADING : a notion of "size" or "dimension" for chains
3) BOUNDARY : a means of cancelling chains of incident grading, satisfying $\partial^2 = 0$.

This lecture will review several homology theories, emphasizing the variety of types of objects counted.

SIMPLICIAL HOMOLOGY

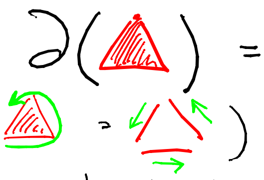
Let X = simplicial complex

CHAINS: $C_*^{simp} X$ generated by [oriented] simplices of X

GRADING: dimension of simplex

BOUNDARY: it is what you think...

(oriented if need be :

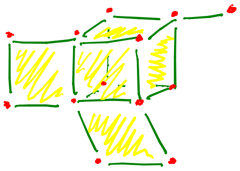


formal sum of edges in C_1

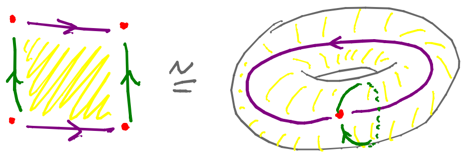
FACT: $H_*^{simp} X$ is independent of the simplicial structure on X

CELLULAR HOMOLOGY

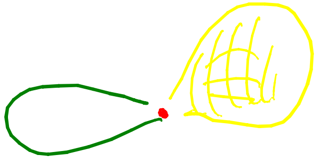
Let X = cell complex, built from cells of various dimension, in any manner that admits a reasonable notion of inductive gluing...



CUBICAL COMPLEX



IDENTIFICATION SPACE



CW-COMPLEX

CHAINS: $C_*^{cell} X$ generated by [oriented] cells of X

GRADING: dimension of cell

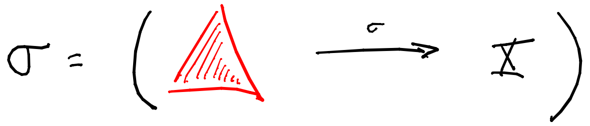
BOUNDARY: it is what you think...

FACT: $H_*^{cell} X$ is independent of the cellular structure on X

SINGULAR HOMOLOGY

Let X = any topological space

CHAINS: $C_*^{sing} X$ generated by MAPS of [oriented] simplices into X

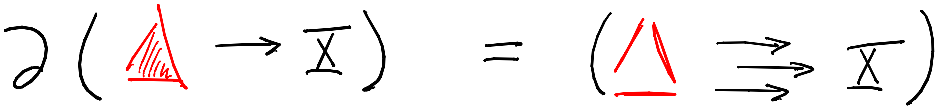


↖ the Platonic 2-simplex

Note that the image of σ in X is potentially ugly, hence the moniker "singular"

GRADING: dimension of the DOMAIN of σ, not its image in X

BOUNDARY: the boundary of a map of a simplex is the induced maps on the faces...



abstract sum of maps

PROBLEM: $C_n X$ is no longer finitely generated, unless X is a finite set

HOWEVER: $C_n X$ is so large that it contains, e.g., all possible simplicial structures on X, as well as all possible deformations thereof.

Two FOUNDATIONAL RESULTS:

THEOREM: H_*^{sing} is an invariant of homotopy equivalence

this is a reasonable result given the sheer size of C_*^{sing}. The key tool in this proof is an algebraic device known as a "CHAIN HOMOTOPY"

THEOREM: $H_*^{sing} \cong H_*^{cell}$ whenever the space has a well-defined cellular (or simplicial) structure

this uses induction on dimension, along with an algebraic tool called the "5-LEMMA"

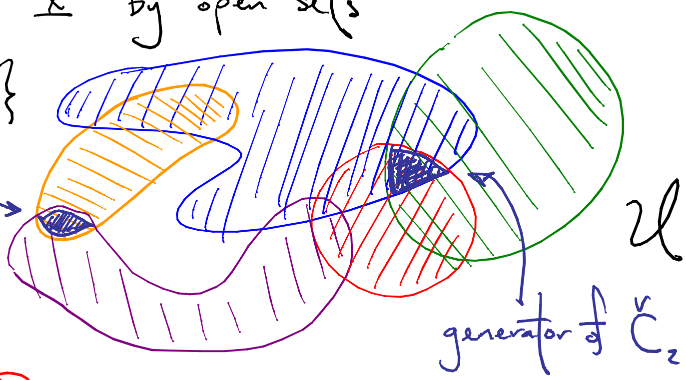
ČECH HOMOLOGY

Let $\mathcal{U} = \{U_\alpha\}$ be a cover of X by open sets

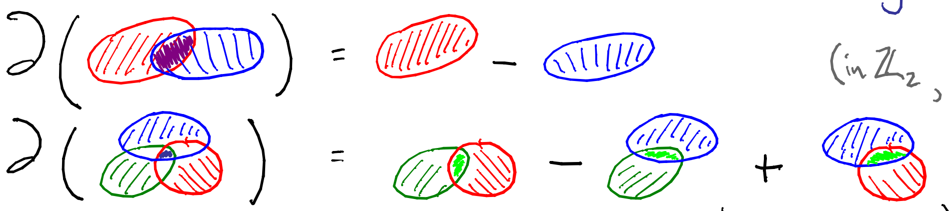
CHAINS: non-empty intersections of $\{U_\alpha\}$

GRADING: "depth" of intersection

This is a generator of $\check{C}_1$



BOUNDARY:



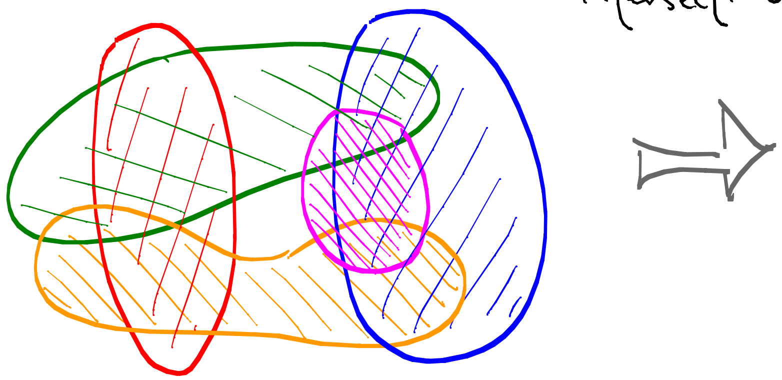
(in $\mathbb{Z}_2$, $+1=-1$ whew!)

(where the +/- are determined by an ordering on the index set $\alpha = \{1,2,\ldots\}$)

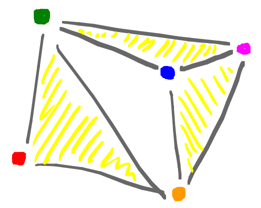
The signs on the boundary operator $\partial : \check{C}_k \to \check{C}_{k-1}$ are arranged so that $\partial \circ \partial = 0$ and the resulting homology is well-defined.

THEOREM: If $\mathcal{U}$ is ACYCLIC -- if $H_*(U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}) \cong H_*(\text{point})$ for all non-empty intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$, then $\check{H}_* \cong H_*(\bigcup_\alpha U_\alpha)$.

One often passes through an intermediate simplicial structure known as the NERVE of the cover -- a (k-1) simplex of $\mathcal{N}(\mathcal{U})$ is a non-empty intersection of k distinct elements of $\mathcal{U}$.



cover $\mathcal{U}$



nerve $\mathcal{N}(\mathcal{U})$

The NERVE LEMMA of Leray says that if all nonempty intersections in $\mathcal{U}$ are contractible sets, then $\mathcal{N}(\mathcal{U})$ is homotopy equivalent to $\bigcup_\alpha U_\alpha$.

MORSE HOMOLOGY

The structure of this homology theory is quite different. Let X be a smooth, compact manifold. Morse theory studies the topology of X as it relates to the calculus of functionals $f : X \to \mathbb{R}$.

- Choose a (Riemannian) metric for X
- Choose a smooth function $f : X \to \mathbb{R}$ ("height function")
- Critical points of f are points of X on which $\nabla f = 0$
- f is a MORSE FUNCTION if all critical points are nondegenerate

ASIDE some terminology...

HESSIAN: $\mathcal{H}f_p = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p \right]_{i,j}$ = matrix of 2nd derivatives of f

p is a nondegenerate critical point if $Df_p = 0$ & $\mathcal{H}f_p$ has no zero eigenvalues.

The UNSTABLE MANIFOLD of a critical point p is the set of points that converge to p under the gradient flow as time → $-\infty$

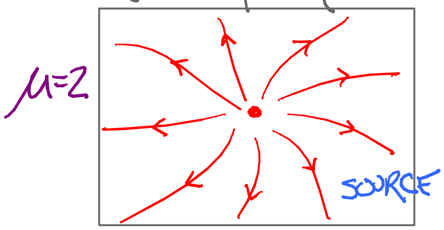
$$W^u(p) = \{ q \in X : \lim_{t \to -\infty} x(t) = p, \text{ where } x(0) = q \}$$

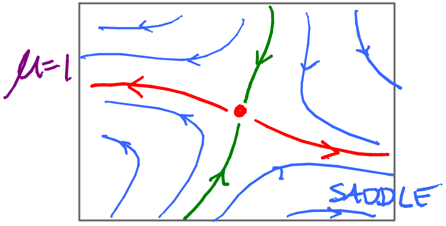
CHAINS: Generators of $MC_*(f)$ are critical points of f

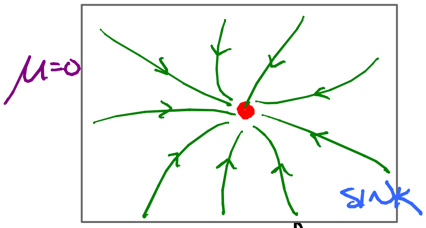
GRADING: MORSE INDEX of $p \in \mathrm{CRIT}(f)$ is

$$\mu(p) = \#\text{ negative eigenvalues of } \mathcal{H}f_p$$
$$= \dim W^u(p)$$

Intuition: The Morse index is the "degree of instability" of the critical point under the flow of $-\nabla f$.







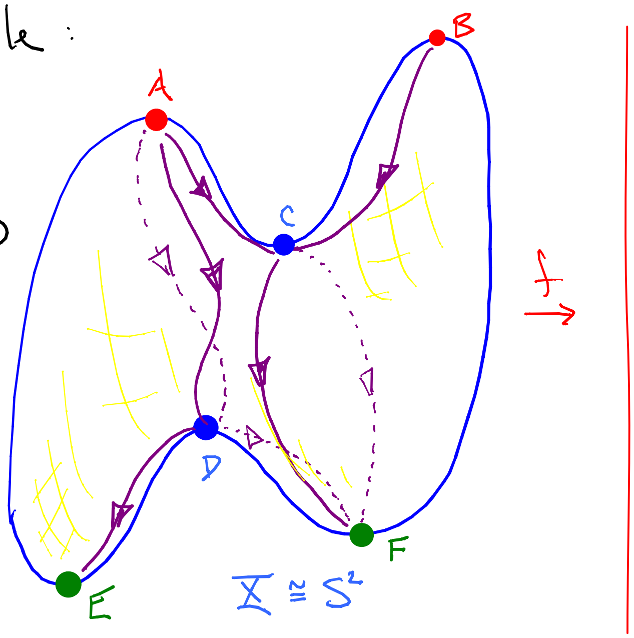
BOUNDARY: In $\mathbb{Z}_2$ coefficients, ∂ counts mod 2 the number of flowlines of $-\nabla f$ which connect critical points of incident μ.

It's best work mod 2 and do an example:

$MC_*(f)$ has 6 generators:

$$\cdots \to 0 \to \underset{\mu=2}{F_A \oplus F_B} \underset{\begin{bmatrix}1&1\\0&0\end{bmatrix}}{\underset{\uparrow}{\to}} \underset{\mu=1}{F_C \oplus F_D} \underset{\begin{bmatrix}0&1\\0&1\end{bmatrix}}{\underset{\uparrow}{\to}} \underset{\mu=0}{F_E \oplus F_F} \to 0$$

(Here, $F = \mathbb{Z}_2$ = field of 2 elements...)



The homology of f is illustrative:

$$MH_2(f) = \ker\begin{bmatrix}1&1\\0&0\end{bmatrix} \cong F$$ with generator $[A+B]$

$$MH_1(f) = \ker\begin{bmatrix}0&1\\0&1\end{bmatrix} \Big/ \operatorname{im}\begin{bmatrix}1&1\\0&0\end{bmatrix} \cong 0$$

$$MH_0(f) = F_E \oplus F_F \Big/ \operatorname{im}\begin{bmatrix}0&1\\0&1\end{bmatrix} \cong F$$ with $[E]=[F]$ as generators

That $MH_*(f) \cong H_*^{sing} S^2$ is no coincidence:

THEOREM: For X a compact manifold, $MH_*(f) \cong H_*^{sing}(X; \mathbb{Z}_2)$

The proof of this theorem is not so mysterious: the isomorphism comes via cellular homology using unstable manifolds as a cell structure on X: $p \in \mathrm{CRIT}(f) \longleftrightarrow W^u(p)$

$$MH_*(f) \xleftrightarrow{\cong} H_*^{cell} X \xleftrightarrow{\cong} H_*^{sing} X$$

independence of f follows.

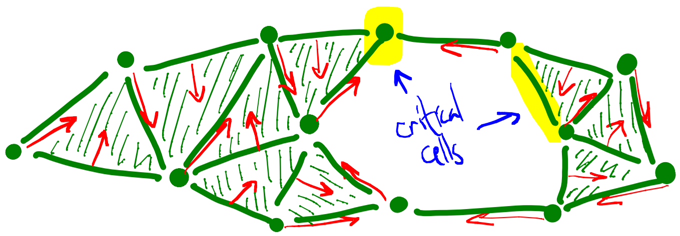
I've hidden a few details: one needs to perturb f and/or the gradient operator (via the metric) to ensure a good ∂-operator. On the other hand, I've not told you about CONLEY-MORSE theory, which counts fairly arbitrary invariant sets of arbitrary continuous vector fields on an arbitrary locally-compact metric space -- one can dispense with the usual "nondegenerate" conditions.

COMBINATORIAL MORSE HOMOLOGY

The fact that Morse homology is related to singular/cellular homology should be a hint that one can count more general notions of "critical objects" with "flowlines" as a boundary operator. Indeed several current research projects (Floer theories, contact homology, symplectic field theory, ...) rely on this principle. Here is a simple, combinatorial instantiation:

Let X = simplicial or cell complex

A COMBINATORIAL VECTOR FIELD on X is a collection V of pairs $V = \{(\tau_\alpha < \sigma_\alpha)\}$ where τ_α is a face of σ_α, and each cell of X lies in at most one pair. A CRITICAL CELL of X is one not listed in V.



CHAINS: generated by critical cells of V

GRADING: dimension of cells

A FLOWLINE of V is a sequence of V-paired cells

$$\boxed{\tau_0 < \sigma_0} > \boxed{\tau_1 < \sigma_1} > \boxed{\tau_2 < \sigma_2} > \cdots > \boxed{\tau_{n-1} < \sigma_{n-1}} > \tau_n$$

"is a face of" in V usual "is a face of" in X

V is a GRADIENT FIELD if there are no flowlines with $\tau_{n>0} = \tau_0$.

$MC_*(V)$ generated by critical cells of a gradient combinatorial field V

$\partial : MC_k(V) \to MC_{k-1}(V)$ counts the number of paths (mod 2, let's say)

$$\sigma > \boxed{\tau_0 < \sigma_0} > \boxed{\tau_1 < \sigma_1} > \boxed{\tau_2 < \sigma_2} > \cdots > \boxed{\tau_{n-1} < \sigma_{n-1}} > \tau$$

critical k-cell critical $(k-1)$-cell

THEOREM: $MH_*(V) \cong H_*^{cell} X$, independent of V

This is one piece of evidence that Morse theory does not really need all those conditions about smooth manifolds, nondegenerate Hessians, etc. For more on this subject, see papers of Forman & recent text of Koslov.

Given a multiplicity of (largely) equivalent homology theories, one proceeds to play one theory off another for gain:

LEMMA: If (C_*, ∂) is a finite-dimensional chain complex and H_* its homology, then

$$\chi(C_*) = \sum_{k=0}^{\infty} (-1)^k \dim C_k = \sum_{k=0}^{\infty} (-1)^k \dim H_k = \chi(H_*)$$

proof: Via linear algebra & the definitions:

1) $\dim C_n = \dim Z_n + \dim B_n$
2) $\dim Z_n = \dim B_n + \dim H_n$

Thus,

$$\begin{aligned} \chi(C_*) &= \sum_k (-1)^k \left(\dim Z_k + \dim B_{k-1}\right) && \text{via (1)} \\ &= \sum_k \underbrace{(-1)^k}_{} \left(\dim H_k + \underbrace{\dim B_k + \dim B_{k-1}}_{\text{telescoping sum}}\right) && \text{via (2)} \\ &= \sum_k (-1)^k \dim H_k = \chi(H_*) \end{aligned}$$

QED.

COR: The Euler characteristic is a homotopy invariant of finite cell complexes.

proof: If X and Y are homotopy equivalent finite cell complexes, then

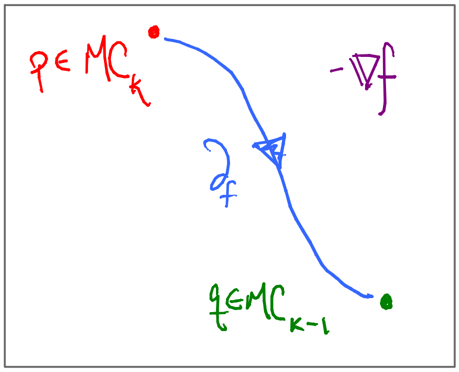
$$\chi(X) = \chi(C_*^{cell} X) = \chi(H_*^{cell} X) = \chi(H_*^{sing} X) = \chi(H_*^{sing} Y) = \chi(H_*^{cell} Y) = \chi(C_*^{cell} Y) = \chi(Y)$$

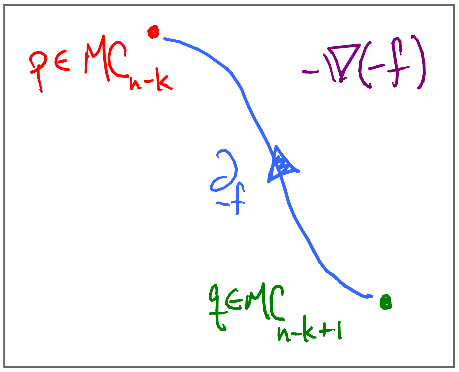
QED.

THEOREM: [POINCARÉ DUALITY] For X a compact manifold of dimension n,

$$H_k(X; \mathbb{R}) \cong H_{n-k}(X; \mathbb{R})$$

proof: $H_k(X;\mathbb{R}) \cong MH_k(f) \cong MH_k(-f) \cong MH_{n-k}(f) \cong H_{n-k}(X;\mathbb{R})$





$$\begin{tikzcd} MC_k(f) \arrow[r, "\partial"] \arrow[d, leftrightarrow, "\cong"'] & MC_{k-1}(f) \arrow[d, leftrightarrow, "\cong"] \\ MC_{n-k}(-f) & MC_{n-k+1}(-f) \arrow[l, "\partial"'] \end{tikzcd}$$

QED

Cor: A compact connected manifold X has a FUNDAMENTAL CLASS, a generator of $H_{\dim(X)}(X;\mathbb{Z}_2)$

proof: $\dim H_{\dim X}(X;\mathbb{Z}_2) \underset{\substack{\uparrow \\ \text{duality}}}{=} \dim H_0(X;\mathbb{Z}_2) \underset{\substack{\uparrow \\ \text{connectivity}}}{=} 1$

QED

This fundamental class, $[X] \in H_{\dim X} X$, is the basis of INTERSECTION THEORY

Cor: Any compact odd-dimensional manifold has $\chi = 0$.

proof:

$$\begin{aligned} \chi(X) &= \sum_k (-1)^k \dim H_k(X;\mathbb{Z}_2) \\ &= \sum_k (-1)^k \dim H_{n-k}(X;\mathbb{Z}_2) \qquad n = \dim X \\ &= \sum_k (-1)^{n-k} \dim H_{n-k}(X;\mathbb{Z}_2) \\ &= (-1)^n \chi(X) \end{aligned}$$

QED

PART 3 : SEQUENCES & SERIES

This lecture fills in the algebraic tools requisite to subsequent applications in data, statistics, systems, and sensing. The guiding principle is that it is not the dimensions of the homology groups that matter so much as DIAGRAMS -- sequences of homology groups with linear maps between them.

From now on, H_* will denote the most sensible homology theory in context (singular, cellular), usually with "convenient" coefficients unspecified...

FUNCTORIALITY

Homology is FUNCTORIAL : given any map $f : X \to Y$, there is an induced map on homology $H_* f : H_* X \to H_* Y$ taking, e.g., a cycle $[\zeta] \in H_k X$ to the cycle $[f \circ \zeta] \in H_k Y$. (One shows that this is well-defined).

The induced map $H_* f$ respects

① IDENTITIES : $H_*(id : X \to X)$ is $id : H_* X \to H_* X$

② COMPOSITION : $H_*(g \circ f) = H_* g \circ H_* f$

Here's a simple application of functoriality....

PROP : There does not exist a RETRACTION of D^n to ∂D^n -- a map from the closed n-disk to its boundary that is the identity on ∂D^n.

proof: Such a retraction gives a diagram

inclusion

$$\begin{tikzcd}
D^n \arrow[r, "r"] & \partial D^n & & 0 \arrow[d, phantom, "\shortparallel"] & & 0 \arrow[d, phantom, "\neq"] \\
 & & & H_{n-1} D^n \arrow[rr, "Hr"] & & H_{n-1} \partial D^n \\
\partial D^n \arrow[uu, "i"] \arrow[uur, "id"'] \arrow[rr, Rightarrow, "\text{apply } H_{n-1}"] & & 0 \neq & H_{n-1} \partial D^n \arrow[u, "Hi"] \arrow[urr, "H(id) = id"'] & &
\end{tikzcd}$$

Commutativity and the computation $H_{n-1}(\partial D^n) \cong H_{n-1}(S^{n-1}) \neq 0$ does it. QED

COR : [BROUWER FIXED POINT THEOREM] Any continuous map $f : D^n \to D^n$ has a fixed point.

proof : One constructs a retraction $r : D^n \to \partial D^n$ from a fixed-point-free map. QED

RELATIVE HOMOLOGY

One helpful construct is to count chains in a space relative to some subspace.

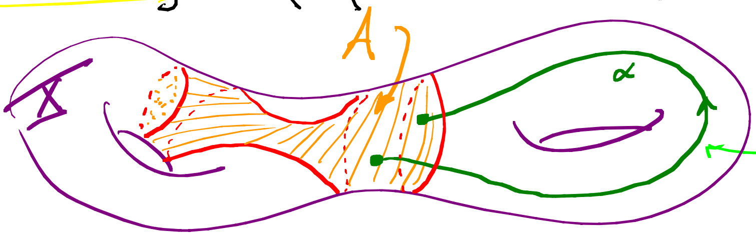
Given $A \subset X$ (a subspace, or a subcomplex in cellular settings...)

CHAINS: "relative" chains $C_*(X, A)$ are equivalence classes of chains in X modulo chains in A:

$$C_k(X, A) = C_k(X) / C_k(A)$$

GRADING: same as for $C_* X$

BOUNDARY: restrict $\partial : C_k(X, A) \to C_{k-1}(X, A)$ in natural way



relative 1-cycle in $C_1(X, A)$
$\partial\alpha \subset A$

The REDUCED HOMOLOGY of X, $\tilde{H}_* X$, is the homology of X relative to a point $\{pt\} \in X$: $\tilde{H} X \cong H_*(X, \{pt\})$. This is a useful construct in many theorems, including...

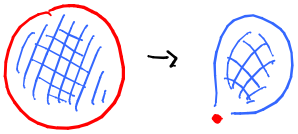
THEOREM: [EXCISION] For A a subcomplex of X,

$$H_*(X, A) \cong \tilde{H}_*(X/A)$$

collapse A to a point

Thus, e.g., $H_*(D^n, \partial D^n) \cong \tilde{H}_*(D^n/\partial D^n)$

$$\cong \tilde{H}_* S^n \cong \begin{cases} \mathbb{F} & * = n \\ 0 & \text{else} \end{cases}$$

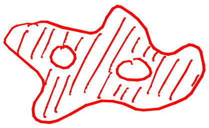
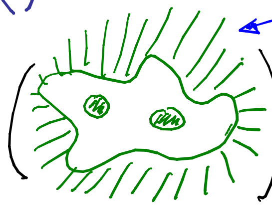


Reduced homology also puts in an appearance in

THEOREM: [ALEXANDER DUALITY] For $A \subset \mathbb{R}^n$ a nonempty compact cell cpx,

$$H_k(\mathbb{R}^n \setminus A) \cong \tilde{H}_{n-k-1}(A)$$

this component is trivial in $\tilde{H}_0$

So, e.g., H_1 () $\cong \tilde{H}_0$ () which has dimension = 2

Diagrammatic arguments in homology are greatly influenced by the notion of EXACTNESS -- a means of regulating (linear) algebraic data.

A sequence of vector spaces and linear transformations

$$\cdots \to V_{i+1} \xrightarrow{\phi_{i+1}} V_i \xrightarrow{\phi_i} V_{i-1} \xrightarrow{\phi_{i-1}} \cdots$$

is EXACT if $\text{KER}\ \phi_i = \text{IM}\ \phi_{i+1}$ for all i.

The relevance of exactness is not immediately evident: note, however, than an exact sequence is a chain complex ($\phi_i \circ \phi_{i+1} = 0\ \forall i$) whose homology $\ker \phi_i / \text{im}\ \phi_{i+1}$ is trivial for all i

Exactness is helpful for doing "inference":

EXAMPLE: Given $\cdots \to 0 \xrightarrow{0} A \xrightarrow{j} B \xrightarrow{0} 0 \to \cdots$

Exactness implies that
1) $\ker j = 0$ (exactness at A)
2) $\text{im}\ j = B$ (exactness at B)

$\Longrightarrow$ $j : A \xrightarrow{\cong} B$ ISOMORPHISM

The following are examples of "SHORT EXACT SEQUENCES"

1) For $A \subset X$,

$$0 \to C_nA \xrightarrow[\uparrow \atop \text{inclusion}]{i} C_nX \xrightarrow[\uparrow \atop \text{quotient}]{j} C_n(X,A) \to 0$$

2) For $\{A, B\}$ an open cover of X,

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_nA \oplus C_nB \xrightarrow{\psi} C_nX \to 0$$

$$\gamma \overset{\phi}{\longmapsto} (\gamma, -\gamma)$$

$$(\alpha, \beta) \overset{\psi}{\longmapsto} \alpha + \beta$$

One checks that these are exact...

These are exceptional in that they are examples of exact sequences of CHAIN COMPLEXES, graded and with a boundary operator. One of the most important tools in topology is a "braiding" of the resulting homologies:

Consider a short exact sequence of chain complexes:

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \longrightarrow 0$$

This creates a large diagram with boundary maps ∂ $(=\partial^A, \partial^B, \partial^C)$

This commutes!

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & {} \arrow[d] \\
0 \arrow[r] \arrow[d] & A_{n+1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A_n \arrow[r, "i"] \arrow[d, "\partial"] & B_n \arrow[r, "j"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A_{n-1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n-1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{n-1} \arrow[r] \arrow[d, "\partial"] & 0 \arrow[d] \\
{} & {} & {} & {} & {}
\end{tikzcd}$$

SNAKE LEMMA: Any short exact sequence of chain complexes $0 \to A_* \to B_* \to C_* \to 0$ induces a long exact sequence on the homology:

$$\cdots \to H_nA \xrightarrow{Hi} H_nB \xrightarrow{Hj} H_nC \xrightarrow{\partial} H_{n-1}A \xrightarrow{Hi} H_{n-1}B \xrightarrow{Hj} \cdots$$

Here, Hi and Hj are induced by i and j respectively. The map $\partial : H_nC \to H_{n-1}A$ is the map which is induced by ∂ as follows:

$[c] \in H_nC \implies c \in C_n$ and $\partial c = 0$

$\implies c = jb$ some $b \in B_n$

but $j\partial b = \partial j b = \partial c = 0$ since $j\partial = \partial j$

$\implies \partial b = i(a)$ for some $a \in A_{n-1}$

$\Big\}\ \partial[c] = [a]$

In "diagram form"

$$\begin{tikzcd}
 & & b \arrow[r, mapsto] \arrow[dd, mapsto] & c \arrow[r, mapsto] \arrow[dd, mapsto] & 0 \\
 & A_n \arrow[r] \arrow[dd] & \boxed{B_n} \arrow[r] \arrow[dd] & \boxed{C_n} \arrow[r] \arrow[d] & 0 \\
 & a \arrow[r, mapsto] & \partial b \arrow[r, mapsto] & 0 = \partial c & \\
0 \arrow[r] & \boxed{A_{n-1}} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} &
\end{tikzcd}$$

The snake lemma is proved via DIAGRAM-CHASING: a method not unlike driving in rush hour traffic. It's slow & dull, but eventually you get there.

e.g.: to show that a is a cycle: i is 1:1. therefore, $i(\partial a) = 0 \iff \partial a = 0$. but $i(\partial a) = \partial(ia) = \partial(\partial b) = 0$ by exactness & commutativity. ✓

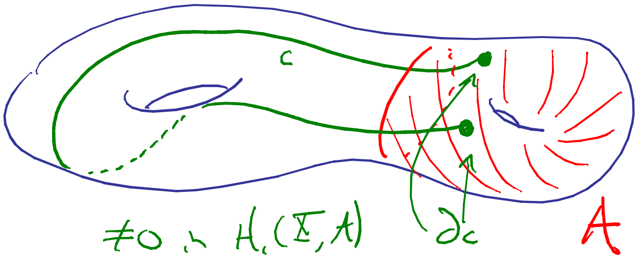
LONG EXACT SEQUENCE OF A PAIR

Given $A \subset X$, the short exact sequence $0 \to C_*A \to C_*X \to C_*(X,A) \to 0$ yields a long exact sequence on homology:

$$\boxed{\cdots \to H_nA \to H_nX \to H_n(X,A) \xrightarrow{\partial} H_{n-1}A \to H_{n-1}X \to \cdots}$$

LES OF PAIR

Here, ∂ takes a relative n-cycle $[c]$ and returns $[\partial c] \in H_{n-1}A$.



This can be used to compute $\tilde{H}_* S^k$

For convenience, we will work in reduced homology throughout.

① $\tilde{H}_n(D^k) \cong \tilde{H}_n(\{pt\}) \cong 0$ for all n (homotopy invariance)

② $\tilde{H}_n(S^0) \cong \begin{cases} \mathbb{F} & n=0 \\ 0 & \text{else} \end{cases}$ since $S^0 = \bullet \quad \bullet$

③ Consider the sequence of the pair $(D^k, \partial D^k)$, where $\partial D^k \cong S^{k-1}$,

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_n D^k \arrow[r] \arrow[d, phantom, "\parallel"] & \tilde{H}_n(D^k, \partial D^k) \arrow[r] \arrow[d, phantom, "\wr\parallel"] & \tilde{H}_{n-1} \partial D^k \arrow[r] \arrow[d, phantom, "\wr\parallel"] & \tilde{H}_{n-1} D^k \arrow[r] \arrow[d, phantom, "\parallel"] & \cdots \\
& 0 & \tilde{H}_n(\underbrace{D^k/\partial D^k}_{\cong S^k}) & \tilde{H}_{n-1} S^{k-1} & 0 &
\end{tikzcd}$$

$$\boxed{0 \to \tilde{H}_n S^k \xrightarrow{\cong} \tilde{H}_{n-1} S^{k-1} \to 0}$$

EXACTNESS YIELDS ISOMORPHISM

Thus $\tilde{H}_n S^k = \begin{cases} \mathbb{F} & n=k \\ 0 & \text{else} \end{cases}$ by induction on $H_* S^0$

The use of relative homology and the long exact sequence of the pair is also at the heart of a classical application of homology:

THEOREM: [INVARIANCE OF DOMAIN] If $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ open and homeomorphic, then $m = n$.

proof: Let $h: U \to V$ be a homeomorphism. Choose $p \in U$ and build a diagram of pairs of spaces:

$$\begin{tikzcd}
(U, U-\{p\}) \arrow[r, "\text{EXCISION}"'] \arrow[d, "h", "\cong"'] & (\mathbb{R}^m, \mathbb{R}^m-\{0\}) \arrow[d] \\
(V, V-\{h(p)\}) \arrow[r, "\text{EXCISION}"'] & (\mathbb{R}^n, \mathbb{R}^n-\{0\})
\end{tikzcd}$$

"localization"

When we pass to homology, the left, top, & bottom maps are $\cong$

$$\begin{tikzcd}
H_*(U, U-\{p\}) \arrow[r, "\cong"'] \arrow[d, "\cong"] & H_*(\mathbb{R}^m, \mathbb{R}^m-\{0\}) \arrow[d] \\
H_*(V, V-\{h(p)\}) \arrow[r, "\cong"'] & H_*(\mathbb{R}^n, \mathbb{R}^n-\{0\})
\end{tikzcd}$$

What is $H_*(\mathbb{R}^m, \mathbb{R}^m-\{0\})$? We can't collapse to a quotient, since the subset $\mathbb{R}^m-\{0\}$ is not a closed subcomplex. Let's use the sequence:

$$\to \underbrace{\tilde{H}_k \mathbb{R}^m}_{0} \to H_k(\mathbb{R}^m, \mathbb{R}^m-\{0\}) \xrightarrow[\cong]{} \tilde{H}_{k-1}(\underbrace{\mathbb{R}^m-\{0\}}_{\cong S^{m-1} \times \mathbb{R}}) \to \underbrace{\tilde{H}_{k-1} \mathbb{R}^m}_{0}$$

(polar coords)

Thus, $H_k(\mathbb{R}^n, \mathbb{R}^m-\{0\}) \cong \begin{cases} \mathbb{F} & k=m-1 \\ 0 & \text{else} \end{cases}$

$\Longrightarrow m=n$ in the (commutative) diagram above...

Finally, the long exact sequence of the pair is key in an inductive proof of the equivalence of singular & cellular homology:

THEOREM: For X a cell complex, $H_*^{cell} X \cong H_*^{sing} X$

(where the isomorphism comes from regarding cells as maps into X...)

proof: Let $X^{(k)}$ denote the "k-skeleton" of X -- the union of all cells of dimension $\leq k$. We induct on k: $k=0$ is obvious.

STRATEGY: Use the pair $(X^{(k+1)}, X^{(k)})$ and consider the long exact sequences in both homology theories:

$$\begin{tikzcd}
{} \arrow[r] & H^{cell}_{n+1}(X^{(k+1)}, X^{(k)}) \arrow[r] \arrow[d] & H^{cell}_{n} X^{(k)} \arrow[r] \arrow[d] & H^{cell}_{n} X^{(k+1)} \arrow[r] \arrow[d] & H^{cell}_{n}(X^{(k+1)}, X^{(k)}) \arrow[r] \arrow[d] & H^{cell}_{n-1} X^{(k)} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^{sing}_{n+1}(X^{(k+1)}, X^{(k)}) \arrow[r] & H^{sing}_{n} X^{(k)} \arrow[r] & H^{sing}_{n} X^{(k+1)} \arrow[r] & H^{sing}_{n}(X^{(k+1)}, X^{(k)}) \arrow[r] & H^{sing}_{n-1} X^{(k)} \arrow[r] & {}
\end{tikzcd}$$

The diagram is commutative & the rows are exact. The vertical arrows are induced by thinking of a cellular chain $\alpha \in C^{cell}_n$ as a singular chain.

OBSERVE:

① By induction the maps $\downarrow$ from $H^{cell}_* X^{(k)} \to H^{sing}_* X^{(k)}$ are $\cong$

② We know that $H_*(X^{(k+1)}, X^{(k)}) \cong \tilde{H}_*(X^{(k+1)}/X^{(k)})$. This quotient is a BOUQUET OF $(k+1)$-SPHERES, $\vee S^{k+1}$, one for each $(k+1)$-cell in X. Thus, after a little computation, $H_{k+1}(X^{(k+1)}, X^{(k)}) \cong \mathbb{F} \oplus \dots \oplus \mathbb{F}$, one copy of $\mathbb{F}$ for each $(k+1)$-cell; and $H_*(X^{(k+1)}, X^{(k)}) \cong 0 \ \ \forall * \neq k+1$.
Thus, the vertical maps $\downarrow$ from $H^{cell}_*(X^{(k+1)}, X^{(k)}) \to H^{sing}_*(X^{(k+1)}, X^{(k)})$ are $\cong$.

The proof ends (as many in this game do) by appealing to the

THEOREM [5-LEMMA] In any commutative diagram of exact sequences:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & A_4 \arrow[r] \arrow[d] & A_5 \arrow[d] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

If the four "outermost" vertical maps $\downarrow$ are isomorphisms, then so is the middle map $\downarrow$

This completes the proof. (many proofs in homology theory follow this way...)

Here is one last "classical" application of relative homology that emphasizes the importance of MAPS on homology as opposed to homology itself:

The LEFSCHETZ INDEX of $f : X \to X$ is

$$\tau f = \sum_{k=0}^{\infty} (-1)^k \, \mathrm{TR}\left(H_k f : H_k X \to H_k X \right)$$

trace of a matrix is independent of basis...

Note that, e.g. $\tau(\mathrm{id} : X \to X) = \sum_k (-1)^k \, \mathrm{TR}\left(H_k \mathrm{id}\right)$

$$= \sum_k (-1)^k \dim H_k X \quad = \chi(X)$$

generalizes Euler characteristic

THEOREM: For X a finite compact cell complex, $f : X \to X$ has a fixed point if $\tau f \neq 0$.

proof (sketch) The proof requires a few steps (given without proofs...)

① One may refine the cell structure on X and approximate f by a CELLULAR map -- it takes cells to cells

② If, by hypothesis $f : X \to X$ is fixed-point-free, then the cellular approximation to f (which we still call f by abuse) takes each cell to a different cell -- there are no fixed cells.

③ Using linear algebra & telescoping sums, one shows

$$\tau f = \sum_k (-1)^k \, \mathrm{TR} \; H_k f : H_k X \circlearrowleft \; = \sum_k (-1)^k \, \mathrm{TR} \; \underbrace{H_k f : H_k\left(X^{(k)}, X^{(k-1)}\right) \circlearrowleft}_{\text{induced map on relative } H_*}$$

DIAGRAM:

$$\begin{tikzcd}
{} \arrow[r] & H_k X^{(k)} \arrow[r] \arrow[d] & H_k\left(X^{(k)}, X^{(k-1)}\right) \arrow[r] \arrow[d] & H_k X^{(k-1)} \arrow[r] \arrow[d] & {} \arrow[d, phantom, "\text{\} induced by } f"] \\
{} \arrow[r] & H_k X^{(k)} \arrow[r] & H_k\left(X^{(k)}, X^{(k-1)}\right) \arrow[r] & H_k X^{(k-1)} \arrow[r] & {}
\end{tikzcd}$$

Observe: If f fixed no cell of X, $\mathrm{TR} \; H_k f : H_k\left(X^{(k)}, X^{(k-1)}\right) \circlearrowleft \; = 0 \; \forall k$

QED

COR: [HAIRY BALL THEOREM] S^n has a nonzero vector field iff n odd.

proof: $(\Longleftarrow)$ direct construction (generalized rotation)

$(\Longrightarrow)$ The time $\varepsilon > 0$ map of the flow of a vector field $V \neq 0$ is a fixed-point-free map of S^n. Compute the Lefschetz index:

$$\tau = \sum_{k=0}^{\infty} (-1)^k \, \mathrm{TR} \; H_k f \; = \underbrace{1}_{k=0} - 0 + 0 - \dots + \underbrace{(-1)^n}_{k=n} \quad > 0 \text{ if } n \text{ even}$$

QED

MAYER-VIETORIS SEQUENCE

There are many other exact sequences available. The short exact sequence of a 2-element cover $\{A,B\}$ of X

$$0 \to C_*(A\cap B) \to C_*A \oplus C_*B \to C_*(A\cup B) \to 0$$

yields

$$\boxed{\to H_n(A\cap B) \to H_nA \oplus H_nB \to H_n(A\cup B) \xrightarrow{\partial} H_{n-1}(A\cap B) \to H_{n-1}A \oplus H_{n-1}B \to}$$

MAYER-VIETORIS LES.

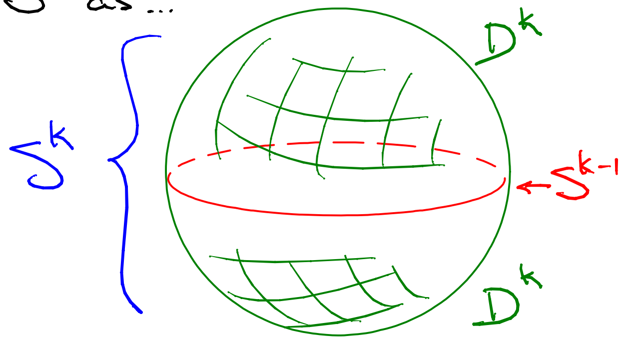
Example: Spheres, again: Decompose S^k as...

Let $A \cong D^k$ (northern hemisphere)

$B \cong D^k$ (southern hemisphere)

$A\cap B \cong S^{k-1}$ (equator)

$A\cup B \cong S^k$



M-V:

$$\longrightarrow \underbrace{\tilde H_n D^k}_{0} \oplus \underbrace{\tilde H_n D^k}_{0} \to \tilde H_n S^k \underset{\substack{\cong \\ \text{via exactness}}}{\longrightarrow} \tilde H_{n-1} S^{k-1} \to \underbrace{\tilde H_{n-1} D^k}_{0} \oplus \underbrace{\tilde H_{n-1} D^k}_{0} \to \cdots$$

So again, we see that $\tilde H_n S^k \cong \tilde H_{n-1} S^{k-1}$

The Mayer-Vietoris sequence is a crucial ingredient in one proof of...

THEOREM: [JORDAN CURVE] Consider an embedding h of $S^k \hookrightarrow S^n$ for $k<n$. Then

$$\tilde H_*(S^n - h(S^k)) \cong \begin{cases} \mathbb{F} & * = n-k-1 \\ 0 & \text{else} \end{cases}$$

PART 4 : SENSOR NETWORKS I

The goal of this lecture is to demonstrate the use of algebraic-topological tools in a particular context -- sensor networks. Throughout, note how diagrammatic arguments facilitate proofs.

NON-LOCALIZED NETWORKS

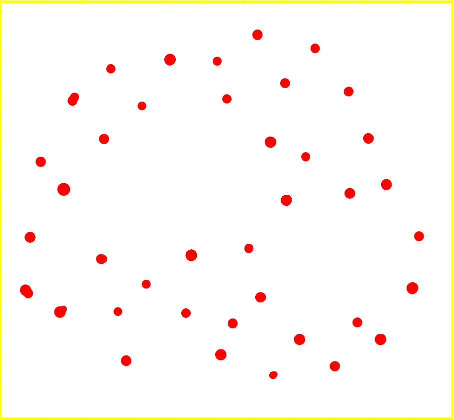
There are innumerable papers on networks – communications, sensor, social, etc. This lecture will concern networks that have an underlying geometry [nodes reside at a particular, usually fixed, location] which is strongly correlated to communication links between nodes. Nearby nodes can communicate; far-off nodes cannot [directly] communicate.

In many current instantiations, sensor nodes have fixed, known, global coordinates, either by fiat (video cameras are mounted & don't wander off...) or via GPS. For such systems, computation geometry methods (Voronoi diagrams, etc.) are efficacious. This lecture concerns non-localized networks.

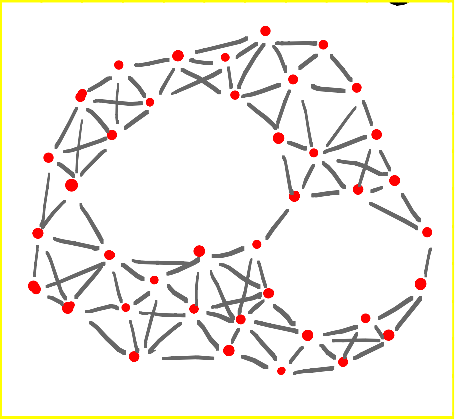
We assume that each sensor knows its (unique) identity and broadcasts such to all nearby nodes. For simplicity, assume symmetric communications and ignore all the intricacies of signal interference, signal bounce, etc.

The result is a COMMUNICATIONS GRAPH : vertices ⟷ nodes

edges ⟷ communication links



NODES



COMMUNICATIONS

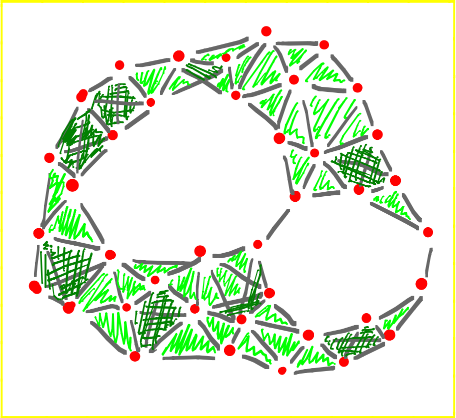
It's clear that this network has "holes" in it, but how do you make that notion precise... ?

CHAIN COMPLEX

$$\begin{tikzcd} C_0 & C_1 \arrow[l, "\partial"'] & \cdots \arrow[l, "\partial"'] \end{tikzcd}$$

The graph methods ubiquitous in network theory ignore the higher-order terms...

The **FLAG COMPLEX** associated to a graph Γ is the largest simplicial complex whose 1-skeleton is Γ. One fills in simplices wherever possible.



FLAG COMPLEX

$$C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow C_3 \leftarrow \cdots$$

nodes pairs triples quads

"cliques" of nodes in pairwise communication

It is clear from this cartoon that H_* of the flag complex is related to how well the network "covers" the domain...

COVERAGE PROBLEMS

Coverage problems are ubiquitous in sensor networks. Assume that each node x_α senses & "covers" thereby a neighborhood $\mathcal{U}_\alpha$. Among the many coverage problems in the literature, one finds:

BLANKET COVERAGE: Does $\bigcup_\alpha \mathcal{U}_\alpha$ contain the domain D of interest?

BARRIER COVERAGE: Does $\bigcup_\alpha \mathcal{U}_\alpha$ form a barrier – a separation between two regions of D one wants to isolate?

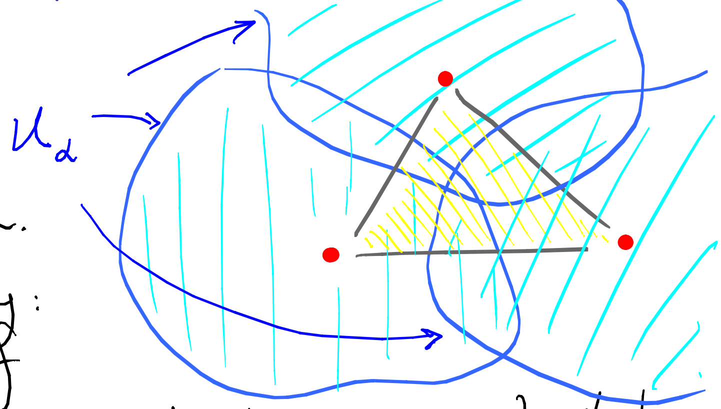
SWEEPING COVERAGE: If the nodes x_α move in time, does $\bigcup_{\alpha,t} \mathcal{U}_\alpha(t)$ contain D?

OPTIMAL COVERAGE: What is the smallest number/optimal locations of nodes to cover D (or separate or sweep...)

To begin, let's assume a given collection of nodes $\{x_\alpha\}$ in the plane $\mathbb{R}^2$, along with a communications graph. If we knew all the coverage regions $\{\mathcal{U}_\alpha\}$ and their intersection lattice, we could use Čech homology and the Nerve Lemma to look for holes as a means of solving blanket coverage. Alas, such data is not readily knowable in realistic networks.

Let us therefore make a somewhat restrictive though reasonable assumption:

ASSUME: For any 3 nodes in pairwise communication, the coverage regions together contain the convex hull of the nodes in $\mathbb{R}^2$

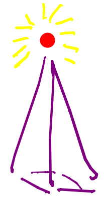


This is reasonable if there is some decent correlation between locations of nodes, communication links, and coverage.

We do not specify the sensor modality: one could work in the context of

- RADAR
- ACOUSTIC
- OPTICAL

⋮



Or, the nodes might not be sensors at all. They could represent wireless hotspots, in which case coverage means you can access the network from any location; or, if the nodes are beacons and the coverage regions are visibility domains, then one might want blanket coverage for robot navigation via beacon-following....

THE HOMOLOGICAL CRITERION FOR COVERAGE

Assume the following:

1. $\{x_\alpha\}$ = set of nodes in $\mathbb{R}^2$ which broadcast unique ID's and establish a communications network.
2. Triples of nodes in pairwise communication have their convex hull in $\mathbb{R}^2$ contained in the union of their coverage domains.
3. Let γ = cycle in the communications graph -- the FENCE -- whose embedded image in $\mathbb{R}^2$ bounds a polygonal domain $D \subset \mathbb{R}^2$.

Denote by $\mathcal{R}$ the flag complex of the communications network.
The fence cycle γ is a 1-cycle in $Z_1(\mathcal{R})$.

[All work in the remainder of this lecture is joint with Vin de Silva]

THEOREM: $D \subset \bigcup_\alpha \mathcal{U}_\alpha$ if, equivalently

① $[\gamma] = 0$ in $H_1 \mathcal{R}$

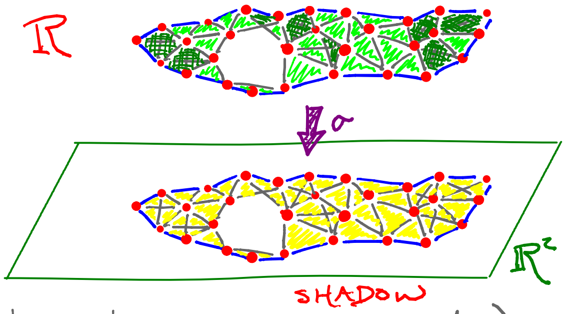
② $\exists\, [\alpha] \neq 0$ in $H_2(\mathcal{R}, \gamma)$ with $\partial\alpha \neq 0$.

HOMOLOGICAL BLANKET COVERAGE CRITERION

proof: The flag complex $\mathcal{R}$ is not a subset of $\mathbb{R}^2$ -- it is usually of dimension > 2. However, there is an obvious projection map of $\mathcal{R}$ to $\mathbb{R}^2$ as follows:

$$\sigma : \mathcal{R} \longrightarrow \mathbb{R}^2$$

$$\boxed{\text{simplex of } \mathcal{R}} \longmapsto \boxed{\text{convex hull of nodes in simplex}}$$



(Note: we don't know "coordinates" of x_α, but σ encodes unknown geometric data)

The map σ takes the pair $(\mathcal{R}, \gamma) \longrightarrow (\mathbb{R}^2, \partial D)$ with the restriction $\sigma : \gamma \to \partial D$ a homeomorphism by definition. Therefore, we have a commutative diagram of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & H_2\mathcal{R} \arrow[r] \arrow[d] & H_2(\mathcal{R}, \gamma) \arrow[r, "\partial"] \arrow[d, "H\sigma"'] & H_1(\gamma) \arrow[r, "H_i"] \arrow[d, "H\sigma"] & H_1\mathcal{R} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_2\mathbb{R}^2 \arrow[r] & H_2(\mathbb{R}^2, \partial D) \arrow[r, "\partial"] & H_1(\partial D) \arrow[r] & H_1\mathbb{R}^2 \arrow[r] & \cdots
\end{tikzcd}$$

Observe:

① By exactness of the upper row, $[\gamma] = 0$ in $H_1\mathcal{R}$ iff $H_i : H_1\gamma \to H_1\mathcal{R}$ is the zero map iff $\exists\, [\alpha] \neq 0$ in $H_2(\mathcal{R}, \gamma)$ with

$$0 \neq \partial[\alpha] = [\partial\alpha].$$ Thus the claimed equivalence.

② $H\sigma : \underset{\mathbb{F}}{\underset{\cong}{H_1\gamma}} \longrightarrow \underset{\mathbb{F}}{\underset{\cong}{H_1\partial D}}$ is an isomorphism, since $\sigma : \gamma \to \partial D$ is a homeomorphism

③ We conclude that $H\sigma \circ \partial\,[\alpha] \neq 0 \in H_1\partial D$

We derive a contradiction by showing that $H\sigma\,[\alpha] = 0$ if $D \not\subset \bigcup_\alpha \mathcal{U}_\alpha$. Assume $\exists\, p \in D \setminus \bigcup_\alpha \mathcal{U}_\alpha$. Then $p \notin$ image $\sigma(\mathcal{R}) \subset \mathbb{R}^2$. Thus, the map $H\sigma$ factors:

$$\begin{tikzcd}
 & H_2(\mathcal{R}, \gamma) \arrow[r, "\text{ONTO}"] \arrow[ld] \arrow[dd] & H_1\gamma \arrow[r] \arrow[dd, "\cong"] & {} \\
H_2(\mathbb{R}^2 - \{p\}, \partial D) \arrow[rd] & & & \\
 & H_2(\mathbb{R}^2, \partial D) \arrow[r, "\cong"] & H_1\partial D \arrow[r] & {}
\end{tikzcd}$$

Lemma: for $p \in D$

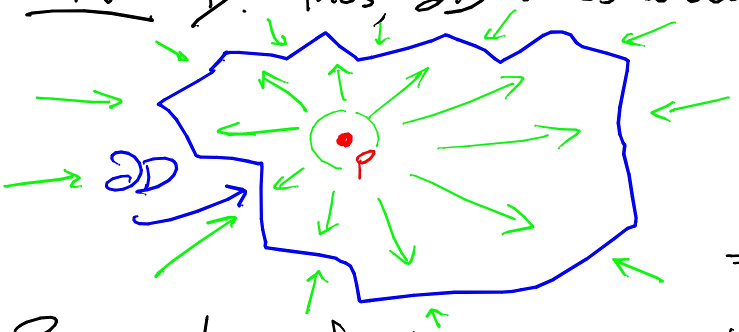
$$H_2(\mathbb{R}^2 - \{p\}, \partial D) = 0$$

Commutativity completes the proof.

proof of lemma: This is a simple exercise: you should try using Mayer-Vietoris or Alexander duality. Here is a proof using the long exact sequence of the pair $(\mathbb{R}^2-\{p\}, \partial D)$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(\mathbb{R}^2-\{p\}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_2(\mathbb{R}^2-\{p\},\partial D) \arrow[r] \arrow[d, phantom, "?"] & H_1\partial D \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_1(\mathbb{R}^2-\{p\}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \cdots \\
& 0 & & \mathbb{F} & \mathbb{F} &
\end{tikzcd}$$

The crucial step is the observation that the map $H_1\partial D \to H_1(\mathbb{R}^2-\{p\})$ is induced by the inclusion $i:\partial D \hookrightarrow \mathbb{R}^2-\{p\}$ and that p lies WITHIN D. Thus, ∂D winds around p:



$\mathbb{R}^2-\{p\}$ deforms to ∂D fixing ∂D.

$\Longrightarrow \quad H_1\partial D \xrightarrow{\cong} H_1(\mathbb{R}^2-\{p\})$

By exactness of the sequence, $H_2(\mathbb{R}^2-\{p\},\partial D)=0$ QED

Exercise: prove the Lemma using Mayer-Vietoris or Alexander duality...

POWER CONSERVATION

Clearly, one could state a simpler coverage criterion by requiring $H_1\mathcal{R}\cong 0$, since $[\gamma]\in H_1\mathcal{R}$ must of necessity be nullhomologous. Why do I prefer computing $[\alpha]\in H_2(\mathcal{R},\gamma)$? POWER.

COR: If $[\alpha]\neq 0 \in H_2(\mathcal{R},\gamma)$ with $\partial\alpha\neq 0$, then D is covered by those nodes implicated in α.

The other nodes can be placed into sleep mode: very useful.

proof:

$$\begin{tikzcd}
0 \arrow[d, phantom, sloped, "\cong"] & H_2(\mathcal{R},\gamma) \arrow[dl] \arrow[r] \arrow[dd] \arrow[ddr, bend left, "\neq 0"] & H_1\gamma \arrow[dd] \\
H_2(\mathbb{R}^2-p,\partial D) \arrow[dr] & & \\
& H_2(\mathbb{R}^2,\partial D) \arrow[r] & H_1\partial D
\end{tikzcd}$$

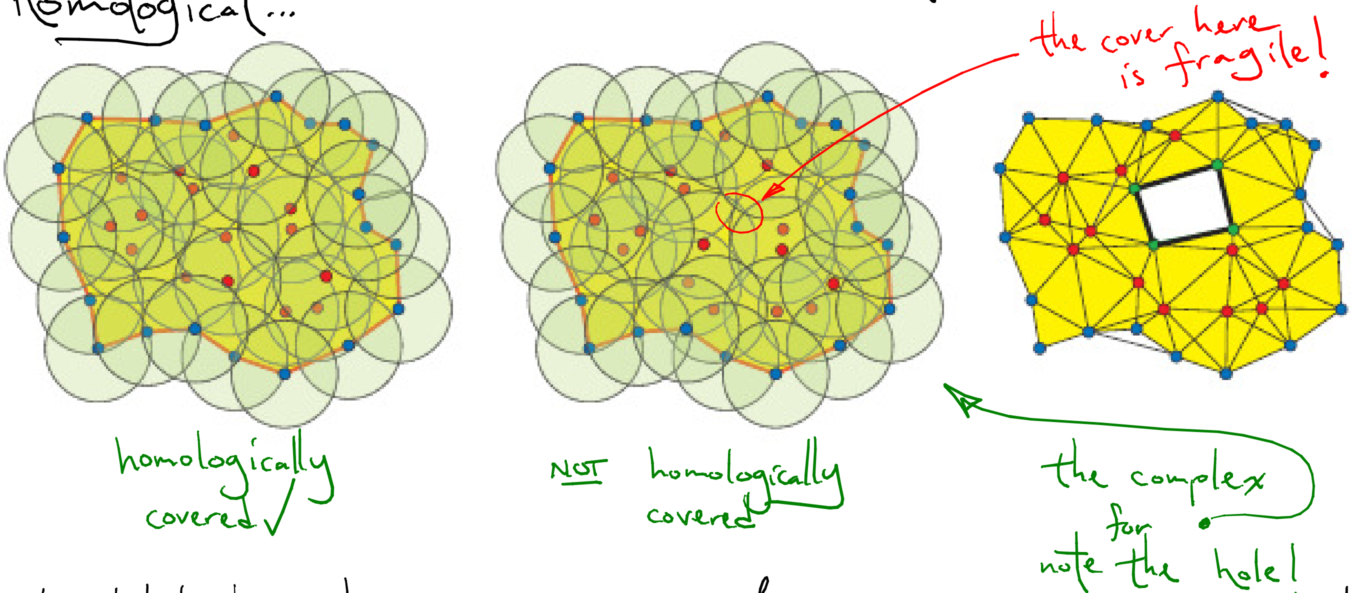
This factors through $\mathbb{R}^2-p$ if the cover sets for α do not cover p.

QED

It is remarkable that the homology automatically returns a potentially parsimonious cover...

COMMENTS

1. This is not an if & only if statement: since the input to the problem is the communication graph (and NOT the impossible-to-compute Čech complex!), there is no way to certify coverage in "borderline" cases -- the coverage needs to be sufficiently redundant in order to be homological...

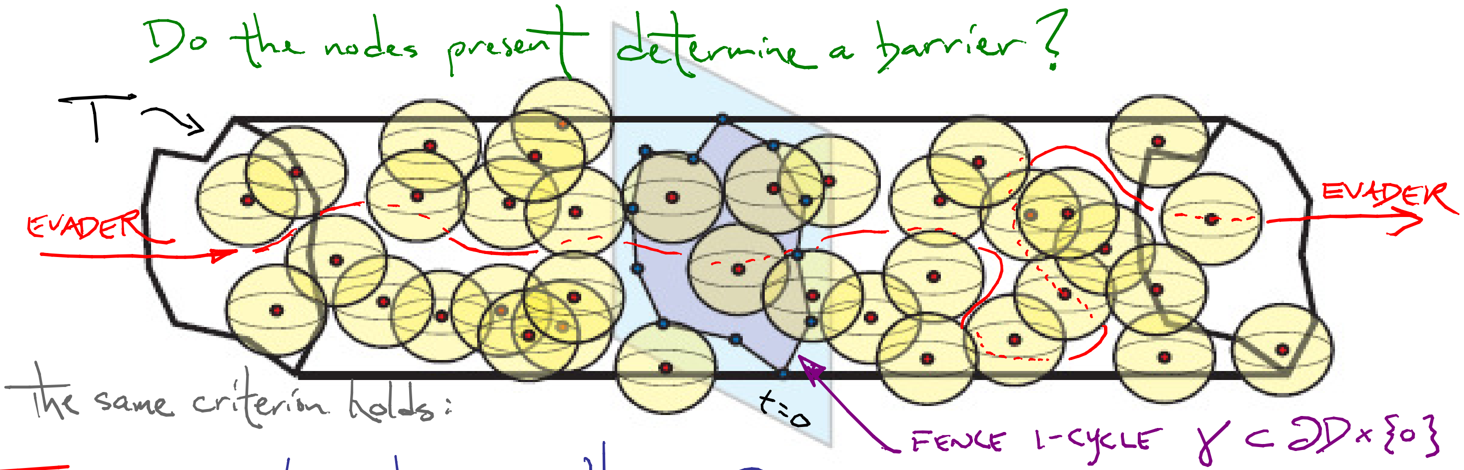


2. As stated, this criterion is CENTRALIZED & requires nodes to upload data to a central computer. It would be beneficial to have a more local, distributed means of verification.
3. In practice, communication signals bounce, fade in & out, interfere, and generally cause no end of woe: yes, we ignore all that for the sake of a clean theorem.
4. Lots of other researchers have looked at coordinate-free coverage using graph theory, probability, percolation, and geometry: homological methods are largely complementary to these efforts.

3-D BARRIER COVERAGE

Consider the setting where nodes lie in a 3-d "tunnel" $T = D \times \mathbb{R}$, where $D \subset \mathbb{R}^2$ is a polygonal domain defined by a 1-cycle γ in the communications network. How can one determine whether an "evader" can trace a surreptitious path from one end of T to the other?

Do the nodes present determine a barrier?



The same criterion holds:

THEOREM: No continuous path $p : \mathbb{R} \to T$ with $\lim_{t \to \pm\infty} p(t) \subset D \times \{\pm\infty\}$ exists if $\exists\ [\alpha] \in H_2(\mathcal{R}, \gamma)$ with $\partial\alpha \neq 0$.

(again, $\mathcal{R}$ = flag complex of comm. network and prior assumptions on coverage regions & convex hulls hold...)

proof: Almost the same!

$$\begin{tikzcd}
 & {[\alpha] \neq 0} \arrow[r] & {[\gamma] \neq 0} \\
 & H_2(\mathcal{R}, \gamma) \arrow[r] \arrow[dl] \arrow[dd] \arrow[ddr, bend left, "\neq 0"] \arrow[ddr, bend right, "=0"'] & H_1\gamma \arrow[dd, "\cong"] \\
H_2(\mathbb{R}^3 \setminus im(p), \partial D \times \{0\}) \arrow[dr] \arrow[d, phantom, sloped, "\wr"] & & \\
0 & H_2(\mathbb{R}^3, \partial D \times \{0\}) \arrow[r] & H_1(\partial D \times \{0\})
\end{tikzcd}$$

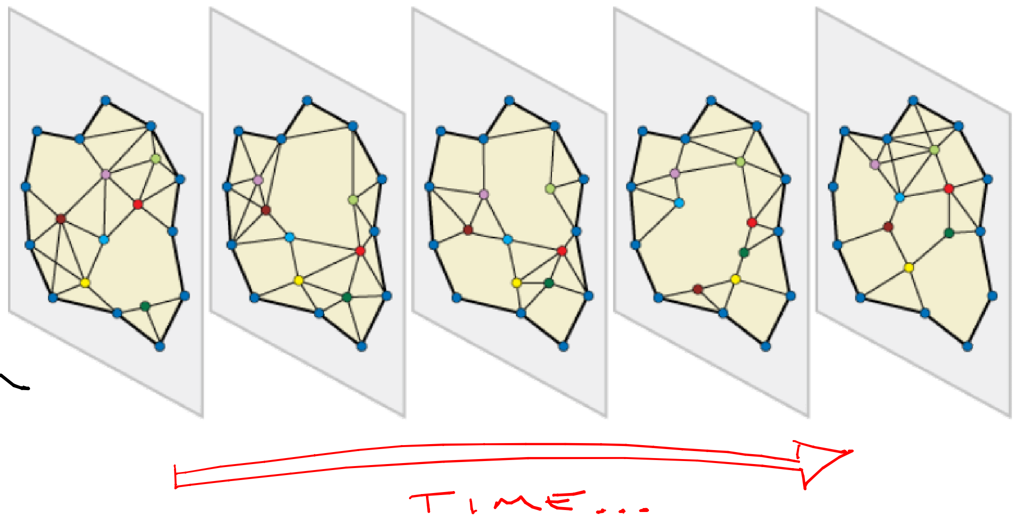
by similar computation as before... EXERCISE!

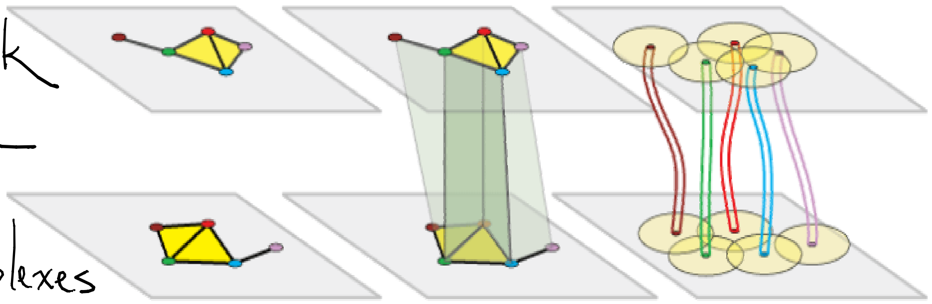
CONTRADICTION.

PURSUIT-EVASION & MOBILE NODES

Consider now the context in which a domain $D \subset \mathbb{R}^2$ has (as before) a fixed cycle γ of sensors defining ∂D; in the interior, there is a set of nodes which move about forming & breaking links.

Can a clever evader avoid detection by maneuvering through the network holes?





IDEA: Sample the network at (sufficiently small) time steps; then identify cells in incident flag complexes with the same vertex ID's.

This creates an AMALGAMATED complex, $\mathcal{AR}$. There is a well-defined 1-cycle γ since we assume that $\partial D \cong \gamma$ is fixed in time. Not surprisingly, the criterion is the same: there is no way for the evader to win if $\exists\, [\alpha] \in H_2(\mathcal{AR}, \gamma)$ with $\partial\alpha \neq 0$.

DOMAINS WITH HOLES

Here is a fun exercise in some simple techniques, the details of which an interested reader should follow...

Given $D \subset \mathbb{R}$ a connected domain with boundary γ having several components, how can one certify coverage of D from the homology of $\mathcal{R}$? Saying $H_1\mathcal{R} = 0$ is not likely!

Decompose $\gamma = \gamma^+ \amalg \gamma^-$ into the "outer" boundary γ^+ and the inner boundary component(s) γ^-.

THEOREM: D is covered [under previous assumptions...] whenever $\exists\, [\alpha] \neq 0$ in $H_2(\mathcal{R}, \gamma)$ with $\partial\alpha \neq 0$ on γ^+.

proof sketch: Decompose ∂D into $\partial D^+ \amalg \partial D^-$ corresponding to the cycles $\gamma = \gamma^+ \amalg \gamma^-$. Any TRIPLE of sets $X \supset A \supset B$ leads to

S.E.S. $\quad 0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$

L.E.S. $\quad \cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots$

Consider the triples $(\mathcal{R}, \gamma, \gamma^-) \xrightarrow{\sigma} (\mathbb{R}^2, \partial D, \partial^- D)$. Voilà:

$$\begin{tikzcd}
 & H_2(R,\gamma) \arrow[r] \arrow[dl, "\text{if } \exists p\in D \text{ not in cover...}"'] \arrow[d, "H\sigma"] \arrow[dr, bend left, "\neq 0"] & H_1(\gamma,\gamma^-) \arrow[r, "\cong"', "\text{EXCISION}"] \arrow[d, "H\sigma"] & H_1\gamma^+ \arrow[d, "\cong"] \\
H_2(\mathbb{R}^2-p,\partial D) \arrow[r] & H_2(\mathbb{R}^2,\partial D) \arrow[r] & H_1(\partial D,\partial^- D) \arrow[r, "\cong", "\text{EXCISION}"'] & H_1\partial^+ D
\end{tikzcd}$$

It's not true that $H_2(\mathbb{R}^2-p,\partial D)=0$; we instead show that the map $H_2(\mathbb{R}^2-p,\partial D)\longrightarrow H_1(\partial D,\partial^- D)$ is the zero map. Consider the L.E.S. of the triple $(\mathbb{R}^2-p,\partial D,\partial^- D)$:

$$\cdots\longrightarrow H_2(\mathbb{R}^2-p,\partial D)\xrightarrow{\partial} H_1(\partial D,\partial^- D)\xrightarrow{Hj} H_1(\mathbb{R}^2-p,\partial^- D)\rightarrow\cdots$$

we want to show this is zero

suffice to show this is injective $\Leftarrow$ I ♡ EXACTNESS!

Here it is:

$$\begin{tikzcd}
 & H_1(\partial D,\partial^- D) \arrow[r, "Hj"] & H_1(\mathbb{R}^2-p,\partial^- D) & \\
\text{EXACT} & H_1\partial D \arrow[u] \arrow[r] & H_1(\mathbb{R}^2-p) \arrow[u] & \text{EXACT} \\
 & H_1\partial^- D \arrow[u] \arrow[r, leftrightarrow, "\cong"'] & H_1\partial^- D \arrow[u] &
\end{tikzcd}$$

Exercise: use this to argue that Hj is injective, completing the proof.

PART 5 : NETWORK COVERAGE

We continue with coverage problems in sensor networks, incorporating a bit more geometry into the framework. Again, the goal is to demonstrate the use of simple algebraic-topological techniques, especially those which are diagrammatic in nature.

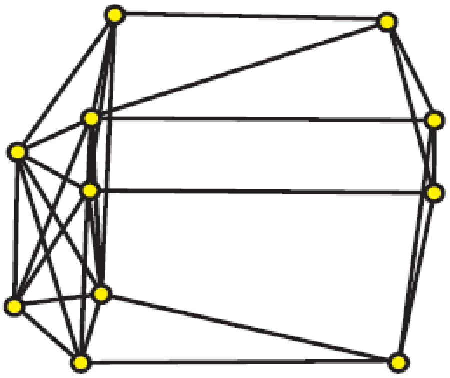
VIETORIS-RIPS COMPLEXES

One approach to ad hoc wireless networks follows the route used in point cloud data -- communication is assumed to be exactly correlated to metric distance.

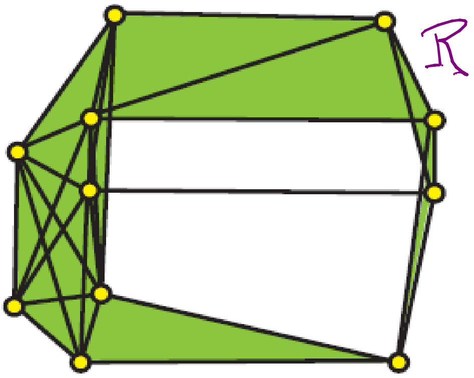
Consider a set of nodes $\mathcal{X}=\{x_\alpha\}$ in $\mathbb{R}^n$; the UNIT DISC GRAPH is the graph whose edges are present between nodes whose distance in (Euclidean) $\mathbb{R}^n$ is ≤ 1. Much has been written on the structure and computability of unit disc graphs.

If, instead of a "unit" disc graph we fix a communications radius r_c, the resulting r_c-disc graph has as its flag complex the Vietoris-Rips complex of the point-set. We denote this complex $\mathcal{R}$, the radius of communication being understood.

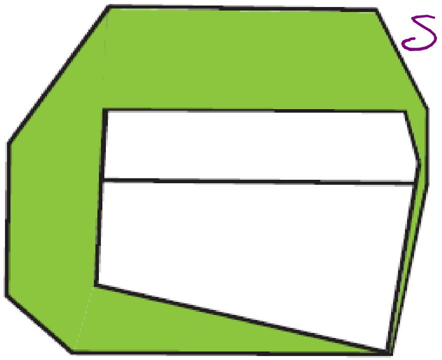
A simple starting point: What is the possible topology of Rips complexes of points in $\mathbb{R}^n$? And how does it compare with what it "models" in $\mathbb{R}^n$?



UNIT DISC GRAPH $\subset \mathbb{R}^2$



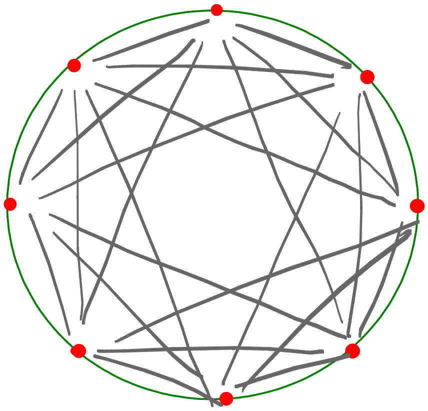
RIPS COMPLEX



SHADOW $\subset \mathbb{R}^2$

Define the shadow of a Vietoris-Rips complex $\mathcal{R}$ to be the image $\mathcal{S}$ of the map $\sigma : \mathcal{R} \to \mathbb{R}^n$ taking a k-simplex of $\mathcal{R}$ to the convex hull of the points defining the simplex in $\mathbb{R}^n$.

OBSERVATIONS:



1. $\dim \mathcal{R} > \dim \mathcal{S}$ in general
2. $\mathcal{R}$ is not homotopy equivalent to $\mathcal{S}$ in general, for any $n>1$. Consider a set of equally-spaced points on a circle of diameter $1+\varepsilon$: the Vietoris-Rips complex is a cross-polytope boundary; homotopic to a sphere.

$$\mathcal{R}_\varepsilon \simeq S^3$$
$$\mathcal{S} \cong D^2$$

3. $H_0\sigma : H_0\mathcal{R} \to H_0\mathcal{S}$ is always an isomorphism.
4. For points in dimension ≥ 4, there are examples where the shadow map $\sigma : \mathcal{R} \to \mathcal{S}$ does not preserve H_1 (due to V. de Silva).

However, for points in $\mathbb{R}^2$, the Vietoris-Rips complex has some rigidity:

THEOREM [CHAMBERS-DESILVA-ERIKSON-G] For points in $\mathbb{R}^2$, the shadow map preserves fundamental group: $\pi_1\sigma : \pi_1\mathcal{R} \to \pi_1\mathcal{S}$ is an $\cong$.

ASIDE: FUNDAMENTAL GROUP π_1

The fundamental group of a space X is the set of homotopy classes of loops in X (relative to a fixed basepoint $x_0 \in X$), with composition of loops, tail-to-head as a group operation. The fundamental group $\pi_1(X)$ is independent of the basepoint for X connected. Like homology, π_1 is functorial, and $f : X \to Y$ induced a homomorphism $\pi_1 f : \pi_1 X \to \pi_1 Y$. The relation to homology is clean: $H_1 X$ is the abelianization of π_1, as the Horewicz Theorem asserts. π_1 is in general uncomputable. Bummer.

The proof of this theorem is hypoelegant. But if you can tolerate case analysis, you are rewarded with some insight into Rips complexes

Cor: Rips complexes of planar point sets have free fundamental group.

We conclude that there is no torsion in H_1, thanks to Hurewicz. This guarantees the safety of computing H_1 in finite-field coefficients. Nice!

It is an open (and likely challenging) problem to determine if $\pi_1 \sigma$ is an isomorphism for point-sets in $\mathbb{R}^3$: it fails to be in $\mathbb{R}^4$.

The unit-disc graph construction is both fairly common and fairly derided in the sensor networks community. The sharp 0-1 cutoff based on exact distance is quite unrealistic. One common approach to managing uncertainty in communication links is to establish a pair of radii $r_s < r_w$ (r_s = "STRONG" signal; r_w = "WEAK" signal), with the model that nodes within distance r_s definitely establish links; nodes separated by distance r_w definitely fail to establish links; and nodes of intermediate distance may or may not have a link (perhaps with some probability). These are called QUASI-UNIT-DISC GRAPHS. We complete to the flag complex and call it a QUASI-VIETORIS-RIPS COMPLEX. In the sensor networks world, this is considered a realistic model.

What effect does a small amount of uncertainty have on the topology of the quasi-Rips complex?

THEOREM [C.D.E.G.] Given any finitely presented group G, there is a quasi-Rips complex Q of points in $\mathbb{R}^2$ with $r_w - r_s$ arbitrarily small with the property that $\pi_1 Q \cong G * F_n$ for some n.

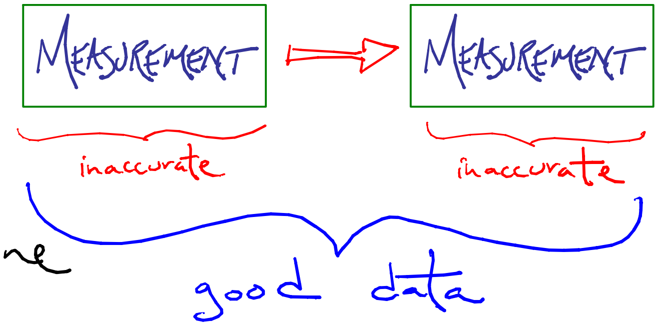
free group on n letters

free product

Otherwise said, any uncertainty in measurements can lead to topological noise at the level of π_1. Even on the level of H_1, torsion can arise, frustrating computation.

This is bad news: $\mathcal{R}$ + uncertain links = BAD TOPOLOGY

However, we've seen before in the context of data sets that the persistent topology of a pair of measurement compensates for one bad measurement



PERSISTENCE & FENCES

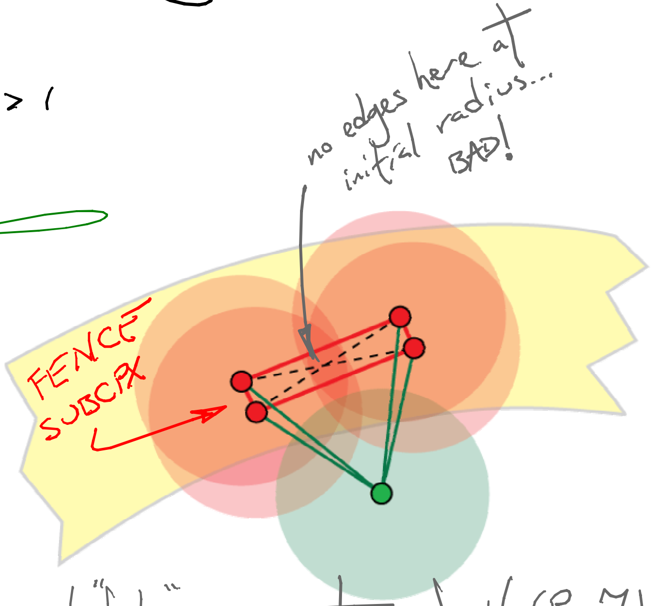
Let us return to the problem of coverage of a domain D by coverage sets U_α associated to nodes x_α in D. We will make some restrictive and unphysical assumptions

- round-ball communications domains
- round-ball coverage regions
- domain with a nice, smooth boundary

On the other hand, we will work with

- domains $D \subset \mathbb{R}^n$ any $n > 1$
- no dedicated fence cycle γ.

FENCE DETECTION:



A "fake" generator of $H_2(\mathcal{R},\mathcal{F})$

Consider $D \subset \mathbb{R}^n$ a connected domain. Assume that nodes $\{x_\alpha\}$ in D can detect when they are "near" ∂D, within a FENCE RADIUS r_f.

Nodes near ∂D identify themselves as FENCE NODES and assemble into a FENCE SUBCOMPLEX $\mathcal{F} \subset \mathcal{R}$

IDEA: The homological coverage criterion should be in $H_n(\mathcal{R},\mathcal{F})$.

FAIL! There are "fake" cycles in $\mathcal{F}$ which do not wind about ∂D

We compensate by considering two communications radii: a STRONG and a WEAK signal radius.

This gives strong & weak Rips complexes and fence subcomplexes which fit into a diagram:

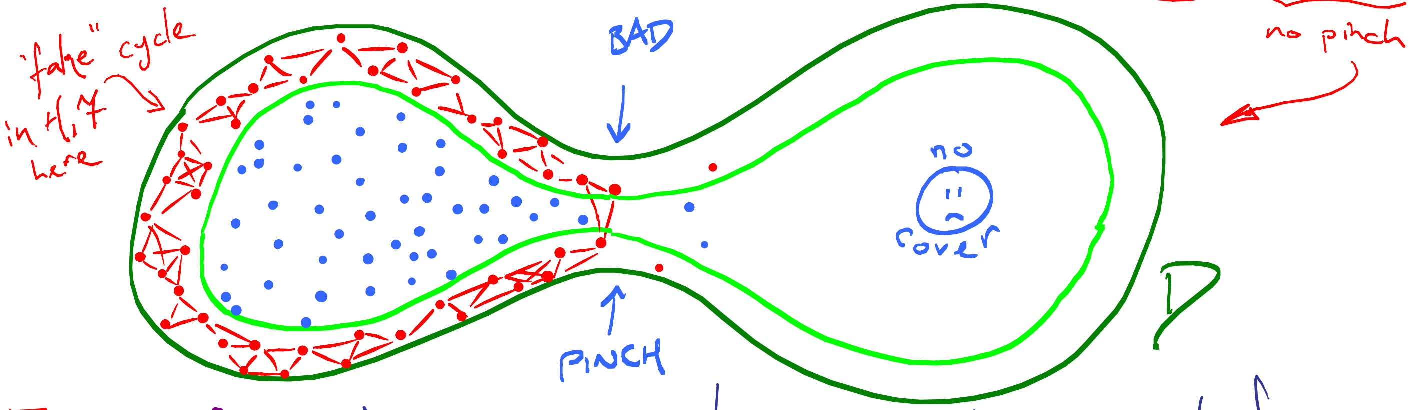
$$\begin{tikzcd}
\mathcal{F}_s \arrow[r, hook] \arrow[d] & \mathcal{R}_s \arrow[d] \\
\mathcal{F}_w \arrow[r, hook] & \mathcal{R}_w
\end{tikzcd}$$

H_* of this diagram gives coverage data.

ASSUMPTIONS :

COV.; STRONG; WEAK

- all coverage & communication is defined on balls of radii r_c, r_s, r_w
- inequalities on radii : $r_c \geq r_s\sqrt{2}$; $r_w \geq r_s\sqrt{10}$
- D = compact domain in $\mathbb{R}^n$ with "nice" boundary (satisfying some techical bounds on internal and external injectivity radii)
- nodes detect when they are within r_f of ∂D ("fence radius")
- ∂D has a collar $N_{\partial D}$ of radius $r_f + r_s/\sqrt{2}$, and $\pi_0(D - N_{\partial D}) = 1$.



THEOREM : [DG] Under above assumptions, $D \setminus N_{\partial D}$ is covered if the inclusion $H_n i : H_n(\mathcal{R}_s, \mathcal{F}_s) \to H_n(\mathcal{R}_w, \mathcal{F}_w)$ is nonvanishing.

(ie., ∃ a persistent class matching the homological criterion)

proof : This provides a good illustration of the use of exactness ...

Consider the "shadow" mapping $\sigma : (\mathcal{R}_s, \mathcal{F}_s) \to (\mathbb{R}^n, N_{\partial D})$. One must tune the radii and size of $N_{\partial D}$ to ensure that $\sigma(\mathcal{F}_s) \subset N_{\partial D}$.
The usual diagram from the long exact sequence of the pairs applies:

If ∃ $p \in D - N_{\partial D}$ not covered....

$$\begin{tikzcd}
 & H_n(\mathcal{R}_s, \mathcal{F}_s) \arrow[r] \arrow[d] \arrow[dl] & H_{n-1}\mathcal{F}_s \arrow[d] \\
H_n(\mathbb{R}^n - p, N_{\partial D}) \arrow[r] & H_n(\mathbb{R}^n, N_{\partial D}) \arrow[r] & H_{n-1}N_{\partial D}
\end{tikzcd}$$

EXERCISE: this is = 0

So if the composition

$$\begin{tikzcd}
H_n(\mathcal{R}_s, \mathcal{F}_s) \arrow[r] & {} \arrow[d] \\
 & H_{n-1}N_{\partial D}
\end{tikzcd}$$

is ≠ 0, we have a contradiction.

♯. We no longer have so much control over the "fence" in this setting. Indeed, it is possible to have generators for $H_{n-1}\mathcal{F}_s$ which do not go all-the-way-around ∂D -- this means that the composition $H_n(R_s,\mathcal{F}_s) \to H_{n-1}\mathcal{F}_s \to H_{n-1}N_{\partial D}$ may send $[\alpha] \mapsto 0$, killing the usual argument via commutativity.

♯ at first you don't succeed, try, try another diagram:

We have the inclusion $(R_s,\mathcal{F}_s) \hookrightarrow (R_w,\mathcal{F}_w)$, and the hypothesis that the homology of this inclusion is nonvanishing at grading n. We need to "refine" this inclusion by inserting a "midrange" complex pair $(R_s,\mathcal{F}_s) \hookrightarrow (R_m,\mathcal{F}_m) \hookrightarrow (R_w,\mathcal{F}_w)$ with communications at radius

$$r_m = r_s\sqrt{\frac{7n-5+2\sqrt{2n(n-1)}}{2n}}$$

(yes, that is an unpleasant formula, and, no, I won't explain why this is the right number. Suffice to say $r_s < r_m < r_w$, and one does NOT need to COMPUTE $(R_m,\mathcal{F}_m)$ -- it just is.)

Inclusions and long exact sequences yield a diagram:

$$\begin{tikzcd}
 & {[\alpha] \neq 0} \arrow[rrr, bend left, "\text{SENDS } [\alpha] \text{ TO } 0 \text{ IN THIS CASE}"] & & & \\
H_n(R_s) \arrow[r] \arrow[d] & \boxed{H_n(R_s,\mathcal{F}_s)} \arrow[r] \arrow[d] \arrow[dd, bend left=60, "\text{HYPOTHESIS}"] & H_{n-1}\mathcal{F}_s \arrow[d] \arrow[dr] & \\
H_n(R_m) \arrow[r] \arrow[d] & H_n(R_m,\mathcal{F}_m) \arrow[r] \arrow[d] & H_{n-1}\mathcal{F}_m \arrow[r] \arrow[d] & H_{n-1}(N_{\partial D}) \\
H_n(R_w) \arrow[r] & \boxed{H_n(R_w,\mathcal{F}_w)} \arrow[r] & H_{n-1}\mathcal{F}_w \arrow[ur] & \\
 & H_i[\alpha] \neq 0 & &
\end{tikzcd}$$

PICKY LEMMA: If $[\alpha] \neq 0$ in $H_n(R_s, \mathcal{F}_s)$ goes to $0 \in H_{n-1}N_{\partial D}$, then $[\partial\alpha] = 0$ in $H_{n-1}\mathcal{F}_m$.

(this uses all the bounds on the sizes of r_s, r_m, and r_f ....)

SQUEEZE LEMMA: Vietoris-Rips complexes factor through Čech complexes

$$\begin{tikzcd}
R_\varepsilon \arrow[rr] \arrow[dr] & & R_{\varepsilon'} \\
& \check{C}_{\varepsilon'} \arrow[ur] &
\end{tikzcd}$$

for $\frac{\varepsilon'}{\varepsilon} \geq \sqrt{\frac{2n}{n+1}}$

(these are optimal constants!)

(n = dimension of $\mathbb{R}^n$)

Given these lemmatta:

$$\begin{tikzcd}
& & {[\alpha] \neq 0} \arrow[rr, mapsto, bend left, "① \text{ SENT TO by picky lemma}"] & & \\
& H_n(R_s) \arrow[r] \arrow[d] & \boxed{H_n(R_s, \mathcal{F}_s)} \arrow[r] \arrow[d] & H_{n-1}\mathcal{F}_s \arrow[d] & \\
\text{② by exactness } \exists\, [\beta] \neq 0 \arrow[rrrr] \arrow[d] & & & & 0 \\
& H_n(R_m) \arrow[r] \arrow[d] \arrow[dl] & H_n(R_m, \mathcal{F}_m) \arrow[r] \arrow[d] & H_{n-1}\mathcal{F}_m \arrow[d] & \\
H_n\check{C} \arrow[dr] & & & & \\
\text{③ by Squeeze Lemma} \arrow[dr] & H_n(R_w) \arrow[r] & \boxed{H_n(R_w, \mathcal{F}_w)} \arrow[r] & H_{n-1}\mathcal{F}_w & \\
& Hi[\beta] \arrow[r] & Hi[\alpha] \neq 0 & &
\end{tikzcd}$$

HYPOTHESIS

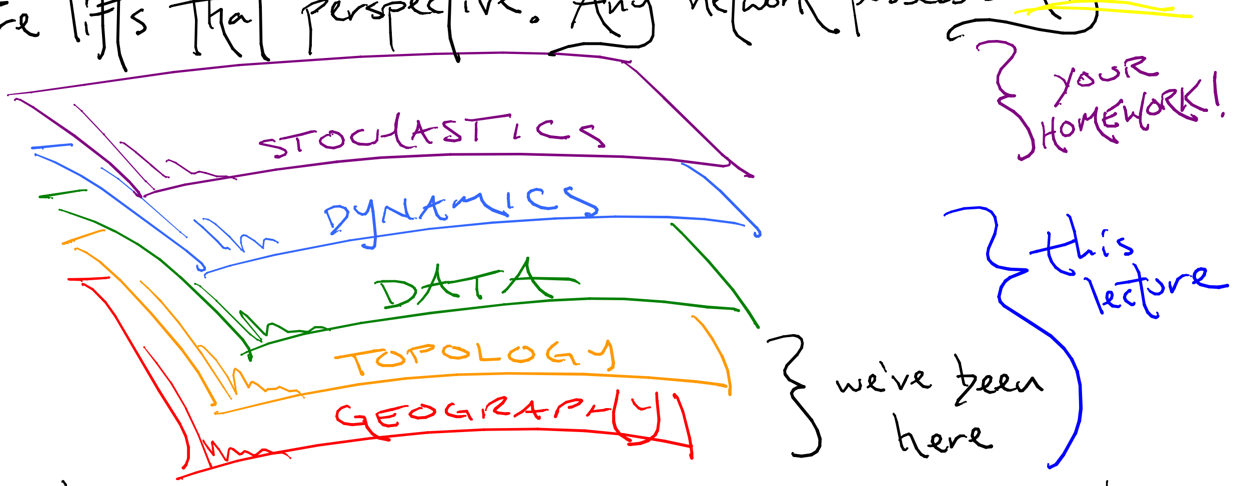
EXERCISE: For $A \subset \mathbb{R}^n$, $H_nA = 0$.

End of proof: if $[\alpha] \neq 0 \in H_n(R_s, \mathcal{F}_s)$ dies in $H_{n-1}N_{\partial D}$, then $[\partial\alpha] = 0$ in $H_{n-1}\mathcal{F}_m$. By exactness $[\alpha] \neq 0 \in H_n(R_m, \mathcal{F}_m)$ is the image of a nonzero class $[\beta] \in H_nR_m$. The inclusion of $[\beta]$ to R_w factors through the Čech complex, which by the Nerve Lemma is homotopic to a subset of $\mathbb{R}^n$. This set has vanishing H_n. Thus, $[\beta] \mapsto 0$ in H_nR_w. Commutativity implies $[\alpha] = 0 \in H_n(R_w, \mathcal{F}_w)$ CONTRADICTION!

PART 6 : EULER CALCULUS

We have focused on the correlation between the topology of a communication or sensor network and the spatial or geographic distribution of nodes. As with point-cloud data sets, the network completes to a chain complex whose homology reveals global structure.

This lecture lifts that perspective. Any network possesses layers



One can think of the data in a (sensor, ...) network as residing "above" the nodes. This is suggestive language evocative of today's algebraic-topological tools...

BUNDLES & SHEAVES

There are numerous ways that topogists (& algebraic geometers) have devised to stack "data" on top of a structure. Many of these ideas are evocative of a manifold, in which local structure (Euclidean) is specified, and global structure can be patched together.

VECTOR BUNDLE : a "total space" E, a "base" manifold B, and a projection $p: E\to B$ with fibers $p^{-1}(b)\cong V$, a vector space. Thus, E splits into a local cross-product of V with neighborhoods of B. E.g., a vector field is a certain map $\mathcal{X}: M \to T_*M$ to the TANGENT BUNDLE of a manifold M.

COVERING SPACE: a "cover" $\tilde{X}$; a "base" X; and a projection $p: \tilde{X} \to X$ with fibers $p^{-1}(x)$ a discrete set of points. Thus, $\tilde{X}$ is locally homeomorphic to X.

FIBER BUNDLE: a "total" space E; a base space B, and a projection $p: E \to B$ with fibers $\cong F$, some fixed space. Thus E is locally $\{\text{neighborhood in } B\} \times F$.

All of the above are classical structures for making "twisted" spaces from simpler spaces (e.g. a Klein bottle is a bundle over S^1 with fiber $\cong S^1$; and it has a torus as a covering space with fiber $\cong$ two points.) [FUN EXERCISE]

SHEAVES are yet another means of assigning "product"-like structure to local neighborhoods of a "base." Roughly speaking, a sheaf over a space B is a mapping $\mathcal{F}: \left\{\begin{matrix}\text{open sets}\\ \text{in } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{(algebraic)}\\ \text{objects}\end{matrix}\right\}$ in a manner that respects two operations:

- RESTRICTION - an inclusion $U \subset V$ induces $\mathcal{F}(V) \to \mathcal{F}(U)$
- OVERLAPS - if $\mathcal{F}(U)$ and $\mathcal{F}(V)$ agree on $\mathcal{F}(U \cap V)$, then there is a unique gluing to $\mathcal{F}(U \cup V)$.

(FINE PRINT: the sheaf assigns open sets to objects of a category... groups, rings, sets, modules, & more are common models for the "sheaf data." The cartoonish definition given here is just the beginning of a rich subject...)

EXAMPLE: $C^k(U)$ = set (ring) of k-times differentiable functions on an open set $U \subset X$ is a sheaf, since we know how to canonically restrict and glue smooth functions.

THE POINT: Sheaves permit the integration of local data across a "cover."

Network data should be "sheafified"

EULER CHARACTERISTIC, REDUX

Recall the Euler characteristic: for a finite cell complex X,

$$\chi(X) = \sum_{\sigma \text{ cell}} (-1)^{\dim \sigma}$$

By the algebraic lemma that $\chi(C_*) = \chi(H_*(C_*))$ for any chain complex C_*, we have that

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k \dim H_k X \quad \Leftarrow \text{homotopy invariant}$$

LEMMA: For A, B finite subcomplexes of a cell complex X,

$$\boxed{\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cup B)}$$

proof: the sequence

$$0 \to C_*(A \cap B) \to C_*A \oplus C_*B \to C_*(A \cup B) \to 0$$

is exact, and thus a chain complex with vanishing homology. Apply χ vertically & horizontally. QED

OBSERVE: χ resembles a measure on subcomplexes...

The idea, going back to Blaschke, Hadwiger, ..., is to view χ as a "scale-invariant" measure on reasonably nice sets [polyhedra, originally]. One writes, for $A \subset X$ subcomplex

$$\int_X \mathbb{1}_A \, d\chi := \chi(A)$$

indicator function of A

We cannot proceed as in integration theory to build σ-algebras as χ is only well-defined for spaces with a finite cell division, and taking limits is problematic. However, we can consider:

$$CF(X) = \{ h : X \to \mathbb{Z} \ : \ h = \sum_{\substack{\sigma \\ \text{cell}}} \underset{\substack{\text{some} \\ c_\sigma \in \mathbb{Z}}}{c_\sigma \mathbb{1}_\sigma} \}$$

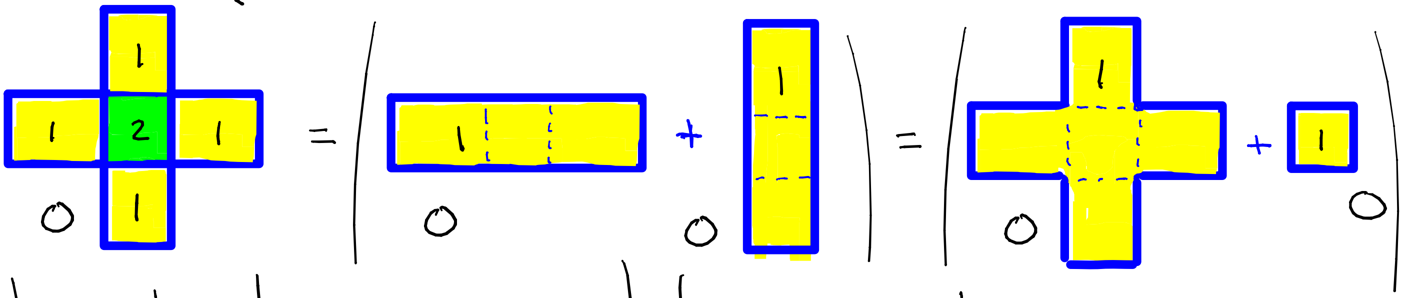
CONSTRUCTIBLE FUNCTIONS on X

(n.b.: each σ is compact in X and we will assume that all such h are compactly supported...)

DEF: The Euler integral is the homomorphism $\int_X \cdot \, d\chi : CF(X) \to \mathbb{Z}$ taking

$$\sum_\sigma c_\sigma \mathbb{1}_\sigma \longmapsto \sum_\sigma c_\sigma \overset{\overset{1}{\shortparallel}}{\chi(\sigma)} = \sum_\sigma c_\sigma$$

This is, note, a function of $h \in CF(X)$ and not of the decomposition of h. This is important.



The integral can be computed in several ways:

LEMMA: For $h \in CF(X)$

$$\int_X h \, d\chi = \sum_{s=-\infty}^{\infty} s \, \chi\{h=s\} = \sum_{s=0}^{\infty} \chi\{h>s\} - \chi\{h<-s\}$$

proof:

$$h = \sum_{s=-\infty}^{\infty} s \mathbb{1}_{\{h=s\}} = \sum_{s=0}^{\infty} s \left(\mathbb{1}_{\{h \geq s\}} - \mathbb{1}_{\{h>s\}} \right) + \sum_{s=0}^{-\infty} s \left(\mathbb{1}_{\{h \leq s\}} - \mathbb{1}_{\{h<s\}} \right) = \sum_{s=0}^{\infty} \mathbb{1}_{\{h>s\}} - \mathbb{1}_{\{h<-s\}}$$ QED

The second formula -- in terms of excursion sets -- tends to be numerically stabler.

EXERCIZE: $\int_{A \cup B} h \, d\chi = \int_A h \, d\chi + \int_B h \, d\chi - \int_{A \cap B} h \, d\chi$

DATA AGGREGATION

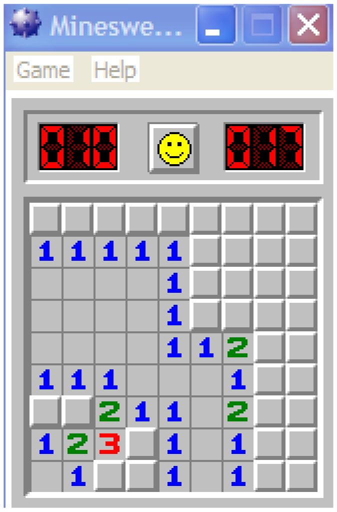
Let's consider a problem of aggregating redundant sensor data; this data will be "counting" or $\mathbb{Z}$-valued data, and thus amenable to the Euler calculus on $CF(X)$.

Consider a situation in which a finite # of TARGETS reside in a domain W. Sensors count the # of targets "nearby", but do not:

- measure distance; bearing;
- identify targets

(this in particular is power-intensive & hard.)

Perhaps you've seen such a sensor field before... oh, maybe not...

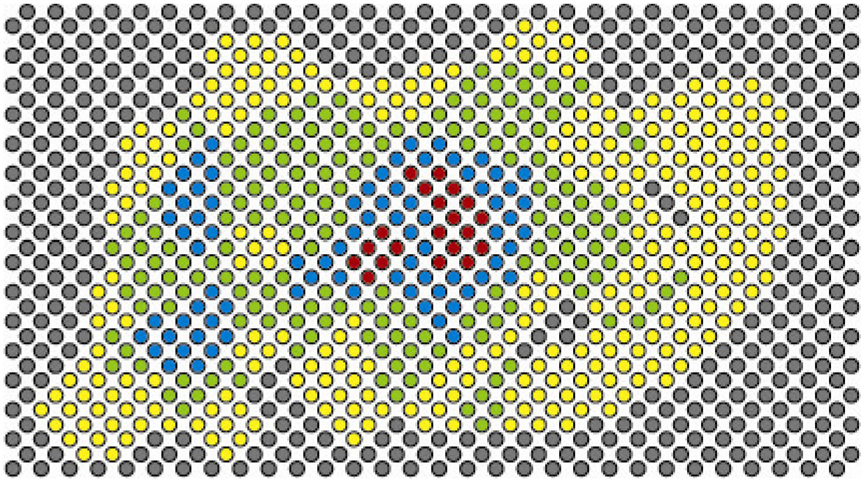


Much of the data is redundant: two sensors may detect the same target. How do you AGGREGATE REDUNDANCY?

This problem becomes much harder when the "counting sensors" are an ad hoc / non-localized network, and when the distance to the target is (unlike in minesweeper!) UNKNOWN.

● = 0
● = 1
● = 2
● = 3
● = 4

How many targets ?



A little formalism: lets model everything in terms of spaces and constructible functions (pretending that everything is as simplicial as need be for the moment).

W = TARGET SPACE; where a finite # of targets are

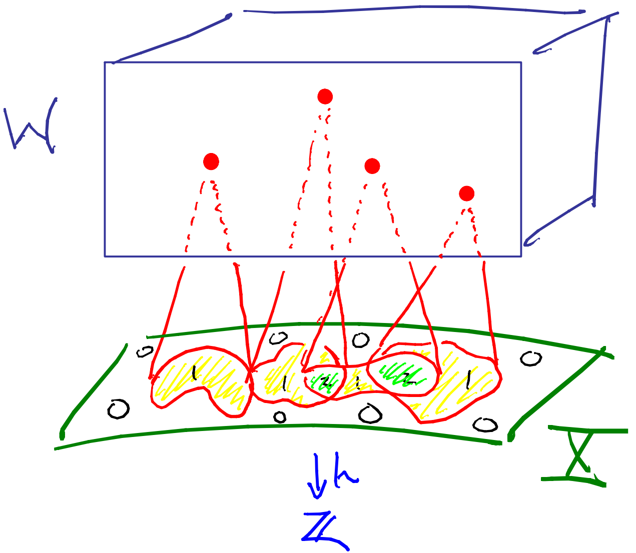
X = SENSOR SPACE; we assume such density among sensors as to be a top'l space

The sensors count targets according to "detection" or "impact", modeled as follows. Each target α has a SUPPORT $\mathcal{U}_\alpha \subset X$ -- the set of sensors in X which detect or sense the target.

The SENSOR FIELD is a counting function $h : X \to \mathbb{Z}$

$h(x)$ = # targets x "sees"

$= \#\{\alpha : x \in \mathcal{U}_\alpha\}$



One uses $d\chi$ to aggregate redundant data in h:

THEOREM: [B-G] If $h \in CF(X)$ counts targets whose supports have $\chi(\mathcal{U}_\alpha) = N \neq 0$ $\forall \alpha$, then

$$\boxed{\#\text{TARGETS} = \frac{1}{N}\int_X h\, d\chi}$$

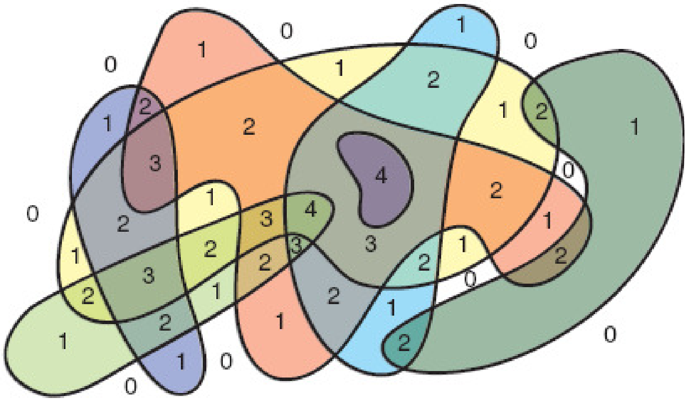
The important feature of this result is that you do not need to know target support sizes or geometry: the topology alone suffices.

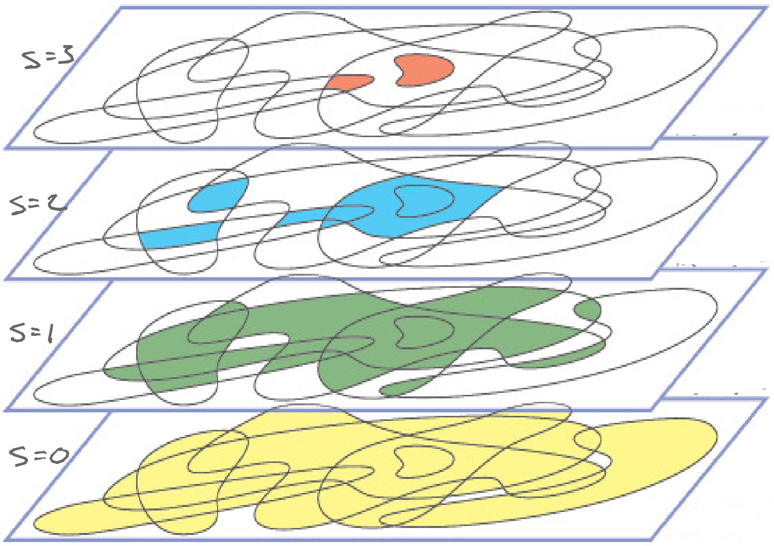
proof: (a trivial application of Euler calculus)

$$\int_{X} h\,d\chi = \int_{X}\Big(\sum_{\alpha}\mathbb{1}_{U_\alpha}\Big)\,d\chi = \sum_{\alpha}\Big(\int_{X}\mathbb{1}_{U_\alpha}\,d\chi\Big) = \sum_{\alpha}\chi(U_\alpha) = N\#\alpha$$

QED

EXAMPLE:





$$\chi(U_\alpha) = 1 \quad \forall\alpha$$

$$\int_{\mathbb{R}^2} h\,d\chi = \sum_{s=0}^{\infty}\chi\{h>s\} = -1+3+3+2 = 7$$ ☺

TAME TOPOLOGY, SHEAVES, & $d\chi$

Let us step back a moment and clean up some details. As stated, $CF(X)$ consists of integer-valued labelling of the cells of a fixed finite cell structure on X. That is too restrictive. Instead of giving into the temptation of saying, "it all works out in the end," let's be rigorous in creating a fuller CF.

We need to have all "measurable" sets/functions for $d\chi$ have some finiteness properties if we want to have χ defined. No Cantor sets, etc! We could restrict attention to some class of "tame" sets guaranteed to have good finiteness properties (e.g., semi-algebraic sets). A more "platform-independent" option is to work with an O-MINIMAL STRUCTURE: see the wonderful book by Van den Dries, "Tame Topology & O-Minimal Structures."

An o-minimal structure is a collection $\mathcal{O} = \{\mathcal{O}_n\}_{n=1}^{\infty}$, where each $\mathcal{O}_n$ is a Boolean algebra (w.r.t. $\cup$, $\cap$, and complements) of subsets of $\mathbb{R}^n$, satisfying axioms:

- PRODUCTS : $A \in \mathcal{O}_n$, $B \in \mathcal{O}_m \Longrightarrow A \times B \in \mathcal{O}_{m+n}$
- PROJECTIONS : $A \in \mathcal{O}_n \Longrightarrow p(A) \in \mathcal{O}_{n-1}$, $p : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$
- FINITENESS : $\mathcal{O}_1$ consists of all FINITE unions of points and intervals

(fine print: some definitions require $\mathcal{O}$ to contain all algebraic curves...)

Elements of $\mathcal{O}$ are called TAME or DEFINABLE sets.

Examples... include semi-linear sets; semi-algebraic sets; subanalytic sets, & more (the "more" is a deep result...).

A "tame" or "definable" map between tame sets is $f : X \to Y$ with the graph of f, $\Gamma_f \subset X \times Y$ a tame set. Of course! One says that X, Y are definably equivalent if $\exists$ an invertible tame map $f : X \to Y$. CAVEAT: Definable maps may not be continuous!

THEOREM : ("TRIANGULATION THEOREM")

All tame sets are definably equivalent to a finite disjoint union of open simplices: $X \cong \coprod_\alpha \sigma_\alpha$

COR: All tame sets have a well-defined Euler characteristic via $\chi(X) = \sum_\alpha (-1)^{\dim \sigma_\alpha}$.

This allows us to define, for a fixed o-minimal structure and $X \subset \mathbb{R}^n$

$$CF(X) = \{ h : X \to \mathbb{Z} \; : \; h^{-1}(c) \in \mathcal{O} \; \forall c \in \mathbb{Z} \}$$

(I will implicitly assume compact supports for simplicity)

Given the triangulation theorem, the Euler integral $\int_X \cdot \, d\chi : CF(X) \to \mathbb{Z}$ is well-defined. One thus thinks of $\mathcal{O}$ as a "σ-algebra" of measurable sets, and CF the set of χ-integrable functions.

Let's bump it up a notch: $CF(X)$ is a SHEAF where to each $U \subset X$ open we assign $CF(U) = \{ h : U \to \mathbb{Z} \; ; \; h^{-1}(c) \in \mathcal{O} \; \forall c \}$. One notes that the restriction & gluing properties hold, and voilà.

"So what?" the heckler says... In defense of formalism, sheaf theory has an extensive, deep history, resulting in the recognition of certain universal patterns. In particular, there are canonical operations one can associate to sheaves [Grothendieck]. One such operation allows one to "push" a sheaf over X to a sheaf over Y via $F: X \to Y$.

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, dash] & Y \arrow[d, dash] \\
CF_X \arrow[r, "F_*"'] & CF_Y
\end{tikzcd}$$

This is called the "pushforward" or "direct image" of F.

($\exists$ fine print... be careful of compactness, etc.)

Observation: [MacPherson; Kashiwara; 1970's] $\int - d\chi$ is the pushforward on constructible functions.

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, dash] & Y \arrow[r, "\text{SMASH!}"] \arrow[d, dash] & \{pt\} \arrow[d, dash] \\
CF_X \arrow[r, "F_*"'] \arrow[rr, bend right=40, "\int_X - d\chi"'] & CF_Y \arrow[r, "\int_Y - d\chi"'] & CF_{\{pt\}} \cong \mathbb{Z}
\end{tikzcd}$$

integer-valued functions over a point...

since the pushforward respects inclusions and gluings implicit in sheaves....

COR: by commutativity of pushforwards, for $F: X \to Y$,

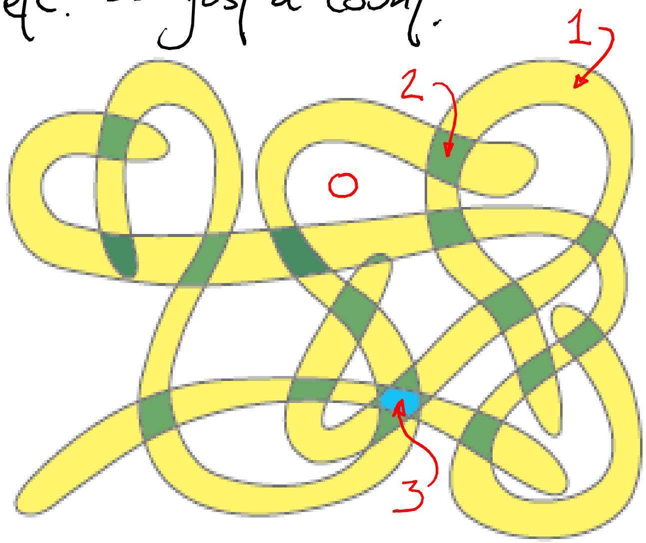
$$\boxed{\int_X h \, d\chi = \int_Y \left(\int_{F^{-1}(y)} h \, d\chi \right) d\chi}$$

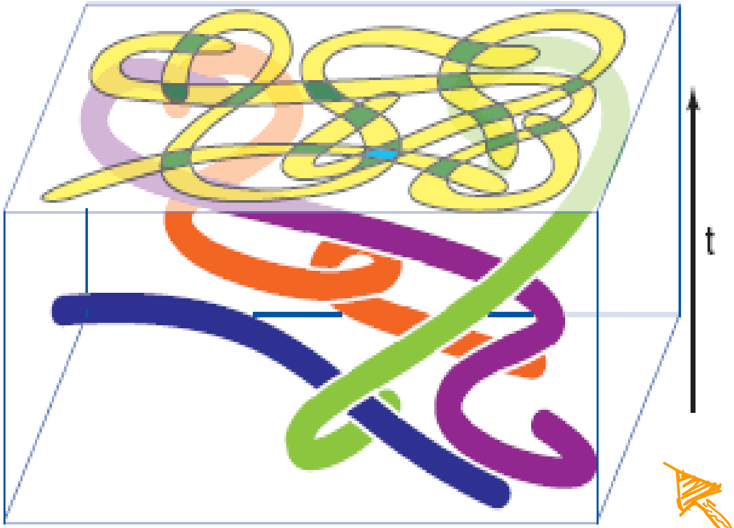
THE FUBINI THEOREM

To be fair, when you know that you want to prove this theorem, you could do it without sheaves (use o-minimal Hardt theorem + fact that $\chi(A \times B) = \chi(A)\chi(B)$.) But, sheaf formalism tells you what to prove: nice.

APPLICATIONS OF FUBINI

Consider a situation where acoustic/vibration sensors are embedded in asphalt, and each sensor registers a count of how many times a vehicle comes rumbling past, with no record of time, direction, etc. -- just a count.





If the vehicles don't make any "loops", then the sensor field is a union of $\mathbb{1}_{\mathcal{U}_\alpha}$, where $\mathcal{U}_\alpha$ = "trace" of vehicle α, and $\mathcal{U}_\alpha$ is contractible with $\chi(\mathcal{U}_\alpha) = 1 \ \forall \alpha$. Thus, $\#\alpha = \int_{\mathbb{R}^2} h \, d\chi$.

What happens if $\mathcal{U}_\alpha$ is not contractible in $\mathbb{R}^2$?

Note that it is contractible in space-time $\mathbb{R}^2 \times \mathbb{R}$, and that the sensor field on $\mathbb{R}^2$ is

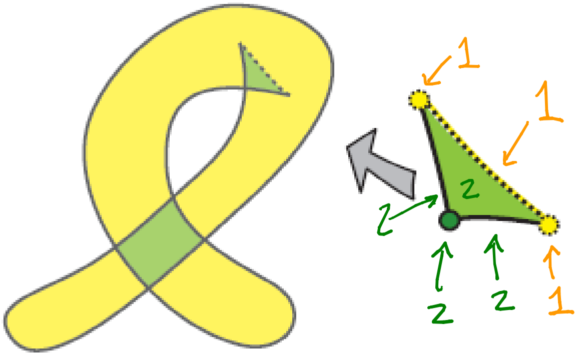
$$h(x) = \text{\# vehicles that drove near } x \text{ over time} = \int_{p^{-1}(x)} \sum_\alpha \mathbb{1}_{V_\alpha} \, d\chi$$

(where V_α = trace of vehicle α in $\mathbb{R}^2 \times \mathbb{R}$, and $p : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is projection)

$$\int_{\mathbb{R}^2} h \, d\chi = \int_{\mathbb{R}^2} \left(\int_{p^{-1}(x)} \sum_\alpha \mathbb{1}_{V_\alpha} \, d\chi \right) d\chi \underset{\text{FUBINI}}{=} \int_{\mathbb{R}^3} \sum_\alpha \mathbb{1}_{V_\alpha} \, d\chi = \sum_\alpha \chi(V_\alpha) = \#\alpha$$

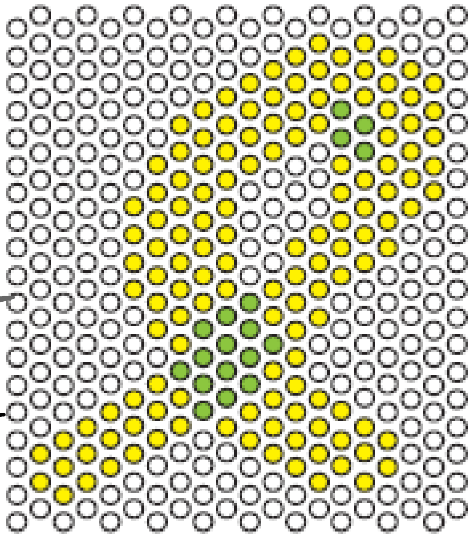
QED

This is efficacious when all intersections of fibers $\{x\}\times\mathbb{R}$ intersect the "tubes" $V_\alpha \subset \mathbb{R}^3$ in *compact* intervals. This can go wrong if a vehicle executes a *tight* turn & creates a cusp singularity.



The Fubini theorem, of course, does not fail. Rather, it becomes impossible to evaluate the integral numerically due to subtle changes on strata of higher codimension.

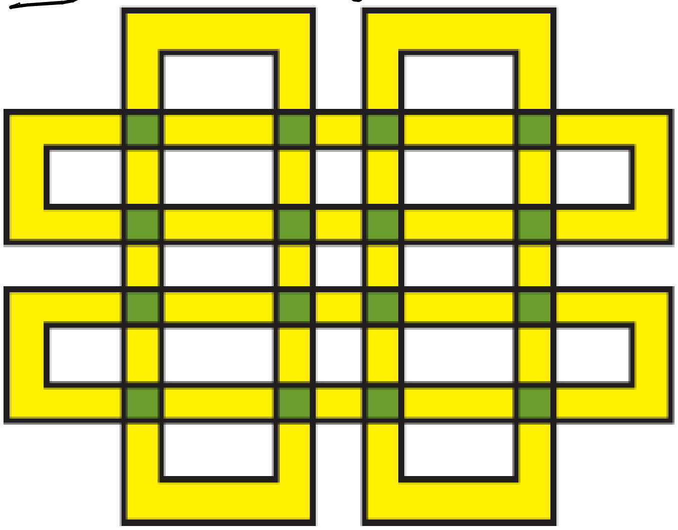
This brings up the subtle issue of numerical integration. The good news is that in *casting* solutions in integral form, we can use the perspectives/ideas of numerical analysis.



However, given a discrete sampling of the cusp, there seems to be no way to get the integral to evaluate to 1...

(Things work well with upper-semicontinuous *integrands*, which the cusp is *not*...)

On the other hand, we can use this to our *advantage*. Recall the condition $\chi(\mathcal{U}_\alpha) = N \neq 0$ in the target enumeration theorem. There is a problem:

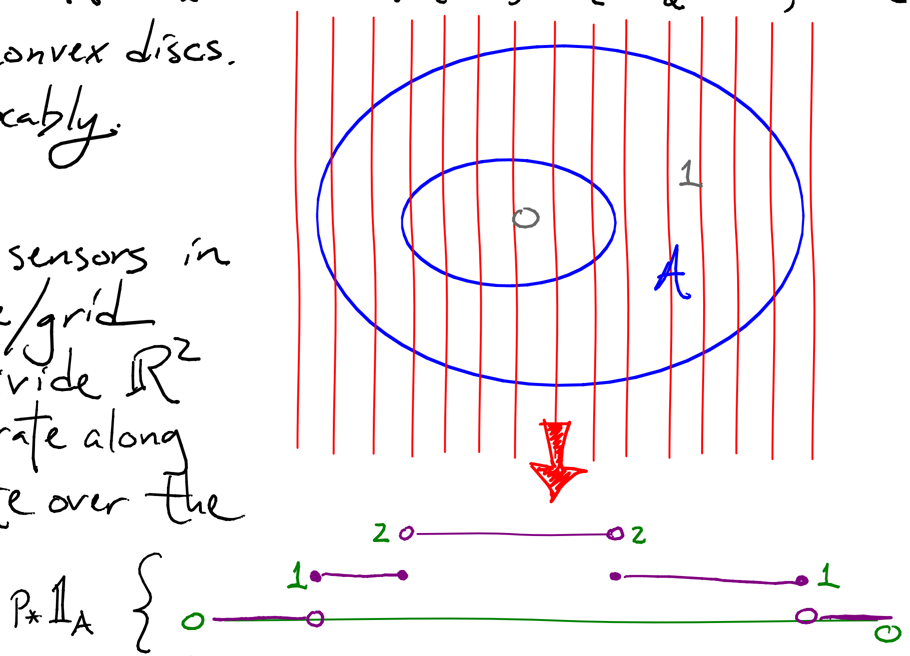


How many annuli comprise this integrand: 4? 2? 6? 8? 10? 12? COULD BE...

But observe that if you use "simple" annuli, the right answer is 4...

Let's assume $h = \sum_\alpha \mathbb{1}_{U_\alpha}$ with U_α annuli of the form $O_\alpha - \mathring{I}_\alpha$, where $O_\alpha \supset I_\alpha$ are compact convex discs. Note that $\int_{\mathbb{R}^2} h\,d\chi = 0$, irrevocably.

Assume further that we have sensors in $X = \mathbb{R}^2$ with a good lattice/grid structure, so that we can divide $\mathbb{R}^2$ into lines. Now, we integrate along the lines <u>first</u>; then integrate over the <u>projected image</u>...



$$\int_{\mathbb{R}} P_* \mathbb{1}_A \, d\chi = \sum_s s\,\chi\{s\text{-level set}\} = 1 \cdot \chi\{\bullet\!\!-\!\!- \quad -\!\!-\!\!\bullet\} + 2 \cdot \chi\{\circ\!\!-\!\!-\!\!\circ\}$$

$$= 1 \cdot 2 + 2 \cdot (-1) = 0$$ OF COURSE....

However, any finitely-sampled set of lines will return an upper semi-continuous approximation. The integral of the resulting projection would therefore be

$$\int_{\mathbb{R}} \overline{(P_* \mathbb{1}_A)} \, d\chi = 1 \cdot \chi\{\bullet\!\!-\!\!\circ \quad \circ\!\!-\!\!\bullet\} + 2 \cdot \chi\{\bullet\!\!-\!\!-\!\!\bullet\} = 2$$

The "numerical error" is a boon: linearity gives

$$\#\text{annuli, given } h = \sum_\alpha \mathbb{1}_{A_\alpha} \quad = \frac{1}{2} \int_{\mathbb{R}} \overline{P_* h} \, d\chi$$

upper semi-cont extension

induced projection along affine lines

TOWARD NUMERICAL ANALYSIS

Perhaps the biggest challenge is the development of numerical integration for $d\chi$. Since everything is $\mathbb{Z}$-valued, there's no such thing as a "small" error. Also a single point has Euler measure $= 1$ ... <u>every</u> point matters!

I close with one result for $\mathbb{R}^2$ which makes numerical integration relatively easy and provides guarantees.

THEOREM: [B-G] For $h \geq 0$ in $CF(\mathbb{R}^2)$ & upper semi-continuous

$$\int_{\mathbb{R}^2} h \, d\chi = \sum_{s=0}^{\infty} \beta_0\{h > s\} - \beta_0\{h \leq s\} + 1$$

0^{th}- BETTI NUMBER

proof: For $A \subset \mathbb{R}^2$ compact, nonempty:

$$\chi(A) = \sum_{k=0}^{\infty} (-1)^k \dim H_k A = \beta_0(A) - \beta_1(A)$$

$[\beta_i = \text{Betti \#'s}]$

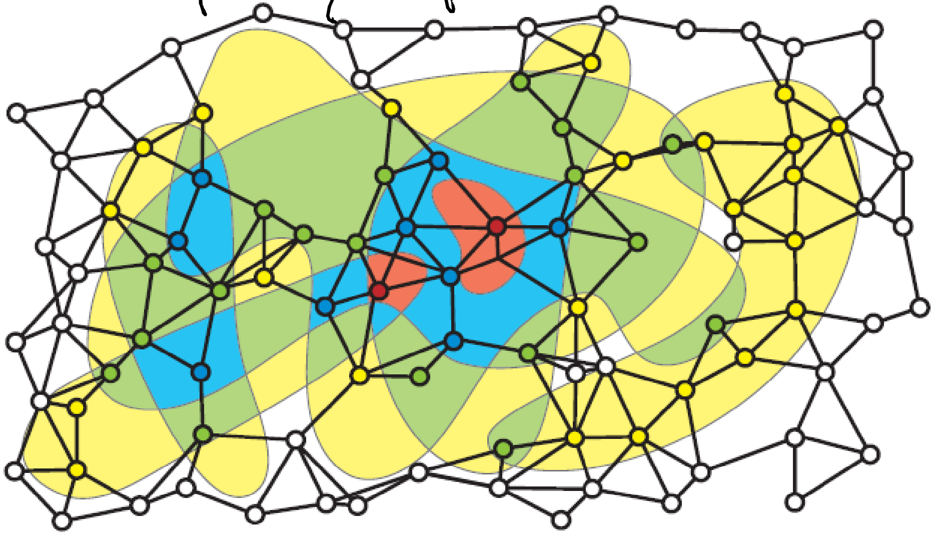
$$= \beta_0 A - (\beta_0(\mathbb{R}^2 - A) - 1)$$

via Alexander duality

Apply this to $\int h \, d\chi = \sum_{s=0}^{\infty} \chi\{h > s\}$

QED

This is a wonderful formula: it says that if you have a discrete sampling of $h \in CF(\mathbb{R}^2)$, then all you need to compute $\int h \, d\chi$ accurately is a means of counting connected components of excursion sets of h. This allows one to compute via a non-localized network of samples -- an ad hoc net. A triangulation is not requisite.



PART 7 : EULER INTEGRATION

Consider a discrete set of nodes $\mathcal{X} = \{x_\alpha\} \subset \mathbb{R}^n$ and a triangulation of $\mathbb{R}^n$ using $\mathcal{X}$ as vertices. If $h \in CF$ is known only on $\mathcal{X}$, how to estimate $\int h \, d\chi$?

NUMERICAL ANALYSIS

easy! take piecewise-linear extension

$$\Rightarrow \int h \, d\chi \cong \int h_{PL} \, d\chi$$

Not so fast... $\int \_ \, d\chi$ is defined only for data which takes values in a discrete set : $\mathbb{R}$-valued integrands = illegal.

$\mathbb{R}$-VALUED EULER INTEGRATION

(G.-C. ROTA & B. CHEN & ... extended $d\chi$ to $\mathbb{R}$-valued integrands, with the feature that $\int h \, d\chi = 0$ $\forall h$ continuous. That's a bug.)

Here is a simplistic attempt :

$$\text{CONSIDER...} \quad \lim_{n\to\infty} \overset{\text{renormalize by } n}{\frac{1}{n}} \int \overbrace{\lfloor nh \rfloor}^{\text{step function w/ } \mathbb{Z}\text{-values}} \, d\chi$$

"floor" = least integer function

LEMMA: For any $h \in \mathrm{Def}(X)$ = { definable $\mathbb{R}$-valued functions }

$$\boxed{\int_X h \lfloor d\chi \rfloor = \lim_{n\to\infty} \frac{1}{n} \int_X \lfloor nh \rfloor \, d\chi}$$

is well-defined

so is

$$\boxed{\int_X h \lceil d\chi \rceil = \lim_{n\to\infty} \frac{1}{n} \int_X \lceil nh \rceil \, d\chi}$$

but they not equal in general. Oops.

EXERCISE: Both reduce to $\int d\chi$ on $CF(\mathbb{X}) \subset DEF(\mathbb{X})$

EXERCISE: Show that, for h an affine function on an open simplex σ,

$$\int_\sigma h\lfloor d\chi\rfloor = (-1)^{\dim\sigma}\inf_\sigma h \quad ; \quad \int_\sigma h\lceil d\chi\rceil = (-1)^{\dim\sigma}\sup_\sigma h$$

(do it directly using step functions ... remember: σ is OPEN)

N.B.: By the triangulation theorem for o-minimal structures, any $h \in DEF(\mathbb{X})$ admits a definable equivalence from $\mathbb{X}$ to a disjoint union of open simplices, on which h is affine. Thus, after the above exercises, you will have proved $\int\lfloor d\chi\rfloor$, $\int\lceil d\chi\rceil$ converge.

What do these integrals measure? How do you compute them? Are they appropriate for numerical analysis?

PROPOSITIONS: For $h \in DEF(\mathbb{X})$

$$\left.\begin{aligned}\int_{\mathbb{X}} h\lfloor d\chi\rfloor &= \int_{s=0}^{\infty} \chi\{h \geq s\} - \chi\{h < -s\}\, ds\\ \int_{\mathbb{X}} h\lceil d\chi\rceil &= \int_{s=0}^{\infty} \chi\{h > s\} - \chi\{h \leq -s\}\, ds\end{aligned}\right\} \text{cf. } \sum_s \chi\{h>s\} - \chi\{h<-s\}$$

$$\left.\begin{aligned}\int_{\mathbb{X}} h\lfloor d\chi\rfloor &= \lim_{\varepsilon\to0^+} \frac{1}{\varepsilon}\int_{\mathbb{R}} s\,\chi\{s \leq h < s+\varepsilon\}\, ds\\ \int_{\mathbb{X}} h\lceil d\chi\rceil &= \lim_{\varepsilon\to0^+} \frac{1}{\varepsilon}\int_{\mathbb{R}} s\,\chi\{s < h \leq s+\varepsilon\}\, ds\end{aligned}\right\} \text{cf. } \sum_s s\,\chi\{h=s\}$$

$$\left.\begin{aligned}\int_{\mathbb{X}} h\lfloor d\chi\rfloor &= \int_{\mathbb{R}} s\,\chi\{h=s\}\lfloor d\chi\rfloor\\ \int_{\mathbb{X}} h\lceil d\chi\rceil &= \int_{\mathbb{R}} s\,\chi\{h=s\}\lceil d\chi\rceil\end{aligned}\right\} \text{generalization of Fubini to } \mathbb{X}\times\mathbb{R}\to\mathbb{R}$$

Here's the truly interesting feature of this theory:

THEOREM: [B-G] For $h \in \mathrm{DEF}^\circ(X)$, consider the STRATIFIED MORSE INDEX of the graph of h in $X \times \mathbb{R}$ relative to the Morse function $\pi : X \times \mathbb{R} \to \mathbb{R}$ (the graph axis):

$$(I^* h)(x) = \lim_{\varepsilon' \ll \varepsilon \to 0^+} \chi\left(\overline{B_\varepsilon(x)} \cap \{h < h(x) + \varepsilon'\}\right)$$

Then $I^* h \in CF(X)$ and

$$\boxed{\int_X h \lfloor d\chi \rfloor = \int_X I^* h \, d\chi}$$

Correspondingly, $I_* h$ is the stratified Morse index relative to the "flipped" function $X \times \mathbb{R} \to -\mathbb{R}$. (Replace $\{h < h(x) + \varepsilon'\}$ with $\{h > h(x) - \varepsilon'\}$ above). Then $\int_X h \lceil d\chi \rceil = \int_X I_* h \, d\chi$.

COR: For $h \in \mathrm{DEF}\, X$ a Morse function on a manifold, $\dim(X) = n$

$$\int_X h \lfloor d\chi \rfloor = \sum_{p \in \mathrm{CRIT}\, h} (-1)^{\overbrace{n-\mu(p)}} h(p)$$

MORSE "CO-INDEX"

$$\int_X h \lfloor d\chi \rfloor = \sum_{p \in \mathrm{CRIT}\, h} (-1)^{\mu(p)} h(p)$$

MORSE INDEX

} alternating sum of critical values

Thus, e.g., on a 1-d manifold, $\int h \lfloor d\chi \rfloor = \frac{1}{2}\,\mathrm{TOTVAR}(h)$.

We have lifted χ from spaces to definable functions by way of Morse theory.

NOW THE BAD NEWS...

COR: $$\int h\,\lfloor d\chi\rfloor = -\int -h\,\lceil d\chi\rceil$$ (POINCARE DUALITY)

EXERCISE: prove this directly

COR: $\int_X - \lfloor d\chi\rfloor : \mathrm{Def}\,X \to \mathbb{R}$ is not linear.
$\lceil d\chi\rceil$ too!

Nuts. But this is unavoidable - Morse data is non-linear.

EXAMPLE:

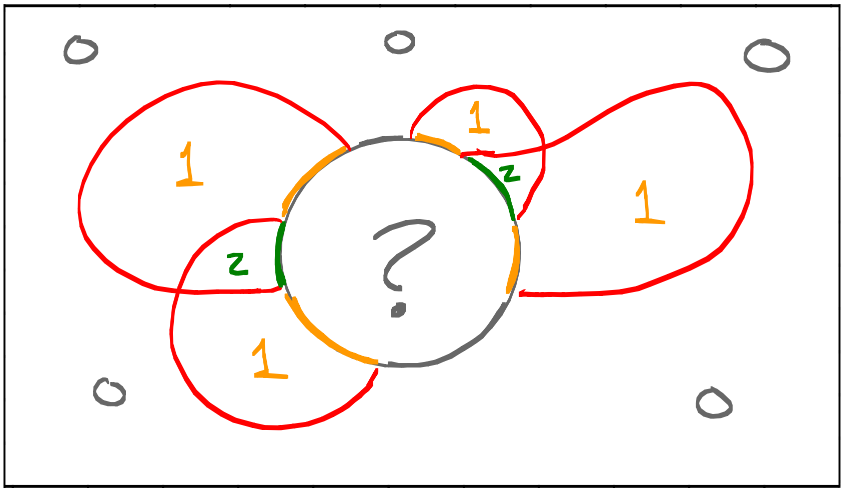
$$\int_{[0,1]} x\,\lfloor d\chi\rfloor + \int_{[0,1]} (1-x)\,\lfloor d\chi\rfloor = 1 + 1 = 2$$

$$\neq \int_{[0,1]} 1\,\lfloor d\chi\rfloor = 1$$

COR: · Fubini fails on $\mathrm{Def}\,X$
· $\int f*g\,\lfloor d\chi\rfloor \neq \int f\,\lfloor d\chi\rfloor \int g\,\lfloor d\chi\rfloor$ } in general

It's funny how hard it gets to prove stuff when the integral operator is not linear. Dang.

Nevertheless, I believe that $\lfloor d\chi\rfloor$ & $\lceil d\chi\rceil$ is the right theory for developing numerical analysis for χ, precisely since it allows for non-integer integrals. One enticing area of inquiry relates to expected values for $\int h\,d\chi$ in the presence of incomplete information. What follows is a suggestive cartoon...



If you have a "hole" D in the domain on which the integrand is unknown, can you estimate $\int h\,d\chi$ in a reasonable & principled manner?

CONJECTURE: Harmonic interpolations over holes yield "good" approximations to the expected integral.

[FINE PRINT: Assume $h = \sum \mathbb{1}_{\sigma_\alpha}$, σ_α homeomorphic to closed balls, none of which live entirely within D. Yes, I did not say what I mean by "good"...]

CARTOON: Consider D = disc in $\mathbb{R}^2$ with $h|_{\partial D}$ fixed in CF.

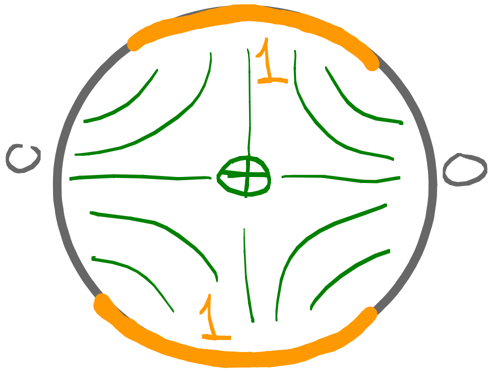
Take harmonic interpolation...

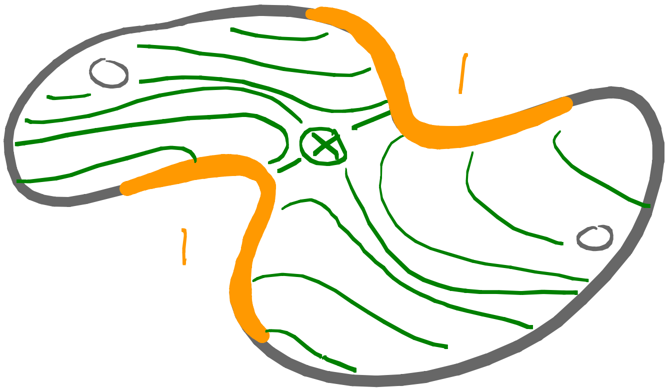
A harmonic interpolation will have maxima/minima on the ∂D, and (in the above, a single) saddle(s) on the interior. For a saddle of height $0 < c < 1$, the Morse formula for the integral over D yields

$$\int_{D^2} \tilde{h}\lfloor dX \rfloor = \sum_{\text{maxima}} + \sum_{\text{minima}} - \sum_{\text{saddles}}$$

$$1+1 \quad + \quad 0 \quad - C \qquad = 2 - C$$

This seems reasonable as an expected value for # targets





symmetric case

non-symmetric case

$C = \frac{1}{2}$, $\int \tilde{h}\lfloor dX \rfloor = \frac{3}{2}$

CAN'T TELL IF 1 or 2

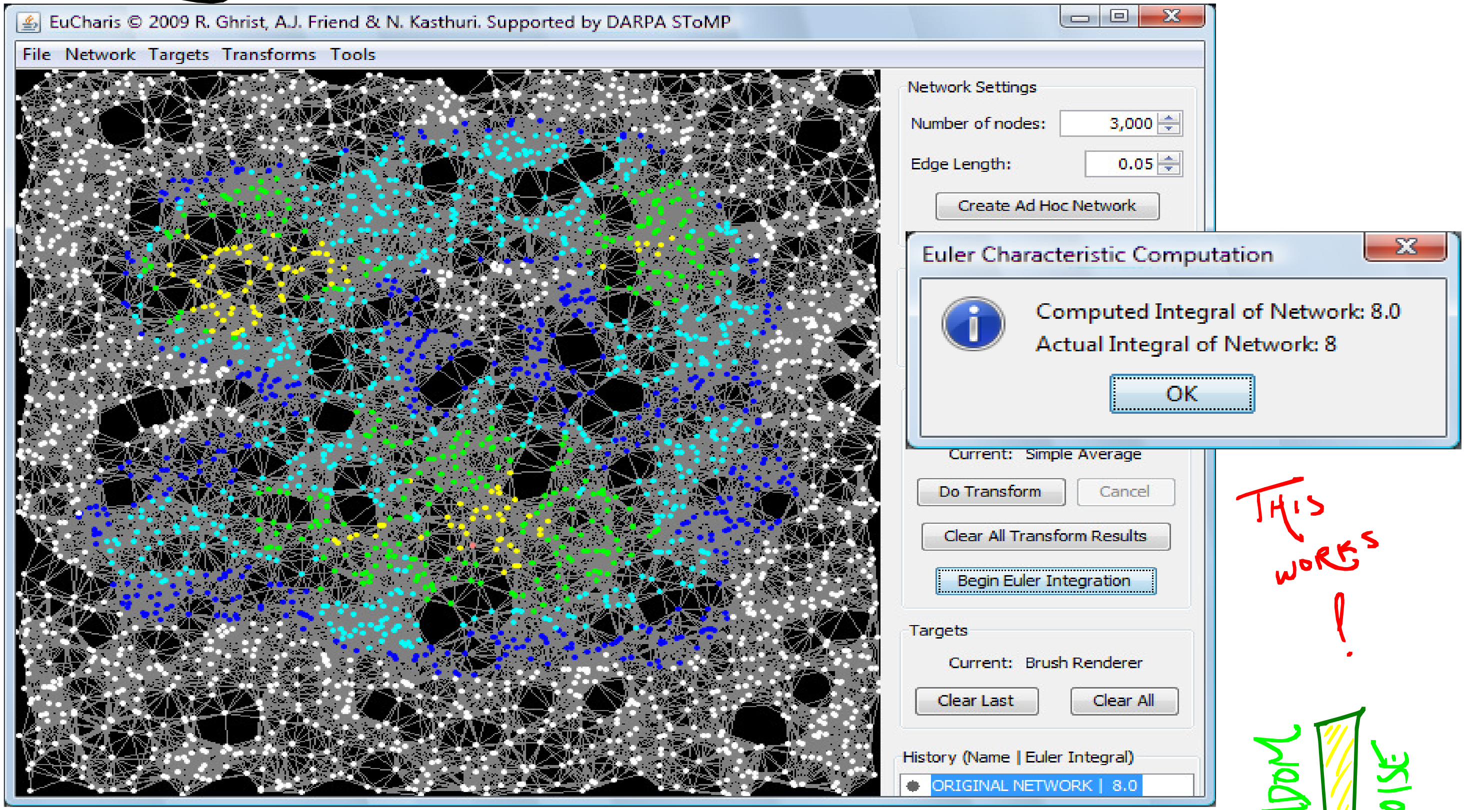
$0 \ll C < 1$

$1 \ll \int \tilde{h}\lfloor dX \rfloor < 2$

Better still, one can associate "confidence measures" to sensor counts which act as weights in a distributed averaging scheme (as a way of computing harmonic interpolants.

NOISE & NUMERICAL COMPUTATION

Noise mitigation is one distinct advantage of working with $\mathbb{R}$-valued integrands. First: noise kills $\int d\chi$: point-errors are highly perturbative, since $\chi(\{pt\}) = 1$!



EuCharis © 2009 R. Ghrist, A.J. Friend & N. Kasthuri. Supported by DARPA SToMP

Network Settings

Number of nodes: 3,000

Edge Length: 0.05

Euler Characteristic Computation

Computed Integral of Network: 8.0

Actual Integral of Network: 8

Current: Simple Average

Targets

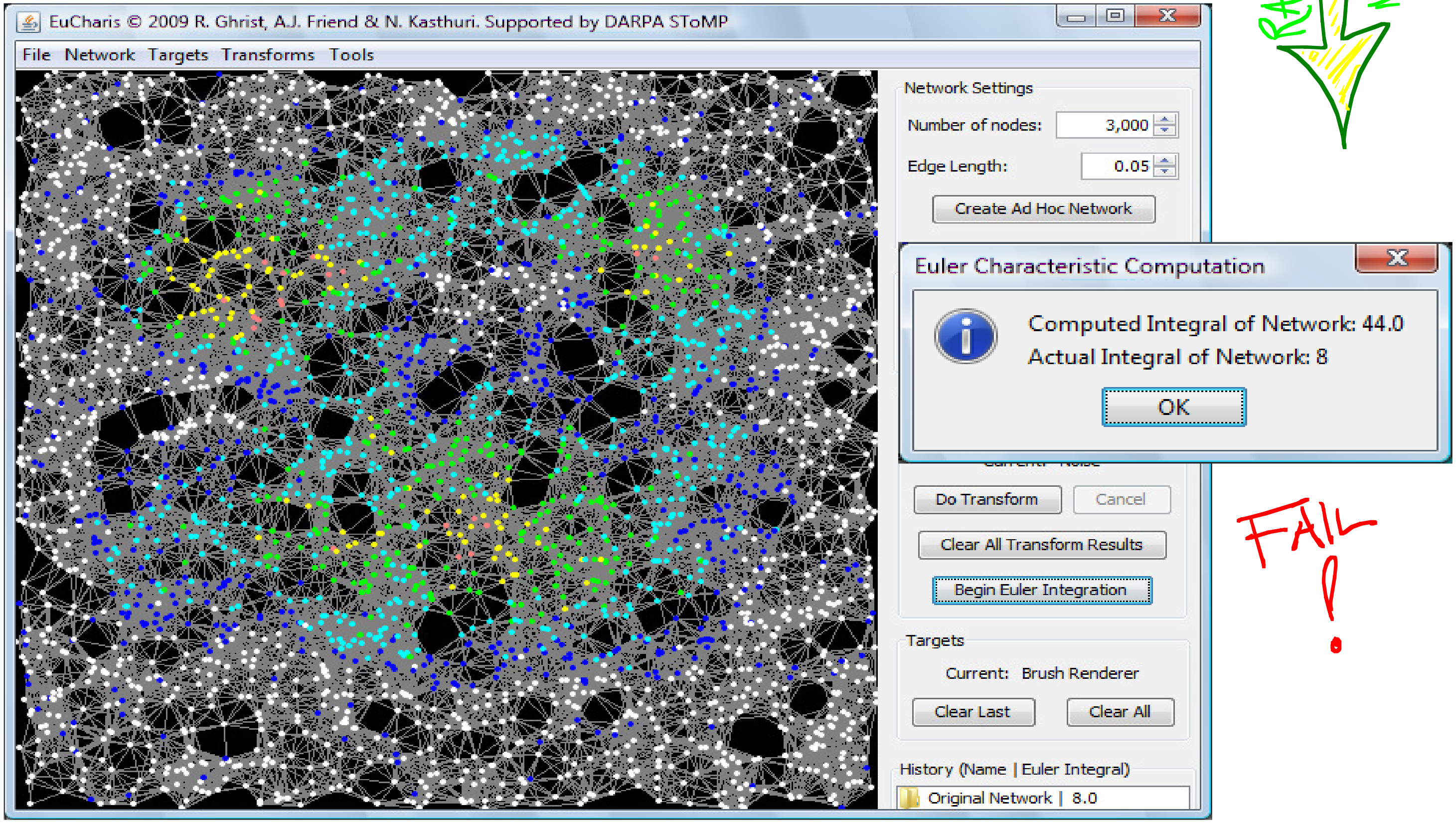
Current: Brush Renderer

History (Name | Euler Integral)

ORIGINAL NETWORK | 8.0

THIS WORKS!

RANDOM NOISE



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Network Settings

Number of nodes: 3,000

Edge Length: 0.05

Euler Characteristic Computation

Computed Integral of Network: 44.0

Actual Integral of Network: 8

Current: Noise

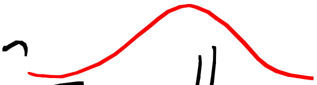
Targets

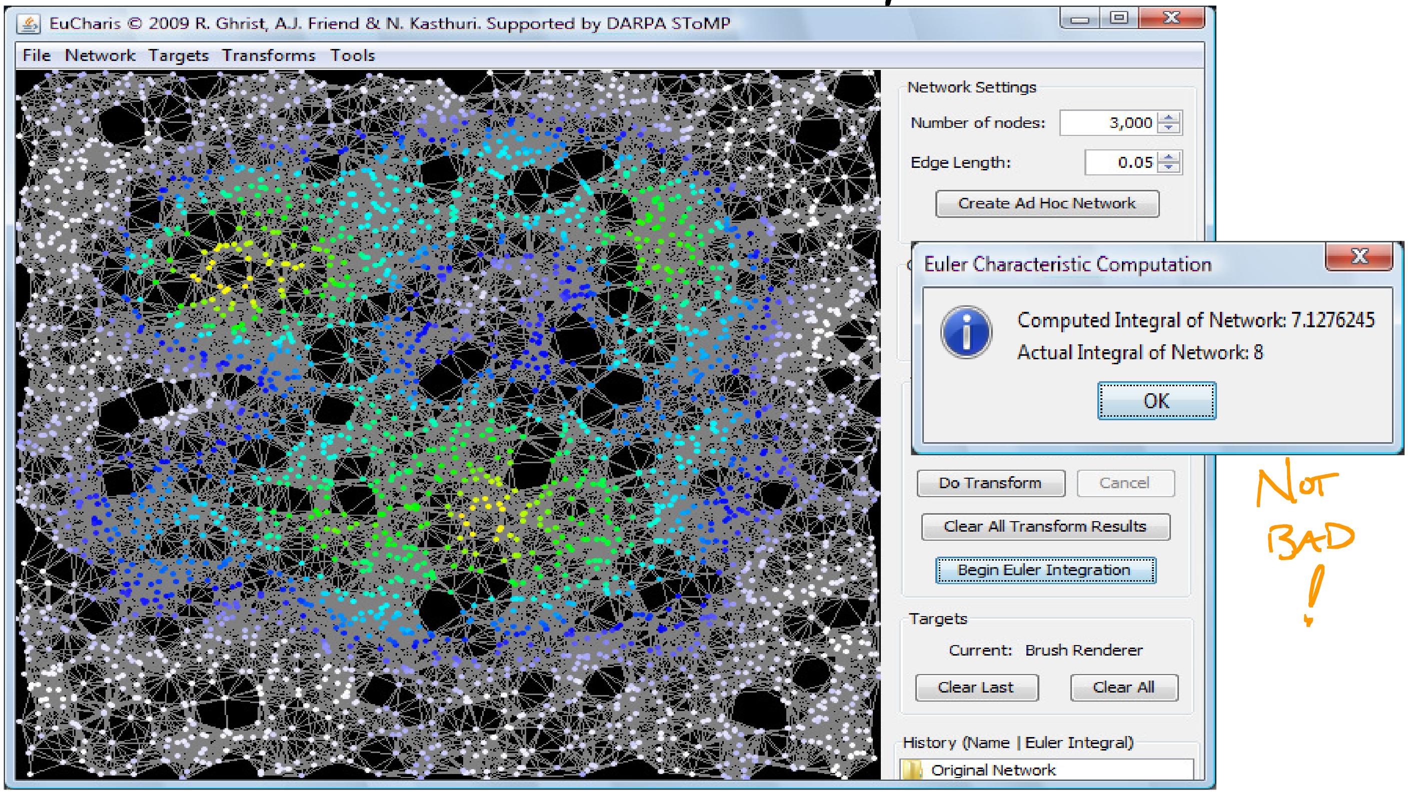
Current: Brush Renderer

History (Name | Euler Integral)

Original Network | 8.0

FAIL!

If we convolve the sensor field with a Gaussian  we pass from CF to DEF; but that's ok! Especially since the integral with respect to $\lfloor d\chi \rfloor$ works...



EuCharis © 2009 R. Ghrist, A.J. Friend & N. Kasthuri. Supported by DARPA SToMP

Number of nodes: 3,000

Edge Length: 0.05

Euler Characteristic Computation

Computed Integral of Network: 7.1276245

Actual Integral of Network: 8

Current: Brush Renderer

Not BAD!

There is much more to say about numerical computation of integrals, in $d\chi$, $\lfloor d\chi \rfloor$, & $\lceil d\chi \rceil$. The salient point is that we do not know much. Despite a long history for $\int d\chi$ in the sheaf-theoretic and integral-geometric literature, almost nobody seems to have considered the problem of numerical integration. There is much to be done.

PART 8: EULER TRANSFORMS

Although I suspect Euler integrals will find applications in a number of areas, it is perhaps signal/radar processing that is the most natural receptor.

INTEGRAL TRANSFORMS

Integration with respect to $d\chi$ really is integration, complete with Fubini, etc. As such, one should be able to perform all manner of integral transforms. Indeed, this was noted in a prescient paper of Schapira from 1985. It remains to build a full-fledged theory of TOPOLOGICAL SIGNAL PROCESSING. Some basic transforms include...

CONVOLUTION: The definition is straightforward. On $\mathbb{R}^n$

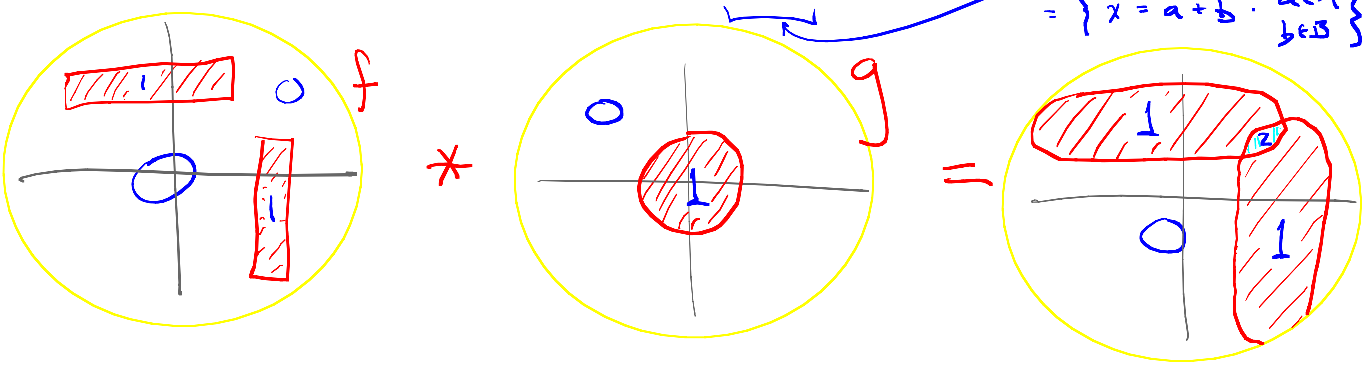
$$(f*g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, d\chi(y)$$

Convolution is commutative, associative, and satisfies

$$\int f*g \, d\chi = \int f \, d\chi \int g \, d\chi$$

It generalizes Minkowski sum for convex sets: if A, B are convex, then

$$\mathbb{1}_A * \mathbb{1}_B = \mathbb{1}_{\underbrace{A+B}_{\text{Minkowski sum } = \{x = a+b \,:\, a\in A,\ b\in B\}}}$$



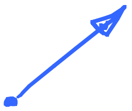
FOURIER TRANSFORM:

Given a "frequency vector" $\xi \in (\mathbb{R}^n)^*$ (the dual space) and $h : \mathbb{R}^n \to \mathbb{Z}$ constructible, let the Fourier Transform of h be

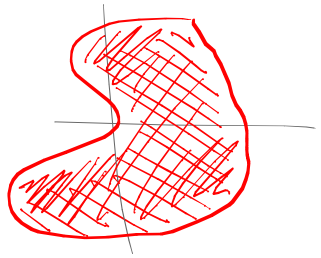
$$(\mathcal{F}h)(\xi) = \int_{s=-\infty}^{\infty} \Big(\int_{\underbrace{\xi^{-1}(s)}_{\text{hyperplane}}} h \, d\chi \Big) \underbrace{ds}_{\text{Lebesgue integral in } s}$$

This is best interpreted in terms of integrating over "isospectral" sets $\{\xi \cdot x = s\}$

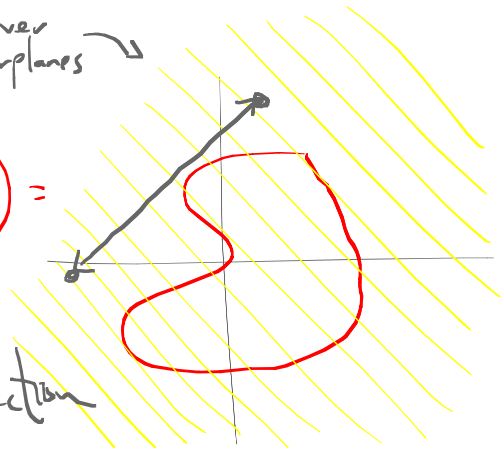
$\xi =$



$A =$



$(\mathcal{F}\mathbb{1}_A)(\xi) =$ "width" of A in ξ direction



This is a "global" version of the MICROLOCAL Fourier transform in sheaf theory

---

BESSEL TRANSFORM:

In this transform, the isospectral sets are concentric spheres.

$$(\mathbb{B}h)(x) = \int_{s=0}^{\infty} \Big(\int_{\partial B_s(x)} h \, d\chi \Big) ds \longleftarrow \text{sphere of radius } s \text{ about } x$$

It's clear that as $\|x\| \to \infty$, the Bessel transform converges to the Fourier transform. For $x \in \mathbb{R}^n - \{0\}$

$$\lim_{c \to +\infty} (\mathbb{B}h)(cx) = (\mathcal{F}h)\Big(\frac{x}{\|x\|}\Big) \qquad x \neq 0$$

I ♡ CARTOON PROOFS

INDEX THEOREM FOR $\mathbb{B}$, $\mathcal{F}$

If A is a closed ball in $\mathbb{R}^n$, then $(\mathcal{F}\mathbb{1}_A)(\xi)$ seems to give the "width" of A while $(\mathbb{B}\mathbb{1}_A)(x)$ is, for $x\in A$, related to the distance from x to the center of A. An Index Theorem for $\mathbb{B}$ & $\mathcal{F}$ makes this intuition precise.

THEOREM: Let $A\subset\mathbb{R}^{2n}$ be the closure of an open tame set. Then the Fourier and Bessel transforms of $\mathbb{1}_A$ are computed via the following integrals on ∂A:

$\xi:\mathbb{R}^{2n}\to\mathbb{R}$

$$\boxed{\begin{aligned}(\mathcal{F}\mathbb{1}_A)(\xi) &= \int_{\partial A}\xi\,\lfloor d\chi\rfloor = \int_{\mathrm{CRIT}\,\xi|_{\partial A}}\xi\, I^*d\chi\\ (\mathbb{B}\mathbb{1}_A)(x) &= \int_{\partial A}d_x\,\lfloor d\chi\rfloor = \int_{\mathrm{CRIT}\,d_x|_{\partial A}}d_x\, I^*d\chi\end{aligned}}$$

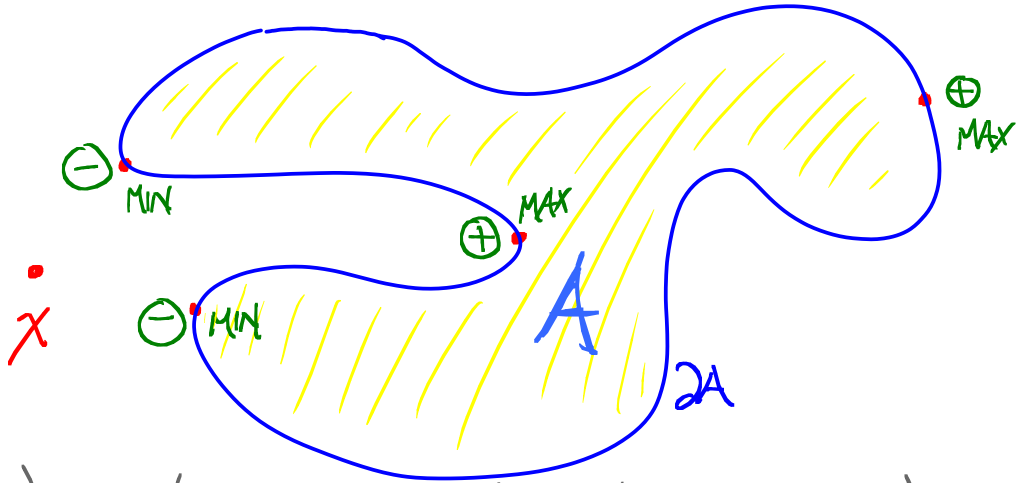
distance-to-x

Note the appearance of $\lfloor d\chi\rfloor$. This is a manifestation of Stokes. For odd-dimensional domains, one has to partition ∂A into regions ∂A^+ ∂A^- based on how the normal to ∂A points with respect to the point x or covector ξ:

$$(\mathcal{F}\mathbb{1}_A)(\xi) = \int_{\partial A^+}\xi\,\lfloor d\chi\rfloor - \int_{\partial A^-}\xi\,\lceil d\chi\rceil$$

$$(\mathbb{B}\mathbb{1}_A)(x) = \int_{\partial A^+}d_x\,\lfloor d\chi\rfloor - \int_{\partial A^-}d_x\,\lceil d\chi\rceil$$

EXAMPLE: Consider $\mathbb{1}_A$ where $A \subset \mathbb{R}^2$ is...



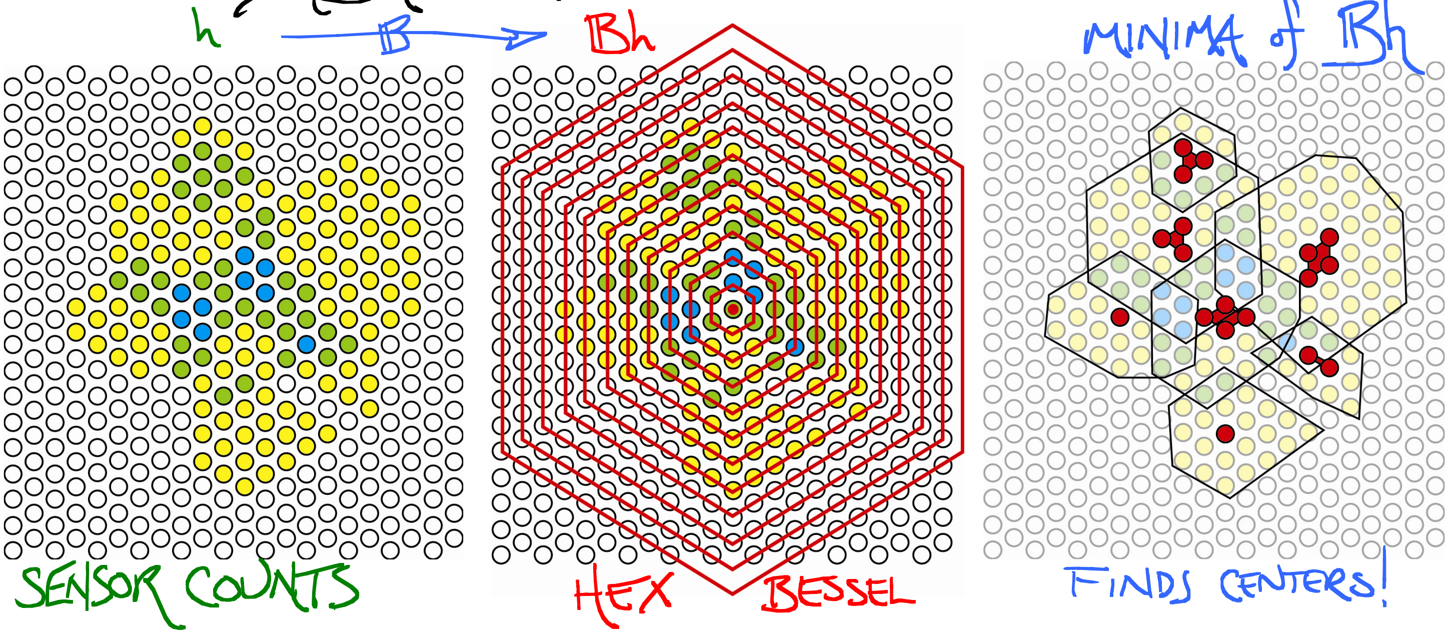
$$(\mathbb{B}\mathbb{1}_A)(x) = (\text{SUM OF MAXIMA}) - (\text{SUM OF MINIMA})$$

The index theorem helps with numerical computation by giving closed-form expressions for $\mathbb{B}\mathbb{1}_A$.

COR: For $A = B_p(r) \subset \mathbb{R}^{2n}$, the Bessel transform is

$$(\mathbb{B}\mathbb{1}_A)(x) = \max_{y\in\partial A}\|x-y\| - \min_{y\in\partial A}\|x-y\|$$

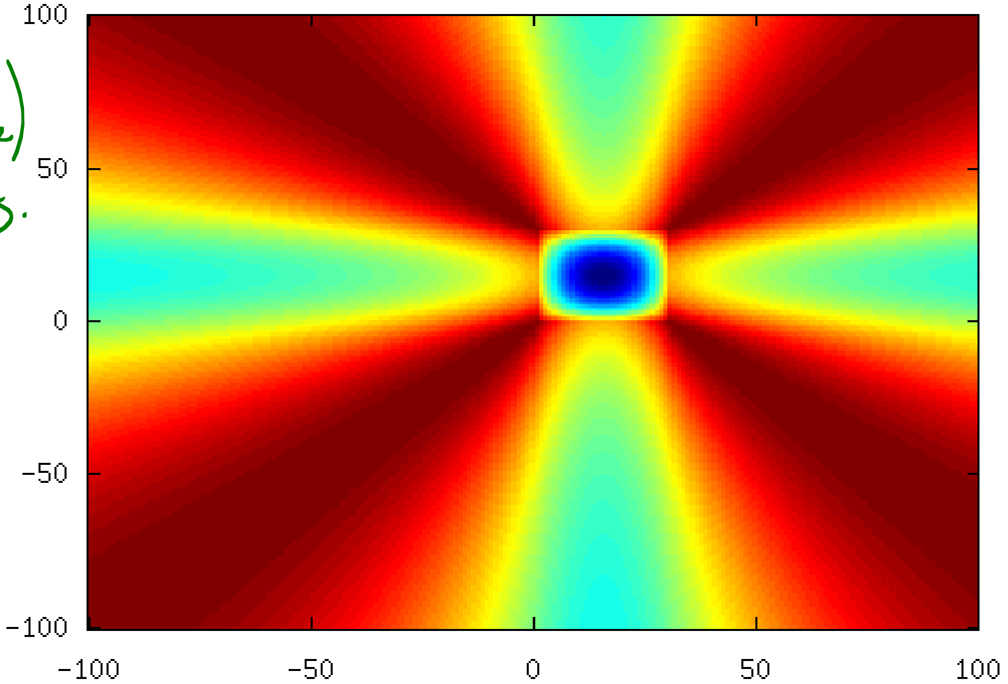
Thus, $\mathbb{B}\mathbb{1}_A$ vanishes at $x = p$ and is a non-decreasing function of distance-to-center. This motivates using $\mathbb{B}$ to LOCATE target centers based on sensor counts...



The Bessel transform can be used with any definable continuous norm: $\ell_2 \to$ circles; $\ell_\infty \to$ squares; see above for HEX-BESSEL. By tuning the "waveform" of the Bessel transform, it is possible to do SHAPE-DETECTION -- to discriminate targets of a certain shape from a cluttered field.

(see work-in-progress with Michael Robinson, Penn)

The principal challenges here involve SIDELOBES:

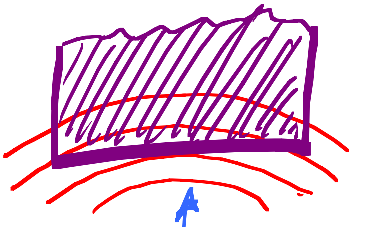


CIRCULAR (ℓ_2) BESSEL TRANS. of $\mathbb{1}_A$

A = SQUARE in $\mathbb{R}^2$

SIDELOBES

You can see why these occur: it's really a waveform mismatch. Interference of these sidelobes gives nonlinear interference problems.



TANGENCY = critical point of d_x

¿BAD NEWS? No, not really. The radar signals processing community has been dealing with this for years, often via hacks. Maybe X-calculus can borrow from (then repay) this community.

TOPOLOGICAL RADON TRANSFORMS

The classical Radon transform integrates over hyperplanes. There is a lovely topological version due to Schapira...

Let $S \subset W \times X$ (support of the transform)

The RADON TRANSFORM with respect to S is the operator...

$$R_S : CF(W) \to CF(X)$$

$$h \longmapsto (P_X)_*\big((P_W)^* h \, \mathbb{1}_S\big)$$

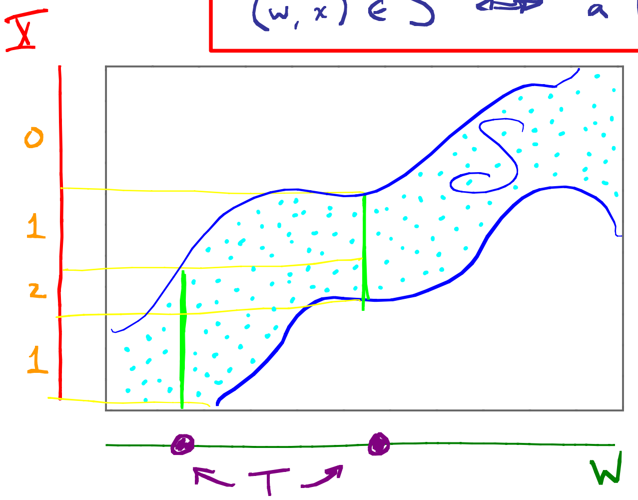
where $\left.\begin{array}{l} P_X : W \times X \to X \\ P_W : W \times X \to W \end{array}\right\}$ projections, and $_*$, * are pullbacks and pushforwards respectively.

EXAMPLE: Let W = workspace with $T \subset W$ a discrete set of targets. The SENSING RELATION for sensors in X is the subset $S \subset W \times X$ given by:

SENSING

$(w,x) \in S \iff$ a target at w is seen by a sensor at x

RELATION



Then $\mathbb{1}_T$ is transformed by R_S to the sensor counting function on X.

The target enumeration result of the previous Part 6 therefore has an interpretation on the level of integral transforms:

$$\int_X - \, d\chi \circ R_S = N \int_W - \, d\chi \quad \text{when} \quad \chi(S^{-1}(w)) = N.$$

RADON INVERSION

Schapira proved a remarkable formula about when a Radon transform R_S is invertible by another $R_{S'}$:

THEOREM [Schapira] Let $S \subset W \times X$ and $S' \subset X \times W$ be relations with fibers S_ω, S'_ω for $\omega \in W$ satisfying

$$\chi(S_\omega \cap S'_\omega) = \mu \quad \text{for all } \omega \in W$$
$$\chi(S_\omega \cap S'_v) = \lambda \quad \text{for all } \omega \neq v \text{ in } W$$

Then

$$(R_{S'} \circ R_S)h = (\mu - \lambda)h + \lambda\left(\int_W h \, d\chi\right)\mathbb{1}_W$$

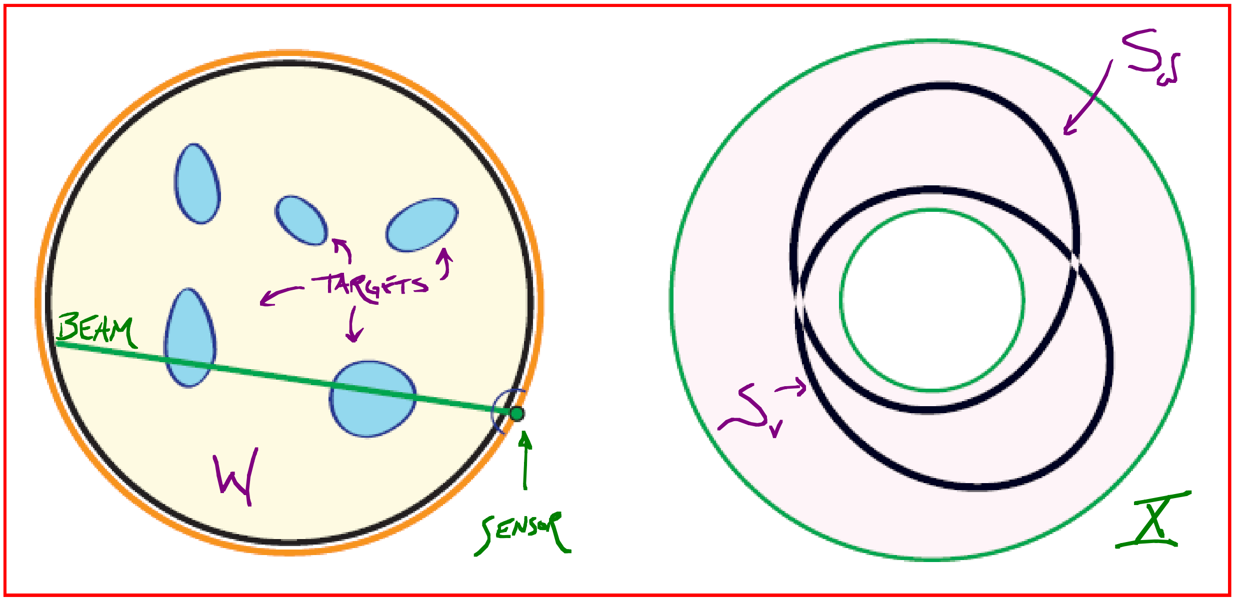
As a consequence, if $\lambda \neq \mu$, then one can recover h from $R_S h$ up to a constant.

This has significant implications in counting-sensor networks. The sensor counting function is $R_S \mathbb{1}_T$ for $T \subset W$ the targets. Given the right kernel S', apply a Radon transform via S' determines the exact location and shapes of the targets: all from anonymous counting.

EXAMPLE: Let $W = D^2$ a disc, with several convex target sets inside. The disc is surrounded by sensor beams which count the number of targets crossing the beam. The sensor space is

$$X = S^1 \times [0,\pi], \text{ and the sensor relation is } S \subset W \times X$$

location on ∂W ↗ ↖ angle of beam (does the beam at x hit ω?)



This is a self-dual sensor relation. The fibers $S_w = \{$all beams that intersect a target at $\omega\}$ is a topological circle. Thus,

$$\mu = \chi(S_w \cap S'_\omega) = \chi(S_\omega) = \chi(S^1) = 0$$

$$\lambda = \chi(S_\omega \cap S'_v) = \chi(2 \text{ points}) = 2$$

(2 distinct points in W are seen by exactly two beams)

Thus, $$R_{S'} \circ R_S 1_T = -2\, 1_T + 2 \,\#\text{TARGETS}$$

This is one of many examples of Radon inversion, which is one of many examples of Euler integration to sensor networks and signal processing...

EPILOGUE: QUO VADIS?

These notes cannot give a comprehensive account; I hope they are comprehensible and will inspire the interested reader to more & better work.

At this time, there are several other lines of research in progress on Topological Sensor Networks, not to mention the exciting activity in Topological Data Analysis.

For the latest updates, please see

www.math.upenn.edu/~ghrist

or contact the author.

Thank you!